FiNANCE FOR ENERGY
MARKET RESEARCH CENTRE

Nonzero-sum Stochastic Differential Games with Impulse Controls and Applications to Retail Energy Markets

Rene Aïd, Matteo Basei, Giorgia Callegaro,
Luciano Campi, Tiziano Vargiolu

Working Paper
RR-FiME-16-05

May 2016

Nonzero-sum stochastic differential games with impulse controls and applications to retail energy markets

René Aïd* Matteo Basei† Giorgia Callegaro†
Luciano Campi‡ Tiziano Vargiolu†

May 3, 2016

Abstract

We study the notion of Nash equilibrium in a general nonzero-sum impulse game for two players. The main mathematical contribution of the paper is a verification theorem which provides, under some regularity conditions, the system of quasi-variational inequalities identifying the value functions and the optimal strategies of the two players.

As an application, we propose a model for the competition among retailers in electricity markets. We first consider a simplified one-player setting, where we obtain a quasi-explicit expression for the value function and the optimal control. Then, we turn to the two-player case and we provide a detailed heuristic analysis of the retail impulse game, conducted along the lines of the verification theorem obtained in the general setting. This allows to identify reasonable candidates for the intervention and continuation regions of both players and their strategies.

Keywords: stochastic differential game, impulse control, Nash equilibrium, quasi-variational inequality, retail electricity market.

1 Introduction

In this article, we study a general two-player nonzero-sum stochastic differential game with impulse controls. More specifically, after setting the general framework, we investigate the notion of Nash equilibrium and identify the corresponding system of quasi-variational inequalities (QVIs). Moreover, we propose within this setting a model for competition in retail electricity markets and give a detailed analysis of its properties in both one-player and two-player cases.

Regarding general nonzero-sum impulse games, we consider a problem where two players can affect a continuous-time stochastic process X by discrete-time interventions which consist in shifting X to a new state (when none of the players intervenes, we assume X to diffuse according to a standard SDE). Each intervention corresponds to a cost for the intervening player and to a gain for the opponent. The strategy of player $i \in \{1,2\}$ is determined by a couple $\varphi_i = (A_i, \xi_i)$, where A_i is a fixed subset of $\mathbb{R}^d$ and ξ_i is a continuous function: player i intervenes if and only if the process X exits from A_i and, when this happens, she shifts the process from state x to state $\xi_i(x)$. Once the strategies $\varphi_i = (A_i, \xi_i)$, $i = 1, 2$, and a starting point x have been chosen, a couple of impulse controls $u_i(x; \varphi_1, \varphi_2) = \{(\tau_{i,k}, \delta_{i,k})\}_{1 \le k \le M_i}$ is uniquely defined: $\tau_{i,k}$ is the k-th intervention time of player i and $\delta_{i,k}$ is the corresponding impulse. Each player aims at maximizing her payoff, defined as follows: for every x belonging to some fixed subset $S \subseteq \mathbb{R}^n$ and every couple of strategies (φ_1, φ_2) we set

$$J^i(x; \varphi_1, \varphi_2) := \mathbb{E}_x \bigg[\int_0^{\tau_S} e^{-\rho_i s} f_i(X_s) ds + \sum_{1 \le k \le M_i \,:\, \tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} \phi_i\Big(X_{(\tau_{i,k})^-}, \delta_{i,k}\Big) + \sum_{1 \le k \le M_j \,:\, \tau_{j,k} < \tau_S} e^{-\rho_i \tau_{j,k}} \psi_i\Big(X_{(\tau_{j,k})^-}, \delta_{j,k}\Big) + e^{-\rho_i \tau_S} h_i\big(X_{(\tau_S)^-}\big) \mathbb{1}_{\{\tau_S < +\infty\}} \bigg], \quad (1.1)$$

*EDF Lab & Finance for Energy Markets Research Centre.
†Department of Mathematics, University of Padova.
‡Department of Statistics, London School of Economics and Political Science. E-mail: L.Campi@lse.ac.uk

where $i, j \in \{1, 2\}$, $i \neq j$ and τ_S is the exit time of X from S. The couple $(\varphi_1^*, \varphi_2^*)$ is a Nash equilibrium if $J^1(x; \varphi_1^*, \varphi_2^*) \geq J^1(x; \varphi_1, \varphi_2^*)$ and $J^2(x; \varphi_1^*, \varphi_2^*) \geq J^2(x; \varphi_1^*, \varphi_2)$, for every couple of strategies φ_1, φ_2.

The game just described is connected to the following system of QVIs, where $i, j \in \{1, 2\}$ with $j \neq i$ and $\mathcal{M}_i, \mathcal{H}_i$ are suitable intervention operators defined in Section 2.2:

$$\begin{aligned} & V_i = h_i, && \text{in } \partial S, \\ & \mathcal{M}_j V_j - V_j \leq 0, && \text{in } S, \\ & \mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\}, \\ & \max\big\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\big\} = 0, && \text{in } \{\mathcal{M}_j V_j - V_j < 0\}. \end{aligned} \tag{1.2}$$

The main mathematical result of this paper is the Verification Theorem 2.9: if two functions V_i, with $i \in \{1, 2\}$, are a solution to (1.2), have polynomial growth and satisfy the regularity condition

$$V_i \in C^2(\mathcal{D}_j \setminus \partial \mathcal{D}_i) \cap C^1(\mathcal{D}_j) \cap C(S), \tag{1.3}$$

where $j \in \{1, 2\}$ with $j \neq i$ and $\mathcal{D}_j = \{\mathcal{M}_j V_j - V_j < 0\}$, then they coincide with the value functions of the game and a characterization of the Nash strategy is possible. We stress here that even if stochastic differential games have been widely studied in the last decades, the case of nonzero-sum impulse games has never been considered, to the best of our knowledge, from a QVI perspective. Indeed, related former works only address zero-sum stopping games [10], the corresponding nonzero-sum problems [2] (with only two, very recent, explicit examples in [7] and [9]) and zero-sum impulse games [8]. We notice that the QVI formulated in [8] for zero-sum impulse games are obtained as a particular case of our framework.[1] Only the two papers [5, 6] deal with some nonzero-sum stochastic differential games with impulse controls using an approach based on backward stochastic differential equations and the maximum principle.

The second contribution of our paper is an application of the general setting to competition in retail electricity markets. Since electricity market deregulation started twenty years ago, electricity retail markets have been mainly studied from the point of view of the regulation: Joskow and Tirole [11] study the effect of the lack of hourly meters in households on retail competition, while von der Fehr and Hansen [13] analyse the switching process of consumers in the Norwegian market.

Here, we are interested in the rationale behind the price policy of electricity retailers for which an illustration is given in Figure 1.1 in the case of the UK electricity markets. Retailers tend to increase the household price when the wholesale price increases and to decrease the household price when the wholesale price decreases. Since retailers change their price nearly at the same moment (moments differ only by a few weeks), one can wonder if these changes are optimal or result in a non-competitive behaviour. This question is the reason why the British energy regulator launched an inquiry on energy retailers in 2014[2].

In this paper, we propose to model the competition between two electricity retailers within the general setting of nonzero-sum impulse games, where it is rational for the retailers to increase or decrease their retail prices at discrete moments depending on the evolution of the wholesale price and of the competitor's choice.

In our model, we assume that retailers buy the energy on a single wholesale market without distinguishing the purchases on the forward market from those on the spot market. Moreover, we suppose that retailers have the same sourcing cost (the price of power on the wholesale market) but may have different fixed cost (i.e. different amount of commercials). We also suppose, for tractability reason, that the structure cost of each retailer is quadratic in her respective market share. Finally, retailers sell electricity to their final consumers at a fixed price (possibly different for each retailer). Both retailers' objective is to maximize their total expected discounted profits.

[1] Notice that, within the subclass of zero-sum impulse games, the results in [8] are more general than ours as they hold true under much less regularity assumptions, requiring the use of viscosity solutions.

[2] The headline findings of the assessment were: *(...) Possible tacit co-ordination: The assessment has not found evidence of explicit collusion between suppliers. However, there is evidence of possible tacit coordination reflected in the timing and size of price announcements and new evidence that prices rise faster when costs rise than they reduce when costs fall. Although tacit coordination is not a breach of competition law, it reduces competition and worsens outcomes for consumers.* Published on Ofgem website on June 26th, 2014, at the address: www.ofgem.gov.uk/press-releases/ofgem-refers-energy-market-full-competition-investigation.

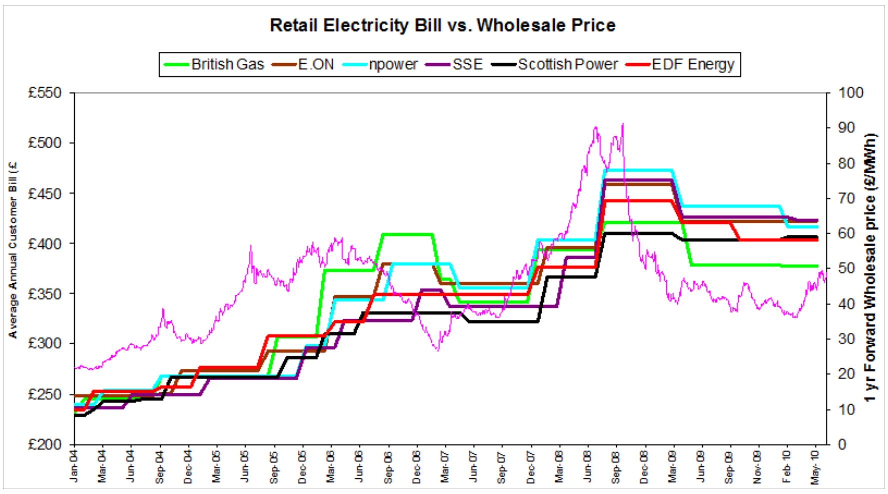

Figure 1.1: Retail electricity bill compared to wholesale price in the UK (source Ofgem). Retail electricity bill is given by the average bill of a household consuming on average 3.3 MWh per year.

Their instantaneous profits are composed of three parts: sale revenue (market share times retail price), sourcing cost (market share times wholesale market price), and structure cost. The wholesale market price evolution is assumed to follow an arithmetic Brownian motion for the sake of simplicity. This is also partly justified by the fact that negative spot prices for electricity are more and more frequent on various national European markets.

A last important feature of our model is that retailers cannot transfer continuously the variations of their sourcing cost to their clients. Instead, they can only change their prices in discrete time. Whenever a retailer changes her price, she faces a fixed cost. Indeed, each time a retailer decides to change her price, she has to advertise it and to inform all her actual clients about that change. Therefore, the problem naturally formulates as a nonzero-sum stochastic impulse control game.

Under the guidance provided by the verification theorem established in the general setting, we provide a detailed analysis of Nash equilibria of the retail impulse game. We focus on Nash equilibria where each retailer keeps her price constant as long as the spread between her price and the wholesale price belongs to some region in the plane (called non-intervention or continuation region). We conjecture that the non-intervention region of retailer i consists of a ribbon in the plane, which is delimited by two curves. When the difference between her retail price and the wholesale price hits the boundary of the non-intervention region, the optimal intervention policy consists in instantaneously changing the retail price in order to come back to the interior of this region. Within this class of Nash equilibria, we obtain a system of algebraic equations that the parameters characterizing the equilibrium have to satisfy.

The outline of the paper is the following. Section 2 rigorously formulates the general impulse stochastic games, defines Nash equilibria, provides the associated system of QVIs and the corresponding verification theorem. In Section 3.1 we consider the retail management problem in a simple one-player framework, while in Section 3.2 we study the two-player model. Finally, Section 4 concludes.

2 Nonzero-sum stochastic impulse games

In this section we consider a general class of two-player nonzero-sum stochastic differential games with impulse controls: after a rigorous formalization (see Section 2.1), we define a suitable differential problem for the value functions of such games (see Section 2.2) and prove a verification theorem (see Section 2.3).

2.1 Formulation of the problem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space whose filtration satisfies the usual conditions of right-continuity and $\mathbb{P}$-completeness. Let $\{W_t\}_{t\geq 0}$ be a k-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Brownian motion and let S be an open subset of $\mathbb{R}^d$. For every $t \geq 0$ and $y \in S$ we denote by $Y^{t,y}$ a solution to the problem

$$dY_s^{t,y} = b(Y_s^{t,y})ds + \sigma(Y_s^{t,y})dW_s, \qquad s \geq t, \tag{2.1}$$

with initial condition $Y_t^{t,y} = y$ and where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are given continuous functions. We will later provide precise conditions ensuring that the process $Y^{t,y}$ is well-defined.

We consider two players, that will be indexed by $i \in \{1, 2\}$. Equation (2.1) models the underlying process when none of the players intervenes; conversely, if player i intervenes with impulse $\delta \in Z_i$, the process is shifted from its current state x to a new state $\Gamma^i(x, \delta)$, where $\Gamma^i : \mathbb{R}^d \times Z_i \to S$ is a continuous function and Z_i is a fixed subset of $\mathbb{R}^{l_i}$, with $l_i \in \mathbb{N}$. Each intervention corresponds to a cost for the intervening player and to a gain for the opponent, both depending on the state x and the impulse δ.

The action of the players is modelled via discrete-time controls: an impulse control for player i is a sequence

$$u_i = \{(\tau_{i,k}, \delta_{i,k})\}_{1\leq k\leq M_i}, \tag{2.2}$$

where $M_i \in \mathbb{N} \cup \{\infty\}$ denotes the number of interventions of player i, $\{\tau_{i,k}\}_k$ are non-decreasing stopping times (the intervention times) and $\{\delta_{i,k}\}_k$ are Z_i-valued $\mathcal{F}_{\tau_{i,k}}$-measurable random variables (the corresponding impulses).

As usual with multiple-control games, we assume that the behaviour of the players, modelled by impulse controls, is driven by strategies, which are defined as follows.

Definition 2.1. *A strategy for player $i \in \{1, 2\}$ is a pair $\varphi_i = (A_i, \xi_i)$, where A_i is a fixed open subset of $\mathbb{R}^d$ and ξ_i is a continuous function from $\mathbb{R}^d$ to Z_i.*

Strategies determine the action of the players in the following sense. Once $\varphi_i = (A_i, \xi_i)$, $i \in \{1, 2\}$, and a starting point $x \in S$ have been chosen, a pair of impulse controls, which we denote by $u_i(x; \varphi_1, \varphi_2)$, is uniquely defined by the following procedure:

- player i intervenes if and only if the process exits from A_i, in which case the impulse is given by $\xi_i(y)$, where y is the state;
- if both the players want to act, player 1 has the priority;
- the game ends when the process exits from S. (2.3)

In the following definition we provide a rigorous formalization of the controls associated to a pair of strategies and the corresponding controlled process, which we denote by $X^{x;\varphi_1,\varphi_2}$. Moreover O denotes a generic subset of $\mathbb{R}^d$.

