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Testing the Nullity of Coefficients of a GARCH Model with Exogenously-Driven Volatility

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Nazim Regnard *

Abstract. This paper establishes the asymptotic properties of the quasi-maximum likelihood estimator (QMLE) of a GARCH(1,1) process with time-varying coefficients driven by an exogenous variable, when some true coefficients may be null. The QMLE is shown to be consistent. Its asymptotic distribution is a projection of a normal vector distribution onto a convex cone. Furthermore, the QMLE is shown to converge to its asymptotic distribution locally uniformly. We then consider the problem of testing that one or several coefficients are equal to zero. The null distribution and the local asymptotic powers of the Wald, Rao-Score and Quasi-Likelihood Ratio tests are derived. The results are derived under mild conditions, that do not require the existence of moments of the observed process. The results are illustrated by numerical simulations. The framework developed here allows some intercepts to be null for certain regimes.

Keywords. Asymptotic efficiency of tests, Boundary, GARCH, Time-varying models, Chi bar distribution, non-stationary processes, Quasi Maximum Likelihood Estimation, Local alternatives.

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1 Introduction

Since the seminal papers of Engle (1982) and Bollerslev (1986), the class of generalized autoregressive conditionally heteroskedastic (GARCH) models was extended to fit apparent high persistence of the volatility dynamic. Indeed, as put forward e.g. by Lamoureux and Lastrapes (1990), and Mikosch and Starica (2004), this persistence might be spuriously caused by parameters varying through the sample, corresponding to different volatility regimes. Different classes of regime switching GARCH models were developed to cope with this problem. Markov-switching (MS) GARCH models have been proposed, in the spirit of the class developed by Hamilton (1989) for ARMA models; see e.g. Hamilton and Susmel (1994), Gray (1996), Dueker (1997), Haas, Mittnik and Paolella (2004). In this approach, the regime process is endogenously determined by the data. However, when one exogenous variable influences importantly the regime switching, a relevant approach could consist in imposing it as generating the regimes. When the exogenous variable driving the regime is periodic, an approach with periodic coefficients might be relevant. These models have been proposed by Basawa and Lund (2001), among others, for ARMA Models and in the GARCH framework by Bollerslev and Ghysels (1996) through their Periodic GARCH (PGARCH). However, periodic models are not suitable when the change of dynamics occurs at irregular dates. Time series models in which the coefficients are subordinated to an exogenous process have been recently proposed and analyzed for the conditional mean by Francq and Gautier (2004a, 2004b) (see also Dalhaus (1997), Bibi and Francq (2003), Azrak and Mélard (2006)) and for the GARCH framework by Regnard and Zakoïan (2010) (hereafter RZ) who introduce a GARCH(1,1) with regimes subordinated to an exogenous variable. They have proved consistency and asymptotic normality of the Quasi Maximum Likelihood Estimator (QMLE) under mild assumptions. Regnard and Zakoïan (2011) have applied this class of Model with parameters indexed by temperature level to natural gas prices. In RZ, the asymptotic behaviour of the QMLE is derived when the true value is *strictly nonnegative*. In this paper, we investigate the behaviour of the QMLE when this assumption is relaxed.

The contribution of this paper is threefold. First, we derive the asymptotic properties of the QMLE when some true coefficients are null under assumptions that do not require the existence of moments of the observed process. In the GARCH(p, q) framework, similar problems are adressed in Andrews (2001) for GARCH(1,1), and more recently in Francq and Zakoïan (2007) in the general GARCH(p, q) $(p, q \geq 1)$. In the standard GARCH(p, q) framework, when some true parameters are null, this problem is non trivial because the QMLE is positively constrained, and thus can not be asymptotically Gaussian. Francq and Zakoïan (2007) derived its asymptotic distribution as a projection of a multivariate Gaussian distribution onto the positive quadrant for the metric defined by the asymptotic covariance matrix of the score vector. The difficulties faced in our framework are of same nature. Second, we study under assumptions not requiring the existence of moments of the observed process the asymptotic properties of the QMLE under a sequence of local alternatives converging to the true value. This problem is adressed in Francq and Zakoïan (2009) for the standard GARCH(p, q). They provide an asymptotic distribution of the QMLE under local alternatives. We will find similar results. Third, we will derive under (i) and (ii) the behaviour of three tests statistics : the Wald, the Rao-Score (or Lagrange Multiplier) Test and the Likelihood Ratio statistics. Previous works focused on testing nullity of some coefficients of ARCH and GARCH models. Andrews (2001) considered testing conditional homoscedasticity against a GARCH(1,1) model. Hong (1997) and Hong and Lee (2001) proposed tests for ARCH effects using spectral density estimators at frequency zero of a squared regression residual series. For other works, see Dufour et al. (2004) and more recently Francq and Zakoïan (2009).

In the large literature developed on estimation of switching regime GARCH models, works only focus, to our knowledge, on estimation when all parameters are strictly nonnegative. We refer to Francq and Zakoïan (2008) for estimation of MS-GARCH by generalized method of moments, Hamilton and Susmel (1994) by QMLE (see also Francq, Roussignol and Zakoïan (2001)) and Aknouche and Bibi (2009) for estimation of PGARCH by QMLE. These works were made in the framework of true coefficients all strictly nonnegative and do not treat asymptotic properties of consistent estimators, when some of them are null. In the standard GARCH framework, tests of nullity only concern the nullity of persistence coefficients, not of the intercept term. Indeed, it must be chosen strictly nonnegative to ensure non degeneracy of the volatility. Switching regimes GARCH models possess one similar intercept by regime. Actually, it is possible to define a switching regime volatility with some null intercept which would be non degenerated. This case might thus constitute a case of interest. To our knowledge, none of the GARCH literature have adressed the problem of testing the nullity of an intercept in the class of MS GARCH or Periodic GARCH models. The results presented here will constitute the first results to our knowledge on testing nullity of coefficients of a regime-switching GARCH model, developing a mathematical framework allowing to test for nullity of intercepts of some regimes.

The paper is organized as follows. Section 2 recalls the model and the results from RZ on estimation when the true parameter is strictly non negative. Section 3 presents results on the asymptotic behaviour of the QMLE (i) when the parameter is on the boundary and (ii) under sequences of local alternatives. Section 4 presents the asymptotic behaviour of the three tests mentioned above under (i) and (ii). Numerical simulations are provided in Section 5. Proofs of theorem are gathered in Section 7.

2 The model

In this section, after presenting in a first part the model and some of its specific probability properties, we recall in a second part some results of estimation by QMLE from RZ.

2.1 Probabilistic properties of the model

We consider the time-varying coefficients GARCH(1,1) model

$$\epsilon_t = \sigma_t \eta_t, \quad \sigma_t^2 = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2, \quad t \in \mathbb{Z} \tag{2.1}$$

where (η_t) is a sequence of independent and identically distributed (iid) centered variables with unit variance; (s_t) is the realization of a process (S_t) with values in a finite set $E = \{e_1, \ldots, e_d\}$; the functions $\omega(\cdot), \alpha(\cdot), \beta(\cdot)$ are defined on E with values in $\mathbb{R}^+$.

In each "regime" (i.e. for each value of s_t) the volatility of Model (2.1) is that of a standard GARCH$(1,1)$ model. In the particular case where (S_t) is periodic, we retrieve the periodic-GARCH model introduced by Bollerslev and Ghysels (1996). The model is closely related to Markov Switching (MS) models. In both models, different regimes are allowed for the volatility. Moreover in Model (2.1), (s_t) can be the realization of a Markov chain. However, a crucial difference is that in MS models, the change of regime is governed by a latent unobservable variable. Another important difference concerns the probabilistic properties of the solutions. Under appropriate conditions, MS models admit stationary solutions. In our model, the change of regime is observed and, in general, the model has a *nonstationary* solution. The model (2.1) is appropriate for a certain type of non stationarity in the volatility. A particular case of interest is the PGARCH, in which the s_t follow a purely periodic path. The model of this paper does not assume periodicity for (S_t) but can accommodate situations where the volatility coefficients fluctuate in time with a certain regularity .

The elementary probabilistic properties of this model such as conditions of existence of (ϵ_t) or its moments have been established by RZ. Another important property concerns the conditions of non degeneracy of (σ_t^2) i.e. conditions ensuring:

$$\sigma_t^2 > 0 \quad a.s. \quad \text{for all} \quad t \in \mathbb{Z}. \tag{2.2}$$

In the standard GARCH framework, (2.2) is equivalent to $\omega(.) = \omega > 0$. We will see hereafter that in the framework of model (2.1), the inequality $\omega(.) > 0$ stays a sufficient condition for (2.2) to hold, but is not a necessary one.

After recalling the conditions of existence of ϵ_t studied in RZ, we present hereafter (without proofs here for brevity) the conditions of non degeneracy of model (2.1) compatible with nullity of $\omega(.)$ for certain regimes.

Conditions of existence of ϵ_t Assuming

A0: (s_t) is a realization of a process (S_t) which is stationary, ergodic, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ as (η_t), and independent of (η_t)

and letting

$$\pi_j = P(S_t = e_j),\ j = \{1, \ldots, d\} \quad \text{and} \quad a(x, y) = \alpha(x)y^2 + \beta(x),$$

RZ established that if

$$\gamma_0 := \sum_{j=1}^{d} \pi_j E\{\log a(e_j, \eta_0)\} < 0, \qquad (2.3)$$

the model (2.1) admits a non anticipative non explosive solution (ϵ_t). RZ also discussed probabilistic properties of this process, such as the existence of moments at order $m \in \mathbb{N}$. It will be convenient to introduce the process

$$\epsilon_{S,t} = \sigma_{S,t}\eta_t, \quad \sigma^2_{S,t} = \omega(S_t) + \alpha(S_t)\epsilon^2_{S,t-1} + \beta(S_t)\sigma^2_{S,t-1}, \quad t \in \mathbb{Z}. \qquad (2.4)$$

Under the condition (2.3), this model admits a strictly stationary solution $(\epsilon_{S,t})$, obtained by replacing the sequence (s_t) by (S_t) in (2.1). Model (2.1) and (2.4) have closely related probabilistic properties. For instance, their condition of existence are exactly the same. However, the conditions for the existence of their moments differ significantly.

Conditions of non degeneracy The quantity $\sigma^2_{S,t}$ may be used to explicit an equivalent condition to (2.2). Indeed, under assumption **A0**, an adaptation of Lemma 1 in Francq and Gautier (2004a) allows to show that

$$\text{if} \quad \mathbb{P}(\sigma^2_{S,t} = 0) = 0 \quad \text{then} \quad \mathbb{P}(\sigma^2_t = 0) = 0, \quad \forall t \in \mathbb{Z}.$$

Thus (2.2) is equivalent to $\mathbb{P}(\sigma^2_{S,t} = 0) = 0$. In view of the following expansion of (2.4):

$$\sigma^2_{S,t} = \omega(S_t) + \sum_{n=1}^{+\infty} a(S_t, \eta_{t-1}) \dots a(S_{t-n+1}, \eta_{t-n})\omega(S_{t-n}),$$

we can state the following assumptions on $\{\omega(.), \alpha(.), \beta(.), (S_t), (\eta_t)\}$ allowing to ensure $\mathbb{P}(\sigma^2_{S,t} = 0) = 0$ and consequently (2.2) :

B1: $\omega(.) > 0$,

B2: For all i, $\mathbb{P}(\alpha(e_i)\eta^2_t + \beta(e_i) = 0) = 0$ and $\mathbb{P}(\omega(S_{-k}) = 0, k \geq 0) = 0$.

The assumption **B2** is compatible with a function $\omega(.)$ that may be null for certain regimes. For instance, it is satisfied when $\omega(e_1) > 0$ and $\omega(e_j) = 0$, for $j \geq 2$ and (S_t) drawing from an iid Bernoulli distribution with $0 < \mathbb{P}(S_t = e_i) < 1$ for all i. Note however that both assumptions preclude $\omega(.) = 0$.

2.2 QML Estimation

Turning to the estimation of the model (2.1), let θ denote the vector of parameters, $\theta = (\omega(e_1), \dots, \omega(e_d),\ \alpha(e_1), \dots, \alpha(e_d), \beta(e_1), \dots, \beta(e_d))'$, with true value θ_0. The parameter is assumed to belong to a parameter space $\Theta \subset [0, +\infty[^{3d}$. The sequence (s_t) is observed, and the order d is known a priori. Let $(\epsilon_1, \dots, \epsilon_n)$ be a realization of length n of the non anticipative solution (ϵ_t). Conditionally on initial values $\tilde{\epsilon}_0$ and $\tilde{\sigma}_0^2$ the Gaussian quasi-likelihood is given by

$$L_n(\theta) = L_n(\theta; \epsilon_1, \dots, \epsilon_n) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi\tilde{\sigma}_t^2}} \exp\left(-\frac{\epsilon_t^2}{2\tilde{\sigma}_t^2}\right),$$

where the $\tilde{\sigma}_t^2$ are defined recursively, for $t \geq 2$, by

$$\tilde{\sigma}_t^2 = \tilde{\sigma}_t^2(\theta) = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\tilde{\sigma}_{t-1}^2.$$

with $\tilde{\sigma}_1^2 = \omega(s_1) + \alpha(s_1)\tilde{\epsilon}_0^2 + \beta(s_1)\tilde{\sigma}_0^2$. A QMLE of θ is defined as any measurable solution $\hat{\theta}_n$ of

$$\hat{\theta}_n = \arg\max_{\theta \in \Theta} L_n(\theta) = \arg\min_{\theta \in \Theta} \tilde{\mathbf{l}}_n(\theta), \tag{2.5}$$

where

$$\tilde{\mathbf{l}}_n(\theta) = n^{-1} \sum_{t=1}^{n} \tilde{\ell}_t, \qquad \text{and} \quad \tilde{\ell}_t = \tilde{\ell}_t(\theta) = \frac{\epsilon_t^2}{\tilde{\sigma}_t^2} + \log \tilde{\sigma}_t^2.$$

Indexing the true parameter values by 0, we introduce the following assumptions.

A0': for all $\theta \in \Theta$, $\omega(.) > 0$,

A1: $\theta_0 \in \Theta$ and Θ is compact,

A2: $\sum_{j=1}^{d} \pi_j E\{\log a_0(e_j, \eta_0)\} < 0$ and $\prod_{j=1}^{d} \overline{\beta}_j^{\pi_j} < 1$, where $\overline{\beta}_j = \sup_{\theta \in \Theta} \beta(e_j)$,

A3: There exist $r, \rho \in (0,1)$ and $C > 0$ such that

$$\forall i > 0, \quad E\left\{a_0^r(S_t, \eta_{t-1}) \dots a_0^r(S_{t-i}, \eta_{t-i-1})\right\} < C\rho^{i+1},$$

A4: η_t^2 has a nondegenerate distribution with $E\eta_t^2 = 1$,

A5: For all i, $\alpha_0(e_i) + \beta_0(e_i) \neq 0$ and there exist $\ell \in \{1, \dots, d\}$ and $k > 0$ such that $\alpha_0(e_\ell)\mathbb{P}(S_{t-k} = e_\ell, S_t = e_i) > 0$,

A6: θ_0 belongs to the interior of Θ,

A7: there exists $\nu > 0$ such that $E\eta_t^{4+\nu} < \infty$.

Defining $\kappa_\eta = E\eta_t^4 < \infty$, and for any $\theta \in \Theta$,

$$\sigma_{S,t}^2 = \sigma_{S,t}^2(\theta) = c_{S,t}(\theta) + \sum_{i=0}^{\infty} \beta(S_t) \dots \beta(S_{t-i}) c_{S,t-i-1}(\theta), \tag{2.6}$$

where $c_{S,t}(\theta) = \omega(S_t) + \alpha(S_t)\epsilon_{S,t-1}^2$. We also define by $(\sigma_t^2)_t = \{\sigma_t^2(\theta)\}_t$ the solution of

$$\sigma_t^2 = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2$$

given, similarly to (2.6), by

$$\sigma_t^2 = c_t + \sum_{i=0}^{\infty} \beta(s_t) \dots \beta(s_{t-i}) c_{t-i-1}(\theta), \tag{2.7}$$

where $c_t(\theta) = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2$. Note that under **A2**, the a.s. convergence of the sum is ensured and consequently $\sigma_{S,t}^2$ and σ_t^2 are well defined. Then RZ (see Theorem 6 and 7 in RZ) shows the following result.

Theorem 2.1 *Let $\hat{\theta}_n$ be a sequence of QML estimators satisfying (2.5). Then*

- **i)** *under* **A0**-**A5** *and* **A0'**, *almost surely* $\hat{\theta}_n \to \theta_0$, *as* $n \to \infty$,
- **ii)** *under* **A0**-**A7** *and* **A0'**, $\sqrt{n}(\hat{\theta}_n - \theta)$ *is asymptotically* $\mathcal{N}(0, (\kappa_\eta - 1)J^{-1})$ *distributed, where*

$$J := E\left(\frac{1}{\sigma_{S,t}^4(\theta_0)} \frac{\partial \sigma_{S,t}^2(\theta_0)}{\partial \theta} \frac{\partial \sigma_{S,t}^2(\theta_0)}{\partial \theta'}\right)$$

is a positive-definite matrix.

A crucial step in the proof of this theorem is to show that the L^2-norm of the vector $\frac{1}{\sigma_{S,t}^2}\frac{\partial \sigma_{S,t}^2(\theta_0)}{\partial \theta}$ and the L^1-norm of the matrix $\frac{1}{\sigma_{S,t}^4}\frac{\partial \sigma_{S,t}^2(\theta_0)}{\partial \theta}\frac{\partial \sigma_{S,t}^2(\theta_0)}{\partial \theta'}$ are finite. A bound for these norms was shown to be of the form Kc^{-1}, where K is a constant and $c > 0$ is the smallest component of θ_0. Obviously, the proof breaks down when one or several components of θ_0 are equal to zero. In the next section we will allow the true parameter to belong to the boundary of Θ i.e. $\partial\Theta = \{\theta_0 \in \Theta : \theta_{0i} = 0, \quad \text{for some} \quad i > 0\}$.

Comments:

1. Note that Theorem 2.1 allows to use Quasi Likelihood Ratio to assess equality of coefficients of regimes when $(\omega_0(.), \alpha_0(.), \beta_0(.)) > 0$. It is a major difference with the MS framework approach where likelihood ratio (LR) can not be used because, under the null hypothesis, some of the parameters of the model would be unidentified. In MS framework, one would rely on alternative criteria related to the fitting of the data or Bayesian framework (see Hamilton (2008) for a survey on MS models). This lack of identification under the null hypothesis does not exist in the framework of Model (2.1).

2. Note that the assumptions **A0'** and **A1** allow to ensure **B1** is verified and thus that the true volatility process is non degenerated.

3. RZ discussed conditions ensuring assumption **A3** in the particular case where (S_t) is a Markov Chain process. These conditions involve the persistence of (S_t) through its transition matrix. Actually, it is possible to define a very simple and large, but restrictive, condition to ensure assumption **A3** which does not rely on dependence structure of (S_t):

   Lemma 2.1 *Assumption* **A3** *is verified if for any* $i \in \{1, \ldots, d\}$

   $$E(\log(\alpha_0(e_i)\eta_t^2 + \beta_0(e_i))) < 0. \tag{2.8}$$

3 Asymptotic behaviour of $\hat{\theta}_n$ when θ_0 is on the boundary

In the following section, we investigate asymptotic properties of the QMLE when some of the coefficients are null. More precisely, we study the behaviour of $\hat{\theta}_n$ when some values of functions $\omega(.), \alpha(.)$ may be null, that is when **A6** is in failure. We study in a first part the consistency of $\hat{\theta}_n$ to θ_0 and in a second part the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$.

3.1 Consistency of the QMLE when θ_0 is on the boundary

In this section, we prove the consistency of $\hat{\theta}_n$ when **A0'** is replaced by a weaker assumption allowing $\omega_0(.)$ and $\alpha_0(.)$ to reach 0. Let us define for any $I \subset \{1, \dots, d\}$,

$$E(I) = \{e_i, \quad i \in I\} \quad , \mathcal{E} = \{A \subset \{1, \dots, d\}, \quad A \neq \emptyset, \quad A \neq \{1, \dots, d\}\}.$$

We will need the following assumption :

A0”: for all $I \subsetneq \{1, \dots, d\}$,with $\mathcal{E}_{I,t} = \{S_j \in E(I), \quad \forall j \in \{1, \dots, t\}\}$:

$$\mathbb{P}(\mathcal{E}_{I,t})^{1/t} \to_{t\to\infty} a(I) \in [0, 1), \quad \text{with} \quad \mathbb{P}(\mathcal{E}_{I,t}) \overset{t\to\infty}{=} O(a(I)^t).$$

The particular case of (S_t) drawn from an iid Bernoulli distribution with $0 < \mathbb{P}(S_t = e_i) < 1$ for all i satisfies it. It can also be satisfied by (S_t) drawn from a homogeneous Markov Chain (see hereafter for an example). Let us define

$$\Theta(I) = \{\theta \in [0, \infty[^{3d}, \quad \omega(e_i) > 0 \quad \forall i \notin I, \quad \alpha(e_i) > 0 \quad \forall i \in I, \quad \beta(.) > 0\}.$$

The following assumption allows parameter values such that $\omega(.)$ reaches 0 for certain regimes :

A0”': $\Theta \subseteq \Gamma =]0, \infty[^d \times [0, \infty[^{2d} \cup \{\cup_{I \in \mathcal{E}} \Theta(I)\}$.

We will need the following assumption which constrains the distribution of (η_t) around 0:

A8: $E|\eta_t|^{-2s} < \infty$ for some $0 < s < 1$.

We are now in position to state our consistency result:

Theorem 3.1 *Let $\hat{\theta}_n$ be a sequence of QML estimators satisfying (2.5). Then under* **A0-A5***,* **A0”***,* **A0”'***,* **A8** *almost surely* $\hat{\theta}_n \to \theta_0$*, as* $n \to \infty$.

Comments:

1. Theorem 3.1 extends the part i) of Theorem 2.1 to θ_0 such that the true value $\omega_0(.)$ may be null. When $\omega_0(.) > 0$, assumptions **A0”**, **A0”'**, **A8** are not necessary. In a sense, this theorem has only interest when $\omega_0(.)$ may cancel.

2. Assumption **A8** is a mild assumption, however precluding the case of distributions of η_t such that $\mathbb{P}(\eta_t^2 = 0) > 0$. It is satisfied in particular when $f_{\eta^2}(x)$, the density function of (η_t^2), is such that

$$f_{\eta^2}(x) \overset{x \to 0}{=} O(x^{-u})$$

for some $0 < u < 1$. This class of distributions obviously includes the most classical nonnegative distributions used in the literature (such as the $\chi^2(.)$ or the square of Student distribution) and in particular density bounded at the neighborhood of 0.