Definition 2.2. *Let $x \in S$ and let $\varphi_i = (A_i, \xi_i)$ be a strategy for player $i \in \{1, 2\}$. Let $\widetilde{\tau}_0 = 0, x_0 = x, \widetilde{X}^0 = Y^{\widetilde{\tau}_0,x_0}, \alpha_0^S = \infty$ and consider the conventions* $\inf \emptyset = \infty$ *and* $[\infty, \infty[= \emptyset$. *For every $k \in \mathbb{N}, k \geq 1$, we define, by induction,*

$$\begin{aligned}
&\alpha_k^O = \inf\{s > \widetilde{\tau}_{k-1} : \widetilde{X}_s^{k-1} \notin O\}, && \textit{[exit time from } O \subseteq \mathbb{R}^d\textit{]}\\
&\widetilde{\tau}_k = (\alpha_k^{A_1} \wedge \alpha_k^{A_2} \wedge \alpha_k^S)\mathbb{1}_{\{\widetilde{\tau}_{k-1}<\alpha_{k-1}^S\}} + \widetilde{\tau}_{k-1}\mathbb{1}_{\{\widetilde{\tau}_{k-1}=\alpha_{k-1}^S\}}, && \textit{[intervention time]}\\
&m_k = \mathbb{1}_{\{\alpha_k^{A_1}\leq\alpha_k^{A_2}\}} + 2\,\mathbb{1}_{\{\alpha_k^{A_2}<\alpha_k^{A_1}\}}, && \textit{[index of the player interv. at } \widetilde{\tau}_k\textit{]}\\
&\widetilde{\delta}_k = \xi_{m_k}\big(\widetilde{X}_{\widetilde{\tau}_k}^{k-1}\big)\mathbb{1}_{\{\widetilde{\tau}_k<\infty\}}, && \textit{[impulse]}\\
&x_k = \Gamma^{m_k}\big(\widetilde{X}_{\widetilde{\tau}_k}^{k-1}, \widetilde{\delta}_k\big)\mathbb{1}_{\{\widetilde{\tau}_k<\infty\}}, && \textit{[starting point for the next step]}\\
&\widetilde{X}^k = \widetilde{X}^{k-1}\mathbb{1}_{[0,\widetilde{\tau}_k[} + Y^{\widetilde{\tau}_k,x_k}\mathbb{1}_{[\widetilde{\tau}_k,\infty[}. && \textit{[contr. process up to the k-th interv.]}
\end{aligned}$$

Let $\bar{k} \in \mathbb{N} \cup \{\infty\}$ be the index of the last significant intervention, and let $M_i \in \mathbb{N} \cup \{\infty\}$ be the number of interventions of player i:

$$\bar{k} := \sup\big\{k \in \mathbb{N} : \mathbb{P}(\widetilde{\tau}_k = \alpha_k^S) < 1 \textit{ and } \mathbb{P}(\widetilde{\tau}_k = \infty) < 1\big\},$$
$$M_i := \textstyle\sum_{1\leq k\leq\bar{k}} \mathbb{1}_{\{m_k=i\}}(k).$$

For $i \in \{1,2\}$ and $1 \le k \le M_i$, let $\eta(i,k) = \min\{l \in \mathbb{N} : \sum_{1\le h \le l} \mathbb{1}_{\{m_h = i\}} = k\}$ (index of the k-th intervention of player i) and let

$$\tau_{i,k} := \widetilde{\tau}_{\eta(i,k)}, \qquad \delta_{i,k} := \widetilde{\delta}_{\eta(i,k)}. \tag{2.4}$$

Finally, the controls $u_i(x;\varphi_1,\varphi_2)$, $i \in \{1,2\}$, the controlled process $X^{x;\varphi_1,\varphi_2}$ and the exit time from S are defined by

$$\begin{aligned} u_i(x;\varphi_1,\varphi_2) &:= \{(\tau_{i,k},\delta_{i,k})\}_{1\le k\le M_i}, \\ X^{x;\varphi_1,\varphi_2} &:= \widetilde{X}^{\bar{k}}, \\ \tau_S^{x;\varphi_1,\varphi_2} &= \inf\{s>0 : X_s^{x;\varphi_1,\varphi_2} \notin S\}. \end{aligned}$$

To shorten the notations, we will simply write X and τ_S. Notice that player 1 has priority in the case of contemporary intervention (i.e., if $\alpha_k^{A_1} = \alpha_k^{A_2}$). In the following lemma we give a rigorous formulation to the properties outlined in (2.3).

Lemma 2.3. *Let $x \in S$ and let $\varphi_i = (A_i, \xi_i)$ be a strategy for player $i \in \{1,2\}$.*

- *The process X admits the following representation (with the convention $[\infty,\infty[=\emptyset$):*

$$X_s = \sum_{k=0}^{\bar{k}-1} Y_s^{\widetilde{\tau}_k, x_k} \mathbb{1}_{[\widetilde{\tau}_k,\widetilde{\tau}_{k+1}[}(s) + Y_s^{\widetilde{\tau}_{\bar{k}}, x_{\bar{k}}} \mathbb{1}_{[\widetilde{\tau}_{\bar{k}},\infty[}(s), \tag{2.5}$$

- *The process X is right-continuous. More precisely, X is continuous and satisfies Equation (2.1) in $[0,\infty[\setminus \{\tau_{i,k} : \tau_{i,k} < \infty\}$, whereas X is discontinuous in $\{\tau_{i,k} : \tau_{i,k} < \infty\}$, where we have*

$$X_{\tau_{i,k}} = \Gamma^i\big(X_{(\tau_{i,k})^-}, \delta_{i,k}\big), \qquad \delta_{i,k} = \xi_i\big(X_{(\tau_{i,k})^-}\big), \qquad X_{(\tau_{i,k})^-} \in \partial A_i. \tag{2.6}$$

- *The process X never exits from the set $A_1 \cap A_2$.*

Proof. We just prove the first property in (2.6), the other ones being immediate. Let $i \in \{1,2\}$, $1 \le k \le M_i$ with $\tau_{i,k} < \infty$ and set $\sigma = \eta(i,k)$, with η as in Definition 2.2. By (2.4), (2.5) and Definition 2.2, we have

$$\begin{aligned} X_{\tau_{i,k}} = X_{\widetilde{\tau}_\sigma} = Y_{\widetilde{\tau}_\sigma}^{\widetilde{\tau}_\sigma, x_\sigma} = x_\sigma = \Gamma^i\big(\widetilde{X}_{\widetilde{\tau}_\sigma}^{\sigma-1}, \widetilde{\delta}_\sigma\big) \\ = \Gamma^i\big(\widetilde{X}_{(\widetilde{\tau}_\sigma)^-}^{\sigma-1}, \widetilde{\delta}_\sigma\big) = \Gamma^i\big(X_{(\widetilde{\tau}_\sigma)^-}, \widetilde{\delta}_\sigma\big) = \Gamma^i\big(X_{(\tau_{i,k})^-}, \delta_{i,k}\big), \end{aligned}$$

where in the fifth equality we have used the continuity of the process $\widetilde{X}^{\sigma-1}$ in $[\widetilde{\tau}_{\sigma-1},\infty[$ and in the next-to-last equality we exploited the fact that $\widetilde{X}^{\sigma-1} \equiv X$ in $[0,\widetilde{\tau}_\sigma[$. □

Each player aims at maximizing her payoff, consisting of four discounted terms: a running payoff, the costs due to her interventions, the gains due to the opponent's interventions and a terminal payoff. More precisely, for each $i \in \{1,2\}$ we consider $\rho_i > 0$ (the discount rate) and continuous functions $f_i : \mathbb{R}^d \to \mathbb{R}$ (the running payoff), $h_i : \mathbb{R}^d \to \mathbb{R}$ (the terminal payoff) and $\phi_i : \mathbb{R}^d \times Z_i \to \mathbb{R}$, $\psi_i : \mathbb{R}^d \times Z_j \to \mathbb{R}$ (the interventions' costs and gains), where $j \in \{1,2\}$ with $j \ne i$. The payoff of player i is defined as follows.

Definition 2.4. *Let $x \in S$, let (φ_1,φ_2) be a pair of strategies and let τ_S be defined as in Definition 2.2. For each $i \in \{1,2\}$, provided that the right-hand side exists and is finite, we set*

$$\begin{aligned} J^i(x;\varphi_1,\varphi_2) := \mathbb{E}_x\bigg[\int_0^{\tau_S} e^{-\rho_i s} f_i(X_s)ds + \sum_{1\le k\le M_i : \tau_{i,k}<\tau_S} e^{-\rho_i \tau_{i,k}} \phi_i\Big(X_{(\tau_{i,k})^-}, \delta_{i,k}\Big) \\ + \sum_{1\le k \le M_j : \tau_{j,k}<\tau_S} e^{-\rho_i \tau_{j,k}} \psi_i\Big(X_{(\tau_{j,k})^-}, \delta_{j,k}\Big) + e^{-\rho_i\tau_S} h_i\big(X_{(\tau_S)^-}\big)\mathbb{1}_{\{\tau_S<+\infty\}}\bigg], \end{aligned} \tag{2.7}$$

where $j \in \{1,2\}$ with $j \ne i$ and $\{(\tau_{i,k},\delta_{i,k})\}_{1\le k\le M_i}$ is the impulse control of player i associated to the strategies φ_1, φ_2.

As usual in control theory, the subscript in the expectation denotes conditioning with respect to the available information (hence, it recalls the starting point). Notice that in the summations above we do not consider stopping times which equal τ_S (since the game ends in τ_S, any intervention is meaningless).

In order for J^i in (2.7) to be well defined, we now introduce the set of admissible strategies in $x \in S$.

Definition 2.5. *Let $x \in S$ and $\varphi_i = (A_i, \xi_i)$ be a strategy for player $i \in \{1,2\}$. We use the notations of Definition 2.2 and we say that the pair (φ_1, φ_2) is x-admissible if:*

1. *for every $k \in \mathbb{N} \cup \{0\}$, the process $Y^{\tilde{\tau}_k, x_k}$ exists and is uniquely defined;*
2. *for $i \in \{1,2\}$, the following random variables are in $L^1(\Omega)$:*
$$\begin{aligned} &\int_0^{\tau_S} e^{-\rho_i s} |f_i|(X_s) ds, &\qquad& e^{-\rho_i \tau_S} |h_i|(X_{(\tau_S)^-}), \\ &\sum_{\tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} |\phi_i|(X_{(\tau_{i,k})^-}, \delta_{i,k}), && \sum_{\tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} |\psi_i|(X_{(\tau_{i,k})^-}, \delta_{i,k}); \end{aligned} \tag{2.8}$$
3. *for each $k \in \mathbb{N}$, the random variable $\|X\|_\infty = \sup_{t \geq 0} |X_t|$ is in $L^k(\Omega)$:*
$$\mathbb{E}_x[\|X\|_\infty^k] < \infty; \tag{2.9}$$
4. *if $\tau_{i,k} = \tau_{i,k+1}$ for some $i \in \{1,2\}$ and $1 \leq k \leq M_i$, then $\tau_{i,k} = \tau_{i,k+1} = \tau_S$;*
5. *if there exists $\lim_{k \to +\infty} \tau_{i,k} =: \eta$ for some $i \in \{1,2\}$, then $\eta = \tau_S$.*

We denote by $\mathcal{A}_x$ the set of the x-admissible pairs.

Thanks to the first and the second conditions in Definition 2.5, the controls $u_i(x; \varphi_1, \varphi_2)$ and the payoffs $J^i(x; \varphi_1, \varphi_2)$ are well-defined. The third condition will be used in the proof of the Verification Theorem 2.9. As for the fourth and the fifth conditions, they prevent each player to exercise twice at the same time and to accumulate the interventions before τ_S.

We conclude the section with the definition of Nash equilibria and value functions for our problem.

Definition 2.6. *Given $x \in S$, we say that $(\varphi_1^*, \varphi_2^*) \in \mathcal{A}_x$ is a Nash equilibrium of the game if*
$$\begin{aligned} J^1(x; \varphi_1^*, \varphi_2^*) &\geq J^1(x; \varphi_1, \varphi_2^*), &\qquad& \forall \varphi_1 \text{ s.t. } (\varphi_1, \varphi_2^*) \in \mathcal{A}_x, \\ J^2(x; \varphi_1^*, \varphi_2^*) &\geq J^2(x; \varphi_1^*, \varphi_2), && \forall \varphi_2 \text{ s.t. } (\varphi_1^*, \varphi_2) \in \mathcal{A}_x. \end{aligned}$$
Finally, the value functions of the game are defined as follows: if $x \in S$ and a Nash equilibrium $(\varphi_1^*, \varphi_2^*) \in \mathcal{A}_x$ exists, we set for $i \in \{1,2\}$
$$V_i(x) := J^i(x; \varphi_1^*, \varphi_2^*).$$

2.2 The quasi-variational inequality problem

We now introduce the differential problem satisfied by the value functions of our games: this will be the key-point of the verification theorem in the next section.

Let us consider an impulse game as in Section 2.1. Assume that the corresponding value functions V_1, V_2 are defined for each $x \in S$ and that for $i \in \{1,2\}$ there exists a (unique) function δ_i from S to Z_i such that
$$\{\delta_i(x)\} = \underset{\delta \in Z_i}{\arg\max} \big\{ V_i(\Gamma^i(x,\delta)) + \phi_i(x,\delta) \big\}, \tag{2.10}$$
for each $x \in S$. We define the four intervention operators by
$$\begin{aligned} \mathcal{M}_i V_i(x) &= V_i\big(\Gamma^i(x, \delta_i(x))\big) + \phi_i\big(x, \delta_i(x)\big), \\ \mathcal{H}_i V_i(x) &= V_i\big(\Gamma^j(x, \delta_j(x))\big) + \psi_i\big(x, \delta_j(x)\big), \end{aligned} \tag{2.11}$$

for $x \in S$ and $i, j \in \{1, 2\}$, with $i \neq j$. Notice that $\mathcal{M}_i V_i = \max_\delta \{V_i(\Gamma^i(\cdot, \delta)) + \phi_i(\cdot, \delta)\}$.

The functions in (2.10) and (2.11) have an immediate and intuitive interpretation. Let x be the current state of the process; if player i (resp. player j) intervenes with impulse δ, the present value of the game for player i can be written as $V_i(\Gamma^i(x, \delta)) + \phi_i(x, \delta)$ (resp. $V_i(\Gamma^j(x, \delta)) + \psi_i(x, \delta)$): we have considered the value in the new state and the intervention cost (resp. gain). Hence, $\delta_i(x)$ in (2.10) is the impulse that player i would use in case she wants to intervene.

Similarly, $\mathcal{M}_i V_i(x)$ (resp. $\mathcal{H}_i V_i(x)$) represents the value of the game for player i when player i (resp. player $j \neq i$) takes the best immediate action and behaves optimally afterwards. Notice that it is not always optimal to intervene, so $\mathcal{M}_i V_i(x) \leq V_i(x)$, for each $x \in S$, and that player i should intervene (with impulse $\delta_i(x)$, as already seen) if and only if $\mathcal{M}_i V_i(x) = V_i(x)$. Hence, we have an heuristic formulation for the Nash equilibria, provided that an explicit expression for V_i is available. The verification theorem will give a rigorous proof to this heuristic argument. We now characterize the value functions V_i.

Assume $V_1, V_2 \in C^2(S)$ (weaker conditions will be given later) and define

$$\mathcal{A}V_i = b \cdot \nabla V_i + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^t D^2 V_i\right),$$

where b, σ are as in (2.1), σ^t denotes the transpose of σ and $\nabla V_i, D^2 V_i$ are the gradient and the Hessian matrix of V_i, respectively. We are interested in the following quasi-variational inequalities (QVIs) for V_1, V_2, where $i, j \in \{1, 2\}$ and $i \neq j$:

$$V_i = h_i, \qquad \text{in } \partial S, \tag{2.12a}$$
$$\mathcal{M}_j V_j - V_j \leq 0, \qquad \text{in } S, \tag{2.12b}$$
$$\mathcal{H}_i V_i - V_i = 0, \qquad \text{in } \{\mathcal{M}_j V_j - V_j = 0\}, \tag{2.12c}$$
$$\max\left\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\right\} = 0, \qquad \text{in } \{\mathcal{M}_j V_j - V_j < 0\}. \tag{2.12d}$$

Notice that there is a small abuse of notation in (2.12a), as V_i is not defined in ∂S, so that (2.12a) means $\lim_{y \to x} V_i(y) = h_i(x)$, for each $x \in \partial S$.

We now provide some intuition behind conditions (2.12a)-(2.12d). First of all, the terminal condition is obvious. Moreover, as we already noticed, (2.12b) is a standard condition in impulse control theory. For 2.12c, if player j intervenes (i.e., $\mathcal{M}_j V_j - V_j = 0$), by the definition of Nash equilibrium we expect that player i does not lose anything: this is equivalent to $\mathcal{H}_i V_i - V_i = 0$. On the contrary, if player j does not intervene (i.e., $\mathcal{M}_j V_j - V_j < 0$), then the problem for player i becomes a classical one-player impulse control one, hence V_i satisfies $\max\left\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\right\} = 0$. In short, the latter condition says that $\mathcal{A}V_i - \rho_i V_i + f_i \leq 0$, with equality in case of non-intervention (i.e., $\mathcal{M}_i V_i - V_i < 0$).

Remark 2.7. *The functions V_i can be unbounded. Indeed, this is the typical case when the penalties depend on the impulse: when the state diverges to infinity, one player has to pay a bigger and bigger cost to push the process back to the continuation region. This corresponds to a strictly decreasing value function (whereas the value of the game is strictly increasing for the competitor, who gains from the opponent's intervention). As a comparison, we recall that in one-player impulse problems the value function is usually bounded from above. Finally, we notice that the operator $\mathcal{A}V_i$ appears only in the region $\{\mathcal{M}_j V_j - V_j < 0\}$, so that V_i needs to be of class C^2 only in such region (indeed, this assumption can be slightly relaxed, as we will see). This represents a further difference with the one-player case, where the value function is asked to be twice differentiable almost everywhere in S, see [12, Thm. 6.2].*

2.2.1 A reduction to the zero-sum case

A verification theorem will be provided in the next section. Here, as a preliminary check on the problem we propose, we show that we are indeed generalizing the system of QVIs provided in [8], where the zero-sum case is considered. We show that, if we assume

$$\begin{gathered} f := f_1 = -f_2, \quad \phi := \phi_1 = -\psi_2, \quad \psi := \psi_1 = -\phi_2, \quad h := h_1 = -h_2, \\ Z := Z_1 = Z_2, \quad \Gamma := \Gamma^1 = \Gamma^2, \quad V := V_1 = -V_2, \end{gathered} \tag{2.13}$$

then the problem in (2.12) collapses into the one considered in [8]. To shorten the equations, we assume $\rho_1 = \rho_2 = 0$ (this makes sense since in [8] a finite-horizon problem is considered). First of all, we define

$$\widetilde{\mathcal{M}}V(x) := \sup_{\delta \in Z} \big\{ V(\Gamma(x,\delta)) + \phi(x,\delta) \big\},$$
$$\widetilde{\mathcal{H}}V(x) := \inf_{\delta \in Z} \big\{ V(\Gamma(x,\delta)) + \psi(x,\delta) \big\},$$

for each $x \in S$. It is easy to see that, under the conditions in (2.13), we have

$$\mathcal{M}_1 V_1 = \widetilde{\mathcal{M}}V, \qquad \mathcal{M}_2 V_2 = -\widetilde{\mathcal{H}}V, \qquad \mathcal{H}_1 V_1 = \widetilde{\mathcal{H}}V, \qquad \mathcal{H}_2 V_2 = -\widetilde{\mathcal{M}}V,$$

so that problem (2.12) writes

$$\begin{aligned}
&V = h, && \text{in } \partial S, && (2.14\text{a})\\
&\widetilde{\mathcal{M}}V \le V \le \widetilde{\mathcal{H}}V, && \text{in } S, && (2.14\text{b})\\
&\mathcal{A}V + f \le 0, && \text{in } \{V = \widetilde{\mathcal{M}}V\}, && (2.14\text{c})\\
&\mathcal{A}V + f = 0, && \text{in } \{\widetilde{\mathcal{M}}V < V < \widetilde{\mathcal{H}}V\}, && (2.14\text{d})\\
&\mathcal{A}V + f \ge 0, && \text{in } \{V = \widetilde{\mathcal{H}}V\}. && (2.14\text{e})
\end{aligned}$$

Simple computations, reported below, show that problem (2.14) is equivalent to

$$\begin{aligned}
&V = h, && \text{in } \partial S, && (2.15\text{a})\\
&\widetilde{\mathcal{M}}V - V \le 0, && \text{in } S, && (2.15\text{b})\\
&\min\{\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\}, \widetilde{\mathcal{H}}V - V\} = 0, && \text{in } S, && (2.15\text{c})
\end{aligned}$$

which is exactly the problem studied in [8], as anticipated. We conclude this section by proving the equivalence of (2.14) and (2.15).

Lemma 2.8. *Problems* (2.14) *and* (2.15) *are equivalent.*

Proof. Step 1. We prove that (2.14) implies (2.15). The only property to be proved is (2.15c). We consider three cases.

First, assume $V = \widetilde{\mathcal{M}}V$. Since $\mathcal{A}V + f \le 0$ and $\widetilde{\mathcal{M}}V - V = 0$, we have $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} = 0$, which implies (2.15c) since $\widetilde{\mathcal{H}}V - V \ge 0$. Then, assume $\widetilde{\mathcal{M}}V < V < \widetilde{\mathcal{H}}V$. Since $\mathcal{A}V + f = 0$ and $\widetilde{\mathcal{M}}V - V < 0$, we have $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} = 0$, which implies (2.15c) since $\widetilde{\mathcal{H}}V - V > 0$. Finally, assume $V = \widetilde{\mathcal{H}}V$. Since $\mathcal{A}V + f \ge 0$ and $\widetilde{\mathcal{M}}V - V \le 0$, we have $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} \ge 0$, which implies (2.15c) since $\widetilde{\mathcal{H}}V - V = 0$.

Step 2. We prove that (2.15) implies (2.14). The only properties to be proved are (2.14c), (2.14d) and (2.14e). We assume $\widetilde{\mathcal{M}}V < \widetilde{\mathcal{H}}V$ (the case $\widetilde{\mathcal{M}}V = \widetilde{\mathcal{H}}V$ being immediate) and consider three cases.

First, assume $V = \widetilde{\mathcal{M}}V$. Since $\widetilde{\mathcal{H}}V - V > 0$, from (2.15c) it follows that $\max\{\mathcal{A}V + f, 0\} = 0$, which implies $\mathcal{A}V + f \le 0$. Then, assume $\widetilde{\mathcal{M}}V < V < \widetilde{\mathcal{H}}V$. Since $\min\{\max\{\alpha, \beta\}, \gamma\} \in \{\alpha, \beta, \gamma\}$ for every $\alpha, \beta, \gamma \in \mathbb{R}$, and since $\widetilde{\mathcal{M}}V - V < 0 < \widetilde{\mathcal{H}}V - V$, from (2.15c) it follows that $\mathcal{A}V + f = 0$. Finally, assume $V = \widetilde{\mathcal{H}}V$. From (2.15c) it follows that $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} \ge 0$, which implies $\mathcal{A}V + f \ge 0$ since $\widetilde{\mathcal{M}}V - V < 0$. $\square$

2.3 A verification theorem

We provide here the main mathematical contribution of this paper, which is a verification theorem for the problem formalized in Section 2.1.

Theorem 2.9 (**Verification theorem**)**.** *Let all the notations and working assumptions in Section 2.1 be in force and let V_i be a function from S to $\mathbb{R}$, with $i \in \{1, 2\}$. Assume that* (2.10) *holds and set $\mathcal{D}_i := \{\mathcal{M}_i V_i - V_i < 0\}$, with $\mathcal{M}_i V_i$ as in* (2.11)*. Moreover, for $i \in \{1, 2\}$ assume that:*

- V_i *is a solution to* (2.12a)-(2.12d)*;*

- $V_i \in C^2(\mathcal{D}_j \setminus \partial\mathcal{D}_i) \cap C^1(\mathcal{D}_j) \cap C(S)$ *and it has polynomial growth;*

- $\partial\mathcal{D}_i$ *is a Lipschitz surface and* V_i *has locally bounded derivatives near* $\partial\mathcal{D}_i$*.*

Finally, let $x \in S$ *and assume that* $(\varphi_1^*, \varphi_2^*) \in \mathcal{A}_x$*, where*

$$\varphi_i^* = (\mathcal{D}_i, \delta_i),$$

with $i \in \{1,2\}$*, the set* $\mathcal{D}_i$ *is as above and the function* δ_i *is as in* (2.10)*. Then,*

$$(\varphi_1^*, \varphi_2^*) \text{ is a Nash equilibrium and } V_i(x) = J^i(x; \varphi_1^*, \varphi_2^*) \text{ for } i \in \{1,2\}.$$

Remark 2.10. *Basically, we are saying that the Nash strategy is characterized as follows: player i intervenes if and only if the controlled process exits from the region* $\{\mathcal{M}_i V_i - V_i < 0\}$ *(equivalently, if and only if* $\mathcal{M}_i V_i(x) = V_i(x)$*, where* x *is the current state). When this happens, his impulse is* $\delta_i(x)$*.*

Remark 2.11. *In the case of such (candidate) optimal strategies, we notice that the properties in Lemma 2.3 imply what follows (the notation is heavy, but it will be crucial to understand the proof of the theorem):*

$$(\mathcal{M}_1 V_1 - V_1)\big(X_s^{x;\varphi_1^*,\varphi_2}\big) < 0, \tag{2.16a}$$

$$(\mathcal{M}_2 V_2 - V_2)\big(X_s^{x;\varphi_1,\varphi_2^*}\big) < 0, \tag{2.16b}$$

$$\delta_{1,k}^{x;\varphi_1^*,\varphi_2} = \delta_1\left(X^{x;\varphi_1^*,\varphi_2}_{\left(\tau_{1,k}^{x;\varphi_1^*,\varphi_2}\right)^-}\right), \tag{2.16c}$$

$$\delta_{2,k}^{x;\varphi_1,\varphi_2^*} = \delta_2\left(X^{x;\varphi_1,\varphi_2^*}_{\left(\tau_{2,k}^{x;\varphi_1,\varphi_2^*}\right)^-}\right), \tag{2.16d}$$

$$(\mathcal{M}_1 V_1 - V_1)\left(X^{x;\varphi_1^*,\varphi_2}_{\left(\tau_{1,k}^{x;\varphi_1^*,\varphi_2}\right)^-}\right) = 0, \tag{2.16e}$$

$$(\mathcal{M}_2 V_2 - V_2)\left(X^{x;\varphi_1,\varphi_2^*}_{\left(\tau_{2,k}^{x;\varphi_1,\varphi_2^*}\right)^-}\right) = 0, \tag{2.16f}$$

for every φ_1, φ_2 *strategies such that* $(\varphi_1, \varphi_2^*), (\varphi_1^*, \varphi_2) \in \mathcal{A}_x$*, every* $s \geq 0$ *and every* $\tau_{i,k}^{x;\varphi_1,\varphi_2^*}$*,* $\tau_{i,k}^{x;\varphi_1^*,\varphi_2} < \infty$*.*

Proof. By Definition 2.6, we have to prove that

$$V_i(x) = J^i(x; \varphi_1^*, \varphi_2^*), \qquad V_1(x) \geq J^1(x; \varphi_1, \varphi_2^*), \qquad V_2(x) \geq J^2(x; \varphi_1^*, \varphi_2),$$

for every $i \in \{1,2\}$ and (φ_1, φ_2) strategies such that $(\varphi_1, \varphi_2^*) \in \mathcal{A}_x$ and $(\varphi_1^*, \varphi_2) \in \mathcal{A}_x$. We show the results for V_1 and J^1, the arguments for V_2 and J^2 being symmetric.

Step 1: $V_1(x) \geq J^1(x; \varphi_1, \varphi_2^*)$*.* Let φ_1 be a strategy for player 1 such that $(\varphi_1, \varphi_2^*) \in \mathcal{A}_x$. Here we will use the following shortened notation:

$$X = X^{x;\varphi_1,\varphi_2^*}, \qquad \tau_{i,k} = \tau_{i,k}^{x;\varphi_1,\varphi_2^*}, \qquad \delta_{i,k} = \delta_{i,k}^{x;\varphi_1,\varphi_2^*}.$$

Thanks to the regularity assumptions and by standard approximation arguments, it is not restrictive to assume $V_1 \in C^2(\mathcal{D}_2) \cap C(S)$ (see [12, Thm. 3.1]). For each $r > 0$ and $n \in \mathbb{N}$, we set

$$\tau_{r,n} = \tau_S \wedge \tau_r \wedge n,$$

where $\tau_r = \inf\{s > 0 : X_s \notin B(0,r)\}$ is the exit time from the ball with radius r. We apply Itô's formula to the function $(t, X_t) \mapsto e^{-\rho_1 t} V_1(X_t)$, integrate in the interval $[0, \tau_{r,n}]$ and take the conditional expectations (the initial point and the stochastic integral are integrable, so that the other terms are integrable too by equality): we get

$$
\begin{aligned}
V_1(x)=\mathbb{E}_x\biggl[-\int_0^{\tau_{r,n}} e^{-\rho_1 s}(\mathcal{A}V_1-\rho_1 V_1)(X_s)ds - \sum_{\tau_{1,k}<\tau_{r,n}} e^{-\rho_1\tau_{1,k}}\Bigl(V_1\bigl(X_{\tau_{1,k}}\bigr)-V_1\bigl(X_{(\tau_{1,k})^-}\bigr)\Bigr)\\
-\sum_{\tau_{2,k}<\tau_{r,n}} e^{-\rho_1\tau_{2,k}}\Bigl(V_1\bigl(X_{\tau_{2,k}}\bigr)-V_1\bigl(X_{(\tau_{2,k})^-}\bigr)\Bigr)+e^{-\rho_1\tau_{r,n}}V_1\bigl(X_{\tau_{r,n}}\bigr)\biggr].
\end{aligned}
\tag{2.17}
$$

We now estimate each term in the right-hand side of (2.17). As for the first term, since $(\mathcal{M}_2V_2 - V_2)(X_s) < 0$ by (2.16b), from (2.12d) it follows that

$$
(\mathcal{A}V_1-\rho_1V_1)(X_s)\le -f_1(X_s), \tag{2.18}
$$

for all $s\in[0,\tau_S]$. Let us now consider the second term: by (2.12b) and the definition of $\mathcal{M}_1V_1$ in (2.11), for every stopping time $\tau_{1,k}<\tau_S$ we have

$$
\begin{aligned}
V_1\bigl(X_{(\tau_{1,k})^-}\bigr) &\ge \mathcal{M}_1V_1\bigl(X_{(\tau_{1,k})^-}\bigr)\\
&= \sup_{\delta\in Z_1}\bigl\{V_1\bigl(\Gamma^1\bigl(X_{(\tau_{1,k})^-},\delta\bigr)\bigr)+\phi_1\bigl(X_{(\tau_{1,k})^-},\delta\bigr)\bigr\}\\
&\ge V_1\bigl(\Gamma^1\bigl(X_{(\tau_{1,k})^-},\delta_{1,k}\bigr)\bigr)+\phi_1\bigl(X_{(\tau_{1,k})^-},\delta_{1,k}\bigr)\\
&= V_1\bigl(X_{\tau_{1,k}}\bigr)+\phi_1\bigl(X_{(\tau_{1,k})^-},\delta_{1,k}\bigr).
\end{aligned}
\tag{2.19}
$$

As for the third term, let us consider any stopping time $\tau_{2,k}<\tau_S$. By (2.16f) we have $(\mathcal{M}_2V_2 - V_2)\bigl(X_{(\tau_{2,k})^-}\bigr)=0$; hence, the condition in (2.12c), the definition of $\mathcal{H}_1V_1$ in (2.11) and the expression of $\delta_{2,k}$ in (2.16d) imply that

$$
\begin{aligned}
V_1\bigl(X_{(\tau_{2,k})^-}\bigr) &= \mathcal{H}_1V_1\bigl(X_{(\tau_{2,k})^-}\bigr)\\
&= V_1\bigl(\Gamma^2\bigl(X_{(\tau_{2,k})^-},\delta_2\bigl(X_{(\tau_{2,k})^-}\bigr)\bigr)\bigr)+\psi_1\bigl(X_{(\tau_{2,k})^-},\delta_2\bigl(X_{(\tau_{2,k})^-}\bigr)\bigr)\\
&= V_1\bigl(\Gamma^2\bigl(X_{(\tau_{2,k})^-},\delta_{2,k}\bigr)\bigr)+\psi_1\bigl(X_{(\tau_{2,k})^-},\delta_{2,k}\bigr)\\
&= V_1\bigl(X_{\tau_{2,k}}\bigr)+\psi_1\bigl(X_{(\tau_{2,k})^-},\delta_{2,k}\bigr).
\end{aligned}
\tag{2.20}
$$

By (2.17) and the estimates in (2.18)-(2.20) it follows that

$$
\begin{aligned}
V_1(x)\ge\mathbb{E}_x\biggl[\int_0^{\tau_{r,n}} e^{-\rho_1 s}f_1(X_s)ds+\sum_{\tau_{1,k}<\tau_{r,n}} e^{-\rho_1\tau_{1,k}}\phi_1\bigl(X_{(\tau_{1,k})^-},\delta_{1,k}\bigr)\\
+\sum_{\tau_{2,k}<\tau_{r,n}} e^{-\rho_1\tau_{2,k}}\psi_1\bigl(X_{(\tau_{2,k})^-},\delta_{2,k}\bigr)+e^{-\rho_1\tau_{r,n}}V_1(X_{\tau_{r,n}})\biggr].
\end{aligned}
$$

Thanks to the conditions in (2.8), (2.9) and the polynomial growth of V_1, we can use the dominated convergence theorem and pass to the limit, first as $r\to\infty$ and then as $n\to\infty$. In particular, for the fourth term we notice that

$$
V_1(X_{\tau_{r,n}})\le C(1+|X_{\tau_{r,n}}|^k)\le C(1+\|X\|_\infty^k)\in L^1(\Omega), \tag{2.21}
$$

for suitable constants $C>0$ and $k\in\mathbb{N}$; the corresponding limit immediately follows by the continuity of V_1 in the case $\tau_S<\infty$ and by (2.21) itself in the case $\tau_S=\infty$ (as a direct consequence of (2.9), we have $\|X\|_\infty^k<\infty$ a.s.). Hence, we finally get

$$
\begin{aligned}
V_1(x)\ge\mathbb{E}_x\biggl[\int_0^{\tau_S} e^{-\rho_1 s}f_1(X_s)ds+\sum_{\tau_{1,k}<\tau_S} e^{-\rho_1\tau_{1,k}}\phi_1\bigl(X_{(\tau_{1,k})^-},\delta_{1,k}\bigr)\\
+\sum_{\tau_{2,k}<\tau_S} e^{-\rho_1\tau_{2,k}}\psi_1\bigl(X_{(\tau_{2,k})^-},\delta_{2,k}\bigr)+e^{-\rho_1\tau_S}h_1(X_{(\tau_S)^-})\mathbb{1}_{\{\tau_S<+\infty\}}\biggr]=J^1(x;\varphi_1,\varphi_2^*).
\end{aligned}
$$

Step 2: $V_1(x)=J^1(x;\varphi_1^*,\varphi_2^*)$. We argue as in Step 1, but here all the inequalities are equalities by the properties of φ_1^*. □

When solving the QVI problem, one deals with functions which are piecewise defined, as it will be clear in the next sections. Then, the regularity assumptions in the verification theorem correspond to suitable pasting conditions, leading to a system of algebraic equations. If the regularity conditions are too strong, the system has more equations than parameters, making the application of the theorem more difficult. Hence, a crucial point when stating a verification theorem is to set regularity conditions which allow such a system to actually have a solution. In [1, Section 3.3] a simple example shows that the regularity conditions we impose lead to an algebraic system with as many equations as parameters, so that a solution exists, at least formally.