3. The process (σ_t^2) with true values satisfying assumptions of Theorem 3.1 is non degenerated. Indeed, $\{\omega_0(.), \alpha_0(.), \beta_0(.), (S_t), (\eta_t)\}$ satisfy either **B1** or **B2**. In view of **A0"'**, when $\theta_0 \in \Theta$, either $\theta_0 \in]0, \infty[^d \times [0, \infty[^{2d}$ and thus **B1** is satisfied, or $\theta_0 \in \Theta(I)$ for some $I \in \mathcal{E}$. In this last case, two results follow. First, $\beta_0(.) > 0$ and thus for all i, $\mathbb{P}(\alpha_0(e_i)\eta_t^2 + \beta_0(e_i) = 0) = 0$. Second, when $\theta_0 \in \Theta(I)$ it ensues that $\{j \in \{1, \ldots, d\} \quad \omega_0(e_j) = 0\} \subseteq I$ by definition of $\Theta(I)$. Thus, in view of assumption **A0"**,

$$\mathbb{P}(\omega(S_{-k}) = 0, k \geq 0) \leq \lim_{k \to \infty} \mathbb{P}(\mathcal{E}_{I,k}) = \lim_{k \to \infty} a(I)^k = 0.$$

In view of these two last results, assumption **B2** is ensured when $\theta_0 \in \Theta(I)$, which completes the proof that (σ_t^2) is non degenerated.

4. Note that in view of **A0"'**, for all $\theta \in \Theta$ $\omega(.) + \alpha(.) > 0$. This condition together with the last comment ensures that for all $t \in \{1, \ldots, n\}, \theta \in \Theta$, $\mathbb{P}(c_t(\theta) = 0) = 0$, consequently $\mathbb{P}(\tilde{\sigma}_t(\theta) = 0) = 0$ in view of $\tilde{\sigma}_t(\theta) \geq c_t(\theta)$, and finally $\tilde{\ell}_n(\theta)$ is finite almost surely.

3.2 Asymptotic distribution of the QMLE when θ_0 is on the boundary

Here, we consider the asymptotic distribution of the QMLE when θ_0 may lie on the boundary. The positivity condition, namely $\omega_0(.), \alpha_0(.), \beta_0(.) > 0$ is crucial for the asymptotic normality of the QMLE $\hat{\theta}_n$. Obviously, a Gaussian asymptotic distribution for $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is precluded when the components $\hat{\theta}_{in}$ of $\hat{\theta}_n$ are constrained to be nonnegative and $\theta_0 \in \partial\Theta$. If, for instance, $\theta_{0i} = 0$ then $\sqrt{n}(\hat{\theta}_{ni} - \theta_{0i}) = \sqrt{n}\hat{\theta}_n \geq 0$ for all n and the asymptotic distribution of this variable cannot be a standard Gaussian. This problem is not specific to our framework. For instance, it arises in the standard GARCH(p, q) framework, where the QMLE is also positively constrained. Francq and Zakoïan (2007) described the behaviour of the QMLE for a standard GARCH$(p, q), (p, q > 1)$ model with some of the $\alpha_0(e_i), \beta_0(e_j)(1 \leq i \leq q, j \in p)$ being null. In the same spirit as Andrews (1997,1999,2001), they use a quadratic expansion of the quasi likelihood function (QL) around θ_0 to describe this distribution as a projection of a Gaussian onto a convex cone. In the following, we will show that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ follows the same type of distribution using similar lines of proof. A crucial step in the proof of their result is to show that the second derivatives of the QL function is L^1 in a neighborhood $\mathcal{V}(\theta_0)$ of θ_0. In our framework, this condition becomes :

$$E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\| \frac{\partial^2 \ell_{S,t}(\theta)}{\partial\theta\partial\theta'} \right\| < \infty, \tag{3.1}$$

with $\ell_{S,t} = \ell_{S,t}(\theta) = \frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2} + \ln \sigma_{S,t}^2$. Some assumptions must be added to those of Theorem 3.1 to ensure the condition (3.1). Before stating our asymptotic result, we show some necessary properties required to ensure (3.1) that arise in the particular case where (S_t) is an iid process.

3.2.1 Moments assumptions on η_t^2 and $\sigma_{S,t}^2$

To show (3.1), a crucial step is to ensure $\left\| \frac{1}{\sigma_{S,t}^2} \frac{\partial \sigma_{S,t}^2}{\partial\theta} \right\| (\theta)$ as possessing moment at order 3 in some appropriate neighbourhood of θ_0 belonging to the interior of Θ, thus in particular at the point $\theta = \theta_0$. In the particular case where (S_t) is iid, when $\theta_0 \in \partial\Theta$, this property requires the following conditions :

$$E\eta_t^6 < \infty \quad \text{and} \quad E\sigma_{S,t}^{-6}(\theta_0) < \infty. \tag{3.2}$$

To show that these conditions are necessary, let us take the particular example of Model (2.1) with $d = 2$, $\alpha_0(e_1) = \omega_0(e_2) = 0$, $\omega_0(e_1), \alpha_0(e_2), \beta_0(.) > 0$ and (S_t) drawn from an iid Bernouilli distribution with $0 < \mathbb{P}(S_t = e_i) < 1$ for all i. Using

(2.6), it can be shown (see hereafter (7.15) and (7.16)) that :

$$\frac{1}{\sigma^2_{S,t}}\frac{\partial\sigma^2_{S,t}}{\partial\alpha(e_1)}(\theta_0) \geq \frac{\epsilon^2_{S,t-1}\,\mathbb{1}_{S_t=e_1}}{\sigma^2_{S,t}(\theta_0)} = \frac{\eta^2_{t-1}\sigma^2_{S,t-1}(\theta_0)\,\mathbb{1}_{S_t=e_1}}{\omega_0(e_1)+\beta_0(e_1)\sigma^2_{S,t-1}(\theta_0)}, \quad \frac{1}{\sigma^2_{S,t}}\frac{\partial\sigma^2_{S,t}}{\partial\omega(e_2)} \geq \frac{\mathbb{1}_{S_t=e_2}}{\sigma^2_{S,t}(\theta_0)}, \tag{3.3}$$

where the equality follows from $\alpha_0(e_1)=0$. η^2_{t-1} and $\sigma^2_{S,t-1}(\theta_0)$ being independent, $E\{\frac{1}{\sigma^2_{S,t}}\frac{\partial\sigma^2_{S,t}}{\partial\alpha(e_1)}(\theta_0)\}^3<\infty$ implies $E\eta_t^6<\infty$. The first part of (3.2) is thus a necessary condition for (3.1) to hold. Now, conditioning (3.3) on $\{S_t=e_2\}$, we obtain :

$$\begin{aligned}
& E\left(\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial\sigma^2_{S,t}}{\partial\omega(e_2)}(\theta_0)\right\}^3|S_t=e_2\right)\\
\geq\ & E\left(\left\{\frac{1}{\sigma^2_{S,t}(\theta_0)}\right\}^3|S_t=e_2\right) \geq E\left(\left\{\frac{1}{(\alpha_0(e_2)\eta_t^2+\beta_0(e_2))\sigma^2_{S,t-1}(\theta_0)}\right\}^3|S_t=e_2\right)\\
=\ & E\frac{1}{(\alpha_0(e_2)\eta_t^2+\beta_0(e_2))^3}E\frac{1}{\sigma^6_{S,t-1}(\theta_0)}.
\end{aligned}$$

The second inequality relies on $\omega_0(e_2)=0$. The equality follows from assumption **A0** and independence of $\{S_t=e_2\}$ and $\sigma^6_{S,t-1}(\theta_0)$ (as (S_t) is an iid process). By $\beta_0(.)>0$, it ensues that a necessary condition for $\frac{1}{\sigma^2_{S,t}}\frac{\partial\sigma^2_{S,t}}{\partial\omega(e_2)}(\theta_0)$ to possess a moment at order 3 is $E\sigma^{-6}_{S,t}(\theta_0)<\infty$. The second part of (3.2) is thus a necessary condition for (3.1) to hold.

3.2.2 Assumption and main result

Let $\theta_0(\epsilon)$ be the vector obtained by replacing all zero coefficients of θ_0 by ϵ. Let us define for any $I\subset\{1,\dots,d\}$,

$$\Theta(I,b)=\{\theta\in[0,\infty[^{3d},\quad \omega(e_i)>0\quad \forall i\notin I,\quad \alpha(e_i)>0\quad\forall i\in I,\quad \beta(.)>a(I)^{1/b}\}.$$

This space is obviously included in $\Theta(I)$. To state the main result of this subsection, we will need the following assumptions :

A7': $E\eta_t^6<\infty$,

A9: $\Theta\subseteq\Gamma_3=]0,\infty[^d\times[0,\infty[^{2d}\cup\{\cup_{I\in\mathcal{E}}\Theta(I,3)\}$,

A10: $\beta_0(.)>0$,

A11: $\theta_0(\epsilon)$ belongs to the interior of Θ for some $\epsilon>0$.

Assumption **A7'** strengthens assumption **A7** and corresponds in the particular case of (S_t) iid process to the first part of (3.2). Assumption **A9**, which is more restrictive than **A0"'** (in view of $\Gamma_3 \subseteq \Gamma$), is required to ensure in the particular case of (S_t) iid process the second part of (3.2). Assumption **A11** is presented in FZ (see assumption **A6** in FZ), and prevents θ_0 from reaching the upper bound of Θ. This assumption is a technical assumption which is satisfied if Θ is sufficiently large. The definition of Γ_3 allows to make such a choice. The main result of this section is the following.

Theorem 3.2 *Let $\hat{\theta}_n$ be a sequence of QML estimators satisfying (2.5). Then under* **A0-A5**, **A0"**, **A7'**, *and* **A8-A11**

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \lambda^{\Lambda} := \arg\inf_{\lambda \in \Lambda} \{\lambda - Z\}' J \{\lambda - Z\},$$

$$\text{with} \quad Z \sim \mathcal{N}(0, (\kappa_\eta - 1)J^{-1}), \quad \Lambda = \Lambda(\theta_0) = \Lambda_1 \times \ldots \times \Lambda_{3d},$$

where $\Lambda_i = \mathbb{R}$ if $\theta_{0i} \neq 0$ and $\Lambda_i = [0, \infty)$ if $\theta_{0i} = 0$ and in the definition of J, derivatives with respect to θ_i are replaced by right derivatives when i is such that $\theta_{0i} = 0$.

Comments:

1. When θ_0 belongs to the interior of Θ, we retrieve the asymptotic distribution of Theorem 2.1 as in this case $\Lambda = \mathbb{R}^{3d}$ and consequently $\lambda^{\Lambda} = Z$. In this sense, this theorem has only interest when $\omega_0(.)$ or $\alpha_0(.)$ may cancel. In such a case, the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is more complex than a Gaussian.

2. We refer to Francq and Zakoïan (2009) for the presentation of a practical method for computing λ^{Λ}.

3. λ_i^{Λ} may differ significantly from the asymptotic Gaussian distribution Z_i for all i, even for i such that $\theta_{0i} \neq 0$. Thus the confidence intervals of the true parameter components derived from Theorem 2.1 may be false, even concerning the estimation of strictly positive components.

4. We insist on the fact that the moment condition **A7'** is on the iid process, not on (ϵ_t).

5. Note that each $\Theta(I,b)$ allows θ_0 to be such that $\omega_0(e_i)=0 \quad \forall i \in I$. But for such θ_0, we must have $\beta_0(.) > a(I)^{1/3}$. Furthermore, it is obvious that if $I \subseteq J$, $a(I) \leq a(J)$. Consequently, the persistence coefficient function $\beta_0(.)$ increases with the number of regimes e_i such that $\omega_0(e_i)=0$.

3.3 Illustration

We provide here one example to illustrate assumptions **A0"** and **A9** with $d=2$ and thus $\mathcal{E}=\{\{1\},\{2\}\}$. Assume that S_t is drawn from a two state Markov Chain with $\mathbb{P}(S_t=e_1|S_{t-1}=e_1)=p_1<1$ and $\mathbb{P}(S_t=e_2|S_{t-1}=e_2)=p_2<1$, and invariant probability $\pi_i := \mathbb{P}(S_t=e_i)$ for all i. For $t \geq 0$,

$$\mathbb{P}(S_1=e_1, S_2=e_1, \ldots, S_t=e_1) = p_1^t \pi_1.$$

and

$$\mathbb{P}(S_1=e_2, S_2=e_2, \ldots, S_t=e_2) = p_2^t \pi_2.$$

Assumption **A0"** is easily satisfied with $a(\{1\})=p_1<1, \quad a(\{2\})=p_2<1$. Theorem 3.2 can thus be applied, for true values θ_0 belonging to

$$\begin{aligned}\Gamma_3 &= \left(]0,\infty[^d\right) \times [0,\infty[^{2d} \cup \{\theta \in [0,\infty[^{3d}, \omega(e_2)>0, \quad \alpha(e_1)>0, \quad \beta(.)>p_1^{1/3}\} \\ &\quad \cup \{\theta \in [0,\infty[^{3d}, \omega(e_1)>0, \quad \alpha(e_2)>0, \quad \beta(.)>p_2^{1/3}\}.\end{aligned}$$

3.4 Non regularity of the QMLE under local alternatives

Here, we derive the asymptotic behaviour of the QMLE under sequences of local alternatives converging to the value θ_0. Let $\theta_n=\theta_0+\tau/\sqrt{n}$, where $\tau=(\tau_{\omega,e_1},\tau_{\alpha,e_1},\tau_{\beta,e_1},\ldots,\tau_{\omega,e_d},\tau_{\alpha,e_d},\tau_{\beta,e_d})' \in (0,\infty)^{3d}$ is such that $\theta_n \in \Theta$ at least for sufficiently large n.

We need to precisely define the data generating process. Defining $a_n(x,y)=\alpha_n(x)y^2+\beta_n(x)$ and assume that **A2** holds. For n sufficiently large,

$$\sum_{j=1}^{d} \pi_j E\{\log a_n(e_j,\eta_0)\} < 0.$$

In view of (2.3), this condition ensures the definition of the non anticipative solution $(\epsilon_{t,n})_{t\in\mathbb{Z}}$ of

$$\epsilon_{t,n}=\sigma_{t,n}\eta_t, \quad \sigma_{t,n}^2=\omega_0(s_t)+\frac{\tau_{\omega,s_t}}{\sqrt{n}}+(\alpha_0(s_t)+\frac{\tau_{\alpha,s_t}}{\sqrt{n}})\epsilon_{t-1}^2+(\beta_0(s_t)+\frac{\tau_{\beta,s_t}}{\sqrt{n}})\sigma_{t-1}^2, \quad t\in\mathbb{Z},$$

where (η_t) is iid (0,1). Given the observations $\epsilon_{1,n}, \dots, \epsilon_{n,n}$, the QMLE satisfies :

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} n^{-1} \sum_{t=1}^{n} \tilde{\ell}_{t,n}(\theta), \quad \text{where} \quad \tilde{\ell}_{t,n}(\theta) = \tilde{\ell}_t(\theta; \epsilon_{n,n}, \dots, \epsilon_{1,n}) = \frac{\epsilon_{t,n}^2}{\tilde{\sigma}_{t,n}^2} + \log \tilde{\sigma}_{t,n}^2, \tag{3.4}$$

where $\tilde{\sigma}_{t,n}^2 = \tilde{\sigma}_{t,n}^2(\theta)$ is obtained by replacing ϵ_u by $\epsilon_{u,n}$, for any $1 \leq u < t$, in $\tilde{\sigma}_t$ but, for simplicity, with initial values independent of n. Similarly, $\sigma_{t,n}^2$ is defined by replacing ϵ_u by $\epsilon_{u,n}, 1 \leq u < t$, in σ_t^2. Denote $\mathbb{P}_{n,\tau}$ the distribution of $(\epsilon_{t,n})$.

Francq and Zakoïan (2009) described the behaviour of the QMLE for a standard GARCH(p,q), $(p, q > 1)$ model under sequences of local alternatives to assumption that some $\alpha(e_i), \beta(e_j)(1 \leq i \leq q, j \in p)$ being null. They use a quadratic expansion of the QL function around θ_0 to describe this distribution as a projection of a Gaussian distribution with mean τ onto a convex cone. In the following, we will show that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ follows the same type of distribution using similar lines of proof. A crucial step in the proof of this result is to show that the supremum for n larger than some $n_0 > 0$ of the third derivatives of the QL function under the true value θ_n are L^1 in a neighborhood $\mathcal{V}(\theta_0)$ of θ_0 for n_0 sufficiently large. In our framework, it can be shown that this condition requires $\left\| \frac{1}{\sigma_{S,t}^2} \frac{\partial \sigma_{S,t}^2}{\partial \theta} \right\| (\theta_0)$ to possess a fourth-order moment. In view of arguments used in subsection 3.2.1, when (S_t) is an iid process, some necessary conditions arise which are :

$$E\eta_t^8 < \infty \quad \text{and} \quad E\sigma_{S,t}^{-8}(\theta_0) < \infty. \tag{3.5}$$

We will thus need more constraining assumptions than **A7'** and **A9**, which are :

A7": there exists $\mu \in \mathbb{R}^{*+}$ such that $E\eta_t^{8+\mu} < \infty$,

A9': $\Theta \subseteq \Gamma_4 =]0, \infty[^d \times [0, \infty[^{2d} \cup \{\cup_{I \in \mathcal{E}} \Theta(I, 4)\}$.

In the particular case of (S_t) drawn from and iid process, the assumption **A7"** (respectively **A9'**) allows to retrieve the first (resp second) part of (3.5). We are now in position to state the following result about behaviour of QMLE under sequences of local alternatives.

Theorem 3.3 *Let $\theta_0 \in \Theta$ and let $\tau \in \in (0, \infty)^{3d}$. Let $(\hat{\theta}_n)$ be a sequence of QMLE estimators satisfying (3.4). Then:*

i) under assumptions of Theorem 3.1, $\hat{\theta}_n \to \theta_0, \mathbb{P}_{n,\tau} \quad a.s. \quad asn \to \infty$.

ii) under **A0-A5**, **A0"**, **A7"** *and* **A8**, **A9'**, **A10**, **A11** *then under* $\mathbb{P}_{n,\tau}$,

$$\sqrt{n}(\hat{\theta}_n - \theta_n) \stackrel{\mathcal{L}}{=} \lambda^{\Lambda}(\tau) - \tau \quad where \quad \lambda^{\Lambda}(\tau) := arg \inf_{\lambda \in \Lambda} \{\lambda - Z - \tau\}' J \{\lambda - Z - \tau\},$$

$$with \quad Z \sim \mathcal{N}(0, (\kappa_\eta - 1)J^{-1}), \quad \Lambda = \Lambda(\theta_0) = \Lambda_1 \times \ldots \times \Lambda_{3d},$$

where $\Lambda_i = \mathbb{R}$ *if* $\theta_{0i} \neq 0$ *and* $\Lambda_i = [0, \infty)$ *if* $\theta_{0i} = 0$.

Note that assumptions **A7"** and **A9'** are needed to obtain the aymptotic distribution of $\hat{\theta}_n$, not for its consistency to θ_0. As for Theorem 3.2, we insist on the fact that the moment condition **A7"** is on the iid process, not on the observed process (ϵ_t).

4 Testing that some coefficients are equal to zero

We have derived the behaviour of the QMLE when the true parameter is on the boundary under some mild conditions. The practical objective of this paper is to develop a methodology for testing the nullity of a sub vector of θ_0. Given the variety of possible tests we decided to limit ourselves to those of Wald, Rao (or Lagrange Multiplier) and the quasi-likelihood ratio (QLR) because these tests, which are based on the QL function, inherit its robustness property. The behaviour of these 3 tests has been investigated by Francq and Zakoïan (2009) for the standard GARCH$(p,q)(p,q > 1)$ model when testing some of the $\alpha_0(e_i), \beta_0(e_j)(1 \leq i \leq q, j \in p)$ being null. They derived asymptotic distributions of these 3 tests. We will use their result to derive a similar result in our framework.

More precisely, and without loss of generality we consider testing the nullity of the last d_2 coefficients of θ, split into two components as $\theta = (\theta^{(1)}, \theta^{(2)})'$ where $\theta^{(i)} \in \mathbb{R}^{d_i}, d_1 + d_2 = 3d$. The null hypothesis is thus:

$$H_0: \quad \theta^{(2)} = \theta_0^{(2)} = 0_{d_2 \times 1} \quad i.e. \quad K\theta_0 = 0_{d_2 \times 1} \quad \text{with} \quad K = (0_{d_2 \times d_1}, I_{d_2}).$$

We consider local hypotheses of the form

$$H_n(\tau): \quad \theta = \theta_0 + \tau/\sqrt{n} \quad \text{with} \quad K\theta_0 = 0_{d_2 \times 1} \quad \text{and} \quad \tau \in (0, +\infty)^{3d}$$

and let

$$H: \quad \theta_0^{(1)} > 0 \quad i.e. \quad \bar{K}\theta_0 > 0 \quad \text{with} \quad \bar{K} = (0_{d_2 \times d_1}, I_{d_2})$$

denote the maintained assumption. The usual form of the Wald, Rao-score and QLR statistics are given by:

$$\begin{aligned} W_n &= \frac{n}{\hat{\kappa}_\eta - 1}\hat{\theta}_n^{(2)\prime}\{K\hat{J}_n^{-1}K'\}\hat{\theta}_n^{(2)}, \\ R_n &= \frac{n}{\hat{\kappa}_{\eta|2} - 1}\frac{\partial\tilde{\mathbf{l}}_n(\hat{\theta}_{n|2})}{\partial\theta'}\hat{J}_{n|2}^{-1}\frac{\partial\tilde{\mathbf{l}}_n(\hat{\theta}_{n|2})}{\partial\theta}, \\ L_n &= n\{\tilde{\mathbf{l}}_n(\hat{\theta}_{n|2}) - \tilde{\mathbf{l}}_n(\hat{\theta}_n)\}, \end{aligned}$$

where $\hat{\theta}_{n|2}$ denotes the *restricted* (by H_0) estimator of θ_0; $\hat{\kappa}_\eta$, $\hat{\kappa}_{\eta|2}$ denote consistent estimators of κ_η; $\hat{J}_n$, $\hat{J}_{n|2}$ denote consistent estimators of the information matrix J. One rejects the null hypothesis for large values of these statistics. The asymptotic distributions of the 3 tests statistics under both the null $H_0 = H_n(0)$ and the local alternatives are given in the following theorem. Note that taking $\tau = 0$ in the definition of $\lambda^\Lambda(\tau)$ in Theorem 3.3 gives the variable λ^Λ of Theorem 3.2. Therefore we can state $\lambda^\Lambda(0) = \lambda^\Lambda$.

Let $\Omega = K'\{(\kappa_\eta - 1)KJ^{-1}K'\}^{-1}K$. We denote by $\chi^2_k(c)$ the noncentral chi-square distribution with noncentrality parameters c and k degrees of freedom.

Theorem 4.1 *Under $H_n(\tau)$ and H, with $\tau \geq 0$ we have :*

$$\begin{aligned} W_n &\xrightarrow{d} W(\tau) = \lambda^{\Lambda\prime}(\tau)\Omega\lambda^\Lambda(\tau), \\ R_n &\xrightarrow{d} \chi^2_{d_2}\{\tau'\Omega\tau\}, \\ L_n &\xrightarrow{d} L(\tau) = -\frac{1}{2}(\lambda^\Lambda_\tau - Z - \tau)'J(\lambda^\Lambda_\tau - Z - \tau) + \frac{\kappa_\eta - 1}{2}(Z+\tau)'\Omega(Z+\tau) \\ &= -\frac{1}{2}\left\{\inf_{K\lambda\geq 0}\|Z+\tau-\lambda\|^2_J - \inf_{K\lambda=0}\|Z+\tau-\lambda\|^2_J\right\} \end{aligned}$$

under the assumptions of Theorem 3.3 if $\tau > 0$ and under the assumptions of Theorem 3.2 for $\tau = 0$.

This theorem is a straightforward consequence of Theorems 3.2 and 3.3, using the lines of proofs in Francq and Zakoïan (2009). The following proposition establishes that the asymptotic distributions of the latter tests are actually the same.

Proposition 1 *With the assumptions of Theorem 4.1, $W_n \overset{o_P(1)}{=} \frac{2}{\kappa_\eta - 1}L_n$.*

Theorem 4.1 and Proposition 1 can be used to compare the local asymptotic behaviour of the 3 tests. The comparison of tests by means of their local asymptotic powers is generally referred to as the Pitman approach.

Comments:

1. As noted in Francq and Zakoïan (2009), contrary to the standard situation, the asymptotic distributions are not the same. Only the score statistic has the standard χ^2 distribution, which is a consequence of the Gaussian asymptotic distribution of the score vector under $H_n(\tau)$. This implies that the standard Rao score test remains valid whatever the position of θ_0 in Θ. On the contrary, valid tests based on the Wald and LR statistics require the correction of the usual critical values.

2. By Theorem 4.1, the tests of asymptotic level α are defined by the critical regions :
$$\{W_n > w_{1-\alpha}\}, \quad \{R_n > \chi^2_{d_2,1-\alpha}\}, \quad \{L_n > l_{1-\alpha}\},$$
where $w_{1-\alpha}$, $\chi^2_{d_2,1-\alpha}$ and $l_{1-\alpha}$ are the $(1-\alpha)$ quantiles of the distributions of $W(0)$, $\chi^2_{d_2}$ and $L(0)$. The first and third tests are called in Francq and Zakoïan (2009) the *modified* Wald and QLR test.

3. In the particular case where the testing assumption concerns only one component i.e.
$$H_0 : \omega_0(e_i) = 0 \quad (\text{or} \quad H_0 : \alpha_0(e_j) = 0),$$
for some $i, j \in \{1, \ldots, d\}$, it can be shown using Theorem 4.1 and arguments from Francq and Zakoïan (2009)), that
$$W = \frac{2}{\kappa_\eta - 1} L \sim (U^2)\, \mathbb{1}_{U \geq 0} = \frac{1}{2}\delta_0 + \frac{1}{2}\chi^2_1.$$
where $U \sim \mathcal{N}(0,1)$ and δ_0 denotes the Dirac mass at 0. It follows that the tests defined by the critical regions $\{W_n > \chi^2_1(1-2\alpha)\}$ and $\{\frac{2}{\kappa_\eta - 1} L_n > \chi^2_1(1-2\alpha)\}$ have asymptotic level α (for $\alpha \leq 1/2$). The standard Wald test $\{W_n > \chi^2_1(1-\alpha)\}$ has asymptotic level $\alpha/2$. In this sense, the standard Wald test is too conservative. The standard QLR test $\{L_n > \chi^2_1(1-\alpha)\}$ has the same asymptotic level $\alpha/2$ when $\kappa_\eta = 3$.

4. Using arguments from Francq and Zakoïan (2009) together with Theorem 4.1, it can be shown that the Wald and QL tests are locally more powerful than the Rao score Test when testing the nullity of one component.

5 Numerical illustration

In this section, we illustrate the assumptions and the theorems mentioned previously, through three examples of Model (2.1) with d regimes arbitrarily fixed to $d = 2$ for the brevity of the presentation. In the following three examples $n = 20000$, s_t is valued in $\{e_1, e_2\}$ and is a realization of the iid process (S_t) drawn from a Bernouilli distribution with $\mathbb{P}(S_t = e_1) = 1 - \mathbb{P}(S_t = e_2) = 0.9$ and $\eta_t \sim \mathcal{N}(0, 1)$. The first model considered is generated by :

$$\sigma_t^2 = \begin{cases} 0.4 \;+\; 0.1 \;\; \epsilon_{t-1}^2 \;+\; 0.7 \;\; \sigma_{t-1}^2 & \text{if} \quad s_t = e_1, \\ 0 \;+\; 0.3 \;\; \epsilon_{t-1}^2 \;+\; 0.7 \;\; \sigma_{t-1}^2 & \text{if} \quad s_t = e_2. \end{cases} \tag{5.1}$$

θ_0 obviously satisfies the condition of Lemma 2.1 because $\alpha_0(.) + \beta_0(.) \leq 1$ and consequently the assumption **A3** is verified. With notations of subsection 3.3, because first $I_0 = \{i \in \{1, \dots, d\}, \quad \omega_0(e_i) = 0\} = \{2\}$ and thus $a(I_0) = 0.1$, and second $\beta_0(.) > a(I_0)^{1/4} = 0.56$, we get thus $\theta_0 \in \Gamma_4$. Consequently **A9'** is achieved. The other assumptions of Theorem 3.3 being verified, θ_0 satisfies all assumptions needed to apply Theorem 3.2 and Theorem 3.3.

The following 2 regimes model is fitted, with standard deviations of Z defined in Theorem 3.2 in parentheses and the confidence intervals of the $\frac{\lambda_i^\Lambda}{\sqrt{n}}$ at level 10% derived from Theorem 3.2 in brackets. This confidence intervals could not be constructed in practice and are given for the sake of comparison.

$$\sigma_t^2 = \begin{cases} \underset{\substack{(0.03) \\ [-0.05 \quad 0.04]}}{0.36} \;+\; \underset{\substack{(0.007) \\ [-0.01 \quad 0.01]}}{0.09} \;\; \epsilon_{t-1}^2 \;+\; \underset{\substack{(0.02) \\ [-0.02 \quad 0.03]}}{0.72} \;\; \sigma_{t-1}^2 & \text{if} \quad s_t = e_1, \\ \underset{\substack{(0.13) \\ [0 \quad 0.22]}}{0.025} \;+\; \underset{\substack{(0.025) \\ [-0.04 \quad 0.04]}}{0.3} \;\; \epsilon_{t-1}^2 \;+\; \underset{\substack{(0.076) \\ [-0.13 \quad 0.03]}}{0.72} \;\; \sigma_{t-1}^2 & \text{if } s_t = e_2. \end{cases} \tag{5.2}$$

The estimated standard deviations conclude to the significance of all parameters except $\omega_0(e_2)$. Note that the confidence interval of $(\hat{\beta}_n(e_2) - \beta_0(e_2))$ deduced from Theorem 3.2 differs significantly from the one deduced from Theorem 2.1. This example illustrates the comment 3 of section 3.2.2. The second model considered is generated by:

$$\sigma_t^2 = \begin{cases} 0.4 + 0.5\ \epsilon_{t-1}^2 + 0.2\ \sigma_{t-1}^2 & \text{if } s_t = e_1, \\ 0.2 + 0\ \epsilon_{t-1}^2 + 0.5\ \sigma_{t-1}^2 & \text{if } s_t = e_2. \end{cases} \tag{5.3}$$

Similarly to the model (5.1), all assumptions needed to apply Theorem 3.2 and Theorem 3.3 are verified. The estimation step leads to :

$$\sigma_t^2 = \begin{cases} \underset{[-0.02\quad 0.02]}{\underset{(0.01)}{0.4}} + \underset{[-0.025\quad 0.025]}{\underset{(0.015)}{0.49}}\ \epsilon_{t-1}^2 + \underset{[-0.025\quad 0.025]}{\underset{(0.015)}{0.21}}\ \sigma_{t-1}^2 & \text{if } s_t = e_1, \\ \underset{[-0.05\quad 0.05]}{\underset{(0.03)}{0.23}} + \underset{[0\quad 0.025]}{\underset{(0.016)}{0.025}}\ \epsilon_{t-1}^2 + \underset{[-0.067\quad 0.059]}{\underset{(0.04)}{0.42}}\ \sigma_{t-1}^2 & \text{if } s_t = e_2. \end{cases} \tag{5.4}$$

The estimated standard deviations conclude to the significance of all parameters except $\alpha_0(e_2)$. Here again, the confidence interval of $(\hat{\beta}_n(e_2) - \beta_0(e_2))$ deduced from Theorem 3.2 differs significantly from the one deduced from Theorem 2.1. The third model considered here is generated by :

$$\sigma_t^2 = \begin{cases} 0.4 + 0\ \epsilon_{t-1}^2 + 0.7\ \sigma_{t-1}^2 & \text{if } s_t = e_1, \\ 0 + 0.3\ \epsilon_{t-1}^2 + 0.7\ \sigma_{t-1}^2 & \text{if } s_t = e_2. \end{cases} \tag{5.5}$$

Similarly to the model (5.1), all assumptions needed to apply Theorem 3.2 and Theorem 3.3 are verified. The estimation step leads to :

$$\sigma_t^2 = \begin{cases} \underset{[-0.075\quad 0.065]}{\underset{(0.011)}{0.32}} + \underset{[0\quad 0.0085]}{\underset{(0.015)}{0.0001}}\ \epsilon_{t-1}^2 + \underset{[-0.051\quad 0.056]}{\underset{(0.015)}{0.76}}\ \sigma_{t-1}^2 & \text{if } s_t = e_1, \\ \underset{[0\quad 0.27]}{\underset{(0.034)}{0.0003}} + \underset{[-0.05\quad 0.05]}{\underset{(0.015)}{0.27}}\ \epsilon_{t-1}^2 + \underset{[-0.21\quad 0.035]}{\underset{(0.04)}{0.706}}\ \sigma_{t-1}^2 & \text{if } s_t = e_2. \end{cases} \tag{5.6}$$

The estimated standard deviations conclude to the significance of all parameters except $\alpha_0(e_1)$ and $\omega_0(e_2)$. Here, the confidence intervals of $(\hat{\beta}_n(e_2) - \beta_0(e_2))$ and $(\hat{\omega}_n(e_1) - \omega_0(e_1))$ deduced from Theorem 3.2 differ significantly from the ones deduced from Theorem 2.1.

In view of (5.2), (5.4) and (5.6), it is natural to proceed to some tests of nullity. Table 1 provides the P-values of the QLR, Wald and Rao Score tests of nullity for the models (5.1), (5.3) and (5.5). The asymptotic distributions of L_n, W_n and R_n were computed according to Theorem 3.2. At level 5%, QLR and Wald tests accept the three null hypothesis. The Rao Score rejects the null hypothesis for models (5.3) and (5.5) and accepts the null hypothesis $H_0 : \alpha_0(e_1) = \omega_0(e_2) = 0$ for (5.5). The null hypothesis $H_0 : \alpha_0(e_1) = \omega_0(e_2) = 0$ for (5.5) is thus accepted by all tests.

Concerning the null hypothesis of others models, we need to decide on which decisions of QLR tests and Wald tests or Rao Score test we need to rely on. The Pitman analysis consists in following the decision of the most locally asymptotic powerful tests. As noted in section 4, the Rao Score is locally less powerful than the two other tests when testing nullity of one component. In these cases, the Pitman analysis leads thus to accept the null hypothesis.

Figure 1 displays for model (5.5) the local asymptotic power functions of the three tests of the null hypothesis $H_0 : \alpha_0(e_1) = \omega_0(e_2) = 0$ computed after simulations of the asymptotic distributions presented in Theorem 3.3. As noted in Proposition 1, these functions are the same for the Wald and QLR tests for the three models. The Pitman analysis concludes that they beat the Rao Score, since their local asymptotic power is greater.

P-value	Model (5.1)	Model (5.3)	Model (5.5)
	$H_0 : \omega_0(e_2) = 0$	$H_0 : \alpha_0(e_2) = 0$	$H_0 : \alpha_0(e_1) = \omega_0(e_2) = 0$
Wald Test	> 0.05	> 0.05	> 0.05
Rao-Score Test	0.0003	0.0003	> 0.05
QLR Test	> 0.05	> 0.05	> 0.05

Table 1: Tests of nullity for Model (5.1)-Model (5.5). P-values at 5% of the three tests are provided.

6 Conclusion

The paper provided a framework to test the nullity of coefficients for a switching regime GARCH model with coefficients driven by an observed exogenous process.

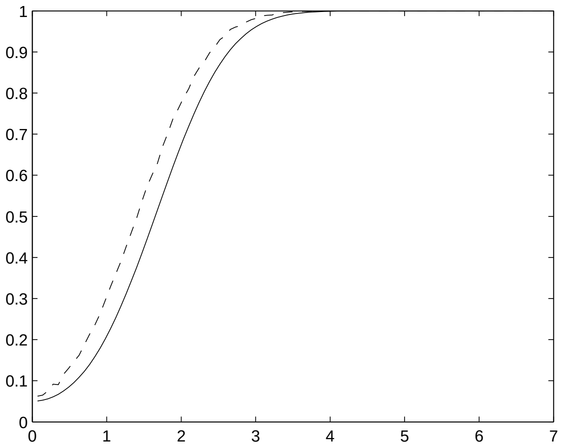

Figure 1: Local asymptotic power functions of QLR and Wald (dashed line),and Rao-Score (full line) test of the null hypothesis $H_0 : \alpha_0(e_1) = \omega_0(e_2) = 0$ for model (5.5) (direction of τ is $(0, 1, 0, 1, 0, 0)$)

This observability of the state variable makes the problem easier to handle than for the Markov Switching processes. The asymptotic behaviour of the QMLE has been derived under (i) the assumption that the true coefficients has components that may cancel and (ii) under sequences of local alternatives converging towards this true value. The asymptotic law of QLR, Wald and Rao-Score tests have benn derived under (i) and (ii), allowing under (i) to test for the nullity of some coefficients and under (ii) to compare tests decisions by leading a Pitman analysis. Numerical simulations illustrating assumptions and results have been proposed. In particular the presence of null components in the true value is shown as invalidating the classical Gaussian significance intervals of estimation obtained by RZ even concerning strictly nonnegative components.

More general remarks can be made regarding the mathematical framework developed in this paper. First, the assumptions are explicit, ie they may be verified either in particular examples either empirically. Second, the results under (i) and (ii) have been derived under assumptions that do not require the existence of moments of the observed process. For this reason, these assumptions are thus mild. Third, they can be qualified as mild also because the regimes dynamics estimable by QMLE are allowed to be very distincts. Fourth, the framework developed allows to test for canceling of some values of $\alpha(.)$. Finally, one specificity of regime switching GARCH models in regards to the standard GARCH models lies in allowing the intercept terms to cancel for certain regimes. This paper shows that the QMLE may be used to test for nullity of some intercepts.

It is hoped that the mathematical framework developed here will help to adress similar problems of nullity tests in others regime-switching GARCH models, such as MS GARCH or Periodic GARCH models.

7 Proofs

Throughout, all expectations are taken with respect to the distribution of (S_t, η_t). For a real random variable X, we define $\|X\|_m = \{E|X|^m\}^{1/m}$ for any $m \in \mathbb{N}$, when $|X|$ possesses a moment at order m. For a random matrix X, $\|X\|_m < \infty$ means that all components of the random variable possess moment of order m. Troughout, the letter K will be used for a positive generic constant, possibly random, whose value is allowed to change throughout the proof. Throughout, all derivatives will be right hand-side derivatives. After presenting some preliminary results and definitions, we give the proofs of Theorem 3.1, Theorem 3.2. The proofs of Theorem 3.3 is omitted in this paper for brevity, but is available from the author. The proofs of results stated in the next subsection will be also omitted for brevity but are available from the author. It will be convenient in our presentation to define the following quantities. Let us define

$$\Omega^*_{S,t}(\theta) = \sum_{i=0}^{\infty} \beta_*^i(\theta)\omega(S_{t-i})$$

with $0 \leq \beta_*(\theta) = \min_{i\in\{1,\ldots,d\}} \beta(e_i)$. Note that the second part of assumption **A2** implies that $\beta_*(\theta) < 1$ for all $\theta \in \Theta$, and thus $\Omega^*_{S,t}$ is a stationary process. Furthermore, in view of (2.6), it is clear that :

$$\sigma^2_{S,t} \geq \Omega^*_{S,t}. \tag{7.1}$$

We will choose $r = s$ without loss of generality, with r defined in **A3** and s defined in **A8**. We start by some technical Lemmas.

Lemma 7.1 *Under* **A0", A0"', A2**, *there exists* $r > 0$ *such that*

$$E\sup_{\theta\in\Theta} \frac{1}{\Omega^{*r}_{S,t}(\theta)} < \infty, \quad \sup_{\theta\in\Theta} E\log^- \Omega^*_{S,t}(\theta) < \infty, \quad \left|E\{\inf_{\theta\in\Theta} \log \Omega^*_{S,t}(\theta)\}\right| < \infty, \tag{7.2}$$

where $\log^-(x) = -\min(\log(x), 0)$.

Lemma 7.2 *Under* **A0"**, **A0"'**, **A2** *and* **A8** *there exists* $r > 0$ *such that*

$$E \sup_{\theta \in \Theta} \frac{1}{c_{S,t}^r(\theta)} < \infty, \quad E \sup_{\theta \in \Theta} \frac{1}{d_{S,t}^r(\theta)} < \infty, \tag{7.3}$$

with $d_{S,t}(\theta) = \omega(S_t) + \alpha(S_t)\Omega_{S,t-1}^*(\theta_0)\eta_{t-1}^2$

r defined in (7.3) and (7.2) will be chosen accordingly to r defined in assumption **A3** and **A8**, without loss of generality. Let us now state a technical Lemma which will be used in the proofs of Theorem 3.2 and 3.3.

Lemma 7.3 *Under assumption* **A0" A2** *, and* **A9** *or* **A9'**, *there exist* $\xi > 0$ *such that*

$$E \sup_{\theta \in \Theta} \frac{1}{\Omega_{S,t}^{*b}(\theta)} < \infty \tag{7.4}$$

with $b = 3 + \xi$ *under* **A9** *and* $b = 4 + \xi$ *under assumption* **A9'**.

It will be convenient to define some processes which are lower-bounds of processes $(\sigma_{S,t}^2(\theta))$. Let us define

$${}^i\sigma_{S,t}^2(\theta) := d_{S,t}(\theta) + \prod_{0 \le h \le i+1} \beta(S_{t-h})\sigma_{S,t-i-2}^2(\theta),{}_i\,\sigma_{S,t}^2(\theta) := c_{S,t}(\theta) + \prod_{0 \le h \le i} \beta(S_{t-h})c_{S,t-i-1}(\theta),$$

$$\sigma_{i,S,t}^2(\theta) := c_{S,t}(\theta) + \prod_{0 \le h \le i} \beta(S_{t-h})\Omega_{S,t-i-1}^*(\theta).$$

These processes have properties stated in the following Lemma (without proof for brevity, but its proof is available from the author).

Lemma 7.4 *For all* $\theta \in \Theta$

$$\max(\sigma_{i,S,t}^2(\theta),{}_i\,\sigma_{S,t}^2(\theta),{}^i\,\sigma_{S,t}^2(\theta)) \le \sigma_{S,t}^2(\theta). \tag{7.5}$$

7.1 Proof of Theorem 3.1

It will be convenient to approximate the sequence $(\tilde{\ell}_t(\theta))$ by a sequence $(\ell_t(\theta))$ which is independent of the initial values. Let

$$\mathbf{l}_n(\theta) = \mathbf{l}_n(\theta; \epsilon_n, \epsilon_{n-1} \ldots,) = n^{-1} \sum_{t=1}^n \ell_t, \quad \ell_t = \ell_t(\theta) = \frac{\epsilon_t^2}{\sigma_t^2} + \ln \sigma_t^2.$$

We will follow the scheme of the proof of Theorem 6 in RZ. Part *i)* remains the same in our framework. We summarize here its result.

Lemma 7.5 *Under Assumption* **A3***:*

$$E\epsilon_{S,t}^{2r} < \infty.$$

Following the scheme of the proof of Theorem 6 in RZ, we need to prove

$$
\begin{array}{ll}
ii) & \lim_{n\to\infty} \sup_{\theta\in\Theta} |\mathbf{l}_n(\theta) - \tilde{\mathbf{l}}_n(\theta)| = 0, \quad a.s., \\
iii) & \sigma_{S,t}(\theta) = \sigma_{S,t}(\theta_0), \quad \forall t,\ P_{\theta_0}\ a.s. \quad \Longrightarrow \quad \theta = \theta_0. \\
iv) & E|\ell_{S,t}(\theta_0)| < \infty \quad \text{and} \quad \text{if} \quad \theta \neq \theta_0, \quad E\ell_{S,t}(\theta) > E\ell_{S,t}(\theta_0), \\
v) & \text{any} \quad \theta \neq \theta_0 \quad \text{has a neighborhood} \quad V(\theta) \quad \text{such that} \\
 & \liminf_{n\to\infty} \inf_{\theta^*\in V(\theta)} \tilde{\mathbf{l}}_n(\theta) > E\ell_{S,1}(\theta_0), \quad a.s.
\end{array}
$$

to obtain the a.s. convergence of $\hat{\theta}_n$ to θ_0. The proof of Theorem 6 *iii)* in RZ ensuring the identifiability of coefficients remains the same, provided that $\sigma_{S,t}^2(\theta_0) > 0$ almost surely. Noting that $\sigma_{S,t}^2(\theta_0) \geq c_{S,t}(\theta_0)$, it is thus clear, under the assumptions of Theorem 3.1, that $E\sigma_{S,t}^{-2r}(\theta_0) \leq Ec_{S,t}^{-r}(\theta_0) < \infty$ (by Lemma 7.2) and thus $\sigma_{S,t}^2(\theta_0) > 0$ almost surely. *iii)* is thus proved.