Moreover, we observe that, unlike one-player control impulse problems, in our verification theorem the candidates are not required to be twice differentiable everywhere. For example, consider the case of player 1: as in the proof we always consider pairs of strategies in the form (φ_1, φ_2), by (2.16b) the controlled process never exits from $\mathcal{D}_2 = \{\mathcal{M}_2 V_2 - V_2 < 0\}$, which is then the only region where the function V_1 needs to be (almost everywhere) twice differentiable in order to apply Itô's formula.

3 Competition in retail energy markets

We now address the optimization problem of energy retailers who wants to maximize their expected profits, by increasing or decreasing the price they charge their customers for the consumption of electricity. In Section 3.1, as a warm-up, we consider a simpler but enlightening one-player version of the problem. In Section 3.2 we will turn to a two-player competitive market and we will focus on a nonzero-sum impulse game, that can be embedded in the setting presented in Section 2, so that results therein will serve us as a guide to perform our analysis.

3.1 The one-player case

The problem we study in this section has a long tradition (see [3]) and it is in particular very similar to the one in [4] (see also the references therein). Nevertheless, we give all the mathematical details (most of them in the Appendix) in order to keep this section self-contained. More precisely, the article [4] solves an optimal control problem of an inventory where the state variable is a mean-reverting process, the running cost is quadratic in the state variable and the switching costs are piecewise linear in the impulse size. Our problem could be seen as a limiting case of theirs when both the proportional switching costs and the mean-reverting part of the state variable tend to zero. We also notice that the running cost in our model is more general than in [4].

Formulation of the problem. Let us consider a retailer who buys energy (electricity, gas, gasoline) on the wholesale market and resells it to final consumers. We address the problem of investigating the retailer's optimal strategy in setting the final price and we model it as an impulse stochastic control problem.

As anticipated, the retailer buys the commodity in the wholesale market. We assume that the continuous-time price of the commodity is modelled by a Brownian motion with drift:

$$S_t = s + \mu t + \sigma W_t, \tag{3.1}$$

for $t \geq 0$, where $S_0 = s > 0$ and $\mu \geq 0, \sigma > 0$ are fixed constants. The standard Brownian motion W is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is equipped with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by W itself and made $\mathbb{P}$-complete (hence right-continuous).

Notice that the retailer has no control on the wholesale price. After buying the energy, the retailer sells it to final consumers. According to the most common contracts in energy markets, the retailer can change the price only after a written communication to all her customers. Then, we model the final price by a piecewise-constant process P. More precisely, we consider an initial price $p > 0$ and a sequence $\{\tau_k\}_{k \geq 1}$ of non-negative random times, which correspond to the retailer's interventions to adjust the price and move P to a new state. If we denote by $\{\delta_k\}_{k \geq 1}$ the corresponding impulses, i.e., $\delta_k = P_{\tau_k} - P_{(\tau_k)^-}$, we have

$$P_t = p + \sum_{\tau_k \leq t} \delta_k, \tag{3.2}$$

for every $t \geq 0$. Let us denote by X the difference or spread between the final price and the wholesale price. In other words, X represents the retailer's unitary income when selling energy (we do not consider, for the moment, the operational costs she faces). By (3.1) and (3.2), we have

$$X_t = P_t - S_t = x - \mu t - \sigma W_t + \sum_{\tau_k \leq t} \delta_k, \tag{3.3}$$

for every $t \geq 0$, where we have set $x = p - s$. We remark that, when the player does not intervene, the process X satisfies the following stochastic differential equation:

$$dX_t = -\mu dt - \sigma dW_t. \tag{3.4}$$

We assume that the retailer's market share at time $t \geq 0$ is a function of X_t, which we denote by $\Phi = \Phi(X_t)$. In our model, we set

$$\Phi(x) = \begin{cases} 1, & x \leq 0, \\ -\frac{1}{\Delta}(x - \Delta), & 0 < x < \Delta, \\ 0, & x \geq \Delta, \end{cases} \tag{3.5}$$

for every $x \in \mathbb{R}$, where $\Delta > 0$ is a fixed constant. In other words, the market share is a truncated linear function of X_t with two thresholds: if $X_t \leq 0$ (in which case the final price of the retailer is lower than the wholesale price) all the customers buy energy from the retailer, whereas if $X_t \geq \Delta$ the retailer has lost all her customers.

At each time $t \geq 0$, the retailer's income from selling the energy is given by $X_t \Phi(X_t)$, but she also has to pay an operational cost, which we assume to be a quadratic function of the market share $\Phi(X_t)$. Hence, the instantaneous payoff is given by

$$R(x) = x\Phi(x) - \frac{b}{2}\Phi^2(x), \tag{3.6}$$

where x is the current state of the process and b is a positive real constant. Moreover, there is a constant penalty $c > 0$ to be paid when the retailer intervenes to adjust P. Finally, we denote by $\rho > 0$ be the discount rate.

To sum up, we consider here the following stochastic impulse control problem.

Definition 3.1. *A control is a sequence $u = \{(\tau_k, \delta_k)\}_{1 \leq k < M}$, where $M \in \mathbb{R} \cup \{+\infty\}$, $\{\tau_k\}_{1 \leq k < M}$ is a non-decreasing non-negative family of stopping times (the intervention times) and $\{\delta_k\}_{1 \leq k < M}$ are real random variables (the corresponding impulses) such that δ_k is $\mathcal{F}_{\tau_k}$-measurable for all $1 \leq k < M$. We denote by $\mathcal{U}$ the set of admissible controls, that is the set of controls u such that*

$$\mathbb{E}\bigg[\sum_{1 \leq k < M} e^{-\rho \tau_k} \bigg] < \infty. \tag{3.7}$$

For each $x \in \mathbb{R}$ and $u \in \mathcal{U}$, we denote by $X^{x;u}$ the process defined in (3.3).

Definition 3.2. *The function V (value function) is defined, for each $x \in \mathbb{R}$, by*

$$V(x) = \sup_{u \in \mathcal{U}} J(x; u),$$

where, for every $u \in \mathcal{U}$

$$J(x; u) := \mathbb{E}_x \bigg[\int_0^{\infty} e^{-\rho t} R(X_t^{x;u}) dt - c \sum_{1 \leq k < M} e^{-\rho \tau_k} \bigg], \tag{3.8}$$

and the function R has been defined in (3.6). *If there exists $u^* \in \mathcal{U}$ such that $V(x) = J(x; u^*)$, we say that u^* is an optimal control in x.*

Notice that the functional J in (3.8) is well-defined, as R is bounded and (3.7) holds. To shorten the notations, we will often omit the dependence on the control and simply write X.

We now list some remarks about the payoff and the penalty of our problem: these properties will be useful for stating and proving our results.

- An explicit expression for the running cost R is

$$R(x) = \begin{cases} x - b/2, & \text{if } x < 0, \\ f(x), & \text{if } 0 \le x \le \Delta, \\ 0, & \text{if } x > \Delta, \end{cases}$$

for every $x \in \mathbb{R}$, where we have set

$$f(x) = -\alpha x^2 + \beta x - \gamma, \qquad \alpha = \frac{1}{\Delta} + \frac{b}{2\Delta^2}, \qquad \beta = 1 + \frac{b}{\Delta}, \qquad \gamma = \frac{b}{2}. \tag{3.9}$$

In particular, we remark that we have $R(x) \ge f(x)$, for every $x \in \mathbb{R}$.

- The function f in (3.9) is a concave parabola:

$$f(x) = -\alpha(x - x_v)^2 + y_v, \tag{3.10}$$

where α is as in (3.9) and the vertex $v = (x_v, y_v)$ is given by:

$$x_v = \frac{\Delta(\Delta + b)}{2\Delta + b}, \qquad y_v = f(x_v) = \frac{\Delta^2}{2(2\Delta + b)}. \tag{3.11}$$

From the retailer's point of view, Equation (3.10) says that x_v is the state which maximizes the payoff $R(x)$, the optimal income being y_v. Notice that the optimal share $\Phi_v := \Phi(x_v)$ is given by

$$\Phi_v = \Phi(x_v) = \frac{\Delta}{2\Delta + b}. \tag{3.12}$$

In particular, if $b = 0$ the optimal share is $1/2$.

- Moreover, we notice that

$$f(x) \ge 0 \text{ if and only if } x \in [x_z, \Delta], \text{ where } x_z = \frac{b\Delta}{2\Delta + b}. \tag{3.13}$$

Equivalently, the payoff $R(X_t)$ is positive if and only if the spread $X_t \in [x_z, \Delta]$. In other words, if we want the income from the energy sale to be higher than the operational costs, we need the spread between the wholesale price and the final price to be greater than x_z.

- Finally, if we consider x_v, y_v, x_z, Φ_v as functions of b, we notice that

$$\begin{array}{llll} x_v(b) \in [\Delta/2, \Delta[, & x_v(0) = \Delta/2, & x_v(+\infty) = \Delta, & x_v' > 0, \\ y_v(b) \in \,]0, \Delta/4], & y_v(0) = \Delta/4, & y_v(+\infty) = 0, & y_v' < 0, \\ x_z(b) \in \,]0, \Delta[, & x_z(0) = 0, & x_z(+\infty) = \Delta, & x_z' > 0, \\ \Phi_v(b) \in]0, 1/2[, & \Phi_v(0) = 1/2, & \Phi_v(+\infty) = 0, & \Phi_v' < 0. \end{array} \tag{3.14}$$

Some intuitive properties of the model are formalized in (3.14): as the operational costs increase the optimal spread x_v increases, the maximal instantaneous income y_v decreases, the region where the payoff is positive gets smaller and the optimal share decreases. In particular, we remark that $\Phi_v \in]0, 1/2[$: for any value of b, it is never optimal to have a market share greater than $1/2$.

Before stating the verification theorem in our one dimensional setting, we introduce the intervention operator $\mathcal{M}$.

Definition 3.3. *Let $V : \mathbb{R} \to \mathbb{R}$ with $\sup_{u \in \mathbb{R}} V(u) \in \mathbb{R}$. The function $\mathcal{M}V$ is defined, for every $x \in \mathbb{R}$, by*

$$\mathcal{M}V(x) = \sup_{\delta \in \mathbb{R}} \{V(x + \delta) - c\}. \tag{3.15}$$

We are now ready to state the verification theorem, which is a classical result in impulse stochastic control theory providing sufficient conditions for the value function. The following proposition is a special case of Theorem 6.2 in [12]. Its proof is therefore omitted.

Proposition 3.4 (**Verification Theorem**). *Let the assumptions and notations of this section hold. Let V be a function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the following conditions:*

- *V is bounded and there exists $x^* \in \mathbb{R}$ such that $V(x^*) = \max_{x\in\mathbb{R}} V(x)$;*
- *$\mathcal{D} = \{\mathcal{M}V - V < 0\}$ is a finite union of intervals;*
- *$V \in C^2(\mathbb{R} \setminus \partial\mathcal{D}) \cap C^1(\mathbb{R})$ and the second derivative of V is bounded near $\partial\mathcal{D}$;*
- *V is a solution to*
$$\max\{\mathcal{A}V - \rho V + R, \mathcal{M}V - V\} = 0, \tag{3.16}$$
where $\mathcal{A}V = (\sigma^2/2)V'' - \mu V'$ is the generator associated to Equation (3.4).

Let $x \in \mathbb{R}$ and let $u^*(x) = \{(\tau_k^*(x), \delta_k^*(x))\}_{1\le k<\infty}$, where the variables (τ_k^*, δ_k^*) (we omit the dependence on x to shorten the notations) are recursively defined by

$$\tau_k^* = \inf\big\{t > \tau_{k-1}^* : (\mathcal{M}V - V)\big(X_t^{x;u_k^*}\big) = 0\big\},$$
$$\delta_k^* = x^* - X_{\tau_k^*}^{x;u_k^*},$$

for $k \ge 1$, where we have set $\tau_0^* = \delta_0^* = 0$ and $u_k^*(x) = \{(\tau_j^*, \delta_j^*)\}_{1\le j\le k}$. Assume that $u^*(x) \in \mathcal{U}$. Then,

$$u^*(x) \text{ is an optimal control in } x \text{ and } V(x) = J(x; u^*(x)).$$

Practically, when dealing with a control problem, one first guesses the form of the continuation region and gets a candidate for the value function by solving Equation (3.16). The final step consists in actually apply the Verification Theorem to such a candidate.

Remark 3.5. *If the parameter c is very high, the retailer may loose all her customers without intervening, as the intervention cost would be higher than the loss she is experiencing. However, this situation is clearly not practically admissible (if the costs are too big, a retailer does not even enter the market). So, in order to keep the model close to reality, we will always require the continuation region to be a subset of $]0, \Delta[$:*

$$\mathcal{D} = \{\mathcal{M}V - V < 0\} \subseteq\,]0, \Delta[. \tag{3.17}$$

As a consequence, when dealing with the continuation region, we will consider as the running cost of the problem the restriction of the function $R_{|]0,\Delta[} = f$ (clearly, we cannot substitute R with f in (3.16), as such equation holds for each $x \in \mathbb{R}$).

The key-stone of Proposition 3.4 is Equation (3.16), which implies

$$\mathcal{A}V - \rho V + R = 0, \qquad \text{in } \{\mathcal{M}V - V < 0\}.$$

We now provide an explicit solution to such an equation. By (3.17) we can replace R with f in $\mathcal{D}$; hence, we are interested in solving

$$\mathcal{A}\varphi - \rho\varphi + f = \frac{\sigma^2}{2}\varphi'' - \mu\varphi' - \rho\varphi + f = 0. \tag{3.18}$$

The general solution to (3.18) is given by

$$\varphi_{A_1,A_2}(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x} - k_2 x^2 + k_1 x - k_0, \tag{3.19}$$

where $A_1, A_2 \in \mathbb{R}$ and we have set

$$\begin{gathered} m_{1,2} = \frac{\mu \pm \sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2}, \\ k_2 = \frac{\alpha}{\rho}, \qquad k_1 = \frac{\beta}{\rho} + \frac{2\alpha\mu}{\rho^2}, \qquad k_0 = \frac{\gamma}{\rho} + \frac{\beta\mu + \alpha\sigma^2}{\rho^2} + \frac{2\alpha\mu^2}{\rho^3}, \end{gathered} \tag{3.20}$$

with α, β, γ as in (3.9). Notice that, when $\mu = 0$, we have

$$-k_2 x^2 + k_1 x - k_0 = \frac{f(x)}{\rho} - \frac{\alpha\sigma^2}{\rho^2}.$$

Hence, the polynomial part in (3.19) is, in this case, a concave parabola with vertex in x_v, with x_v as in (3.11); as a consequence, by (3.10) we also have the following representation:

$$\varphi_{A_1,A_2}(x) = A_1 e^{\theta x} + A_2 e^{-\theta x} - k_2(x - x_v)^2 + k_3, \tag{3.21}$$

where, to shorten the notations, we have set

$$\theta = \sqrt{\frac{2\rho}{\sigma^2}}, \qquad k_3 = \frac{f(x_v)}{\rho} - \frac{\alpha\sigma^2}{\rho^2} = \frac{f(x_v)}{\rho} - \frac{2k_2}{\theta^2}. \tag{3.22}$$

We stress that the representation in (3.21) holds only in the case $\mu = 0$.

The solution in the case $\mu = 0$. We will show that the classical verification theorem (Proposition 3.4 above) can be applied, so that a semi-explicit expression for the value function and the optimal control is available. First, we build a candidate for the value function, then, we show that the verification theorem actually applies to such a candidate.