Let us first prove *ii)*. Lines of the proof of Theorem 6 *ii)* in RZ allows to obtain:

$$\sup_{\theta\in\Theta} ||\tilde{\sigma}_t^2 - \sigma_t^2|| \quad \leq \quad K \sup_{\theta\in\Theta} \beta(s_t)\ldots\beta(s_1), \quad \forall t \tag{7.6}$$

Thus, using $\log x \leq x - 1$, almost surely,

$$
\begin{aligned}
\sup_{\theta\in\Theta} |\tilde{\mathbf{l}}_n(\theta) - \mathbf{l}_n(\theta)| &\leq n^{-1} \sum_{t=1}^{n} \sup_{\theta\in\Theta} \left\{ \left| \frac{\tilde{\sigma}_t^2 - \sigma_t^2}{\tilde{\sigma}_t^2 \sigma_t^2} \right| \epsilon_t^2 + \left| \log\left(1 + \frac{\sigma_t^2 - \tilde{\sigma}_t^2}{\tilde{\sigma}_t^2}\right) \right| \right\} \\
\leq \quad & Kn^{-1} \sum_{t=1}^{n} \sup_{\theta\in\Theta} \left\{ \frac{\beta(s_t)\ldots\beta(s_1)\epsilon_t^2}{\tilde{\sigma}_t^2(\theta)\sigma_t^2(\theta)} \right\} + Kn^{-1} \sum_{t=1}^{n} \sup_{\theta\in\Theta} \left\{ \frac{\beta(s_t)\ldots\beta(s_1)}{\sigma_t^2(\theta)} \right\}, \\
\leq \quad & Kn^{-1} \sum_{t=1}^{n} \sup_{\theta\in\Theta} \{\beta(s_t)\ldots\beta(s_1)\} \sup_{\theta\in\Theta} \left\{ \frac{\epsilon_t^2}{c_t^2(\theta)} \right\} \\
+ \quad & Kn^{-1} \sum_{t=1}^{n} \sup_{\theta\in\Theta} \{\beta(s_t)\ldots\beta(s_1)\} \sup_{\theta\in\Theta} \left\{ \frac{1}{c_t(\theta)} \right\}.
\end{aligned} \tag{7.7}
$$

The last inequality follows from $\tilde{\sigma}_t^2(\theta) \geq c_t(\theta)$ and $\sigma_t^2(\theta) \geq c_t(\theta)$. RZ showed (see Proof of Theorem 6 *ii)*) that for t sufficiently large

$$\sup_{\theta\in\Theta} \{\beta(s_t)\ldots\beta(s_1)\} \leq \beta_*^t,$$

for some $0 < \beta_* < 1$. Therefore, the first and second terms in the right hand-side term of (7.7) are bounded, for n large enough, respectively by $Kn^{-1}\sum_{t=1}^n \beta_*^t$ $\sup_{\theta\in\Theta}\left\{\frac{\epsilon_t^2}{c_t^2(\theta)}\right\}$ and $Kn^{-1}\sum_{t=1}^n \beta_*^t \sup_{\theta\in\Theta}\left\{\frac{1}{c_t(\theta)}\right\}$.

By an extension of Lemma 1 in Francq and Gautier (2004a), it will be sufficient to show that $n^{-1}\sum_{t=1}^{n}\beta_*^t \sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t}^2}{c_{S,t}^2(\theta)}\right\}$ and $n^{-1}\sum_{t=1}^{n}\beta_*^t \sup_{\theta\in\Theta}\left\{\frac{1}{c_{S,t}(\theta)}\right\}$ converge to 0, $\mathbb{P}$ a.s. when $n\to\infty$. Using the Cesaro Lemma, it will be sufficient to show that $\beta_*^t \sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t}^2}{c_{S,t}^2(\theta)}\right\}$ and $\beta_*^t \sup_{\theta\in\Theta}\left\{\frac{1}{c_{S,t}(\theta)}\right\}$ converge to 0, a.s. when $t\to+\infty$. For the first term, this result follows from the Borel-Cantelli lemma, because

$$\begin{aligned}\sum_{t=1}^{\infty}\mathbb{P}\left(\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t}^2}{c_{S,t}^2(\theta)}\right\}>\zeta\right) &\leq \sum_{t=1}^{\infty}\frac{\beta_*^{tr/4}}{\zeta^{r/4}}E\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t}^2}{c_{S,t}^2(\theta)}\right\}^{r/4}\\ \leq \left\{E\sup_{\theta\in\Theta}\frac{1}{c_{S,t}^r}\right\}^{1/2} &\left\{E|\epsilon_{S,t}|^r\right\}^{1/2}\sum_{t=1}^{\infty}\frac{\beta_*^{tr/4}}{\zeta^{r/4}}<\infty\end{aligned}$$

for any $\zeta>0$. The first inequality follows from the Markov inequality, the second one from the Cauchy-Schwarz inequality and from the stationarity of $(c_{S,t})$ and $(\epsilon_{S,t})$. The result follows from Lemma 7.5, (7.3) and $\beta_*<1$. The same arguments allow to conclude for the second term because

$$\sum_{t=1}^{\infty}\mathbb{P}\left(\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{1}{c_{S,t}(\theta)}\right\}>\zeta\right) \quad\leq\quad \sum_{t=1}^{\infty}\frac{\beta_*^{tr}}{\zeta^r}E\sup_{\theta\in\Theta}\left\{\frac{1}{c_{S,t}^r(\theta)}\right\}<\infty$$

for any $\zeta>0$, by (7.3) and $\beta_*<1$. Thus *ii)* is proved.

The proof of *iv)* is exactly the same as the one developped in the proof of Theorem 6 *iv)* in RZ provided $E\ell_{S,t}^-(\theta)<\infty$ for all $\theta\in\Theta$. In view of (7.1), as $\log^-(.)$ is a decreasing function, for all $\theta\in\Theta$, $E\ell_{S,t}^-(\theta)\leq E\log^-\sigma_{S,t}^2(\theta)\leq\sup_{\theta\in\Theta}E\log^-\Omega_{S,t}^*(\theta)<\infty$ where the result follows from Lemma 7.1. *iv)* is thus proved.

It remains to show *v)*. For any $\theta\in\Theta$ and any positive integer k, let $V_k(\theta)$ be the open ball with center θ and radius $1/k$. The proof of Theorem 6 *v)* in RZ remains the same, provided that the expectation of $\inf_{\theta^*\in V_k(\theta)\cap\Theta}\ell_{S,t}(\theta^*)<\infty$ exists. This term being larger than $\inf_{\theta\in\Theta}\log\Omega_{S,t}^*(\theta)$, which possesses a finite expectation by Lemma 7.1, this result is thus established.

7.2 Proof of Theorem 3.2

We will follow the scheme of the proof of Theorem 2 in Francq and Zakoïan (2007). First, we prove that the score at $\hat{\theta}_n$ is asymptotically Gaussian then we will prove Theorem 3.2.

In this part, we will use Lemma 7.3 under the assumptions **A0"**, **A2** and **A9**. Thus as stated in Lemma 7.3, (7.4) will be here considered true for $b=3+\xi$.

7.2.1 Asymptotic normality of the score at $\hat{\theta}_n$

Let us define for $\theta \in \Theta$, $a_\theta(x,y) = \alpha(x)y^2 + \beta(x)$. Here we state some properties on the quantities $a_\theta^v(S_t,\eta_{t-1})\dots a_\theta^v(S_{t-i},\eta_{t-i-1})$, $\frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)}$ and $\sigma_{S,t}^2(\theta)$ in the neighborhood of θ_0. This Lemma will be used later when treating the property of the score vector in the neighborhood of θ_0 and in the proof of Theorem 3.3.

Lemma 7.6 *Under the assumptions of Theorem 3.2 ii), for all $m \geq 0$ there exist a neighborhood $\mathcal{V}(\theta_0)$ of θ_0, $0 < v \leq r, 0 \leq \rho^* < 1, \quad C^* \geq 0$ such that :*

$$\forall i > 0, \quad E \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \{a_\theta^v(S_t,\eta_{t-1})\dots a_\theta^v(S_{t-i},\eta_{t-i-1})\} < C^*(\rho^*)^{i+1}. \tag{7.8}$$

where $a_\theta(x,y) = \alpha(x)y^2 + \beta(x)$

$$\left\| \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)} \right\|_m < \infty, \quad E \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \sigma_{S,t}^{2v}(\theta) < \infty. \tag{7.9}$$

Proof of Lemma 7.6

The assumption **A9** allows to define, for any $\delta > 0$ and $\epsilon > 0$ a neighborhood $\mathcal{V}(\theta_0)$ of θ_0 such that $\forall \theta \in \mathcal{V}(\theta_0)\cap\Theta, \quad \forall i \in \{1,\dots,3d\}$:

$$(1-\delta)\theta_i < \theta_{0,i} < (1+\delta)\theta_i \quad \text{if} \quad \theta_{0,i} \neq 0 \quad \text{and} \quad 0 \leq \theta_i < \epsilon \quad \text{if} \quad \theta_{0,i} = 0. \tag{7.10}$$

Let us first prove that $\frac{c_{S,t}(\theta_0)}{c_{S,t}(\theta)} \leq 2(1+\delta)$ for all $\theta \in \mathcal{V}(\theta_0)\cap\Theta$. Letting $i \in \{1,\dots,d\}$, let us consider case (i): $\omega_0(e_i) = 0$, and case (ii) $\alpha_0(e_i) = 0$. First, note that the assumption **A9'** entails that $\omega_0(.) + \alpha_0(.) > 0$. Consequently, in case (i) $\alpha_0(e_i) > 0$ and in case (ii) $\omega_0(e_i) > 0$. Now, for all $\theta \in \mathcal{V}(\theta_0)\cap\Theta$ satisfying case (i), for all t

$$\frac{\omega_0(e_i) + \alpha_0(e_i)\eta_t^2}{\omega(e_i) + \alpha(e_i)\eta_t^2} = \frac{\alpha_0(e_i)\eta_t^2}{\omega(e_i) + \alpha(e_i)\eta_t^2} \leq \frac{\alpha_0(e_i)}{\alpha(e_i)} \leq (1+\delta).$$

The first inequality follows from $\alpha_0(e_i) > 0$ which entails $\alpha(e_i) > 0$ in view of (7.10) and the last one relies on (7.10). Similarly for $\theta \in \mathcal{V}(\theta_0)\cap\Theta$ satisfying case (ii), by similar arguments,

$$\frac{\omega_0(e_i) + \alpha_0(e_i)\eta_t^2}{\omega(e_i) + \alpha(e_i)\eta_t^2} = \frac{\omega_0(e_i)}{\omega(e_i) + \alpha(e_i)\eta_t^2} \leq \frac{\omega_0(e_i)}{\omega(e_i)} \leq (1+\delta).$$

When $\omega_0(e_i), \alpha_0(e_i) > 0$, it is clear that for all $\theta \in \mathcal{V}(\theta_0)\cap\Theta$, $\omega(e_i), \alpha(e_i) > 0$ and thus :

$$\frac{\omega_0(e_i) + \alpha_0(e_i)\eta_t^2}{\omega(e_i) + \alpha(e_i)\eta_t^2} \leq \frac{\omega_0(e_i)}{\omega(e_i)} + \frac{\alpha_0(e_i)}{\alpha(e_i)} \leq 2(1+\delta).$$

These three inequalities allow to entail that $\frac{c_{S,t}(\theta_0)}{c_{S,t}(\theta)} \leq 2(1+\delta)$ for all $\theta \in \mathcal{V}(\theta_0) \cap \Theta$. Now, for $\varsigma \in]0,1[$:

$$\begin{aligned}
&\sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{\sigma^2_{S,t}(\theta_0)}{\sigma^2_{S,t}(\theta)} \\
\leq\ & \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \left\{ \frac{c_{S,t}(\theta_0) + \sum_{k=0}^{\infty} \beta_0(S_t)\dots\beta_0(S_{t-i})c_{S,t-i-1}(\theta_0)}{c_{S,t}(\theta) + \sum_{k=0}^{\infty} \beta(S_t)\dots\beta(S_{t-i})c_{S,t-i-1}(\theta)} \right\} \\
\leq\ & 2(1+\delta) + \sum_{k=0}^{\infty} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \left\{ \frac{\beta_0(S_t)\dots\beta_0(S_{t-i})c_{S,t-i-1}(\theta_0)}{c_{S,t}(\theta) + \beta(S_t)\dots\beta(S_{t-i})c_{S,t-i-1}(\theta)} \right\} \\
\leq\ & K \\
+\ & \sum_{i=0}^{\infty} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \left\{ \frac{\beta_0(S_t)\dots\beta_0(S_{t-i})c_{S,t-i-1}(\theta_0)}{\beta(S_t)\dots\beta(S_{t-i})c_{S,t-i-1}(\theta)} \left(\frac{\beta(S_t)\dots\beta(S_{t-i})c_{S,t-i-1}(\theta)}{c_{S,t}(\theta)} \right)^{\varsigma} \right\} \\
\leq\ & K + K\sum_{i=0}^{\infty} \left(\frac{(1+\delta)^{\varsigma}}{1-\delta} \right)^{i+1} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} c^{\varsigma}_{S,t-i-1}(\theta) \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} c^{-\varsigma}_{S,t}(\theta) \\
& \{\beta_0(S_t)\dots\beta_0(S_{t-i})\}^{\varsigma}.
\end{aligned}$$

The second inequality follows from $\frac{c_{S,t}(\theta_0)}{c_{S,t}(\theta)} \leq 2(1+\delta)$ for all $\theta \in \mathcal{V}(\theta_0) \cap \Theta$. The third one relies on the elementary inequality $x/(y+z) \leq (x/z)(z/y)^{\varsigma}$ for any $\varsigma \in]0,1[$, $x, y, z > 0$. The fourth inequality relies on two arguments. As $\beta_0(.)$ (by assumption **A10**), in view of (7.10), firstly $\beta_0(.)/\beta(.) \leq (1-\delta)^{-1}$ and secondly $\beta(.) \leq \beta_0(.)(1+\delta)$. Choosing $\varsigma = \frac{r}{4m}$ and $\delta > 0$ small enough such that

$$\frac{(1+\delta)^{\varsigma}}{1-\delta}\rho^{1/4m} < 1 \tag{7.11}$$

and using the Minkowski inequality, we obtain

$$\begin{aligned}
\left\| \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{\sigma^2_{S,t}(\theta_0)}{\sigma^2_{S,t}(\theta)} \right\|_m \leq\ & K + K\sum_{i=0}^{\infty} \left(\frac{(1+\delta)^{\varsigma}}{1-\delta} \right)^{i+1} \left\| \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} c^{\varsigma}_{S,t-i-1}(\theta) \right\|_{2m} \\
& \left\| \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} c^{-\varsigma}_{S,t}(\theta) \right\|_{4m} \left\| \{\beta_0(S_t)\dots\beta_0(S_{t-i})\}^{\varsigma} \right\|_{4m} \\
\leq\ & K + K\sum_{i=0}^{\infty} \left(\frac{(1+\delta)^{\varsigma}\rho^{1/4m}}{1-\delta} \right)^{i} < \infty.
\end{aligned}$$

The second inequality follows from the Cauchy-Schwarz inequality. Noting that the components of array of Θ are bounded by a constant $\overline{\theta}$ (ie $\theta_i \leq \overline{\theta}$ for all $\theta \in \Theta$), we easily get using Lemma 7.5, $E\sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} c^r_{S,t}(\theta) \leq \overline{\theta}^r(1 + E\epsilon^{2r}_{S,t-1}) < \infty$. This result together with the stationarity of $(c_{S,t})$, (7.3) and the assumption **A3** allows to retrieve the third inequality. The final result follows from $\frac{(1+\delta)^{\varsigma}}{1-\delta}\rho^{1/4m} < 1$ and the first part of (7.9) is proved.

Now we turn to the proof of (7.8). Let us choose $t \in \mathbb{Z}$, $\theta \in \mathcal{V}(\theta_0) \cap \Theta$ and $i \in \{1, \dots, d\}$. Noting that $\beta_0(.) > 0$ (by the assumption **A10**), $\beta^* := \min_{i \in 1,\dots,d} \beta_0(e_i) > 0$. (7.10) leads to :

$$\frac{a_\theta^u(e_i, \eta_{t-1})}{a_0^u(e_i, \eta_{t-1})} \leq (1+\delta)^u \quad \text{if} \quad \alpha_0(e_i) \neq 0$$

and

$$\frac{a_\theta^u(e_i, \eta_{t-1})}{a_0^u(e_i, \eta_{t-1})} \leq \{\frac{\alpha(e_i)\eta_{t-1}^2}{\beta_0(e_i)} + \frac{\beta(e_i)}{\beta_0(e_i)}\}^u \leq \{\frac{\epsilon \eta_{t-1}^2}{\beta^*} + 1 + \delta\}^u \quad \text{if} \quad \alpha_0(e_i) = 0.$$

The second inequality relies on $\alpha(e_i) \leq \epsilon$ when $\alpha_0(e_i) = 0$, $\beta(e_i) < (1+\delta)\beta_0(e_i)$ and $0 < \beta^* \leq \beta_0(e_i)$ for all $\theta \in \mathcal{V}(\theta_0) \cap \Theta$. Thus using the elementary inequality $(x+y)^u \leq x^u + y^u$ for $0 < u < 1$ on the right hand-side condition, we get $\frac{a_\theta^u(e_i,\eta_{t-1})}{a_0^u(e_i,\eta_{t-1})} \leq (1+\delta)^u + \frac{\epsilon^u}{\beta^*}\eta_{t-1}^{2u}$ in both cases. Choosing $v = r/2$ and using the Cauchy-Schwarz inequality:

$$\begin{aligned} & E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \prod_{h=0}^{i} a_\theta^v(S_{t-h}, \eta_{t-h-1}) \\ \leq & \left\{ E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \prod_{h=0}^{i} \frac{a_\theta^r(S_{t-h}, \eta_{t-h-1})}{a_0^r(S_{t-h}, \eta_{t-h-1})} \right\}^{1/2} \left\{ E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \prod_{h=0}^{i} a_0^r(S_{t-h}, \eta_{t-h-1}) \right\}^{1/2} \\ \leq & \left\{ E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \prod_{h=0}^{i} \{(1+\delta)^r + \frac{\epsilon^r}{\beta^*}\eta_{t-h-1}^{2r}\} \right\}^{1/2} \rho^{\frac{i+1}{2}} \leq \{(1+\delta)^r + \frac{\epsilon^r}{\beta^*} E\eta_t^{2r}\}^{\frac{i+1}{2}} \rho^{\frac{i+1}{2}}. \end{aligned}$$

The second inequality comes from the assumption **A3** together with an application of

$$\max_{i \in \{1,\dots,d\}} \frac{a_\theta^u(e_i, \eta_{t-1})}{a_\theta^u(e_i, \eta_{t-1})} \leq (1+\delta)^u + \frac{\epsilon^u}{\beta^*}\eta_{t-1}^{2u}.$$

Remarking that $(1+\delta)^u + \frac{\epsilon^u}{\underline{\beta}_0^u}\eta_{t-h-1}^{2u}$ does not depend on θ, we can forget the supremum in the right hand-side term of the second inequality. (η_t) being an iid process, we easily get the third inequality. η_t^2 possessing a moment at order 2, thus a moment at order $2r$ ($r < 1$), it is possible to choose $\epsilon, \delta > 0$ according to (7.11) such that:

$$\rho^* := \{(1+\delta)^r + \frac{\epsilon^r}{\beta^*} E\eta_t^{2r}\}^{1/2} \rho^{1/2} < 1 \tag{7.12}$$

and thus define an appropriate ρ^* to retrieve (7.8) together with the first term of (7.9).

The second term of (7.9) is a consequence of the first term of (7.9) and (7.8). The lemma is thus proved.

□

When $\theta_0 > 0$ (being understood componentwise), RZ showed that $\left\| \frac{1}{\sigma^2_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \theta} \right\|$ possesses moments at any order $m \in \mathbb{N}$ in a neighborhood of θ_0 (see Proof of Theorem 7). As noted previously, this property does not hold when θ_0 lies on the boundary. Here, we show that this variable possesses a third-order moment at the true value θ_0 under the assumptions made.

Lemma 7.7 *Under the assumptions of theorem 3.2, there exists a neighborhood $\mathcal{V}(\theta_0)$ of θ_0 such that*

$$\left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^2_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \theta} \right\|_3 < \infty, \quad \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ \frac{1}{\sigma^2_{S,t}} \frac{\partial^2 \sigma^2_{S,t}}{\partial \theta \partial \theta'} \right\} \right\|_3 < \infty, \quad (7.13)$$

$$\left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^4_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \theta} \frac{\partial \sigma^2_{S,t}}{\partial \theta'} \right\|_{3/2} < \infty. \quad (7.14)$$

Proof of Lemma 7.7 First note that (7.14) directly follows from the first term of (7.13) using Cauchy-Schwarz inequality. Differentiating (2.6) we obtain, for $k \in \{1 \ldots, d\}$,

$$\frac{\partial \sigma^2_{S,t}}{\partial \omega(e_k)} = \mathbb{1}_{S_t = e_k} + \sum_{i=0}^{\infty} \beta(S_t) \ldots \beta(S_{t-i}) \mathbb{1}_{S_{t-i-1}=e_k}, \quad (7.15)$$

$$\frac{\partial \sigma^2_{S,t}}{\partial \alpha(e_k)} = \epsilon^2_{S,t-1} \mathbb{1}_{S_t = e_k} + \sum_{i=0}^{\infty} \beta(S_t) \ldots \beta(S_{t-i}) \epsilon^2_{S,t-i-2} \mathbb{1}_{S_{t-i-1}=e_k}, \quad (7.16)$$

and

$$\frac{\partial \sigma^2_{S,t}}{\partial \beta(e_k)} = \sum_{i=0}^{\infty} \left\{ \sum_{j=0}^{i} \prod_{h \neq j, 0 \leq h \leq i} \beta(S_{t-h}) \mathbb{1}_{S_{t-j}=e_k} \right\} c_{S,t-i-1}(\theta). \quad (7.17)$$

Let us now turn to prove the second term of (7.13). Let us denote

$$\beta^{**} = \inf_{\theta \in \mathcal{V}(\theta_0) \cap \Theta, i \in \{1, \ldots, \delta\}} \beta(e_i).$$

In view of the assumption **A10**, $\mathcal{V}(\theta_0)$ can be chosen appropriately to get $\beta^{**} > 0$. We will choose here $\mathcal{V}(\theta_0)$ such that (7.9) can be true for $m = 8$ thus for any $m \leq 8$, by the Hôlder inequality.

We first prove the first term of (7.13). Note that by the positivity of the coefficients involved in (7.15)-(7.17), the derivatives of $\sigma^2_{S,t}$ are nonnegative. It can be shown as in RZ that the first-order derivatives with respect to $\alpha(e_k)$ (respectively $\omega(e_k)$) for $k \in \{1, \ldots, d\}$ such that $\alpha_0(e_k) \neq 0$ (respectively $\omega_0(e_k) \neq 0$) are bounded on $\mathcal{V}(\theta_0)$ by $\frac{1+\delta}{\alpha_0(e_k)}$ (respectively $\frac{1+\delta}{\omega_0(e_k)}$), with notations of Lemma 7.6. They thus possess a moment at any order. Turning to the derivative with respect to $\omega(e_k)$ for $k \in \{1, \dots, d\}$ such that $\omega_0(e_k) = 0$, using (7.5)

$$\frac{1}{\sigma^2_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \omega(e_k)}(\theta) \leq \frac{1 + \beta(S_t)}{\sigma^2_{S,t}(\theta)} + \sum_{i=1}^{\infty} \frac{\beta(S_t) \dots \beta(S_{t-i})}{\sigma^2_{i,S,t}(\theta)}. \tag{7.18}$$

Let us choose $M > 0$ such that $\frac{1}{3+\xi} + \frac{1}{M} = \frac{1}{3}$ with $\xi > 0$ defined in Lemma 7.3. Using the Minkowski inequality, we have with v defined in Lemma 7.6:

$$\begin{aligned}
& \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^2_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \omega(e_k)} \right\|_3 \\
\leq & \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1 + \beta(S_t)}{\sigma^2_{S,t}(\theta)} \right\|_3 \\
+ & \sum_{i=1}^{\infty} \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\Omega^*_{S,t-i-1}(\theta)} \frac{\beta(S_t) \dots \beta(S_{t-i}) \Omega^*_{S,t-i-1}(\theta)}{c_{S,t}(\theta) + \beta(S_t) \dots \beta(S_{t-i}) \Omega^*_{S,t-i-1}(\theta)} \right\|_3 \\
\leq & K + \sum_{i=1}^{\infty} \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\Omega^{*(1-\frac{r}{2M})}_{S,t-i-1}(\theta)} \right\|_{3+\xi} \\
& \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \beta^{\frac{v}{2M}}(S_t) \dots \beta^{\frac{v}{2M}}(S_{t-i}) \right\|_{2M} \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{c^{v/2M}_{S,t}(\theta)} \right\|_{2M} \\
\leq & K + K \sum_{i=1}^{\infty} \rho^{*i/2M} < \infty.
\end{aligned} \tag{7.19}$$

The second inequality follows firstly from the elementary inequality $\frac{x}{1+x} \leq x^s$ for all $x \geq 0$, $0 < s < 1$, secondly from $E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^6_{S,t}(\theta)} \leq E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\Omega^{*3}_{S,t}(\theta)} < \infty$ (by (7.4)), and thirdly from the applications of the Hölder inequality $\|XY\|_3 \leq \|X\|_{3+\xi} \|Y\|_M$ together with the Cauchy-Schwarz inequality. Now, first, $\sup_{\theta \in \Theta} \Omega^{*-1}_{S,t}(\theta)$ possessing moment at order $(3 + \xi)$ by (7.4), it possesses thus moment at order $(1 - \frac{r}{2M})(3 + \xi)$ by the Hôlder inequality. Second, $\sup_{\theta \in \Theta} c^{-1}_{S,t}(\theta)$, possessing moment at order r by (7.3), possesses moment at order $v \leq r$. These two arguments combinated with the elementary inequalities

$$\sup_{\theta \in \Theta} \Omega^{*-1}_{S,t}(\theta) \geq \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \Omega^{*-1}_{S,t}(\theta) \quad \text{and} \quad \sup_{\theta \in \Theta} c^{-1}_{S,t}(\theta) \geq \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} c^{-1}_{S,t}(\theta),$$

the stationarity of $(\Omega^*_{S,t}(\theta))$ and (7.8) lead to the third inequality. The result follows from $\rho^* < 1$. Turning to the derivative with respect to $\alpha(e_k)$ for $k \in \{1, \dots, d\}$ such that $\alpha_0(e_k) = 0$, by (7.16) and using the inequality (7.5), we have for $0 < u < 1$, for

$\theta \in \mathcal{V}(\theta_0) \cap \Theta$:

$$\begin{aligned}
& \frac{1}{\sigma^2_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \alpha(e_k)}(\theta) \\
\leq\ & \frac{\epsilon^2_{S,t-1}}{\sigma^2_{S,t}(\theta)} + \sum_{i=0}^{\infty} \frac{\beta(S_t)\ldots\beta(S_{t-i})\epsilon^2_{S,t-i-2}\,\mathbb{1}_{S_{t-i-1}=e_k}}{{}^{i}\sigma^2_{S,t}(\theta)} \qquad (7.20)\\
\leq\ & K\eta^2_{t-1}\frac{\sigma^2_{S,t-1}(\theta_0)}{\sigma^2_{S,t-1}(\theta)} + \sum_{i=0}^{\infty}\eta^2_{t-i-2}\frac{\beta(S_t)\ldots\beta(S_{t-i})\sigma^2_{t-i-2}(\theta_0)}{d_{S,t}(\theta)+\beta^{**}\beta(S_t)\ldots\beta(S_{t-i})\sigma^2_{S,t-i-2}(\theta)}\\
\leq\ & K\eta^2_{t-1}\frac{\sigma^2_{S,t-1}(\theta_0)}{\sigma^2_{S,t-1}(\theta)} + \sum_{i=0}^{\infty}\frac{\eta^2_{t-i-2}}{\beta^{**}}\frac{\sigma^2_{S,t-i-2}(\theta_0)}{\sigma^2_{S,t-i-2}(\theta)}\left\{\frac{\beta^{**}\beta(S_t)\ldots\beta(S_{t-i})\sigma^2_{S,t-i-2}(\theta)}{d_{S,t}(\theta)}\right\}^u. \qquad (7.21)
\end{aligned}$$

The second inequality follows from the obvious inequality $\sigma^2_{S,t}(\theta) \geq \beta^{**}\sigma^2_{S,t-1}(\theta)$ with $\beta^{**} > 0$ and $\beta(S_{t-i-1}) \geq \beta^{**} > 0$ for $\theta \in \mathcal{V}(\theta_0) \cap \Theta$. The last one relies on the elementary inequality $\frac{x}{1+x} \leq x^s$ for any $x \geq 0$ and $u \in [0,1]$. Taking supremum on (7.21) for $\theta \in \mathcal{V}(\theta_0) \cap \Theta$, choosing $u = v/24$ (with v defined in Lemma 7.6) and using the Minkowski inequality, we obtain:

$$\begin{aligned}
& \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\sigma^2_{S,t}}\frac{\partial\sigma^2_{S,t}}{\partial\alpha(e_k)}(\theta)\right\|_3 \\
\leq\ & K\left\|\eta^2_{t-1}\right\|_3 \left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\sigma^2_{S,t-1}(\theta_0)}{\sigma^2_{S,t-1}(\theta)}\right\|_3 \\
+\ & K\sum_{i=0}^{\infty}\left\|\eta^2_{t-i-2}\right\|_3\left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\sigma^2_{S,t-i-2}(\theta_0)}{\sigma^2_{S,t-i-2}(\theta)}\right\|_6\left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}(\beta^u(S_t)\ldots\beta^u(S_{t-i}))\right\|_{12}\\
& \left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\sigma^{2u}_{S,t-i-2}(\theta)\right\|_{24}\left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} d^{-u}_{S,t}(\theta)\right\|_{24}\\
\leq\ & K\Big(1+\sum_{i=0}^{\infty}\{E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}(\beta^v(S_t)\ldots\beta^v(S_{t-i}))\}^{\frac{1}{24}}\Big) \leq K\sum_{i=0}^{\infty}\{(\rho^*)^{\frac{1}{24}}\}^i < \infty. \qquad (7.22)
\end{aligned}$$

First, noting that $d_{S,t}$ belongs to $\sigma(\eta_{t-1}, S_t, S_{t-1}, \ldots)$, it is clear that $\frac{\sigma^2_{S,t-i-2}(\theta_0)}{\sigma^2_{S,t-i-2}(\theta)}$ $\left\{\frac{\beta^{**}\beta(S_t)\ldots\beta(S_{t-i})\sigma^2_{S,t-i-2}(\theta)}{d_{S,t}(\theta)}\right\}^u$ are independent of η_{t-i-2} by assumption **A0**. Second **A0** implies also the independence between $\frac{\sigma^2_{S,t-1}(\theta_0)}{\sigma^2_{S,t-1}(\theta)}$ and η_{t-1}. These both arguments together with use of three succesives Cauchy Schwartz inequalities lead to the first inequality. The second one follows from (7.3), together with the strict stationarity of $(\sigma^2_{S,t})$ and $\left(\frac{\sigma^2_{S,t-1}(\theta_0)}{\sigma^2_{S,t-1}(\theta)}\right)$, (7.9) and (7.8). The result comes from $\rho^* < 1$.

We turn now to the derivative with respect to $\beta(e_k)$. We will show that it possesses moments at any order $M \in \mathbb{N}$. Remarking that for $\theta \in \mathcal{V}(\theta_0) \cap \Theta$ $\beta(.) \geq$

$\beta^{**} > 0$ we retrieve as in RZ :

$$\frac{\partial \sigma^2_{S,t}}{\partial \beta(e_k)}(\theta) \leq \frac{1}{\beta(e_k)} \sum_{i=0}^{\infty} (i+1)\beta(S_t)\ldots\beta(S_{t-i})c_{S,t-i-1}(\theta).$$

Now, taking supremum on this inequality for $\theta \in \mathcal{V}(\theta_0) \cap \Theta$, using the Minkowski inequality and the inequality (7.5), we obtain, with v defined in Lemma 7.6 :

$$\begin{aligned}
& \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\sigma^2_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \beta(e_k)} \right\|_M \\
\leq \;& \sum_{i=0}^{\infty} \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\beta(e_k)} \frac{(i+1)\beta(S_t)\ldots\beta(S_{t-i})c_{S,t-i-1}(\theta)}{{}_i\sigma^2_{\mathrm{S,t}}(\theta)} \right\|_M \\
\leq \;& \frac{1}{\beta^{**}} \sum_{i=0}^{\infty} \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{(i+1)\beta(S_t)\ldots\beta(S_{t-i})c_{S,t-i-1}(\theta)}{c_{S,t}(\theta)+\beta(S_t)\ldots\beta(S_{t-i})c_{S,t-i-1}(\theta)} \right\|_M \\
\leq \;& K\sum_{i=0}^{\infty}(i+1) \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \left\{ \frac{\beta(S_t)\ldots\beta(S_{t-i})c_{S,t-i-1}(\theta)}{c_{S,t}(\theta)} \right\}^{v/4M} \right\|_M \\
\leq \;& K\sum_{i=0}^{\infty}(i+1) \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \{\beta(S_t)\ldots\beta(S_{t-i})\}^{v/4M} \right\|_{2M} \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} c_{S,t-i-1}(\theta)^{v/4M} \right\|_{4M} \\
& \left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} c_{S,t}(\theta)^{-v/4M} \right\|_{4M} \\
\leq \;& K\sum_{i=0}^{\infty}(i+1)\{E \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \{\beta(S_t)\ldots\beta(S_{t-i})\}^{v/2}\}^{1/2M} \\
\leq \;& K\sum_{i=0}^{\infty}(i+1)\rho^{*i/4M} < \infty. \qquad (7.23)
\end{aligned}$$

The second inequality relies on $\beta(.) > \beta^{**} > 0$ for $\theta \in \mathcal{V}(\theta_0) \cap \Theta$ and the definition of ${}_i\sigma^2_{\mathrm{S,t}}$. The third one relies the elementary inequality $\frac{x}{1+x} \leq x^{v/12}$. The fourth one is based on the Hölder inequality. The elementary inequality $(a+b)^v \leq a^v + b^v$, for all $a, b \geq 0$, and $r \in (0,1)$, together with Lemma 7.5 and Θ compact thus upper bounded componentwise entail $E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} c^v_{S,t} < \infty$. From this result together with (7.2) and the stationarity of $(c_{S,t})$, we deduce the fifth inequality. The last one relies on (7.8) and the result on $\rho^* < 1$.

The second order derivatives of $\sigma^2_{S,t}$ may be treated in similar way. We omit the proof for brevity. The lemma is thus proved.

□

Lemma 7.8 *Under the assumptions of Lemma 7.7,*

$$E\left\|\frac{\partial \ell_{S,t}(\theta_0)}{\partial \theta}\frac{\partial \ell_{S,t}(\theta_0)}{\partial \theta}\right\| < \infty \quad \textit{and} \quad E\left\|\frac{\partial^2 \ell_{S,t}(\theta_0)}{\partial \theta \partial \theta'}\right\| < \infty.$$

Proof of Lemma 7.8

The first and second derivatives of $\ell_{S,t} = \frac{\epsilon^2_{S,t}}{\sigma^2_{S,t}} + \log \sigma^2_{S,t}$ are given by :

$$\frac{\partial \ell_{S,t}}{\partial \theta} = \left\{1 - \frac{\epsilon^2_{S,t}}{\sigma^2_{S,t}}\right\}\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial \sigma^2_{S,t}}{\partial \theta}\right\}, \tag{7.24}$$

$$\frac{\partial^2 \ell_{S,t}}{\partial \theta \partial \theta'} = \left\{1 - \frac{\epsilon^2_{S,t}}{\sigma^2_{S,t}}\right\}\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial^2 \sigma^2_{S,t}}{\partial \theta \partial \theta'}\right\} + \left\{2\frac{\epsilon^2_{S,t}}{\sigma^2_{S,t}} - 1\right\}\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial \sigma^2_{S,t}}{\partial \theta}\right\}\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial \sigma^2_{S,t}}{\partial \theta'}\right\}. \tag{7.25}$$

In view of Lemma 7.7, $\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial \sigma^2_{S,t}}{\partial \theta}(\theta_0)\right\}$ possesses moment at order 2 and in view of **A7'**, $\left\{1 - \frac{\epsilon^2_{S,t}}{\sigma^2_{S,t}(\theta_0)}\right\} = \{1 - \eta_t^2\}$ possesses a moment at order 2. These both terms being independent, we get that $\frac{\partial \ell_{S,t}}{\partial \theta}(\theta_0)$ possesses moment at order 2 and thus the first result of the Lemma. Using Lemma 7.7, $\left\{1 - \frac{\epsilon^2_{S,t}}{\sigma^2_{S,t}(\theta_0)}\right\}$ and $\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial^2 \sigma^2_{S,t}}{\partial \theta \partial \theta'}(\theta_0)\right\}$ possess moment at order 1 and are independent thus also their product. This lemma also allows to get that $\left\{2\frac{\epsilon^2_{S,t}}{\sigma^2_{S,t}(\theta_0)} - 1\right\}$ and $\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial \sigma^2_{S,t}(\theta_0)}{\partial \theta}\right\}\left\{\frac{1}{\sigma^2_{S,t}}\frac{\partial \sigma^2_{S,t}(\theta_0)}{\partial \theta'}\right\}$ are independent and possess moment at order 1. Thus their product also possesses moment at order 1. $\frac{\partial^2 \ell_{S,t}(\theta_0)}{\partial \theta \partial \theta'}$, which is the sum of these both products, possesses thus moment at order 1. The Lemma is thus proved.

□

Lemma 7.9 *Under the assumptions of Theorem 3.2, J is non-singular and*

$$Var_{\theta_0}\left\{\frac{\partial \ell_{S,t}}{\partial \theta}\right\} = (\kappa_\eta - 1)J.$$

Proof of Lemma 7.9 The proof follows from Lemma 7.8 and the assumptions **A3**-**A4**(see RZ, Proof of Theorem 4.2 *ii)*).

□

The following lemma, together with Theorem 1 i), readily shows that J can be consistently estimated by $\hat{J}_n = \frac{\partial^2 \mathbf{l}_n(\hat{\theta}_n)}{\partial \theta \partial \theta'}$.

Lemma 7.10 *Under the assumptions of Lemma 7.7, for any $\epsilon > 0$, there exists a neighborhood $\mathcal{V}(\theta_0)$ of θ_0 such that, almost surely,*

$$E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\| \frac{\partial^2 \ell_{S,t}(\theta)}{\partial \theta \partial \theta'} \right\| < \infty \tag{7.26}$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\| \frac{\partial^2 \ell_t(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \ell_t(\theta_0)}{\partial \theta \partial \theta'} \right\| \leq \epsilon. \tag{7.27}$$

Proof of Lemma 7.10

In view of (7.25), if we get the existence of $\mathcal{V}(\theta_0)$ such that :

$$\left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 1 - \frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2} \right\} \right\|_3 < \infty, \quad \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 2\frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2} - 1 \right\} \right\|_3 < \infty, \tag{7.28}$$

$$\left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ \frac{1}{\sigma_{S,t}^2} \frac{\partial^2 \sigma_{S,t}^2}{\partial \theta \partial \theta'} \right\} \right\|_2 < \infty, \quad \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma_{S,t}^2} \frac{\partial \sigma_{S,t}^2}{\partial \theta} \right\|_3 < \infty, \tag{7.29}$$

then, by the Cauchy-Schwarz inequality, we will get

$$\begin{aligned} & E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 1 - \frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2} \right\} \left\{ \frac{1}{\sigma_{S,t}^2} \frac{\partial^2 \sigma_{S,t}^2}{\partial \theta \partial \theta'} \right\} \\ \leq & \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 1 - \frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2} \right\} \right\|_2 \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ \frac{1}{\sigma_{S,t}^2} \frac{\partial^2 \sigma_{S,t}^2}{\partial \theta \partial \theta'} \right\} \right\|_2 < \infty. \end{aligned}$$

and using the Hôlder inequality,

$$\begin{aligned} & E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 2\frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2} - 1 \right\} \left\{ \frac{1}{\sigma_{S,t}^2} \frac{\partial \sigma_{S,t}^2}{\partial \theta} \right\} \left\{ \frac{1}{\sigma_{S,t}^2} \frac{\partial \sigma_{S,t}^2}{\partial \theta'} \right\} \\ \leq & \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 2\frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2} - 1 \right\} \right\|_3 \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma_{S,t}^2} \frac{\partial \sigma_{S,t}^2}{\partial \theta} \right\|_3 \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma_{S,t}^2} \frac{\partial \sigma_{S,t}^2}{\partial \theta'} \right\|_3, \end{aligned}$$

and hence we will retrieve (7.26). We will choose here $\mathcal{V}(\theta_0)$ satisfying lemma 7.6 with $m = 8$. This choice allows to get (7.29) by Lemma 7.7. Thus using the assumption **A7'** and the independence between η_t and $\frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)}$ following from the assumption **A0**, it follows

$$\left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2(\theta)} \right\|_3 = \{E\eta_t^6\}^{1/3} \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)} \right\|_3 < \infty.$$

Using the Minkowski inequality for both terms of (7.28), (7.28) is achieved. (7.26) is thus proved. Now, Lemma 1 in Francq and Gautier (2004a) shows that:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\| \frac{\partial^2 \ell_t(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \ell_t(\theta_0)}{\partial \theta \partial \theta'} \right\| = E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\| \frac{\partial^2 \ell_{S,t}(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \ell_{S,t}(\theta_0)}{\partial \theta \partial \theta'} \right\|$$

The expectation exists in view of (7.26), which ensures the convergence. This expectation decreases to 0 when the neighborhood $\mathcal{V}(\theta_0)$ decreases to the singleton $\{\theta_0\}$. Thus (7.27) is proved and the lemma is proved.

□

The following lemma shows that the initial values are asymptotically negligible.

Lemma 7.11 *Under the assumptions of Theorem 3.2,*

$$\left\| n^{-1/2} \sum_{t=1}^{n} \left\{ \frac{\partial \ell_t(\theta_0)}{\partial \theta} - \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \theta} \right\} \right\| \to 0 \tag{7.30}$$

and

$$\sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\| \frac{1}{n} \sum_{t=1}^{n} \left\{ \frac{\partial^2 \ell_t(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \tilde{\ell}_t(\theta)}{\partial \theta \partial \theta'} \right\} \right\| \to_P 0. \tag{7.31}$$

Proof of Lemma 7.11 Following the lines of proof of Theorem 7 *iv)* in RZ, we have almost surely, for t sufficiently large with $\sup_{\theta \in \Theta} \prod_{j=1}^{d} \beta^{\pi_j}(e_j) < \beta_* < 1$,

$$\sup_{\theta \in \Theta} \left\| \frac{\partial \sigma_t^2}{\partial \theta} - \frac{\partial \tilde{\sigma}_t^2}{\partial \theta} \right\| < K \beta_*^t. \tag{7.32}$$

In view of (7.6), $\tilde{\sigma}_t^2(\theta) \geq c_t(\theta)$ and $\sigma_t^2(\theta) \geq c_t(\theta)$, we have

$$\left| \frac{1}{\sigma_t^2(\theta)} - \frac{1}{\tilde{\sigma}_t^2(\theta)} \right| = \left| \frac{\sigma_t^2(\theta) - \tilde{\sigma}_t^2(\theta)}{\sigma_t^2(\theta) \tilde{\sigma}_t^2(\theta)} \right| \leq \frac{K \beta_*^t}{\sigma_t^2(\theta) c_t(\theta)}, \quad \frac{\sigma_t^2(\theta)}{\tilde{\sigma}_t^2(\theta)} \leq 1 + K \frac{\beta_*^t}{c_t(\theta)}. \tag{7.33}$$

Since

$$\frac{\partial \tilde{\ell}_t}{\partial \theta}(\theta) = \left\{ 1 - \frac{\epsilon_t^2}{\tilde{\sigma}_t^2(\theta)} \right\} \left\{ \frac{1}{\tilde{\sigma}_t^2(\theta)} \frac{\partial \tilde{\sigma}_t^2(\theta)}{\partial \theta} \right\} \quad \text{and} \quad \frac{\partial \ell_t}{\partial \theta}(\theta) = \left\{ 1 - \frac{\epsilon_t^2}{\sigma_t^2(\theta)} \right\} \left\{ \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \theta} \right\},$$

we have, using (7.33), the first inequality in (7.32) and the computations detailed in the lines of the proof of Theorem 7 *iv)* in RZ,

$$\left| \frac{\partial \tilde{\ell}_t}{\partial \theta_i}(\theta_0) - \frac{\partial \ell_t}{\partial \theta_i}(\theta_0) \right| \leq K \frac{\beta_*^t}{c_t(\theta_0)} (1 + \eta_t^2) \left| 1 + \left\{ \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \theta_i} \right\} \right|.$$

Using the elementary inequality $(x+y)^u \leq x^u + y^u$, for $x, y \geq 0$ and $0 < u < 1$, it follows that for any $0 < u < 1$:

$$\left| n^{-1/2} \sum_{t=1}^{n} \left\{ \frac{\partial \ell_t(\theta_0)}{\partial \theta_i} - \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \theta_i} \right\} \right|^u \leq K n^{-\frac{u}{2}} \sum_{t=1}^{n} \frac{\beta_*^{ut} (1 + \eta_t^2)^u}{c_t^u(\theta_0)} \left| 1 + \left\{ \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \theta_i} \right\} \right|^u.$$

Let us choose $u < \min(r/2, 1/2)$, Lemma 7.7 entails that $\left|\frac{1}{\sigma^2_{S,t}(\theta_0)}\frac{\partial\sigma^2_{S,t}(\theta_0)}{\partial\theta_i}\right|$ possesses a moment of order 1, thus a moment of order 2u. We get thus using (7.3) that

$$E\frac{1}{c^u_{S,t}(\theta_0)} < \infty, \quad E\frac{1}{c^{2u}_{S,t}(\theta_0)} < \infty, \quad E\left|\frac{1}{\sigma^2_{S,t}(\theta_0)}\frac{\partial\sigma^2_{S,t}(\theta_0)}{\partial\theta_i}\right|^{2u} < \infty, \qquad (7.34)$$

for $u > 0$ sufficiently small. Using the independence between η_t and $(\sigma^2_{S,t}(\theta_0), c_{S,t}(\theta_0))$, the stationarity of $c_{S,t}(\theta_0)$ and $\frac{1}{\sigma^2_{S,t}(\theta_0)}\frac{\partial\sigma^2_{S,t}(\theta_0)}{\partial\theta_i}$:

$$\begin{aligned} & E\left(\sum_{t=1}^{n}\frac{\beta_*^{ut}}{c^u_{S,t}(\theta_0)}(1+\eta_t^2)^u\left|1+\left\{\frac{1}{\sigma^2_{S,t}(\theta_0)}\frac{\partial\sigma^2_{S,t}(\theta_0)}{\partial\theta_i}\right\}\right|^u\right) \\ \leq\ & E((1+\eta_t^2)^u)\left(E\frac{1}{c^u_{S,t}(\theta_0)} + E\frac{1}{c^u_{S,t}(\theta_0)}\left|\frac{1}{\sigma^2_{S,t}(\theta_0)}\frac{\partial\sigma^2_{S,t}(\theta_0)}{\partial\theta_i}\right|^u\right)\sum_{t=1}^{n}\beta_*^{ut/2} \\ \leq\ & K\sum_{t=1}^{\infty}\beta_*^{ut} < \infty. \end{aligned}$$

The second inequality follows from the Cauchy-Schwarz inequality, the existence of moment at order $u < 1$ for η_t^2 and (7.34). The result follows from $\beta_* < 1$. This result shows that $\sum_{t=1}^{n}\frac{\beta_*^{ut}(1+\eta_t^2)^u}{c^u_{S,t}(\theta_0)}\left|1+\left\{\frac{1}{\sigma^2_{S,t}(\theta_0)}\frac{\partial\sigma^2_{S,t}(\theta_0)}{\partial\theta_i}\right\}\right|^u$ converges a.s.. Thus, by a straightforward extension of Lemma 1 in Francq and Gautier (2004a) the right hand-side term of (7.34) converges to 0, a.s. showing (7.30).