We focus on the following case:

$$\mu = 0, \qquad c \le \bar{c}, \tag{3.23}$$

where $\bar{c}$ will be specified later in Proposition 3.7. Since our goal is to apply Proposition 3.4, we first try to find a solution to (3.16), in order to get a candidate $\widetilde{V}$ for V.

It is reasonable to assume that the retailer's continuation region (i.e., when he does not intervene) is in the form $\mathcal{D} =]\underline{x}, \bar{x}[$ and it is included in $]0, \Delta[$ (recall Equation (3.17)). As a consequence, the real axis is heuristically divided into:

$$\mathbb{R}\backslash]\underline{x}, \bar{x}[= \{\mathcal{M}V - V = 0\}, \text{ where the retailer intervenes,}$$
$$]\underline{x}, \bar{x}[= \{\mathcal{M}V - V < 0\}, \text{ where the retailer does not intervene.}$$

Then, the QVI problem (3.16) suggests the following candidate for V:

$$\widetilde{V}(x) = \begin{cases} \varphi(x), & \text{if } x \in]\underline{x}, \bar{x}[, \\ \mathcal{M}\widetilde{V}(x), & \text{if } x \in \mathbb{R}\backslash]\underline{x}, \bar{x}[, \end{cases}$$

where φ is a solution to the equation (recall that $]\underline{x}, \bar{x}[\subseteq]0, \Delta[$, where $R = f$)

$$\mathcal{A}\varphi - \rho\varphi + f = 0,$$

and where the function $\mathcal{M}\widetilde{V}$ (recall Definition 3.15) is given by

$$\mathcal{M}\widetilde{V}(x) = \sup_{\delta \in \mathbb{R}} \{\widetilde{V}(x + \delta) - c\} = \sup_{y \in \mathbb{R}} \{\widetilde{V}(y)\} - c.$$

Heuristically, it is reasonable to assume that the function $\widetilde{V}$ has a unique maximum point x^*, which belongs to the continuation region $]\underline{x}, \bar{x}[$ (where $\widetilde{V} = \varphi$):

$$\max_{y \in \mathbb{R}} \{\widetilde{V}(y)\} = \max_{y \in]\underline{x}, \bar{x}[} \{\varphi(y)\} = \varphi(x^*), \qquad \text{where} \qquad \varphi'(x^*) = 0, \;\; \varphi''(x^*) \le 0, \;\; \underline{x} < x^* < \bar{x}.$$

We recall that an explicit formula for φ has been provided before in this section: in particular, since we are considering the case $\mu = 0$, we can use the formula in (3.21). Moreover, we recall that the parameters in $\widetilde{V}$ must be chosen so as to satisfy the regularity assumptions of the verification theorem: $\widetilde{V}$ has to be continuous and differentiable in $\underline{x}, \bar{x}$. To sum up, the candidate is as follows:

Definition 3.6. *For every $x \in \mathbb{R}$, we set*

$$\widetilde{V}(x) = \begin{cases} \varphi_{A_1,A_2}(x), & \text{in }]\underline{x}, \bar{x}[, \\ \varphi_{A_1,A_2}(x^*) - c, & \text{in } \mathbb{R} \backslash]\underline{x}, \bar{x}[, \end{cases}$$

where φ_{A_1,A_2} is as in (3.21) *and the five parameters $(A_1, A_2, \underline{x}, \bar{x}, x^*)$ satisfy*

$$0 < \underline{x} < x^* < \bar{x} < \Delta, \tag{3.24}$$

and the following conditions:

$$\begin{cases} \varphi'_{A_1,A_2}(x^*) = 0 \text{ and } \varphi''_{A_1,A_2}(x^*) < 0, & \text{(optimality of } x^*\text{)} \\ \varphi'_{A_1,A_2}(\underline{x}) = 0, & (C^1\text{-pasting in } \underline{x}) \\ \varphi'_{A_1,A_2}(\bar{x}) = 0, & (C^1\text{-pasting in } \bar{x}) \\ \varphi_{A_1,A_2}(\underline{x}) = \varphi_{A_1,A_2}(x^*) - c, & (C^0\text{-pasting in } \underline{x}) \\ \varphi_{A_1,A_2}(\bar{x}) = \varphi_{A_1,A_2}(x^*) - c. & (C^0\text{-pasting in } \bar{x}) \end{cases} \tag{3.25}$$

In order to have a well-posed definition, we first need to prove that a solution to (3.25) actually exists.

Since the system cannot be solved directly, we try to make some guesses to simplify it. Consider the structure of the problem: the running cost is symmetric with respect to x_v (see the formulas (3.10) and (3.11)), the penalty is constant, the uncontrolled process is a scaled Brownian motion (recall that $\mu = 0$). Then, we expect the value function to be symmetric with respect to x_v, which corresponds to the choice $A_1 e^{\theta x_v} = A_2 e^{-\theta x_v}$. The same argument suggests to set $(\underline{x} + \bar{x})/2 = x_v$. Finally, as a symmetry point is always a local maximum or minimum point, we expect $x^* = x_v$. In short, our guess is

$$A_1 = A e^{-\theta x_v}, \qquad A_2 = A e^{\theta x_v}, \qquad (\underline{x} + \bar{x})/2 = x_v, \qquad x^* = x_v, \tag{3.26}$$

with $A \in \mathbb{R}$. In particular, we now consider functions in the form

$$\varphi_A(x) = A e^{\theta(x - x_v)} + A e^{-\theta(x - x_v)} - k_2 (x - x_v)^2 + k_3,$$

where $A \in \mathbb{R}$ and the coefficients k_2 and k_3 have been defined in (3.20) and (3.22).

Indeed, an easy check shows that $x^* = x_v$ is a local maximum for φ_A (so that the first condition in (3.25) is satisfied) if and only if $A > 0$. Then, under our guess (3.26), we can equivalently rewrite the system (3.25) as

$$\begin{cases} \varphi'_A(\bar{x}) = 0, \\ \varphi_A(\bar{x}) = \varphi_A(x_v) - c, \end{cases}$$

with $A > 0$ and $\bar{x} > x_v$. Equivalently, we have to solve

$$\begin{cases} A\theta e^{\theta(\bar{x} - x_v)} - A\theta e^{-\theta(\bar{x} - x_v)} - 2k_2(\bar{x} - x_v) = 0, \\ A e^{\theta(\bar{x} - x_v)} + A e^{-\theta(\bar{x} - x_v)} - k_2(\bar{x} - x_v)^2 - 2A + c = 0. \end{cases}$$

In order to simplify the notations, we operate a change of variable and set $\bar{y} = \bar{x} - x_v$, so that we have

$$A\theta e^{\theta \bar{y}} - A\theta e^{-\theta \bar{y}} - 2k_2 \bar{y} = 0, \tag{3.27a}$$
$$A e^{\theta \bar{y}} + A e^{-\theta \bar{y}} - k_2 \bar{y}^2 - 2A + c = 0, \tag{3.27b}$$

where $A > 0$ and $\bar{y} > 0$. Finally, notice that the order condition (3.24) now reads

$$\bar{y} < \Delta - x_v. \tag{3.28}$$

So, in order to prove that $\widetilde{V}$ is well-defined it suffices to show that a solution to (3.27a)-(3.27b)-(3.28) exists and is unique. The proof of the following proposition is in the Appendix.

Proposition 3.7. *Assume $c < \bar{c}$, with $\bar{c} = \xi(\Delta^2/(2\Delta + b))$, where ξ is a suitable function defined in* (A.11)*. Then, the function $\widetilde{V}$ in Definition 3.6 is well-defined, namely there exists a solution*

$$(A_1, A_2, \underline{x}, \bar{x}, x^*)$$

to System (3.25)*, which is given by*

$$A_1 = Ae^{-\theta x_v}, \qquad A_2 = Ae^{\theta x_v},$$
$$x^* = x_v, \qquad \underline{x} = x_v - \bar{y}, \qquad \bar{x} = x_v + \bar{y},$$

where x_v is as in (3.11) *and $(A, \bar{y})$ is the unique solution to* (3.27a)-(3.27b)-(3.28)*.*

We conclude this section with an application of the verification theorem in Proposition 3.4, which yields that the candidate $\widetilde{V}$ defined in the previous section actually corresponds to the value function. Moreover, we characterize the optimal price management policy: the retailer has to intervene if and only if the process hits $\underline{x}$ or $\bar{x}$ and, when this happens, she has to shift X back to the state x^*. The proof of the next result is postponed to the Appendix.

Proposition 3.8. *Let* (3.23) *hold and let $\widetilde{V}$ be as in Definition 3.6. For every $x \in \mathbb{R}$, an optimal control for the problem in Definition 3.2 exists and is given by $u^*(x) = \{(\tau_k^*, \delta_k^*)\}_{1 \le k < \infty}$, where the variables (τ_k^*, δ_k^*) are recursively defined by*

$$\begin{aligned} \tau_k^* &= \inf\Big\{t > \tau_{k-1}^* : X_t^{x;u_k^*} \in \{\underline{x}, \bar{x}\}\Big\}, \\ \delta_k^* &= x^* - X_{\tau_k^*}^{x;u_k^*}, \end{aligned} \tag{3.29}$$

for $k \ge 1$, where we have set $\tau_0^* = \delta_0^* = 0$ and $u_k^* = \{(\tau_j^*, \delta_j^*)\}_{1 \le j \le k}$. Moreover, $\widetilde{V}$ coincides with the value function: for every $x \in \mathbb{R}$ we have

$$\widetilde{V}(x) = V(x) = J(x; u^*(x)).$$

Remark 3.9. *For more details and results on the one-player model including the nonzero drift case ($\mu \neq 0$) as well as the asymptotic analysis of the value function and the optimal strategy as $c \to 0$ we refer to [1].*

3.2 The two-player case

We now extend the one-player model in Section 3.1 to a competitive two-player energy market model, getting a nonzero-sum stochastic game with impulse controls, which is a special case of the general framework in Section 2. After setting the model in Section 3.2.1, we provide in Section 3.2.2 a system of equations to be solved in order to fully determine the value function.

3.2.1 Formulation of the problem

The one-player model in Section 3.1 has the advantage of being mathematically tractable. However, it does not fully reproduce the fierce competition which characterizes modern deregulated energy markets: the interaction between opposing retailers is only implicitly considered (the player's market share decreases as her price rises). Motivated by this fact, we now modify our model by introducing a second player.

Hence, we assume that the retail market is made up of two opponent players, indexed by $i \in \{1, 2\}$. Similarly to Section 3.1, each retailer buys energy at the wholesale price S, with S as in (3.1), and resells it to her customers at a final price P^i, with P^i resembling (3.2):

$$S_t = s + \mu t + \sigma W_t, \qquad P_t^i = p_i + \sum_{\tau_k^i \le t} \delta_k^i, \tag{3.30}$$

for each $t \ge 0$, where s is the initial wholesale price, μ and σ are fixed constants, W is a one-dimensional Brownian motion, p_i is the initial retail price and $\{\tau_k^i, \delta_k^i\}_k$ is the impulse control corresponding to the retailer's interventions on the final price, as in Section 3.1. Notice that the retailers buy the energy from the same provider and that they do not influence the wholesale price.

In order to have a realistic model, the market share of player $i \in \{1,2\}$ depends on the price she asks (as in the one-player case) and on the opponent's pricing choices as well. In particular, if the two retailers fix the same final price then both market shares are equal to 50%, whereas a lower price with respect to the competitor should correspond to an increase in the number of customers. Let $i, j \in \{1,2\}$, with $i \neq j$, and let $\Delta > 0$ be a fixed constant. In our model we assume that the market share of retailer i at time $t \geq 0$ is $\Phi(P_t^i - P_t^j)$, where

$$\Phi(x) = \begin{cases} 1, & x \leq -\Delta, \\ -\frac{1}{2\Delta}(x - \Delta), & -\Delta < x < \Delta, \\ 0, & x \geq \Delta, \end{cases} \tag{3.31}$$

for every $x \in \mathbb{R}$. In other words, the market share of player i is a truncated linear function of $P_t^i - P_t^j$, with two thresholds: if $P_t^i \leq P_t^j - \Delta$ the retailer has the monopoly of the market, whereas if $P_t^i \geq P_t^j + \Delta$ the retailer has lost all his customers.

Remark 3.10. *Notice that the market share function Φ in (3.31) is not the same as in the case of one player (recall Equation (3.5)). This is clearly due to the presence of a second market actor, which leads to an expansion of the domain (from $(0, \Delta)$ to $(-\Delta, \Delta)$) where Φ takes values in $]0, 1[$.*

Extending (3.6) to the present situation, the running payoff of player $i \in \{1,2\}$ at time $t \geq 0$ is

$$(P_t^i - S_t)\Phi(P_t^i - P_t^j) - \frac{b_i}{2}\Phi(P_t^i - P_t^j)^2 = X_t^i \Phi(X_t^i - X_t^j) - \frac{b_i}{2}\Phi(X_t^i - X_t^j)^2 =: R_i(X_t^1, X_t^2), \tag{3.32}$$

where $b_i \geq 0$ is a fixed constant and the process X^i, introduced in order to reduce the dimension of the problem, represents the retailer's unitary income when selling energy:

$$X_t^i := P_t^i - S_t = x_i - \mu t - \sigma W_t + \sum_{\tau_k^i \leq t} \delta_k^i, \tag{3.33}$$

for every $t \geq 0$, where we have used (3.30) and we have set $x_i = p_i - s$. In particular, we remark that in the strip $\{(x_1, x_2) : -\Delta < x_1 - x_2 < \Delta\}$ the payoff R_i is a second-degree polynomial in the variables x_1, x_2:

$$(R_i)_{|\mathcal{D}} = f_i, \tag{3.34}$$

$$f_i(x_1, x_2) = -\frac{b_i}{8} + \left(\frac{b_i}{4\Delta} + \frac{1}{2}\right) x_i - \frac{b_i}{4\Delta} x_j + \left(\frac{b_i}{4\Delta^2} + \frac{1}{2\Delta}\right) x_i x_j - \left(\frac{b_i}{8\Delta^2} + \frac{1}{2\Delta}\right) x_i^2 - \frac{b_i}{8\Delta^2} x_j^2.$$

Finally, let $\rho > 0$ be the discount rate and let c_i be the fixed intervention cost for player $i \in \{1,2\}$.

It is clear that we are dealing with a nonzero-sum impulse game, belonging to the class studied in Section 2, with (on the left-hand side we use the notations in Section 2)

$$S = \mathbb{R}^2, \qquad Z^1 = Z^2 = \mathbb{R}, \qquad \Gamma^1((x_1, x_2); \delta) = (x_1 + \delta, x_2), \qquad \Gamma^2((x_1, x_2); \delta) = (x_1, x_2 + \delta),$$
$$f_i = R_i, \qquad h_i = 0, \qquad \phi_i \equiv -c_i, \qquad \psi_i \equiv 0,$$

and with the following functional to be maximized:

$$J^i(x_1, x_2; \varphi_1, \varphi_2) = \mathbb{E}_{x_1, x_2}\left[\int_0^\infty e^{-\rho t}\left(X_t^i \Phi(X_t^i - X_t^j) - \frac{b_i}{2}\Phi(X_t^i - X_t^j)^2\right) dt - c_i\left(\sum_{1 \leq k < M} e^{-\rho \tau_k^i}\right)\right], \tag{3.35}$$

for every initial state $(x_1, x_2) \in \mathbb{R}^2$ and for every strategy (φ_1, φ_2) (recall Definition 2.1).

3.2.2 Looking for a solution

Our goal is to apply the Verification Theorem 2.9 to the nonzero-sum game in (3.35). Hence, we now try to heuristically find a solution to the corresponding QVI problem, in order to get a pair of candidates $\widetilde{V}_1, \widetilde{V}_2$ for the value functions V_1, V_2.

Candidates. We start with the two equations in (2.12), which we recall for reader's convenience (we do not have any terminal condition as $S = \mathbb{R}^2$):

$$\begin{aligned} &\mathcal{H}_i\widetilde{V}_i - \widetilde{V}_i = 0, && \text{in } \{\mathcal{M}_j\widetilde{V}_j - \widetilde{V}_j = 0\}, \\ &\max\big\{\mathcal{A}\widetilde{V}_i - \rho\widetilde{V}_i + R_i, \mathcal{M}_i\widetilde{V}_i - \widetilde{V}_i\big\} = 0, && \text{in } \{\mathcal{M}_j\widetilde{V}_j - \widetilde{V}_j < 0\}, \end{aligned} \tag{3.36}$$

for $i, j \in \{1, 2\}$, with $i \neq j$.