The proof of (7.31) combines similar arguments and arguments developed in the proof of part *v)* of Theorem 7 *iv)* in RZ. We omit it for brevity.

□

The following Lemma establishes the asymptotic normality of the normalized score.

Lemma 7.12 *Under the assumptions of Theorem,* $J_n = \frac{\partial^2\mathbf{l}_n(\theta_0)}{\partial\theta\partial\theta'}$. *is an a.s. positive definite matrix for sufficiently large n , and*

$$Z_n := -J_n^{-1}\sqrt{n}\frac{\partial\mathbf{l}_n(\theta_0)}{\partial\theta} \to^{\mathcal{L}} Z, \quad \textit{with} \quad Z \sim \mathcal{N}(0, (\kappa_\eta - 1)J^{-1}).$$

Proof of Lemma 7.12 This Lemma corresponds to the part *v)* of the proof of Theorem 2.1 in RZ. Its proof is exactly the same provided $E\left|\lambda'\frac{\partial\ell_{S,t}}{\partial\theta}(\theta_0)\right|^3 < \infty$. In view of Lemma 7.7, (7.24) and **A7'**, it can be easily shown that :

$$E\left|\lambda'\frac{\partial\ell_{S,t}}{\partial\theta}(\theta_0)\right|^3 = E\{1+\eta_t^2\}^3 E\left|\lambda'\frac{1}{\sigma^2_{S,t}}\frac{\partial\sigma^2_{S,t}}{\partial\theta}(\theta_0)\right|^3 < \infty, \quad \forall\lambda \in \mathbb{R}^{3d}.$$

□

7.2.2 Asymptotic distribution of $\hat{\theta}_n$

(7.31), (7.31) and (7.30) together with $Z_n \to^{\mathcal{L}} Z$ (by Lemma 7.12) and $J_n \to J$ almost surely allow to obtain the result of Theorem 3.2, using lines of proofs of Theorem 2 in Francq and Zakoïan (2007) (with exactly same notations).

□

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Testing the nullity of coefficients of a GARCH Model with exogenously-driven volatility : complementary results

A Complements to the proof of Theorem 3.1

Proof of iv). In view of (7.1), as $\log^-(.)$ is a decreasing function, for all $\theta \in \Theta$, $E\ell_{S,t}^-(\theta) \leq E\log^- \sigma_{S,t}^2(\theta) \leq \sup_{\theta\in\Theta} E\log^- \Omega_{S,t}^*(\theta) < \infty$ where the result follows from Lemma 7.1. Thus, $E\ell_{S,t}(\theta)$ is well defined and belongs to $\mathbb{R}\cup\{\infty\}$. Remark that $E\ell_{S,t}(\theta) = \infty$, when, for instance, $\theta = (\omega, 0, \ldots, 0, \omega, 0, \ldots)$ and $E\epsilon_{S,t}^2 = \infty$. However, we will show that $E|\ell_{S,t}(\theta_0)| < \infty$. It remains to show that $E\ell_{S,t}^+(\theta_0) < \infty$. Using the Jensen inequality,

$$E\log\sigma_{S,t}^2(\theta_0) = E\frac{1}{r}\log\{\sigma_{S,t}^2(\theta_0)\}^r \leq \frac{1}{r}\log E\{\sigma_{S,t}^2(\theta_0)\}^r < \infty$$

where the result follows from Lemma 7.5. Therefore

$$E\ell_{S,t}(\theta_0) = E_S\left\{\frac{\sigma_{S,t}^2(\theta_0)\eta_t^2}{\sigma_{S,t}^2(\theta_0)} + \log\sigma_{S,t}^2(\theta_0)\right\} = 1 + E\log\sigma_{S,t}^2(\theta_0) < \infty.$$

Since $\log x \leq x-1, \quad \forall x > 0$ and $\log x = x - 1$ if and only if $x = 1$, we have

$$\begin{aligned} E\ell_{S,t}(\theta) - E\ell_{S,t}(\theta_0) &= E\log\frac{\sigma_{S,t}^2(\theta)}{\sigma_{S,t}^2(\theta_0)} + E\frac{\sigma_{S,t}^2(\theta)}{\sigma_{S,t}^2(\theta_0)} - 1 \\ &\geq E\left\{\log\frac{\sigma_{S,t}^2(\theta)}{\sigma_{S,t}^2(\theta_0)} + \log\frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)}\right\} \geq 0 \end{aligned}$$

with equality if and only if $\frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)} = 1 \quad a.s$, and thus, from *iii)* in RZ, if and only if $\theta = \theta_0$.

Proof of v). For any $\theta \in \Theta$ and any positive integer k, let $V_k(\theta)$ be the open ball with center θ and radius $1/k$. Following exactly the lines of the proof in Francq and Zakoïan (2004), we have

$$\begin{aligned} \liminf_{n\to\infty} \inf_{\theta^*\in V_k(\theta)\cap\Theta} \tilde{\mathbf{l}}_n(\theta^*) &\geq \liminf_{n\to\infty} \inf_{\theta^*\in V_k(\theta)\cap\Theta} \mathbf{l}_n(\theta^*) - \limsup_{n\to\infty}\sup_{\theta\in\Theta} |\mathbf{l}_n(\theta) - \tilde{\mathbf{l}}_n(\theta)| \\ &\geq \liminf_{n\to\infty} n^{-1}\sum_{t=1}^n \inf_{\theta^*\in V_k(\theta)\cap\Theta} \ell_t(\theta^*). \end{aligned}$$

Now, we apply Lemma 1 in Francq and Gautier (2004a). Indeed, the expectation of $\left\{\inf_{\theta^*\in V_k(\theta)\cap\Theta}\ell_{S,t}(\theta^*)\right\}_t$ exists because it is larger than $\inf_{\theta\in\Theta}\log\Omega^*_{S,t}(\theta)$ which possesses a finite expectation by Lemma 7.1. Thus we obtain,

$$\liminf_{n\to\infty} n^{-1}\sum_{t=1}^{n}\inf_{\theta^*\in V_k(\theta)\cap\Theta}\ell_t(\theta^*) = E\inf_{\theta^*\in V_k(\theta)\cap\Theta}\ell_{S,1}(\theta^*).$$

By the Beppo-Levi theorem, when k increases to ∞, $E\inf_{\theta^*\in V_k(\theta)\cap\Theta}\ell_{S,1}(\theta^*)$ increases to $E\ell_{S,1}(\theta)$. Thus *v)* is proved.

B Complements to the proof of Theorem 3.2

B.1 Complements to the proof of Lemma 7.7

Let us now prove the result for second order derivatives of $\sigma^2_{S,t}$. Using (7.15)-(7.17), we get for all $m,k\in\{1,\ldots,d\}$:

$$\frac{\partial^2\sigma^2_{S,t}}{\partial\alpha(e_k)\partial\alpha(e_m)} = \frac{\partial^2\sigma^2_{S,t}}{\partial\omega(e_k)\partial\omega(e_m)} = \frac{\partial^2\sigma^2_{S,t}}{\partial\omega(e_k)\partial\alpha(e_m)} = 0.$$

Turning to $\frac{\partial^2\sigma^2_{S,t}}{\partial\omega(e_k)\partial\beta(e_m)}$, it is clear in view of 7.15

$$\begin{aligned}
&\beta(e_m)\frac{\partial^2\sigma^2_{S,t}}{\partial\omega(e_k)\partial\beta(e_m)}\\
=\;&\beta(e_m)\sum_{i=0}^{\infty}\left\{\sum_{j=0}^{i}\prod_{h\neq j,0\leq h\leq i}\beta(S_{t-h})\,\mathbb{1}_{S_{t-j}=e_m}\right\}\mathbb{1}_{S_{t-i-1}=e_k}\\
\leq\;&\sum_{i=0}^{\infty}(i+1)\beta(S_t)\ldots\beta(S_{t-i})\,\mathbb{1}_{S_{t-i-1}=e_k}\,.
\end{aligned}$$

The right hand-side term of this inequality is similar to (7.15). The arguments use to prove (7.19) allow to show $\left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{\sigma^2_{S,t}(\theta)}\frac{\partial^2\sigma^2_{S,t}}{\partial\omega(e_k)\partial\beta(e_m)}(\theta)\right\|_3\leq\frac{K}{\beta^{**}}<\infty$.

Turning to $\frac{\partial^2\sigma^2_{S,t}}{\partial\alpha(e_k)\partial\beta(e_m)}$:

$$\begin{aligned}
&\beta(e_m)\frac{\partial^2\sigma^2_{S,t}}{\partial\alpha(e_k)\partial\beta(e_m)}\\
=\;&\beta(e_m)\sum_{i=0}^{\infty}\left\{\sum_{j=0}^{i}\prod_{h\neq j,0\leq h\leq i}\beta(S_{t-h})\,\mathbb{1}_{S_{t-j}=e_m}\right\}\epsilon^2_{t-i-2}\,\mathbb{1}_{S_{t-i-1}=e_k}\\
\leq\;&\sum_{i=0}^{\infty}(i+1)\beta(S_t)\ldots\beta(S_{t-i})\epsilon^2_{t-i-2}\,\mathbb{1}_{S_{t-i-1}=e_k}\,.
\end{aligned}$$

The right hand-side term of this inequality is similar to the right hand-side term of (7.16). The arguments used to prove (7.22) allow to show:

$$\left\| \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\sigma^2_{S,t}(\theta)} \frac{\partial^2\sigma^2_{S,t}}{\partial\alpha(e_k)\partial\beta(e_m)}(\theta) \right\|_3 \leq \frac{K}{\beta^{**}} < \infty.$$

Turning to $\frac{\partial^2\sigma^2_{S,t}}{\partial\beta(e_l)\partial\beta(e_k)}$ we get that:

$$\begin{aligned}
&\beta(e_l)\beta(e_k)\frac{\partial^2\sigma^2_{S,t}}{\partial\beta(e_l)\partial\beta(e_k)} \\
=\ &\beta(e_l)\beta(e_k)\sum_{i=0}^{\infty}\left\{\frac{\partial^2\beta(S_t)\dots\beta(S_{t-i})}{\partial\beta(e_l)\partial\beta(e_k)}c_{S,t-i-1}\right\} \\
=\ &\beta(e_l)\beta(e_k)\sum_{i=0}^{\infty}\beta(S_t)\dots\beta(S_{t-i})\left\{\sum_{0\leq j,h\leq i,h\neq j}\frac{1}{\beta(e_k)\beta(e_l)}\mathbb{1}_{\{S_{t-h}=e_h,S_{t-j}=e_l\}}\right\}c_{S,t-i-1} \\
\leq\ &\sum_{i=0}^{\infty} i(i-1)\beta(S_t)\dots\beta(S_{t-i})c_{S,t-i-1}.
\end{aligned}$$

The right hand-side term of this inequality is similar to (7.15). Treating it in the same way as in the proof of (7.23) allows to give $\left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{\sigma^2_{S,t}(\theta)}\frac{\partial^2\sigma^2_{S,t}}{\partial\beta(e_l)\partial\beta(e_k)}(\theta)\right\|_3 \leq \frac{K}{\beta^{**}} < \infty$, so the existence of the second expectation in (7.13) is proved.

B.2 Complements to the proof of Lemma 7.6

Now we turn to the second term of (7.9). Choosing $\delta, \epsilon > 0$ satisfying (7.11) and (7.12), choosing $u = v/2$ with v defined according to (7.8) and using the elementary inequality $(x+y)^u \leq x^u + y^u$ for $0 < u < 1$ on (2.7):

$$\begin{aligned}
&E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\sigma^{2u}_{S,t}(\theta) \\
\leq\ &\sum_{i=0}^{\infty} E\left\{\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\{\beta^u(S_t)\dots\beta^u(S_{t-i})\}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}c^u_{S,t-i-1}(\theta)\right\} \\
\leq\ &\sum_{i=0}^{\infty}\left\{E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\{\beta^v(S_t)\dots\beta^v(S_{t-i})\}\right\}^{1/2}\left\{E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}c^v_{S,t-i-1}(\theta)\right\}^{1/2} \\
\leq\ &K\sum_{i=0}^{\infty}(\rho^*)^{\frac{i+1}{2}} < \infty.
\end{aligned}$$

The second inequality follows from the Cauchy-Schwarz inequality. Using Lemma 7.5, $E\epsilon^{2v}_{S,t}$ by Hölder inequality in view of $v \leq r$. The components of array of Θ being bounded by a constant $\overline{\theta}$ (ie $\theta_i \leq \overline{\theta}$ for all $\theta \in \Theta$), we easily thus get

$E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} c_{S,t}^v(\theta) \leq \overline{\theta}^v(1+E\epsilon_{S,t-1}^{2v}) < \infty$. This result together with the stationarity of $(c_{S,t}(\theta))$ and (7.8) lead to the third inequality. The result follows from $\rho^* < 1$ because δ is chosen according to (7.12). This u and v in (7.8) may be chosen equal without loss of generality. We thus have defined $\mathcal{V}(\theta_0)$ satisfying (7.8) together with (7.9) for all $m \in \mathbb{N}$.

B.3 Complements to the proof of Lemma 7.11

We prove here (7.31). Similarly to (7.32), following the lines of proof of Theorem 7 *iv)* in RZ, we have almost surely, for t sufficiently large with $\sup_{\theta\in\Theta}\prod_{j=1}^d \beta^{\pi_j}(e_j) < \beta_* < 1$,

$$\sup_{\theta\in\Theta}\left\|\frac{\partial^2\sigma_t^2}{\partial\theta\partial\theta} - \frac{\partial^2\tilde{\sigma}_t^2}{\partial\theta\partial\theta'}\right\| < K\beta_*^t, \quad \forall t. \tag{B.1}$$

For any $0 < u < 1$, using inequalities (B.1) and (7.33), and following some computations detailed in the lines of the proof of Theorem 7 *iv)* in RZ, it can be shown that:

$$\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left|\frac{1}{n}\sum_{t=1}^n\left\{\frac{\partial^2\ell_t(\theta)}{\partial\theta_i\partial\theta_j} - \frac{\partial^2\tilde{\ell}_t(\theta)}{\partial\theta_i\partial\theta_j}\right\}\right|^u \leq Kn^{-1}\sum_{t=1}^n \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\beta_*^{ut}}{c_t^u(\theta)}\mathcal{Y}_t^u, \tag{B.2}$$

where

$$\mathcal{Y}_t = \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left\{1+\frac{\epsilon_t^2}{\sigma_t^2}\right\}\left\{1+\frac{1}{\sigma_t^2}\frac{\partial^2\sigma_t^2}{\partial\theta_i\partial\theta_j} + \frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta_i}\frac{\partial\sigma_t^2}{\partial\theta_j}\right\}.$$

Let us define:

$$\mathcal{Y}_{S,t} = \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left\{1+\frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2}\right\}\left\{1+\frac{1}{\sigma_{S,t}^2}\frac{\partial^2\sigma_{S,t}^2}{\partial\theta_i\partial\theta_j} + \frac{1}{\sigma_{S,t}^4}\frac{\partial\sigma_{S,t}^2}{\partial\theta_i}\frac{\partial\sigma_{S,t}^2}{\partial\theta_j}\right\}.$$

Note that $\mathcal{Y}_{S,t}$ involves only non-negative terms, thus is non-negative. The Cauchy-Schwarz and Minkowski inequalities entail that for any $u > 0$,

$$\begin{aligned}\|\mathcal{Y}_{S,t}\|_u \leq & \left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left\{1+\frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2}\right\}\right\|_{2u} \\ & \left\{1+\left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{\sigma_{S,t}^2}\frac{\partial^2\sigma_{S,t}^2}{\partial\theta_i\partial\theta_j}\right\|_{2u} + \left\|\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{\sigma_{S,t}^4}\frac{\partial\sigma_{S,t}^2}{\partial\theta_i}\frac{\partial\sigma_{S,t}^2}{\partial\theta_j}\right\|_{2u}\right\}.\end{aligned}$$

In view of (7.29), the terms involving first and second-order derivatives of $\sigma_{S,t}^2$ and $\frac{\epsilon_{S,t}^2}{\sigma_{S,t}^2}$ possess moments of order u for $u > 0$ sufficiently small. Let us choose $u > 0$

such that $E\mathcal{Y}_{S,t}^{2u} < \infty$ and smaller than $r/2$. Now, in view of the stationarity of $(c_{S,t})$ and $(\mathcal{Y}_{S,t})$:

$$E\sum_{t=1}^{n} \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{\beta_*^{ut}}{c_{S,t}^u(\theta)}\mathcal{Y}_{S,t}^u$$

$$\leq (\sum_{t=1}^{\infty}\beta_*^{ut})E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{c_{S,t}^u(\theta)}\mathcal{Y}_{S,t}^u \leq K\left\{E\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{c_{S,t}^{2u}(\theta)}\right\}^{\frac{1}{2}}\{E\mathcal{Y}_{S,t}^{2u}\}^{\frac{1}{2}} < \infty.$$

The second inequality follows from the Cauchy-Schwarz inequality and $\beta_*^u < 1$. The result relies on $E\mathcal{Y}_{S,t}^{2u} < \infty$ and (7.3). Thus $\sum_{t=1}^{n}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\beta_*^{ut}}{c_{S,t}^u(\theta)}\mathcal{Y}_{S,t}^u$ converges a.s. and thus $\sum_{t=1}^{n}\frac{\beta_*^{ut}}{c_t^u(\theta)}\mathcal{Y}_t^u$ converges a.s. by a straightforward extension of Lemma 1 in Francq and Gautier (2004a). The right hand-side term of (B.2) thus converges to 0 a.s., which shows (7.31).

B.4 Proof of Lemma 7.12

We will use the following Central Limit Theorem (CLT) for martingale differences (see Billingsley, 1995, p.476).

Lemma B.1 *Let $\{(X_t),\mathcal{F}_t\}$ a $\mathbb{R}^d$ ($d\in\mathbb{N}^*$)-martingale difference such that*

a) There exists a $d\times d$ matrix A such that, when $n\to\infty$,

$$\forall\epsilon>0, \lambda\in\mathbb{R}^d, \quad \frac{1}{n}\sum_{t=1}^{n} Var\{\lambda' X_t|\mathcal{F}_{t-1}\}\to\lambda' A\lambda, \quad a.s.$$

b)

$$\forall\epsilon>0, \quad \sum_{t=1}^{n} E\left\{\left(\frac{\lambda' X_t}{\sqrt{n}}\right)^2 \mathbb{1}_{\{|\frac{\lambda' X_t}{\sqrt{n}}|>\epsilon\}}\right\}\to 0.$$

Then

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t \xrightarrow{\mathcal{L}} \mathcal{N}(0,A).$$

Here $X_t = \frac{\partial}{\partial\theta}\ell_t(\theta_0)$ and $\mathcal{F}_t=\sigma(X_{t-j}, j\geq 0)$. Note that by Lemma A.2 in RZ,

$$\frac{1}{n}\sum_{t=1}^{n} Var\{\lambda' X_t|\mathcal{F}_{t-1}\}\to_{n\to\infty}\lambda' Var\left\{\frac{\partial\ell_{S,t}}{\partial\theta}\right\}\lambda = (\kappa_\eta - 1)\lambda' J\lambda.$$

(X_t) being a martingale difference, this result ensures condition a) is satisfied with $A := (\kappa_\eta - 1)J$. In view of Lemma 7.7, (7.24) and **A7'**, it can be easily shown that :

$$E\left|\lambda'\frac{\partial\ell_{S,t}}{\partial\theta}(\theta_0)\right|^3 = E\{1+\eta_t^2\}^3 E\left|\lambda'\frac{1}{\sigma_{S,t}^2}\frac{\partial\sigma_{S,t}^2}{\partial\theta}(\theta_0)\right|^3 < \infty, \quad \forall\lambda\in\mathbb{R}^{3d}.$$

Using Lemma 1 in Francq and Gautier (2004a), we get:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}E\left|\lambda'\frac{\partial}{\partial\theta}\ell_t(\theta_0)\right|^3 = E\left|\lambda'\frac{\partial}{\partial\theta}\ell_{S,t}(\theta_0)\right|^3 < \infty \quad \forall\lambda\in\mathbb{R}^{3d}, \quad a.s..$$

Noting that $\mathbb{1}_{\{|\frac{\lambda'\frac{\partial}{\partial\theta}\ell_t(\theta_0)}{\sqrt{n}}|>\epsilon\}} \leq \frac{1}{\epsilon}\left|\frac{\lambda'\frac{\partial}{\partial\theta}\ell_t(\theta_0)}{\sqrt{n}}\right|$, this result entails:

$$\begin{aligned}\forall\epsilon>0, \qquad & \sum_{t=1}^{n}E\left[\left\{\frac{\lambda'\frac{\partial}{\partial\theta}\ell_t(\theta_0)}{\sqrt{n}}\right\}^2 \mathbb{1}_{\{|\frac{\lambda'\frac{\partial}{\partial\theta}\ell_t(\theta_0)}{\sqrt{n}}|>\epsilon\}}\right] \\ \leq & \frac{1}{\epsilon n^{3/2}}\sum_{t=1}^{n}E\left|\lambda'\frac{\partial}{\partial\theta}\ell_t(\theta_0)\right|^3 \to 0.\end{aligned}$$

We get now all the conditions needed to use Lemma B.1 and obtain

$$\sqrt{n}\frac{\partial \mathbf{l}_n(\theta_0)}{\partial\theta} \xrightarrow{\mathcal{L}} \mathcal{N}(0,(\kappa_\eta-1)J).$$

In view of Lemma 1 in Francq and Gautier (2004a), $J_n\to J$ a.s. and in view of Lemma 7.9, J is non-singular. The conclusion follows from the Slutsky lemma.

B.5 Complements to asymptotic distribution of $\hat{\theta}_n$

Following Andrews (1997, 1999, 2001) and Francq et Zakoïan (2007), we can use the following asymptotic development :

$$\begin{aligned} & \tilde{\mathbf{l}}_n(\theta)-\tilde{\mathbf{l}}_n(\theta_0) \\ = & \frac{\partial\tilde{\mathbf{l}}_n(\theta_0)}{\partial\theta}(\theta-\theta_0)+\frac{1}{2}(\theta-\theta_0)'\left(\frac{\partial^2\tilde{\mathbf{l}}_n(\theta_{ij}^*)}{\partial\theta\partial\theta'}\right)(\theta-\theta_0) \\ = & \frac{\partial\tilde{\mathbf{l}}_n(\theta_0)}{\partial\theta}(\theta-\theta_0)+\frac{1}{2}(\theta-\theta_0)'\left(\frac{\partial^2\tilde{\mathbf{l}}_n(\theta_0)}{\partial\theta\partial\theta'}\right)(\theta-\theta_0)+R_n(\theta) \\ = & -\frac{1}{2n}Z_n'J_n\sqrt{n}(\theta-\theta_0)-\frac{1}{2n}\sqrt{n}(\theta-\theta_0)'J_nZ_n+\frac{1}{2}(\theta-\theta_0)'J_n(\theta-\theta_0) \\ & +R_n(\theta)+R_n^*(\theta) \\ = & \frac{1}{2n}\left\|Z_n-\sqrt{n}(\theta-\theta_0)\right\|_{J_n}^2-\frac{1}{2n}Z_n'J_nZ_n+R_n(\theta)+R_n^*(\theta)\end{aligned}$$

and :

$$\begin{aligned} R_n(\theta) = & \frac{1}{2}(\theta-\theta_0)'\left[\left(\frac{\partial^2\tilde{\mathbf{l}}_n(\theta_{ij}^*)}{\partial\theta\partial\theta'}\right)-\left(\frac{\partial^2\tilde{\mathbf{l}}_n(\theta_0)}{\partial\theta\partial\theta'}\right)\right](\theta-\theta_0), \\ R_n^*(\theta) = & \left\{\frac{\partial\tilde{\mathbf{l}}_n(\theta_0)}{\partial\theta}-\frac{\partial\mathbf{l}_n(\theta_0)}{\partial\theta}\right\}(\theta-\theta_0)+\frac{1}{2}(\theta-\theta_0)'\left\{\frac{\partial^2\tilde{\mathbf{l}}_n(\theta_0)}{\partial\theta\partial\theta'}-J_n\right\}(\theta-\theta_0),\end{aligned}$$

where θ^*_{ij} is between θ and θ_0 and all derivatives are right hand-side derivatives. The strong consistency of $\hat{\theta}_n$ and Lemma 7.10 entail that

$$\frac{\partial^2 \mathbf{l}_n(\theta^*_{ij})}{\partial\theta_i\partial\theta_j} \to J(i,j)$$

for all θ^*_{ij} between $\hat{\theta}_n$ and θ_0. This result together with (7.31) entails that for any sequence θ_n such that $\theta_n - \theta_0 = o_P(1)$:

$$R_n(\theta_n) = o_P(\|\theta_n - \theta_0\|^2_{J_n}). \tag{B.3}$$

(7.31) and (7.30) entail that for any sequence θ_n such that $\theta_n - \theta_0 = o_P(1)$:

$$R^*_n(\theta) \;=\; o_P(n^{-1/2}\,\|\theta_n - \theta_0\|_{J_n}) + o_P(\|\theta_n - \theta_0\|^2_{J_n}). \tag{B.4}$$

The properties (B.3) and (B.4) together with $Z_n \to^{\mathcal{L}} Z$ (by Lemma 7.12) and $J_n \to J$ almost surely allow to obtain the result of Theorem 3.2, using lines of proofs of Theorem 2 in Francq and Zakoïan (2007) (with exactly same notations).

C Proof of Theorem 3.3

Similarly to (2.7), let us define

$$\sigma^2_{t,n}(\theta) = c_{t,n}(\theta) + \sum_{i=0}^{\infty} \beta(s_t)\dots\beta(s_{t-i})c_{t-i-1,n}(\theta),$$

where $c_{t,n}(\theta) = \omega(s_t) + \alpha(s_t)\epsilon^2_{t-1,n}$. Let $\ell_{t,n}(\theta) = \frac{\epsilon^2_{t,n}}{\sigma^2_{t,n}(\theta)} + \log\sigma^2_{t,n}(\theta)$, so that the theoretical and empirical objective functions can still be denoted $\mathbf{l}_n(\theta) = n^{-1}\sum_{t=1}^n \ell_{t,n}(\theta)$ and $\tilde{\mathbf{l}}_n(\theta) = n^{-1}\sum_{t=1}^n \tilde{\ell}_{t,n}(\theta)$. We need to define for any $\theta \in \Theta$,

$$\sigma^2_{S,t,n} = \sigma^2_{S,t,n}(\theta) = c_{S,t,n}(\theta) + \sum_{i=0}^{\infty} \beta(S_t)\dots\beta(S_{t-i})c_{S,t-i-1,n}(\theta), \tag{C.1}$$

where $c_{S,t,n}(\theta) = \omega(S_t) + \alpha(S_t)\epsilon^2_{S,t-1,n}$. Note that under **A2**, the a.s. convergence of the sum is ensured and consequently $\sigma^2_{S,t,n}$ is well defined. It will be convenient to define some processes which are lower-bounds of processes $(\sigma^2_{S,t,n}(\theta))$. Defining $\Omega_{S,t}(\theta) = \omega(S_t) + \sum_{i=0}^{\infty}\prod_{j=0}^{i}\beta(S_{t-j})\omega(S_{t-i-1})$, we are in position to define for all $\theta \in \Theta$,

$$R_{i,S,t}(\theta) := \sigma^2_{S,t}(\theta) - \prod_{0\le h\le i}\beta(S_{t-h})\Omega_{S,t-i-1}(\theta),$$

$$ {}_i\mathrm{R}_{\mathrm{S,t}}(\theta) := \sigma^2_{S,t}(\theta) - \prod_{0\le h\le i} \beta(S_{t-h}) c_{S,t-i-1}(\theta), $$

and

$$ {}^i\mathrm{R}_{S,t}(\theta) := \sigma^2_{S,t}(\theta) - \prod_{0\le h\le i+1} \beta(S_{t-h}) \sigma^2_{S,t-i-2}(\theta). $$

We define $R_{i,t}$ (respectively ${}_i\mathrm{R}_{\mathrm{t}}$ and ${}^i\mathrm{R}_t$) by replacing in the definition of $R_{i,S,t}$ (respectively ${}_i\mathrm{R}_{\mathrm{S,t}}$ and ${}^i\mathrm{R}_{S,t}$) S_t by s_t. We can now define some quantities which are lower bound of $(\sigma^2_{S,t,n}(\theta))$: for $\theta, \theta^* \in \Theta$,

$$ \sigma^2_{i,S,t}(\theta,\theta^*) := R_{i,S,t}(\theta^*) + \prod_{0\le h\le i} \beta(S_{t-h}) \Omega_{S,t-i-1}(\theta^*) $$
$$ {}_i\sigma^2_{\mathrm{S,t,n}}(\theta,\theta^*) :=_i \mathrm{R}_{S,t}(\theta^*) + \prod_{0\le h\le i+1} \beta(S_{t-h}) c_{S,t-i-1,n}(\theta) $$
$$ {}^i\sigma^2_{\mathrm{S,t,n}}(\theta,\theta^*) :=^i \mathrm{R}_{S,t}(\theta^*) + \prod_{0\le h\le i+1} \beta(S_{t-h}) \sigma^2_{S,t-i-2,n}(\theta) $$

and similarly define $\sigma^2_{i,t}$ (respectively ${}_i\sigma^2_{\mathrm{t,n}}$ and ${}^i\sigma^2_{\mathrm{t,n}}$) by replacing in the definition of $\sigma^2_{i,t}$ (respectively ${}_i\sigma^2_{\mathrm{S,t,n}}$ and ${}^i\sigma^2_{\mathrm{S,t,n}}$) S_t by s_t. We now summarize in the following Lemma some useful inequalities throughout the proof.

Lemma C.1 *for all* $\theta \ge \theta^*$

$$ \max(\sigma^2_{i,S,t}(\theta,\theta^*),_i \sigma^2_{S,t,n}(\theta,\theta^*),^i \sigma^2_{S,t,n}(\theta,\theta^*)) \le \sigma^2_{S,t,n}(\theta), \tag{C.2} $$

$$ R_{i,S,t}(\theta) \ge c_{S,t}(\theta), \quad {}_i\mathrm{R}_{S,t}(\theta) \ge c_{S,t}(\theta), \quad {}^i\mathrm{R}_{S,t}(\theta) \ge d_{S,t}(\theta). \tag{C.3} $$

We need to define also for all $\theta \in \Theta$, $\ell_{S,t,n} = \ell_{S,t,n}(\theta) = \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} + \ln \sigma^2_{S,t,n}(\theta)$, with $\sigma^2_{S,t,n}$ defined according to (C.1). The following inequalities, which are straightforward consequences of $\tau > 0$, will be used throughout. For any $n \ge n_0$, and $\theta \ge \theta^* \ge \tilde\theta$ we have :

$$ \epsilon^2_{t,n_0} \ge \epsilon^2_{t,n} \ge \epsilon^2_t \quad \text{and} \quad \sigma^2_{t,n_0}(\theta) \ge \sigma^2_{t,n}(\theta^*) \ge \sigma^2_t(\tilde\theta). \tag{C.4} $$

For any neighborhood $\mathcal{V}(\theta_0)$ satisfying Lemma 7.6, taking n_0 sufficiently large for $\theta_{n_0} \in \mathcal{V}(\theta_0)$, we get that (7.8) leads to:

$$ \forall i > 0, \quad E\left\{a^v_{n_0}(S_t,\eta_{t-1}) \dots a^v_{n_0}(S_{t-i},\eta_{t-i-1})\right\} < C^*(\rho^*)^{i+1}. \tag{C.5} $$

with $C^* > 0$ and $\rho^* < 1$ and v defined according to Lemma 7.6. Assumption **A3** is thus true at θ_n and this condition allows to ensure ϵ_{S,t,n_0} possessing a moment at order $2v$. We refer to the lines of the proof of Lemma 7.5 which is presented in RZ and we can state that :

$$E\epsilon_{S,t,n_0}^{2v} < \infty, \quad E\sigma_{S,t,n_0}^{2v}(\theta_{n_0}) < \infty, \quad \sup_{\theta\in\Theta} Ec_{S,t,n_0}^{v} < \infty, \tag{C.6}$$

with v defined as in Lemma 7.6. Now we state a Lemma similar to Lemma 7.6.

Lemma C.2 *Under the assumptions of Theorem 3.3, there exist $\nu > 0$, a neighborhood $\mathcal{V}(\theta_0)$ of θ_0, $n_0 \in \mathbb{N}$ such that $\theta_{n_0} \in \mathcal{V}(\theta_0)$ and $0 < w \le v$ such that :*

$$\left\| \sup_{n\ge n_0} \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{\sigma_{S,t,n}^2(\theta_n)}{\sigma_{S,t,n}^2(\theta)} \right\|_{4+\nu} < \infty, \quad E \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \sigma_{S,t,n_0}^{2w}(\theta) < \infty. \tag{C.7}$$

Proof of Lemma C.2

The lines of Proof of Lemma 7.6 cannot be used here because $\frac{c_{S,t,n}(\theta_n)}{c_{S,t,n}(\theta)}$ is not bounded by a constant on $\mathcal{V}(\theta_0)\cap\Theta$ (with notations of Lemma 7.6). Let us define for $n \ge n_0$ and $\bar{\alpha}_n = \max_{i\in\{1,\dots,d\}} \alpha_n(e_i)$, $\bar{\omega}_n = \max_{i\in\{1,\dots,d\}} \omega_n(e_i)$, $\underline{\beta}_n = \max_{i\in\{1,\dots,d\}} \beta_n(e_i)$. We will exploit the following expansions of (2.6) at $\theta = \theta_0$ and (C.1) at $\theta = \theta_n$:

$$\sigma_{S,t}^2(\theta_0) = \omega_0(S_t) + a_0(S_t,\eta_{t-1})\sigma_{S,t-1}^2(\theta_0) \tag{C.8}$$

and

$$\begin{aligned}\sigma_{S,t,n}^2 &= \omega_n(S_t) + a_n(S_t,\eta_{t-1})\sigma_{S,t-1}^2 \\ &= \omega_n(S_t) + \sum_{i=1}^{+\infty} a_n(S_t,\eta_{t-1})\dots a(S_{t-i+1},\eta_{t-i})\omega_n(S_{t-i}).\end{aligned} \tag{C.9}$$

where the last equality follows from an obvious recursion. In view of (2.6) for all $i \ge 0$

$$\begin{aligned}\sigma_{S,t}^2(\theta_0) &= c_{S,t}(\theta_0) + \beta_0(S_t)\sigma_{S,t-1}^2(\theta_0) \\ &\ge c_{S,t}(\theta_0) + \beta^* \prod_{h=1}^{i} a_0(S_{t-h},\eta_{t-h-1})\sigma_{S,t-i-1}^2(\theta_0) \\ &\ge c_{S,t}(\theta_0) + \beta^* \prod_{h=1}^{i} a_0(S_{t-h},\eta_{t-h-1})\Omega_{S,t-i-1}^*(\theta_0).\end{aligned} \tag{C.10}$$

For all $i \ge 0$, the inequality $\sigma_{S,t-1}^2(\theta_0) \ge \prod_{h=1}^{i} a_0(S_{t-h},\eta_t)\sigma_{S,t-i-1}^2(\theta_0)$ is a straightforward consequence of (C.8). This argument together with $\beta_0(S_t) \ge \beta^*$ implies the first inequality. The second one comes from $\sigma^2_{S,t-i-1}(\theta_0) \geq \Omega^*_{S,t-i-1}(\theta_0)$. More over note that

$$\frac{a_n(S_t, \eta_{t-1})}{a_0(S_t, \eta_{t-1})} \leq \{1 + \frac{1}{\sqrt{n}\beta^*}(1 + \eta^2_{t-1})\}. \tag{C.11}$$

In view of (C.9) for any n and any $0 < u < 1$:

$$\begin{aligned}
\frac{\sigma^2_{S,t,n}(\theta_n)}{\sigma^2_{S,t}(\theta_0)} &\leq \frac{\omega_n(S_t) + \sum_{i=0}^{\infty}\prod_{h=0}^{i} a_n(S_{t-h}, \eta_{t-h-1})\omega_n(S_{t-i-1})}{\sigma^2_{S,t}(\theta_0)} \\
&\leq \frac{\omega_n(S_t) + a_n(S_t, \eta_{t-1})\omega_n(S_{t-1})}{\sigma^2_{S,t}(\theta_0)} \\
&+ \sum_{i=1}^{\infty} \frac{\prod_{h=0}^{i} a_n(S_{t-h}, \eta_{t-h-1})\omega_n(S_{t-i-1})}{c_{S,t}(\theta_0) + \beta^* \prod_{h=1}^{i} a_0(S_{t-h}, \eta_{t-h-1})\Omega^*_{S,t-i-1}(\theta_0)} \\
&\leq \frac{\omega_n(S_t) + a_n(S_t, \eta_{t-1})\omega_n(S_{t-1})}{\sigma^2_{S,t}(\theta_0)} \\
&+ a_n(S_t, \eta_{t-1}) \sum_{i=1}^{\infty}\prod_{h=1}^{i} \frac{a_n(S_{t-h}, \eta_{t-h-1})}{a_0(S_{t-h}, \eta_{t-h-1})} \frac{\omega_n(S_{t-i-1})}{\Omega^*_{S,t-i-1}(\theta_0)} \\
&\quad \frac{\prod_{h=1}^{i} a_0(S_{t-h})\Omega^*_{S,t-i-1}(\theta_0)}{c_{S,t}(\theta_0) + \beta^* \prod_{h=1}^{i} a_0(S_{t-h})\Omega^*_{S,t-i-1}(\theta_0)} \\
&\leq \frac{\bar{\omega}_n + (\bar{\alpha}_n \eta^2_{t-1} + \bar{\beta}_n)\bar{\omega}_n}{\Omega^*_{S,t}(\theta_0)} \\
&+ K(\bar{\alpha}_n \eta^2_{t-1} + \bar{\beta}_n) \sum_{i=1}^{\infty}\prod_{h=1}^{i} \{1 + \frac{1}{\sqrt{n}\beta^*}(1 + \eta^2_{t-h-1})\} \frac{\prod_{h=1}^{i} a_0^u(S_{t-h})\bar{\omega}_n}{\Omega^{*(1-u)}_{S,t-i-1}(\theta_0) c^u_{S,t}(\theta_0)}
\end{aligned} \tag{C.12}$$

The second inequality follows from (C.10). The fourth one follows from (C.11), inequality $\sigma^2_{S,t}(\theta_0) \geq \Omega^*_{S,t}(\theta_0)$, the elementary inequality $x/(1 + x) \leq x^u$ for any $x \geq 0$, $0 < u < 1$, the definition of $\bar{\alpha}_n$,$\bar{\beta}_n$, $\bar{\omega}_n$ and $\underline{\beta}_n$.

Because μ defined in Assumption **A7"** and ξ defined in (7.4) are strictly nonnegative, we can define $\zeta_1, \zeta_2 > 0$ such that $\zeta_1 < \zeta_2 < \min(\mu/2, \xi)$. It is clear in view of **A7"** and (7.4) that

$$\left\|\eta^2_t\right\|_{4+\zeta_1} \leq \left\|\eta^2_t\right\|_{4+\zeta_2} \leq \left\|\eta^2_t\right\|_{4+\mu/2} < \infty, \tag{C.13}$$

$$\left\|\Omega^{*-1}_{S,t}(\theta_0)\right\|_{4+\zeta_1} \leq \left\|\Omega^{*-1}_{S,t}(\theta_0)\right\|_{4+\zeta_2} \leq \left\|\Omega^{*-1}_{S,t}(\theta_0)\right\|_{4+\xi} < \infty \tag{C.14}$$

by the Hölder inequality (because $\zeta_2 \leq \mu/2$ for the first result and $\zeta_2 \leq \xi$ for the second one). Therefore $\{1 + \frac{1}{\sqrt{n}\beta^*}(1 + \|\eta^2_t\|_{4+\zeta_2})\} \to 1$ when $n \to \infty$.

Defining $M > 0$ such that $\frac{1}{M} + \frac{1}{4+\zeta_2} = \frac{1}{4+\zeta_1}$, because $\rho < 1$, it is possible in view of this last result to choose $n \geq 0$ such that $\{1+\frac{1}{\sqrt{n}\beta^*}(1+\|\eta^2_t\|_{4+\zeta_2})\}\rho^{1/2M} < 1$. Using the Minkowski inequality on the right hand-side term of (C.12), with $u = \frac{r}{2M} < 1$, we get

$$\begin{aligned}
&\left\| \frac{\sigma^2_{S,t,n}(\theta_n)}{\sigma^2_{S,t}(\theta_0)} \right\|_{4+\zeta_1} \\
\leq\ & \{\bar{\omega}_n + (\bar{\alpha}_n \left\|\eta^2_{t-1}\right\|_{4+\zeta_1} + \bar{\beta}_n)\bar{\omega}_n\} \left\| \frac{1}{\Omega^*_{S,t}(\theta_0)} \right\|_{4+\zeta_1} \\
+\ & K \sum_{i=1}^{\infty} \left\| (\bar{\alpha}_n \eta^2_{t-1} + \bar{\beta}_n) \prod_{h=1}^{i} \{1 + \frac{1}{\sqrt{n}\beta^*}(1+\eta^2_{t-h-1})\} \frac{\prod_{h=1}^{i} a_0^u(S_{t-h})\bar{\omega}_n}{\Omega^{*(1-u)}_{S,t-i-1}(\theta_0) c^u_{S,t}(\theta_0)} \right\|_{4+\zeta_1} \\
\leq\ & K + K \sum_{i=1}^{\infty} \left\| (\bar{\alpha}_n \eta^2_{t-1} + \bar{\beta}_n) \prod_{h=1}^{i} \{1 + \frac{1}{\sqrt{n}\beta^*}(1+\eta^2_{t-h-1})\} \frac{1}{\Omega^{*(1-u)}_{S,t-i-1}(\theta_0)} \right\|_{4+\zeta_2} \\
& \left\| \frac{1}{c^u_{S,t}(\theta_0)} \right\|_{2M} \left\| \prod_{h=1}^{i} a_0^u(S_{t-h}) \right\|_{2M} \\
\leq\ & K + K \sum_{i=1}^{\infty} \left\| \frac{1}{\Omega^{*(1-u)}_{S,t-i-1}(\theta_0)} \right\|_{4+\zeta_2} \\
& \prod_{h=1}^{i} \{1 + \frac{1}{\sqrt{n}\beta^*}(1 + \left\|\eta^2_{t-h-1}\right\|_{4+\zeta_2})\} \left\| \prod_{h=1}^{i} a_0^{r/2M}(S_{t-h}) \right\|_{2M} \\
\leq\ & K + K \sum_{i=1}^{\infty} \{1 + \frac{1}{\sqrt{n}\beta^*}(1 + \left\|\eta^2_{t}\right\|_{4+\zeta_2})\}^i \rho^{i/2M} < \infty. \qquad \text{(C.15)}
\end{aligned}$$

The first inequality relies on the independence between η^2_{t-1} and $\Omega^*_{S,t}(\theta_0)$ coming from **A0** and the Minkowski inequality. The second inequality relies on the Hölder inequality $\|XY\|_{4+\zeta_1} \leq \|X\|_{4+\zeta_2} \|Y\|_M$, the Cauchy-Schwarz inequality on the terms inside the sum, (C.13) and (C.14). The third one relies firstly on the independence between the η^2_{t-i} (for all i) and $\Omega^*_{S,t-i-1}(\theta_0)$ coming from **A0** and the independence between η^2_{t-1} and η^2_{t-h-1} for $h \geq 0$, secondly from $\left\|\bar{\alpha}_n \eta^2_{t-1} + \bar{\beta}_n\right\|_{4+\zeta_2} \leq \bar{\alpha}_n \left\|\eta^2_{t-1}\right\|_{4+\zeta_2} + \bar{\beta}_n < \infty$ in view of (C.13) and thirdly from the Minkowski inequality. The last one relies on (C.14) together with the stationarity of process $\Omega^{*(1-u)}_{S,t}(\theta_0)$, **A3** and independence between the η_{t-h-1}. The result follows from $\{1 + \frac{1}{\sqrt{n}\beta^*}(1 + \|\eta^2_t\|_{4+\zeta_2})\}\rho^{1/2M} < 1$.