Remark 3.11. *It is reasonable to assume that a Nash equilibrium does not correspond to a situation where one of the players exits the market. In other words, we heuristically assume that both the continuation regions $\{\mathcal{M}_i\widetilde{V}_i - \widetilde{V}_i < 0\}$, $i \in \{1, 2\}$, are included in the strip $\{(x_1, x_2) : -\Delta < x_1 - x_2 < \Delta\}$, so that in (3.36) we can replace R_i by f_i, thanks to (3.34).*

Now, recall that $\{\mathcal{M}_i\widetilde{V}_i - \widetilde{V}_i = 0\}$ corresponds to the region where player $i \in \{1, 2\}$ intervenes and remember that, in the case of contemporary intervention, player 1 has the priority. As a consequence, we have $\{\mathcal{M}_j\widetilde{V}_j - \widetilde{V}_j < 0, \mathcal{M}_i\widetilde{V}_i - \widetilde{V}_i = 0\} = \{\mathcal{M}_i\widetilde{V}_i - \widetilde{V}_i = 0\}$. Then, the equations in (3.36) can be equivalently rewritten as

$$\widetilde{V}_i = \begin{cases} \mathcal{M}_i\widetilde{V}_i, & \text{in } \{\mathcal{M}_i\widetilde{V}_i - \widetilde{V}_i = 0\}, \\ \varphi_i, & \text{in } \{\mathcal{M}_j\widetilde{V}_j - \widetilde{V}_j < 0, \mathcal{M}_i\widetilde{V}_i - \widetilde{V}_i < 0\}, \\ \mathcal{H}_i\widetilde{V}_i, & \text{in } \{\mathcal{M}_j\widetilde{V}_j - \widetilde{V}_j = 0\}, \end{cases} \tag{3.37}$$

for $i, j \in \{1, 2\}$ and $i \neq j$, where φ_i is a solution to

$$\mathcal{A}\varphi_i - \rho\varphi_i + f_i = -\mu(\partial_{x_1} + \partial_{x_2})\varphi_i + \frac{1}{2}\sigma^2(\partial_{x_1} + \partial_{x_2})^2\varphi_i - \rho\varphi_i + f_i = 0. \tag{3.38}$$

We now conjecture an expression for the three regions in (3.37). It is reasonable to assume that player 1 intervenes if and only if her unitary income X_t^1 exits from a suitable interval, whose boundaries depend on the state variable X_t^2 controlled by her opponent. We denote this interval by $]\underline{x}_1(X_t^2), \bar{x}_1(X_t^2)[$, where $\underline{x}_1, \bar{x}_1$ are suitable functions. Hence, we guess that $\{\mathcal{M}_1\widetilde{V}_1 - \widetilde{V}_1 = 0\}$ (the intervention region of player 1) is given by $\big\{(x_1, x_2) : x_1 \notin]\underline{x}_1(x_2), \bar{x}_1(x_2)[\big\}$. A symmetric argument for the intervention region $\{\mathcal{M}_2\widetilde{V}_2 - \widetilde{V}_2 = 0\}$ of player 2 leads to $\big\{(x_1, x_2) : x_2 \notin]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\}$, but we have to exclude from such region the points where player 1 already intervenes (in case of contemporary intervention, player 1 has the priority). Finally, the common continuation region $\{\mathcal{M}_1\widetilde{V}_1 - \widetilde{V}_1 < 0, \mathcal{M}_2\widetilde{V}_2 - \widetilde{V}_2 < 0\}$ is just the complement of such sets. In short, see also Figure 3.1, we have

$$\begin{aligned} \big\{\mathcal{M}_1\widetilde{V}_1 - \widetilde{V}_1 = 0\big\} &= \big\{(x_1, x_2) : x_1 \in]-\infty, \underline{x}_1(x_2)] \cup [\bar{x}_1(x_2), +\infty[\big\} =: R, \\ \big\{\mathcal{M}_2\widetilde{V}_2 - \widetilde{V}_2 = 0\big\} &= \big\{(x_1, x_2) : x_1 \in]\underline{x}_1(x_2), \bar{x}_1(x_2)[,\ x_2 \in]-\infty, \underline{x}_2(x_1)] \cup [\bar{x}_2(x_1), +\infty[\big\} =: B, \\ \big\{\mathcal{M}_1\widetilde{V}_1 - \widetilde{V}_1 < 0, \mathcal{M}_2\widetilde{V}_2 - \widetilde{V}_2 < 0\big\} &= \big\{(x_1, x_2) : x_1 \in]\underline{x}_1(x_2), \bar{x}_1(x_2)[,\ x_2 \in]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\} =: W. \end{aligned}$$

We recall once more the interpretation: R is the region where player 1 intervenes (red area in the picture), B is the region where player 2 intervenes (blue area in the picture), W is the region where no one intervenes (white area in the picture). By (3.37) we then get

$$\widetilde{V}_1 = \begin{cases} \mathcal{H}_1\widetilde{V}_1, & \text{in } B, \\ \varphi_1, & \text{in } W, \\ \mathcal{M}_1\widetilde{V}_1, & \text{in } R, \end{cases} \qquad \widetilde{V}_2 = \begin{cases} \mathcal{M}_2\widetilde{V}_2, & \text{in } B, \\ \varphi_2, & \text{in } W, \\ \mathcal{H}_2\widetilde{V}_2, & \text{in } R. \end{cases}$$

To go on, we need to estimate φ_i and the operators $\mathcal{M}_i, \mathcal{H}_i$. Let us start with φ_i. The only differential operator in (3.38) is $\partial_{x_1} + \partial_{x_2}$, which suggests the change of variable $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$, so that the PDE becomes a second-order linear ODE in the variable y_1, which is easily solvable for each $y_2 \in \mathbb{R}$ fixed. Then, after reintroducing the original variables, we get the following solution to (3.38):

$$\varphi_i(x_1, x_2) = A_{i,+}(x_1 - x_2)e^{m_+(x_1+x_2)} + A_{i,-}(x_1 - x_2)e^{m_-(x_1+x_2)} + \hat{\varphi}_i(x_1, x_2),$$

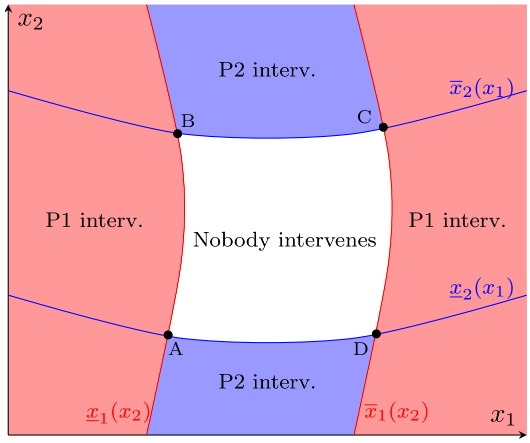

Figure 3.1: Partition of the domain in the three regions R (red), B (blue), W (white) depending on possible players' interventions.

where $A_{i,+/-}$ are suitable real functions, $m_{+/-}$ are the two roots of $\frac{1}{2}\sigma^2 m^2 - \mu m - \rho = 0$ and $\hat{\varphi}_i$ (the particular solution to the corresponding ODE) is

$$\hat{\varphi}_i(x_1, x_2) = -\left(\frac{b_i}{8\rho} + \frac{\mu}{2\rho^2}\right) + \left(\frac{b_i}{4\Delta\rho} + \frac{\mu}{2\Delta\rho^2} + \frac{1}{2\rho}\right) x_i - \left(\frac{b_i}{4\Delta\rho} + \frac{\mu}{2\Delta\rho^2}\right) x_j$$
$$+ \left(\frac{b_i}{4\Delta^2\rho} + \frac{1}{2\Delta\rho}\right) x_i x_j - \left(\frac{b_i}{8\Delta^2\rho} + \frac{1}{2\Delta\rho}\right) x_i^2 - \frac{b_i}{8\Delta^2\rho} x_j^2.$$

We now estimate $\mathcal{M}_i \widetilde{V}_i$ and $\mathcal{H}_i \widetilde{V}_i$, where the operators $\mathcal{M}_i, \mathcal{H}_i$ have been defined in (2.11). Let $(x_1, x_2) \in \mathbb{R}^2$ and let us start from the functions δ_i in (2.10): the definition here reads

$$\{\delta_1(x_1, x_2)\} = \arg\max_{\delta_1 \in \mathbb{R}} \widetilde{V}_1(x_1 + \delta_1, x_2), \qquad \{\delta_2(x_1, x_2)\} = \arg\max_{\delta_2 \in \mathbb{R}} \widetilde{V}_2(x_1, x_2 + \delta_2).$$

Heuristically, it is reasonable to assume that the function $\widetilde{V}_1(\cdot, x_2)$ has a unique maximum point $x_1^*(x_2)$ and that this point belongs to the continuation region (where, by definition, $\widetilde{V}_1 = \varphi_1$); we can argue similarly for $\widetilde{V}_2(x_1, \cdot)$. Thus we get

$$\begin{aligned} x_1 + \delta_1(x_1, x_2) &= x_1^*(x_2), \qquad \text{where} \qquad \{x_1^*(x_2)\} = \arg\max_{y \in \mathbb{R}} \varphi_1(y, x_2), \\ x_2 + \delta_2(x_1, x_2) &= x_2^*(x_1), \qquad \text{where} \qquad \{x_2^*(x_1)\} = \arg\max_{y \in \mathbb{R}} \varphi_2(x_1, y). \end{aligned}$$

Then, by the definition in (2.11) we have

$$\begin{aligned} \mathcal{M}_1 \widetilde{V}_1(x_1, x_2) &= \varphi_1\big(x_1^*(x_2), x_2\big) - c_1, \qquad & \mathcal{H}_1 \widetilde{V}_1(x_1, x_2) &= \varphi_1\big(x_1, x_2^*(x_1)\big), \\ \mathcal{M}_2 \widetilde{V}_2(x_1, x_2) &= \varphi_2\big(x_1, x_2^*(x_1)\big) - c_2, \qquad & \mathcal{H}_2 \widetilde{V}_2(x_1, x_2) &= \varphi_2\big(x_1^*(x_2), x_2\big), \end{aligned}$$

for each $(x_1, x_2) \in \mathbb{R}^2$. We finally get the following (heuristic) candidates for the value functions:

$$\widetilde{V}_1(x_1, x_2) = \begin{cases} \varphi_1\big(x_1, x_2^*(x_1)\big), & \text{in } B, \\ \varphi_1\big(x_1, x_2\big), & \text{in } W, \\ \varphi_1\big(x_1^*(x_2), x_2\big) - c_1, & \text{in } R, \end{cases} \qquad \widetilde{V}_2(x_1, x_2) = \begin{cases} \varphi_2\big(x_1, x_2^*(x_1)\big) - c_2, & \text{in } B, \\ \varphi_2\big(x_1, x_2\big), & \text{in } W, \\ \varphi_2\big(x_1^*(x_2), x_2\big), & \text{in } R. \end{cases}$$

Notice that the derivative of $x_2 \mapsto \varphi_1\big(x_1^*(x_2), x_2\big) - c_1$ (that is $\widetilde{V}_1$ in the region R) is

$$\left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(x_1^*(x_2), x_2\big) \cdot \big(x_1^*\big)'(x_2) + \left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(x_1^*(x_2), x_2\big). \tag{3.39}$$

Conditions. We now collect all the conditions the candidates $\widetilde{V}_1, \widetilde{V}_2$ have to satisfy. We just write the equations for $\widetilde{V}_1$, the ones for $\widetilde{V}_2$ being symmetric. More in detail, we need to set the optimality condition for x_1^* and to impose the regularity required in the assumptions of Theorem 2.9, that is ($\mathcal{D}_i$ is the continuation region of player i)

$$\widetilde{V}_1 \in C^2(\mathcal{D}_2 \setminus \partial\mathcal{D}_1) \cap C^1(\mathcal{D}_2) \cap C^0(\mathbb{R}^2).$$

Let A, B, C, D be the intersections of the four curves $\underline{x}_i, \bar{x}_i$, as in Figure 3.1. As for the C^0 condition, we have to set a C^0-pasting in the boundaries between the three regions, that are the curved segments AD, BC and the two vertical curves. As for the C^1 condition ($\mathcal{D}_2$ is the central horizontal strip), we have to add a C^1-pasting in the segments AB and CD. As for the C^2 condition, it is satisfied by definition.

- *Optimality of x_1^*.* As $x_1^*(x_2)$ is by definition the maximizer of $x_1 \mapsto \varphi_1(x_1, x_2)$, for each $x_2 \in \mathbb{R}$ we have the following first-order condition:

$$\Big(\frac{\partial \varphi_1}{\partial x_1}\Big)\big(x_1^*(x_2), x_2\big) = 0, \qquad x_2 \in \mathbb{R}.$$

- *Continuity.* We first set the continuity on the curve $x_1 = \underline{x}_1(x_2)$ (left vertical curve in the picture). The function $\widetilde{V}_1$ has two different expressions in the central vertical strip, one in the white region W and one in the blue region B, so that we need two separate continuity conditions, one in the segment AB and one outside such segment:

$$\begin{aligned}
&\varphi_1\big(x_1^*(x_2), x_2\big) - c_1 = \varphi_1\big(\underline{x}_1(x_2), x_2\big), && x_2 \in \big[x_2^A, x_2^B\big],\\
&\varphi_1\big(x_1^*(x_2), x_2\big) - c_1 = \varphi_1\Big(\underline{x}_1(x_2), x_2^*\big(\underline{x}_1(x_2)\big)\Big), && x_2 \in \mathbb{R} \setminus \big[x_2^A, x_2^B\big].
\end{aligned}$$

Similarly, for the continuity on the curve $x_1 = \bar{x}_1(x_2)$ (right vertical curve in the picture):

$$\begin{aligned}
&\varphi_1\big(x_1^*(x_2), x_2\big) - c_1 = \varphi_1\big(\bar{x}_1(x_2), x_2\big), && x_2 \in \big[x_2^D, x_2^C\big],\\
&\varphi_1\big(x_1^*(x_2), x_2\big) - c_1 = \varphi_1\Big(\bar{x}_1(x_2), x_2^*\big(\bar{x}_1(x_2)\big)\Big), && x_2 \in \mathbb{R} \setminus \big[x_2^D, x_2^C\big].
\end{aligned}$$

We now set the continuity on the segment AD, which belongs to the curve $x_2 = \underline{x}_2(x_1)$ (lower horizontal curve in the picture):

$$\varphi_1\big(x_1, x_2^*(x_1)\big) = \varphi_1\big(x_1, \underline{x}_2(x_1)\big), \qquad x_1 \in \big]x_1^A, x_1^D\big[.$$

Similarly, for the continuity on the segment BC, which belongs to the curve $x_2 = \bar{x}_2(x_1)$ (upper horizontal curve in the picture):

$$\varphi_1\big(x_1, x_2^*(x_1)\big) = \varphi_1\big(x_1, \bar{x}_2(x_1)\big), \qquad x_1 \in \big]x_1^B, x_1^C\big[.$$

- *Differentiability.* We now set a C^1-pasting on the segment AB, which belongs to the curve $x_1 = \underline{x}_1(x_2)$ (left vertical curve in the picture). As φ_1 is a two-dimensional function, we need to set one condition for each derivative (for $\partial/\partial x_2$ we use (3.39) and notice that the first term is zero because of the optimality condition):

$$\begin{aligned}
&\Big(\frac{\partial \varphi_1}{\partial x_1}\Big)\big(\underline{x}_1(x_2), x_2\big) = 0, && x_2 \in \big[x_2^A, x_2^B\big],\\
&\Big(\frac{\partial \varphi_1}{\partial x_2}\Big)\big(\underline{x}_1(x_2), x_2\big) = \Big(\frac{\partial \varphi_1}{\partial x_2}\Big)\big(x_1^*(x_2), x_2\big), && x_2 \in \big[x_2^A, x_2^B\big].
\end{aligned}$$

Similarly, for the C^1-pasting on the segment DC, which belongs to the curve $x_1 = \bar{x}_1(x_2)$ (right vertical curve in the picture):

$$\begin{aligned}
&\Big(\frac{\partial \varphi_1}{\partial x_1}\Big)\big(\bar{x}_1(x_2), x_2\big) = 0, && x_2 \in \big[x_2^D, x_2^C\big],\\
&\Big(\frac{\partial \varphi_1}{\partial x_2}\Big)\big(\bar{x}_1(x_2), x_2\big) = \Big(\frac{\partial \varphi_1}{\partial x_2}\Big)\big(x_1^*(x_2), x_2\big), && x_2 \in \big[x_2^D, x_2^C\big].
\end{aligned}$$

We can finally collect all the conditions our candidate function φ_1 must satisfy:

$$\begin{cases} \left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(x_1^*(x_2), x_2\big) = 0, & x_2 \in \mathbb{R}, \\ \varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\big(\underline{x}_1(x_2), x_2\big) + c_1, & x_2 \in \big[x_2^A, x_2^B\big], \\ \varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\Big(\underline{x}_1(x_2), x_2^*\big(\underline{x}_1(x_2)\big)\Big) + c_1, & x_2 \in \mathbb{R} \setminus \big[x_2^A, x_2^B\big], \\ \varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\big(\bar{x}_1(x_2), x_2\big) + c_1, & x_2 \in \big[x_2^D, x_2^C\big], \\ \varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\Big(\bar{x}_1(x_2), x_2^*\big(\bar{x}_1(x_2)\big)\Big) + c_1, & x_2 \in \mathbb{R} \setminus \big[x_2^D, x_2^C\big], \\ \varphi_1\big(x_1, x_2^*(x_1)\big) = \varphi_1\big(x_1, \underline{x}_2(x_1)\big), & x_1 \in \big]x_1^A, x_1^D\big[, \\ \varphi_1\big(x_1, x_2^*(x_1)\big) = \varphi_1\big(x_1, \bar{x}_2(x_1)\big), & x_1 \in \big]x_1^B, x_1^C\big[, \\ \left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(\underline{x}_1(x_2), x_2\big) = 0, & x_2 \in \big[x_2^A, x_2^B\big], \\ \left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(\underline{x}_1(x_2), x_2\big) = \left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(x_1^*(x_2), x_2\big), & x_2 \in \big[x_2^A, x_2^B\big], \\ \left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(\bar{x}_1(x_2), x_2\big) = 0, & x_2 \in \big[x_2^D, x_2^C\big], \\ \left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(\bar{x}_1(x_2), x_2\big) = \left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(x_1^*(x_2), x_2\big), & x_2 \in \big[x_2^D, x_2^C\big]. \end{cases}$$

Then, we have to consider the 11 equations above, along with the corresponding ones for φ_2. Therefore if a solution to such system exists, then we have a well-defined candidate and we can safely apply the verification theorem, as we did in the one-player case.

This problem remains still open. In particular, concerning the above system for φ_1, we considered the three test cases when $\bar{x}_1(\cdot), \underline{x}_1(\cdot)$ and $x_1^*(\cdot)$ are: constant, linear and quadratic functions of x_2. In none of these case we could find a satisfactory answer to our problem, which would most probably require the use of viscosity solutions as in [8] in order to go beyond the case of smooth value functions. This is postponed to future research.

3.2.3 The case of a stubborn competitor

We focus here on the case when one of the two players, say player 2, never changes her retail price (in this sense she is a stubborn competitor). Therefore her retail price is constant, i.e. $P_t^2 \equiv p_2$ for every $t \geq 0$. This can be artificially seen as a particular case of the two-player retail game of the previous section, supposing that player 2 has an infinite intervention cost, $c_2 = +\infty$. In other terms, player 2 intervention cost is so high that it is never optimal for her to change the retail price. Moreover, in order to base our intuition on the results we obtained in the one-player case, we assume that the wholesale price is driftless, i.e. $\mu = 0$.

In this situation the objective functional of player 1 (recall Equation (3.35)) is given by

$$J^1(x_1, x_2; \phi_1) \quad = \quad \mathbb{E}_{x_1,x_2}\left[\int_0^\infty e^{-\rho t}\left(X_t^1 \Phi(X_t^1 - X_t^2) - \frac{b_1}{2}\Phi^2(X_t^1 - X_t^2)\right)dt - c_1 \sum_{1 \leq k < M} e^{-\rho \tau_k^1}\right],$$

for every initial state $(x_1, x_2) \in \mathbb{R}^2$ (recall that $x_i = p_i - s$) and every strategy ϕ_1 (recall that player 2 does not intervene). Equivalently, choosing as state variables P^1 and S, the above functional reads

$$J^1(p_1, s; \phi_1) \quad = \quad \mathbb{E}_{p_1,s}\left[\int_0^\infty e^{-\rho t}\left((P_t^1 - S_t)\Phi(P_t^1 - p_2) - \frac{b_1}{2}\Phi^2(P_t^1 - p_2)\right)dt - c_1 \sum_{1 \leq k < M} e^{-\rho \tau_k^1}\right]$$

for every initial state $(p_1, s) \in \mathbb{R}^2$, where $p_1 = P_0^1$ and $s = S_0$, every strategy ϕ_1 and where $p_2 = P_0^2$. The problem is clearly simplified, since only one market actor is playing the game. Nevertheless, the setting is still bi-dimensional, since the state variables are P^1 and S. Notice that the impulses of player 1 modify only the state P^1.

Remark 3.12. *From now on we consider only the maximization problem of player* 1. *Since there is no ambiguity about which player is optimizing her objective, we drop both the subscript and the superscript* 1 *from the notation.*

For every $(p,s) \in \mathbb{R}^2$, player 1 value function $V(p,s)$ is defined as the supremum of $J(p,s;\phi)$ over all admissible strategies ϕ. Analogously to the heuristics in the previous section, it is reasonable to assume that the continuation region is included in the strip $\{(p,s) : p_2 - \Delta < p < p_2 + \Delta, s \in \mathbb{R}\}$ and the candidate $\widetilde{V}$ for the value function V is (recall that player 2 never intervenes because she has infinite intervention cost):

$$\widetilde{V} = \begin{cases} \mathcal{M}\widetilde{V}, & \text{in } \{\mathcal{M}\widetilde{V} - \widetilde{V} = 0\} \\ \varphi, & \text{in } \{\mathcal{M}\widetilde{V} - \widetilde{V} < 0\} \end{cases}, \tag{3.40}$$

where $\mathcal{M}\widetilde{V}(p,s) = \sup_{\delta \in \mathbb{R}} \widetilde{V}(p+\delta, s) - c$ and where φ solves

$$\mathcal{A}\varphi - \rho\varphi + f = \frac{1}{2}\sigma^2 \partial_{ss}\varphi - \rho\varphi + f = 0, \tag{3.41}$$

with

$$f(p,s) = -\frac{1}{2\Delta}(p-s)(p-p_2-\Delta) - \frac{b}{8\Delta^2}(p-p_2-\Delta)^2. \tag{3.42}$$

Hence we expect φ to be of the form

$$\varphi(p,s) = A(p)e^{ms} + B(p)e^{-ms} + \hat{\varphi}(p,s),$$

where A and B are suitable real functions, $m = \sqrt{2\rho}/\sigma$ and $\hat{\varphi}(p,s)$ is a particular solution of the ODE (3.41).

We conjecture that the continuation region is $\mathcal{D} = \{(p,s) \in \mathbb{R}^2 : p \in]\underline{p}(s), \overline{p}(s)[\}$, for suitable functions $\underline{p}, \overline{p}$. This means that whenever player 1 retail price P_t exits the interval $]\underline{p}(S_t), \overline{p}(S_t)[$, she intervenes to push her retail price towards a target price $p^*(S_t)$, where $p^*(s)$, for $s \in \mathbb{R}$, is obtained as the maximizer of the function $p \mapsto f(p,s)$ in (3.42) since, being $\mu = 0$, the optimizer of $\widetilde{V}(\cdot, s)$ is the same as the maximiser of $f(\cdot, s)$ (compare to Proposition 3.7). It is also reasonable to assume that this maximum point is unique and it belongs to the continuation region, so that we have

$$\{\delta(p,s)\} = \arg\max_{\delta \in \mathbb{R}} \widetilde{V}(p+\delta, s) \quad \text{and} \quad p + \delta(p,s) = p^*(s).$$

A simple computation gives

$$p^*(s) = \frac{2\Delta}{4\Delta + b}\left[\left(\frac{b}{2\Delta} + 1\right)(p_2 + \Delta) + s\right]. \tag{3.43}$$

Essentially, each time her price falls outside the continuation region $\mathcal{D}$, she intervenes to push the price towards the target $p^*(s)$. Moreover, intervention costs being fixed for player 1, we guess that $\underline{p}(s)$ and $\overline{p}(s)$ are equidistant from $p^*(s)$ in $\mathcal{D}$ as in the one-player case (see Proposition 3.7), hence $|p^*(s) - \underline{p}(s)| = |p^*(s) - \overline{p}(s)|$ in the continuation region.

Now, notice that $p^*(p_2+\Delta) = p_2 + \Delta$ and that the point $A := (p_2+\Delta, p_2+\Delta)$ belongs to the boundary of the intervention region. Moreover for any price $p \geq p_2 + \Delta$ the market share of player 1 is zero, so that the set $\{(p,s) : p \geq p_2 + \Delta\}$ is contained in the intervention region. Since $\underline{p}$ and $\overline{p}$ are equidistant from p^* at all point in the continuation region, they have to be equidistant from p^* at the point A as well, which implies that $\underline{p} = \overline{p} = p^*$ at the point A.

This situation is summarized in Figure 3.2, where $\underline{p}$ and $\overline{p}$ intersect at the point A, the continuation region is in white (W) and the intervention area is in red (R).

More precisely, we have:

$$\begin{gathered} \left\{\mathcal{M}\widetilde{V} - \widetilde{V} < 0\right\} = \left\{(p,s) : p \in]\underline{p}(s), \overline{p}(s)[, \ s < s_A\right\} = W, \\ \left\{\mathcal{M}\widetilde{V} - \widetilde{V} = 0\right\} = R = W^c, \end{gathered}$$

where we set $s_A := p_2 + \Delta$. So, the candidate function $\widetilde{V}$ is (notice that $\widetilde{V}(p^*(s), s) - c = \varphi(p^*(s), s) - c$):

$$\widetilde{V}(p,s) = \begin{cases} \varphi(p^*(s), s) - c, & \text{in R} \\ \varphi(p,s), & \text{in W.} \end{cases}$$

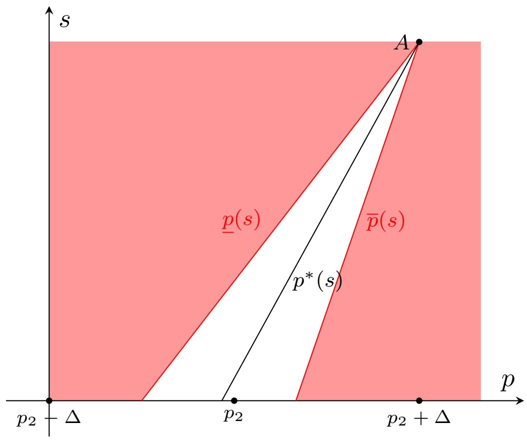

Figure 3.2: Partition of the domain in the two regions R (red) and W (white) depending on possible player 1 interventions in the case of a stubborn competitor.

The regularity conditions of the value function required by the verification theorem are: $V \in C^2(\mathbb{R}^2 \setminus \partial\mathcal{D}) \cap C^1(\mathbb{R}^2)$, so that it suffices to ask optimality of $p^*(s)$ and a C^0 and a C^1-pasting at the frontier of $\mathcal{D}$: for all $s < s_A = p_2 + \Delta$ we have

$$
\begin{aligned}
&\Big(\frac{\partial\varphi}{\partial p}\Big)\big(p^*(s), s\big) = 0,\\
&\varphi\big(p^*(s), s\big) - c = \varphi\big(\underline{p}(s), s\big),\\
&\varphi\big(p^*(s), s\big) - c = \varphi\big(\overline{p}(s), s\big),\\
&\Big(\frac{\partial\varphi}{\partial p}\Big)\big(\underline{p}(s), s\big) = \Big(\frac{\partial\varphi}{\partial p}\Big)\big(p^*(s), s\big) = 0,\\
&\Big(\frac{\partial\varphi}{\partial p}\Big)\big(\overline{p}(s), s\big) = \Big(\frac{\partial\varphi}{\partial p}\Big)\big(p^*(s), s\big) = 0,\\
&\Big(\frac{\partial\varphi}{\partial s}\Big)\big(\underline{p}(s), s\big) = \Big(\frac{\partial\varphi}{\partial s}\Big)\big(p^*(s), s\big),\\
&\Big(\frac{\partial\varphi}{\partial s}\Big)\big(\overline{p}(s), s\big) = \Big(\frac{\partial\varphi}{\partial s}\Big)\big(p^*(s), s\big).
\end{aligned}
$$

As the points $(p^*(s), s)$ belong to the continuation region for each $s < s_A$, we have

$$V(p^*(s_{A^-}), s_{A^-}) = \varphi(p^*(s_{A^-}), s_{A^-}).$$

Moreover, as the point $A = (p^*(s_A), s_A)$ belongs to the intervention region, we have $V(p^*(s_A), s_A) = \varphi(p^*(s_A), s_A) - c$. Since φ is continuous by definition, we deduce that V is not continuous at A, giving one more argument urging the use of viscosity solutions for a rigorous treatment of such models.

Remark 3.13. *From an economical point of view, this situation makes sense: by keeping her retail price constant when the sourcing cost increases, player 2 forces her opponent (player 1) to increase unilaterally her price and thus to lose a bigger and bigger market share until she exits the market. This is a strategy that could be implemented by financially sound players (i.e. able to endure financial losses due to a retail price lower than the sourcing cost) to push their weak competitors out of the market.*

4 Conclusions

In this paper we consider a general two-player nonzero-sum impulse game, whose state variable follows a diffusive dynamics driven by a multi-dimensional Brownian motion. After setting the problem, we provide a verification theorem giving sufficient conditions in order for the solutions of a suitable system of quasi-variational inequalities to coincide with the value functions of the two players. To the best of our knowledge this result is new to the literature on impulse games and it constitutes the major mathematical contribution of the present paper. The general setting is motivated by a model of competition among retailers in electricity markets, which we also treat in both one-player and two-player cases. While in the one-player case we gave a full rigorous treatment of the impulse control problem, in the two-player case we provide a detailed heuristic study of the shape of the value functions and their optimal strategies. Making the heuristics fully rigorous would most probably require the use of viscosity solutions, which looks far from being an easy extension of the methods employed in [8] for zero-sum impulse games. This is left to future research.

References

[1] M. Basei, *Topics in stochastic control and differential game theory, with application to mathematical finance.* Ph.D. Thesis in Mathematics, University of Padova (2016).

[2] A. Bensoussan, A. Friedman, *Nonzero-sum stochastic differential games with stopping times and free boundary problems*, Trans. Amer. Math. Society 231 (1977), no. 2, 275–327.

[3] A. Bensoussan, J. L. Lions, *Contrôle impulsionnel et inéquations quasi variationnelles*, Vol. 1, Dunod, Paris, 1982.

[4] A. Cadenillas, P. Lakner, P. Pinedo, *Optimal control of a mean-reverting inventory*, Operations Research 58(9) (2010), 1697–1710.

[5] D. Chang, H. Wang, Z. Wu, *Maximum principle for non-zero sum differential games of BSDEs involving impulse controls*, Control Conference (CCC), 2013 32nd Chinese, IEEE, 1564–1569.

[6] D. Chang, Z. Wu, *Stochastic maximum principle for non-zero sum differential games of FBSDEs with impulse controls and its application to finance*, Journal of industrial and management optimization, 11.1 (2015), 27–40.

[7] N. Chen, M. Dai, X. Wan, *A Nonzero-Sum Game Approach to Convertible Bonds: Tax Benefit, Bankruptcy Cost, and Early/Late Calls*, Mathematical Finance 23 (2013), no. 1, 57–93.

[8] A. Cosso, *Stochastic differential games involving impulse controls and double-obstacle quasi-variational inequalities*, SIAM J. Control Optim. 51 (2013), no. 3, 2102–2131.

[9] T. De Angelis, G. Ferrari, J. Moriarty. *Nash equilibria of threshold type for two-player nonzero-sum games of stopping.* Preprint (2015), arXiv:1508.03989v1 [math.PR].

[10] A. Friedman, *Stochastic games and variational inequalities*, Arch. Rational Mech. Anal. 51 (1973), no. 5, 321–346.

[11] P. Joskow, J. Tirole, *Electricity retail competition*, RAND Journal of Economics, 37 (2006), no. 4, 799–815.

[12] B.K. Øksendal, A. Sulem, *Applied stochastic control of jump diffusions, Second Edition*, Springer-Verlag, Berlin-Heidelberg, 2007.

[13] N.-H.M. von der Fehr, P.V. Hansen, *Electricity retailing in Norway*, The Energy Journal 13 (2010), no. 1, 25–45.

A Auxiliary results for the one-player case in the retail energy market model

In this appendix we have gathered all auxiliary results and proofs that have been used in the one-player Section 3.1. Proposition 3.7 follows from Lemmas A.1 and A.2.

Lemma A.1. *A solution* $(A, \bar{y}) \in \,]0, +\infty[^2$ *to* (3.27a)-(3.27b) *exists and it is unique.*

Proof. First step. Let us start by Equation (3.27a). For a fixed $A > 0$, we are looking for the strictly positive zeros of the function h_A defined by

$$h_A(y) = A\theta e^{\theta y} - A\theta e^{-\theta y} - 2k_2 y, \tag{A.1}$$

for each $y > 0$. The derivative is

$$h'_A(y) = A\theta^2 e^{\theta y} + A\theta^2 e^{-\theta y} - 2k_2 = \frac{A\theta^2 \left(e^{\theta y}\right)^2 - 2k_2\left(e^{\theta y}\right) + A\theta^2}{e^{\theta y}}.$$

We need to consider two cases, according to the value of A. Let

$$\bar{A} = \frac{k_2}{\theta^2} = \frac{\sigma^2(2\Delta + b)}{4\rho^2\Delta^2}. \tag{A.2}$$

If $A \geq \bar{A}$ we have $h'_A > 0$ in $]0, \infty[$; hence, since $h_A(0) = 0$, Equation (3.27a) does not have any solution in $[0, +\infty[$. On the contrary, if $A < \bar{A}$ we have $h'_A < 0$ in $]0, \widetilde{y}[$ and $h'_A > 0$ in $]\widetilde{y}, \infty[$, for a suitable $\widetilde{y} = \widetilde{y}(A) > 0$; hence, since $h_A(0) = 0$ and $h_A(+\infty) = +\infty$, Equation (3.27a) has exactly one solution $\bar{y} = \bar{y}(A) > 0$ (notice that $\bar{y}(A) > \widetilde{y}(A)$). In short, we have proved that, for a fixed $A > 0$, Equation (3.27a) admits a solution $\bar{y} \in]0, \infty[$ if and only if $A \in]0, \bar{A}[$; in this case the solution is unique and we denote it by $\bar{y} = \bar{y}(A)$.

Finally, we remark that

$$\lim_{A \to 0^+} \bar{y}(A) = +\infty, \qquad \lim_{A \to \bar{A}^-} \bar{y}(A) = 0. \tag{A.3}$$

The first limit follows by $\bar{y}(A) > \widetilde{y}(A)$ and $\lim_{A\to 0^+} \widetilde{y}(A) = +\infty$ (this one by a direct computation of $\widetilde{y}$), whereas the second limit is immediate.

Second step. We now consider Equation (3.27b). For each $A \in \,]0, \bar{A}[$, we define

$$g(A) = -Ae^{\theta\bar{y}(A)} - Ae^{-\theta\bar{y}(A)} + k_2\bar{y}^2(A) + 2A, \tag{A.4}$$

where $\bar{y}(A)$ is well-defined by the first step. We are going to prove that

$$\lim_{A \to 0^+} g(A) = +\infty, \qquad \lim_{A \to \bar{A}^-} g(A) = 0, \qquad g' < 0. \tag{A.5}$$

This concludes the proof: indeed, if (A.5) holds, it follows that the equation $g(A) = c$, which is just a rewriting of (3.27b), has exactly one solution $A \in]0, \bar{A}[$. It is then clear that the pair $(A, \bar{y}(A))$ is a solution to (3.27a)-(3.27b) (the unique one, since uniqueness holds for (3.27b)).

It remains to check (A.5). For the first claim in (A.5), by (3.27a) we can write A as a function of $\bar{y}$,

$$A = \frac{2k_2}{\theta} \frac{\bar{y}(A)}{e^{\theta\bar{y}(A)} - e^{-\theta\bar{y}(A)}}, \tag{A.6}$$

so that g also reads

$$g(A) = k_2\bar{y}^2(A) - \frac{2k_2}{\theta} \frac{e^{\theta\bar{y}(A)} + e^{-\theta\bar{y}(A)} - 2}{e^{\theta\bar{y}(A)} - e^{-\theta\bar{y}(A)}} \bar{y}(A), \tag{A.7}$$

which we rewrite as

$$g(A) = k_2\bar{y}^2(A) - \frac{2k_2}{\theta} \frac{(e^{\theta\bar{y}(A)} - 1)^2}{(e^{\theta\bar{y}(A)})^2 - 1} \bar{y}(A); \tag{A.8}$$

then, by (A.3) we have

$$\lim_{A \to 0^+} g(A) = \lim_{z \to +\infty} \left(k_2 z^2 - \frac{2k_2}{\theta} \frac{(e^{\theta z} - 1)^2}{(e^{\theta z})^2 - 1} z \right) = +\infty.$$

As for the second claim in (A.5), it is immediate by the definition of g and by (A.3). We finally show that the third claim in (A.5) holds. Notice that

$$g'(A) = -e^{\theta\bar{y}(A)} - e^{-\theta\bar{y}(A)} + 2 - \left(A\theta e^{\theta\bar{y}} - A\theta e^{-\theta\bar{y}} - 2k_2\bar{y}\right)\bar{y}'(A).$$

By (3.27a), the coefficient of $\bar{y}'(A)$ is zero; thus, we have

$$g'(A) = -e^{\theta\bar{y}(A)} - e^{-\theta\bar{y}(A)} + 2 = -\frac{(e^{\theta\bar{y}(A)} - 1)^2}{e^{\theta\bar{y}(A)}} < 0, \tag{A.9}$$

which concludes the proof. □

As already noticed in (A.6), using (3.27a) we can write A as a function of $\bar{y}$: for every $\bar{y} > 0$ we have

$$A(\bar{y}) = \frac{2k_2}{\theta}\frac{\bar{y}}{e^{\theta\bar{y}} - e^{-\theta\bar{y}}}. \tag{A.10}$$

We are going to consider the function $\xi := g \circ A$, where g has been defined in (A.4) and A is as in (A.10). In (A.8) we have already computed an expression for ξ, which we recall here: for every $\bar{y} > 0$ we have

$$\xi(\bar{y}) = (g \circ A)(\bar{y}) = k_2\bar{y}^2 - \frac{2k_2}{\theta}\frac{(e^{\theta\bar{y}} - 1)^2}{(e^{\theta\bar{y}})^2 - 1}\bar{y}. \tag{A.11}$$

Lemma A.2. *Let $(A, \bar{y})$ be as in Lemma A.1 and let $\bar{c} = \xi(\Delta^2/(2\Delta + b))$, with ξ as in* (A.11)*. Then, the condition in* (3.28) *is satisfied if and only if $c \leq \bar{c}$.*

Proof. Let g be as in (A.4) and assume, for the moment, that the function A in (A.10) is decreasing. Then, since g is decreasing by (A.5), we deduce that $\xi = g \circ A$ is increasing. Hence, we have $\bar{y} < \Delta - x_v$ if and only if $\xi(\bar{y}) < \xi(\Delta - x_v)$. The conclusion follows since $\xi(\bar{y}) = g(A(\bar{y})) = c$ by (3.27b) and since $\Delta - x_v = \Delta^2/(2\Delta + b)$ by (3.11).

So, we just need to prove that $\bar{y} \mapsto A(\bar{y})$ is decreasing. A direct differentiation in (A.10) leads to an expression whose sign is not easy to estimate. Then, we write $A = A(\bar{y})$ in (3.27a) and differentiate with respect to $\bar{y}$. We get

$$A'(\bar{y})\theta\big(e^{\theta\bar{y}} - e^{-\theta\bar{y}}\big) + A(\bar{y})\theta^2\big(e^{\theta\bar{y}} + e^{-\theta\bar{y}}\big) - 2k_2 = 0,$$

so that, after rearranging, we have

$$A'(\bar{y}) = -\frac{A(\bar{y})\theta^2 e^{\theta\bar{y}} + A(\bar{y})\theta^2 e^{-\theta\bar{y}} - 2k_2}{\theta(e^{\theta\bar{y}} - e^{-\theta\bar{y}})} = -\frac{h'_{A(\bar{y})}(\bar{y})}{\theta(e^{\theta\bar{y}} - e^{-\theta\bar{y}})} < 0, \tag{A.12}$$

where in the numerator we have recognized $h'_{A(\bar{y})}(\bar{y})$, with $h_{A(\bar{y})}$ as in (A.1), and we have $h'_{A(\bar{y})}(\bar{y}) > 0$ since $h_{A(\bar{y})}$ is increasing in $[\tilde{y}, +\infty[\ni \bar{y}$ (see Lemma A.1). □

Lemma A.3. *Let* (3.23) *hold and let $\widetilde{V}$ and x^* be as in Definition 3.6. Then, for every $x \in \mathbb{R}$ we have*

$$\mathcal{M}\widetilde{V}(x) = \varphi_A(x^*) - c.$$

In particular, we have

$$\{\mathcal{M}\widetilde{V} - \widetilde{V} < 0\} =]\underline{x}, \bar{x}[, \qquad \{\mathcal{M}\widetilde{V} - \widetilde{V} = 0\} = \mathbb{R} \setminus]\underline{x}, \bar{x}[. \tag{A.13}$$

Proof. First of all, recall that $\widetilde{V}$ is symmetric with respect to x^* and notice that:

- $\widetilde{V}$ is strictly decreasing in $]x^*, \bar{x}[$ (since we have $\widetilde{V} = \varphi_A$ by definition and $\varphi'_A < 0$ in $]x^*, \bar{x}[$ by the proof of Lemma A.1);
- $\widetilde{V}$ is constant in $[\bar{x}, +\infty[$ by definition of $\widetilde{V}$, with $\widetilde{V} \equiv \varphi_A(x^*) - c$.

Then, we deduce that

$$\max_{y\in\mathbb{R}} \widetilde{V}(y) = \widetilde{V}(x^*) = \varphi_A(x^*), \qquad \min_{y\in\mathbb{R}} \widetilde{V}(y) = \varphi_A(x^*) - c. \tag{A.14}$$

As a consequence, for every $x \in \mathbb{R}$ we have

$$\mathcal{M}\widetilde{V}(x) = \max_{\delta\in\mathbb{R}}\{\widetilde{V}(x + \delta) - c\} = \max_{y\in\mathbb{R}} \widetilde{V}(y) - c = \varphi_A(x^*) - c.$$

By the definition of $\widetilde{V}$, we have

$$\mathcal{M}\widetilde{V}(x) - \widetilde{V}(x) = 0, \qquad \text{in } \mathbb{R}\backslash]\underline{x}, \bar{x}[.$$

Moreover, as $\varphi_A(\bar{x}) = \varphi_A(x^*) - c$ by (3.25) and $\varphi_A(\bar{x}) = \min_{[\underline{x},\bar{x}]} \varphi_A$ by the previous arguments, we have

$$\mathcal{M}\widetilde{V}(x) - \widetilde{V}(x) = \varphi_A(x^*) - c - \varphi_A(x) = \varphi_A(\bar{x}) - \varphi_A(x) < 0, \qquad \text{in }]\underline{x}, \bar{x}[,$$

which concludes the proof. □

We conclude this appendix with the proof of Proposition 3.8.

Proof of Proposition 3.8. We have to check that the candidate $\widetilde{V}$ satisfies all the assumptions of Proposition 3.4. For the reader's convenience, we briefly report these conditions:

(i) $\widetilde{V}$ is bounded and $\max_{x\in\mathbb{R}} \widetilde{V}(x)$ exists;

(ii) $\widetilde{V} \in C^2(\mathbb{R} \setminus \{\underline{x}, \bar{x}\}) \cap C^1(\mathbb{R})$;

(iii) $\widetilde{V}$ satisfies $\max\{\mathcal{A}\widetilde{V} - \rho\widetilde{V} + R, \mathcal{M}\widetilde{V} - \widetilde{V}\} = 0$;

(iv) the optimal control is admissible, i.e., $u^*(x) \in \mathcal{U}$ for every $x \in \mathbb{R}$.

Condition (i) and (ii). The first condition holds by (A.14), whereas the second condition follows by the definition of $\widetilde{V}$.

Condition (iii). We have to prove that for every $x \in \mathbb{R}$ we have

$$\max\{\mathcal{A}\widetilde{V}(x) - \rho\widetilde{V}(x) + R(x), \mathcal{M}\widetilde{V}(x) - \widetilde{V}(x)\} = 0. \tag{A.15}$$

In $]\underline{x}, \bar{x}[$ the claim is true, as $\mathcal{M}\widetilde{V} - \widetilde{V} < 0$ by (A.13) and $\mathcal{A}\widetilde{V} - \rho\widetilde{V} + R = 0$ by definition (recall that here we have $R = f$ and $\widetilde{V} = \varphi_A$, with $\mathcal{A}\varphi_A - \rho\varphi_A + f = 0$). As for $\mathbb{R}\backslash]\underline{x}, \bar{x}[$, we already know by (A.13) that $\mathcal{M}\widetilde{V} - \widetilde{V} = 0$. Then, to conclude we have to prove that

$$\mathcal{A}\widetilde{V}(x) - \rho\widetilde{V}(x) + R(x) \leq 0, \qquad \forall x \in \mathbb{R}\backslash]\underline{x}, \bar{x}[.$$

By symmetry, it is enough to prove the claim for $x \in [\bar{x}, +\infty[$. By the definition of $\widetilde{V}(x)$ and (3.25), in the interval $[\bar{x}, +\infty[$ we have $\widetilde{V} \equiv \varphi_A(x^*) - c = \varphi_A(\bar{x})$; hence, the inequality reads

$$-\rho\varphi_A(\bar{x}) + R(x) \leq 0, \qquad \forall x \in [\bar{x}, +\infty[.$$

As R is decreasing in $[x_v, +\infty[\supseteq [\bar{x}, +\infty[$, it is enough to prove the claim in $x = \bar{x}$:

$$-\rho\varphi_A(\bar{x}) + R(\bar{x}) \leq 0.$$

Since $\mathcal{A}\varphi_A(\bar{x}) - \rho\varphi_A(\bar{x}) + f(\bar{x}) = 0$ and $f(\bar{x}) = R(\bar{x})$, we have

$$-\rho\varphi_A(\bar{x}) + f(\bar{x}) = -\frac{\sigma^2}{2}\varphi_A''(\bar{x}) \leq 0,$$

which is true as $\bar{x}$ is a local minimum of $\varphi_A \in C^\infty(\mathbb{R})$, so that $\varphi_A''(\bar{x}) \geq 0$.

Condition (iv). Let $x \in \mathbb{R}$ and recall (3.7): we have to show that

$$\mathbb{E}_x\left[\sum_{k\geq 1} e^{-\rho\tau_k^*}\right] < \infty.$$

When using the optimal control u^*, the retailer intervenes when the process hits $\underline{x}$ or $\bar{x}$ and shifts the process to $x^* \in]\underline{x}, \bar{x}[$. As a consequence, we can decompose each variable τ_k^* as a sum of suitable exit times from $]\underline{x}, \bar{x}[$. Given $y \in \mathbb{R}$, let ζ^y denote the exit time of the process $y + \sigma W$, where W is a Brownian motion, from the interval $]\underline{x}, \bar{x}[$. Then, we have $\tau_1^* = \zeta^x$ and

$$\tau_k^* = \zeta^x + \sum_{l=1}^{k-1} \zeta_l^{x^*},$$

for every $k \geq 2$, where the variables $\zeta_l^{x^*}$ are independent and distributed as ζ^{x^*}. As a consequence, we have

$$\mathbb{E}_x\left[\sum_{k\geq 2} e^{-\rho\tau_k^*}\right] = \mathbb{E}_x\left[\sum_{k\geq 2} e^{-\rho\left(\zeta^x+\sum_{l=1}^{k-1}\zeta_l^{x^*}\right)}\right] = \mathbb{E}_x\left[e^{-\rho\zeta^x}\sum_{k\geq 2}\prod_{l=1,\dots,k-1} e^{-\rho\zeta_l^{x^*}}\right].$$

By the Fubini-Tonelli theorem and the independence of the variables:

$$\mathbb{E}_x\left[e^{-\rho\zeta^x}\sum_{k\geq 2}\prod_{l=1,\dots,k-1} e^{-\rho\zeta_l^{x^*}}\right] = \mathbb{E}_x\left[e^{-\rho\zeta^x}\right]\sum_{k\geq 2}\prod_{l=1,\dots,k-1}\mathbb{E}_x\left[e^{-\rho\zeta_l^{x^*}}\right].$$

As the variables $\zeta_l^{x^*}$ are identically distributed with $\zeta_l^{x^*} \sim \zeta^{x^*}$, we can conclude:

$$\sum_{k\geq 2}\prod_{l=1,\dots,k-1}\mathbb{E}_x\left[e^{-\rho\zeta_l^{x^*}}\right] = \sum_{k\geq 2}\mathbb{E}_x\left[e^{-\rho\zeta^{x^*}}\right]^{k-1} < \infty,$$

which is a converging geometric series. $\square$

Finance for Energy Market Research Centre

Institut de Finance de Dauphine, Université Paris-Dauphine
1 place du Maréchal de Lattre de Tassigny
75775 PARIS Cedex 16
www.fime-lab.org