Let us now define $\nu > 0$ such that $\nu < \zeta_1$ and $M_\nu > 0$ such that $\frac{1}{M_\nu} + \frac{1}{4+\zeta_1} = \frac{1}{4+\nu}$ and choose $\mathcal{V}(\theta_0)$ defined as in Lemma 7.6 for $m = M_\nu$. Noting that $\sigma^2_{S,t,n}(\theta_n)$ is decreasing with respect to n, we get that

$$\sup_{n \geq n_0} \frac{\sigma^2_{S,t,n}(\theta_n)}{\sigma^2_{S,t}(\theta_0)} \leq \frac{\sigma^2_{S,t,n_0}(\theta_{n_0})}{\sigma^2_{S,t}(\theta_0)}. \qquad \text{(C.16)}$$

Let us choose $n_0 > 0$ sufficiently large to get $\{1 + \frac{1}{\sqrt{n_0}\underline{\beta}_0}(1 + \|\eta_t^2\|_{4+\zeta_2})\}^i \rho^i < 1$. The result (C.15) can thus be used for $n = n_0$. Remarking that $\sigma_{S,t,n}^2(\theta) \geq \sigma_{S,t}^2(\theta)$, using the Hölder inequality:

$$\begin{aligned}
\left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma_{S,t,n}^2(\theta_n)}{\sigma_{S,t,n}^2(\theta)} \right\|_{4+\nu} &\leq \left\| \sup_{n \geq n_0} \frac{\sigma_{S,t,n}^2(\theta_n)}{\sigma_{S,t}^2(\theta_0)} \right\|_{4+\zeta_1} \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)} \right\|_{M_\nu} \\
&\leq \left\| \frac{\sigma_{S,t,n_0}^2(\theta_{n_0})}{\sigma_{S,t}^2(\theta_0)} \right\|_{4+\zeta_1} \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma_{S,t}^2(\theta_0)}{\sigma_{S,t}^2(\theta)} \right\|_{M_\nu} < \infty.
\end{aligned}$$

The second inequality follows from (C.16). The result follows from (7.9) for $m = M_\nu$ together with (C.15) and the first term of (C.7) is proved. The second term of (C.7) can be proved combining (C.6) and the arguments used in the proof of (7.9). The lemma is thus proved.

□

In the following, we will choose $n_0 > 0$ sufficiently large to get $\gamma_{n_0} < 0$ (thus to ensure that ϵ_{S,t,n_0} is a stationary process), $\theta_{n_0} \in \Theta$, (C.5) (thus (C.6)) and (C.7) true and $\mathcal{V}(\theta_0)$ will be chosen according to (C.7).

C.1 Consistency of $\hat{\theta}_n$

Following the scheme of the proof of Theorem 3 in Francq and Zakoïan (2009), we will establish the following intermediate results :

i) $\lim_{n\to\infty} \sup_{\theta\in\Theta} |\mathbf{l}_n(\theta) - \tilde{\mathbf{l}}_n(\theta)| = 0, \quad a.s.,$

ii) $\lim_{n\to\infty} \mathbf{l}_n(\theta_n) = E\ell_{S,t}(\theta_0),$

iii) for any $\theta \neq \theta_0$ there exists a neighborhood $V(\theta)$ such that

$$\liminf_{n\to\infty} \inf_{\theta^* \in V(\theta)} \tilde{\mathbf{l}}_n(\theta^*) > E\ell_{S,t}(\theta_0).$$

First we show **i)**. Let us define $\tilde{\sigma}_{t,n}$ recursively for $t \geq 2$, by

$$\tilde{\sigma}_{t,n}^2 = \tilde{\sigma}_{t,n}^2(\theta) = \omega(s_t) + \alpha(s_t)\epsilon_{t-1,n}^2 + \beta(s_t)\tilde{\sigma}_{t-1,n}^2$$

with $\tilde{\sigma}_{1,n}^2(\theta) = \omega(s_1) + \alpha(s_1)\tilde{\epsilon}_{0,n}^2 + \beta(s_1)\tilde{\sigma}_{0,n}^2$ and for instance $\tilde{\epsilon}_{0,n}^2 = \tilde{\sigma}_{0,n}^2 = \epsilon_{1,n}^2$. Thus :

$$\tilde{\sigma}_{t,n}^2 = c_{t,n} + \sum_{i=0}^{t-3} \beta(s_t)\dots\beta(s_{t-i})c_{t-i-1,n} + \beta_n(s_t)\dots\beta(s_2)\tilde{c}_{1,n} + \beta(s_t)\dots\beta(s_1)\tilde{\sigma}_{0,n}^2.$$

Following the lines of proof of Theorem 6 *ii)* in RZ, for t sufficiently large :

$$\begin{aligned}\sup_{\theta\in\Theta}|\tilde{\sigma}_{t,n}^2-\sigma_{t,n}^2| &= \sup_{\theta\in\Theta}\left|\beta(s_t)\dots\beta(s_2)(\tilde{c}_{1,n}-c_{1,n})+\beta(s_t)\dots\beta(s_1)(\tilde{\sigma}_{0,n}^2-\sigma_{0,n}^2)\right| \\ &\leq \sup_{\theta\in\Theta}\left|\beta(s_t)\dots\beta(s_2)(\tilde{c}_{1,n_0}-c_1)+\beta(s_t)\dots\beta(s_1)(\tilde{\sigma}_{0,n_0}^2-\sigma_0^2)\right| \\ &\leq K\sup_{\theta\in\Theta}\beta(s_t)\dots\beta(s_1)\leq C\beta_*^t,\end{aligned}$$

with $\beta_*<1$. The first inequality logically follows from (C.4) and the second one from **A2** because it is proved in the proof of (**??**). Proceeding as in the proof of Theorem 3.1, we obtain almost surely, using $\log x\leq x-1$, almost surely, for $n\geq n_0$,

$$\begin{aligned}\sup_{\theta\in\Theta}|\tilde{\mathbf{l}}_n(\theta)-\mathbf{l}_n(\theta)| &\leq n^{-1}\sum_{t=1}^n\sup_{\theta\in\Theta}\left\{\left|\frac{\tilde{\sigma}_{t,n}^2-\sigma_{t,n}^2}{\tilde{\sigma}_{t,n}^2\sigma_{t,n}^2}\right|\epsilon_{t,n}^2+\left|\log\left(1+\frac{\sigma_{t,n}^2-\tilde{\sigma}_{t,n}^2}{\tilde{\sigma}_{t,n}^2}\right)\right|\right\} \\ &\leq Kn^{-1}\sum_{t=1}^n\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{t,n}^2}{c_t^2(\theta)}\right\}+Kn^{-1}\sum_{t=1}^n\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{1}{c_t(\theta)}\right\} \\ &\leq Kn^{-1}\sum_{t=1}^n\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{t,n_0}^2}{c_t^2(\theta)}\right\}+Kn^{-1}\sum_{t=1}^n\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{1}{c_t(\theta)}\right\}.\end{aligned}$$

The second inequality follows from $\tilde{\sigma}_{t,n}^2\geq\tilde{\sigma}_t^2\geq c_t$ and $\sigma_{t,n}^2\geq\sigma_t^2\geq c_t$ which are consequences of (C.4) and the definition of σ_t^2. The third inequality follows from (C.4). By an extension of Lemma 1 in Francq and Gautier (2004a), it will be sufficient to show that $Kn^{-1}\sum_{t=1}^n\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t,n_0}^2}{c_{S,t}^2(\theta)}\right\}$ and $Kn^{-1}\sum_{t=1}^n\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{1}{c_{S,t}(\theta)}\right\}$ converge to 0 a.s. as $n\to\infty$. The convergence of the second term is ensured in the proof of (**??**). Using the Cesaro Lemma, it will be sufficient to show that $\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t,n_0}^2}{c_{S,t}^2(\theta)}\right\}$ converges to 0, almost surely when $t\to\infty$. This result follows from the Borel-Cantelli lemma and

$$\begin{aligned}\sum_{t=1}^\infty\mathbb{P}(\beta_*^t\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t,n_0}^2}{\tilde{\sigma}_{S,t}^2\sigma_{S,t}^2(\theta)}\right\}>\zeta) &\leq \sum_{t=1}^\infty\frac{\beta_*^{tv/4}}{\zeta^{v/4}}E\sup_{\theta\in\Theta}\left\{\frac{\epsilon_{S,t,n_0}^2}{c_{S,t}^2(\theta)}\right\}^{v/4} \\ \leq \left\{E\sup_{\theta\in\Theta}\frac{1}{c_{S,t}^v}\right\}^{1/2}\{E|\epsilon_{S,t,n_0}|^v\}^{1/2}\sum_{t=1}^\infty\frac{\beta_*^{tv/4}}{\zeta^{v/4}}<\infty.\end{aligned}$$

for any $\zeta>0$, where the first inequality follows from the Markov inequality and the second one from the Cauchy-Schwarz inequality and the stationarity of $(c_{S,t})$ and (ϵ_{S,t,n_0}). The result follows from (C.6), (7.3) in view of $v\leq r$ and $\beta_*<1$. **i)** is thus proved.

Now we will prove **ii)**. We have

$$l_n(\theta_n)=\frac{1}{n}\sum_{t=1}^n\frac{\epsilon_{t,n}^2}{\sigma_{t,n}^2(\theta_n)}+\log\sigma_{t,n}^2(\theta_n)=\frac{1}{n}\sum_{t=1}^n\eta_t^2+\frac{1}{n}\sum_{t=1}^n\log\sigma_{t,n}^2(\theta_n).$$

In the right hand-side term of the last equality, the first sample mean converges to 1,a.s, and the second one is between $\frac{1}{n}\sum_{t=1}^{n}\log\sigma_t^2(\theta_0)$ and $\frac{1}{n}\sum_{t=1}^{n}\log\sigma_{t,n_0}^2(\theta_{n_0})$. Using Lemma 1 in Francq and Gautier (2004a), these sample means converge a.s. to $E\log\sigma_{S,t}^2(\theta_0)$ and $E\log\sigma_{S,t,n_0}^2(\theta_0)$ respectively, when $n\to\infty$. The latter one decreases to the former one when $n_0\to\infty$, which establishes **ii)**.

It remains to show **iii)**. For any $\theta\in\Theta$ and any positive integer k, let $V_k(\theta)$ be the open ball with center θ and radius $1/k$. Proceeding as in RZ (see Proof of Theorem 6 *v)*), and in view of (C.4), we find

$$\begin{aligned}
\liminf_{n\to\infty}\inf_{\theta^*\in V_k(\theta)\cap\Theta}\tilde{\mathrm{l}}_n(\theta^*) &\geq \liminf_{n\to\infty} n^{-1}\sum_{t=1}^{n}\inf_{\theta^*\in V_k(\theta)\cap\Theta}\ell_{t,n}(\theta^*)\\
&\geq \liminf_{n\to\infty} n^{-1}\sum_{t=1}^{n}\inf_{\theta^*\in V_k(\theta)\cap\Theta}\left(\log\sigma_{t,n}^2+\frac{\epsilon_{t,n}^2}{\sigma_{t,n}^2}\right)(\theta^*)\\
&\geq \liminf_{n\to\infty} n^{-1}\sum_{t=1}^{n}\inf_{\theta^*\in V_k(\theta)\cap\Theta}\left(\log\sigma_t^2+\frac{\epsilon_t^2}{\sigma_{t,n_0}^2}\right)(\theta^*)\\
&\geq E\inf_{\theta^*\in V_k(\theta)\cap\Theta}\left(\log\sigma_{S,t}^2+\frac{\epsilon_{S,t}^2}{\sigma_{S,t,n_0}^2}\right)(\theta^*).
\end{aligned}$$

The last equality follows from Lemma 1 in Francq and Gautier (2004a). In the last equality, the infimum is larger than $-|E\inf_{\theta^*\in\Theta}\log\Omega_{S,t}^*(\theta^*)|$ which is finite by (7.2) and which ensures the existence of this expectation. By the Beppo-Levi theorem, when k and n_0 increase to ∞, $E\inf_{\theta^*\in V_k(\theta)\cap\Theta}\left(\log\sigma_{S,t}^2+\frac{\epsilon_{S,t}^2}{\sigma_{S,t,n_0}^2}\right)(\theta^*)$ increases to $E\ell_{S,t}(\theta)$. In view of $E\ell_{S,t}(\theta)>E\ell_{S,t}(\theta_0)$, for $\theta\neq\theta_0$ which was shown in RZ, **iii)** is proved.

C.2 Asymptotic normality of the score at θ_n

In this part, we will use Lemma 7.3 under the assumptions **A0"**, **A2** and **A9'**. Thus as stated in Lemma 7.3, (7.4) will be here considered true for $b=4+\xi$. We will show here that, when $n\to\infty$,

$$n^{-1/2}\sum_{t=1}^{n}\frac{\partial}{\partial\theta}\tilde{\ell}_{t,n}(\theta_n)\xrightarrow{\mathcal{L}}\mathcal{N}(0,(\kappa_\eta-1)J) \tag{C.17}$$

and

$$n^{-1}\sum_{t=1}^{n}\frac{\partial^2}{\partial\theta_i\partial\theta_j}\tilde{\ell}_{t,n}(\theta_{ij}^*)\xrightarrow{P}J(i,j) \tag{C.18}$$

for any θ^*_{ij} between θ_n and $\hat{\theta}_n$. Following the lines of the proof of Theorem 3 in Francq and Zakoïan (2009), we will show that for some neighborhood $\mathcal{V}(\theta_0)$ of θ_0,

a) $E\left\|\frac{\partial \ell_{S,t,n_0}(\theta_{n_0})}{\partial\theta}\frac{\partial \ell_{S,t,n_0}(\theta_{n_0})}{\partial\theta}\right\| < \infty, \quad E\left\|\frac{\partial^2 \ell_{S,t,n_0}(\theta_{n_0})}{\partial\theta\partial\theta'}\right\| < \infty,$

b) $n^{-1/2}\sum_{t=1}^n \frac{\partial}{\partial\theta}\ell_{t,n}(\theta_n) \xrightarrow{\mathcal{L}} \mathcal{N}(0,(\kappa_\eta - 1)J),$

c)

$$E\sup_{n\geq n_0}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left\|\frac{\partial^2 \ell_{S,t,n}(\theta)}{\partial\theta\partial\theta'}\right\| < \infty,$$

d)

$$\left\|n^{-1/2}\sum_{t=1}^n\left\{\frac{\partial \ell_{t,n}(\theta_n)}{\partial\theta} - \frac{\partial \tilde{\ell}_{t,n}(\theta_n)}{\partial\theta}\right\}\right\| \to 0 \tag{C.19}$$

and

$$\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left\|\frac{1}{n}\sum_{t=1}^n\left\{\frac{\partial^2 \ell_{t,n}(\theta)}{\partial\theta\partial\theta'} - \frac{\partial^2 \tilde{\ell}_{t,n}(\theta)}{\partial\theta\partial\theta'}\right\}\right\| \to_P 0, \tag{C.20}$$

e)

$$n^{-1}\sum_{t=1}^n \frac{\partial^2}{\partial\theta_i\partial\theta_j}\ell_{t,n}(\theta_n) \to J(i,j) \quad a.s.,$$

f) for all $i,j,k \in \{1,\ldots,3d\}$,

$$E\sup_{n\geq n_0}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left|\frac{\partial^3 \ell_{S,t,n}(\theta)}{\partial\theta_i\partial\theta_j\partial\theta_k}\right| < \infty.$$

Francq and Zakoïan (2009) have shown first that (C.17) is consequence of **b)** together with the first part of **d)** and second that **e)**, **f)** together with the second part of **d)** lead to (C.18). Let us now turn to the proofs of **a)**-**f)**.

Since θ_{n_0} belongs to the interior of Θ, in view of the properties established in RZ in the proof of Theorem 2.1 **a)** is a direct consequence of the assumption **A3** satisfied at the point θ_{n_0}, which is verified in view of (C.5). Turning to **b)**, we will use the following Theorem (Lindeberg TCL) (see Billingsley(1995), p.476).

Lemma C.3 *Let for any $n \geq 0$, $\{(X_{t,n}), \mathcal{F}_{t,n}\}$ a $\mathbb{R}^d$-martingale difference such that* *i) There exists a $d \times d$ matrix A such that, when $n \to \infty$,*

$$\forall \lambda \in \mathbb{R}^d, \quad \frac{1}{n}\sum_{t=1}^{n} Var\{\lambda' X_{t,n}|\mathcal{F}_{t-1,n}\} \to \lambda' A \lambda, \quad a.s.$$

ii) $\forall \lambda \in \mathbb{R}^d, \forall \epsilon > 0, \quad \sum_{t=1}^{n} E\left\{\left(\frac{\lambda' X_{t,n}}{\sqrt{n}}\right)^2 \mathbb{I}_{\{|\frac{\lambda' X_{t,n}}{\sqrt{n}}|>\epsilon\}}\right\} \to 0.$

Then $\frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_{t,n} \to \mathcal{N}(0, A)$.

In view of

$$n^{-1/2}\sum_{t=1}^{n}\frac{\partial}{\partial\theta}\ell_{t,n}(\theta_n) = n^{-1/2}\sum_{t=1}^{n}(1-\eta_t^2)\frac{1}{\sigma_{t,n}^2}\frac{\partial\sigma_{t,n}^2}{\partial\theta} := n^{-1/2}\sum_{t=1}^{n} X_{t,n},$$

we will use Lemma C.3. Indeed, recall that $\sigma_{t,n}^2$ and its derivatives are measurable with respect to the σ-field $\mathcal{F}_{t-1}$ generated by the variables $\eta_{t-i}, i > 0$. It follows that for any $n \geq 1$, $\{X_{t,n}, \mathcal{F}_{t-1}\}$ is a *non stationary* martingale difference. Note that even if it is *non stationary*, Lemma C.3 can be applied, when assumption *i)* and *ii)* are satisfied. For all $t \in \mathbb{Z}$ and $n \in \mathbb{N}$, θ_n belongs to the interior of Θ, and thus $X_{t,n}$ possesses moments at order 2 (see RZ). Let $\lambda \in \mathbb{R}^{3d}$, let $x_{t,n} = \lambda X_{t,n}$ and let

$$s_{t,n}^2 = E(x_{t,n}^2|\mathcal{F}_{t-1}) = (\kappa_\eta - 1)\lambda'\frac{1}{\sigma_{t,n}^4}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\lambda.$$

Proof of **b)**: To prove **b)**, considering *i)* and *ii)* in Lemma C.3, it will be sufficient to show that :

$$\frac{1}{n}\sum_{t=1}^{n} s_{t,n}^2 \xrightarrow{\mathbf{P}} (\kappa_\eta - 1)\lambda' J \lambda \tag{C.21}$$

$$\text{and} \quad \frac{1}{n}\sum_{t=1}^{n} E(x_{t,n}^2 \mathbb{1}_{|x_{t,n}^2| \geq n^{1/2}\epsilon}) \to 0, \quad \text{when} \quad n \to \infty, \tag{C.22}$$

for any $\epsilon > 0$. First consider the derivative of $\sigma_{t,n}^2$ with respect to $\beta(e_k)$. We have :

$$\frac{\partial\sigma_{t,n}^2}{\partial\beta(e_k)} = \sum_{i=0}^{\infty}\left\{\sum_{j=0}^{i}\prod_{h\neq j, 0\leq h\leq i}\beta(s_{t-h})\mathbb{1}_{S_{t-j}=e_k}\right\}\{\omega(s_{t-i-1}) + \alpha(s_{t-i-1})\epsilon_{t-i-2,n}^2\}.$$

Turning to derivative with respect to $\beta(e_k)$, and in view of $\theta_0 \leq \theta_n$ together with the inequality (C.2):

$$\begin{aligned}
&\frac{1}{\sigma_{t,n}^2}\frac{\partial\sigma_{t,n}^2}{\partial\beta(e_k)}(\theta_n) \\
&\leq \frac{1}{\beta_n(e_k)}\sum_{i=0}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_n(s_{t-h})c_{t-i-1,n}(\theta_n)}{{}_i\sigma_{\mathrm{t,n}}^2(\theta_n,\theta_0)}\sum_{j=0}^{i}\mathbb{1}_{s_{t-j}=e_k} \\
&\leq \frac{1}{\beta_0(e_k)}\sum_{i=0}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_n(s_{t-h})c_{t-i-1,n}(\theta_n)}{{}_i\mathrm{R}_t(\theta_0)+\prod_{0\leq h\leq i}\beta_n(s_{t-h})c_{t-i-1,n}(\theta_n)}\sum_{j=0}^{i}\mathbb{1}_{s_{t-j}=e_k} \\
&\leq \frac{1}{\beta_0(e_k)}\sum_{i=0}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_{n_0}(s_{t-h})c_{t-i-1,n_0}(\theta_{n_0})}{{}_i\mathrm{R}_t(\theta_0)+\prod_{0\leq h\leq i}\beta_{n_0}(s_{t-h})c_{t-i-1,n_0}(\theta_{n_0})}\sum_{j=0}^{i}\mathbb{1}_{s_{t-j}=e_k} := Y_\beta.
\end{aligned} \tag{C.23}$$

The second inequality follows from $\beta_n(e_k) > \beta_0(e_k)$. Noting that for any constant $a > 0$ and $b > 0$, the function $x \to x/(a+bx)$ is increasing over the positive line and remarking that $\prod_{h\neq j,0\leq h\leq i}\beta_n(s_{t-h})c_{t-i-1,n}(\theta_n) \leq \prod_{h\neq j,0\leq h\leq i}\beta_{n_0}(s_{t-h})\ c_{t-i-1,n_0}(\theta_{n_0})$ which comes from the inequalities (C.4), the last inequality follows. Now, we treat the derivative with respect to $\alpha(e_k)$ for $k \in \{1,\ldots,d\}$ such that $\alpha_0(e_k) = 0$. Let us recall

$$\frac{\partial\sigma_{t,n}^2}{\partial\alpha(e_k)} = \epsilon_{t-1,n}^2 + \sum_{i=0}^{\infty}\beta(s_t)\ldots\beta(s_{t-i})\epsilon_{t-i-2,n}^2\,\mathbb{1}_{s_{t-i-1}=e_k}\,.$$

In view of $\theta_0 \leq \theta_n$ together with the inequality (C.2):

$$\begin{aligned}
&\frac{1}{\sigma_{t,n}^2}\frac{\partial\sigma_{t,n}^2}{\partial\alpha(e_k)}(\theta_n) \\
&\leq \frac{\epsilon_{t-1,n}^2}{\sigma_{t,n}^2(\theta_n)} + \sum_{i=0}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_n(s_{t-h})\epsilon_{t-i-2,n}^2}{{}^i\sigma_{\mathrm{t,n}}^2(\theta_n,\theta_0)}\mathbb{1}_{s_{t-i-1}=e_k} \\
&\leq \eta_{t-1}^2 + \sum_{i=0}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_n(s_{t-h})\eta_{t-i-2}^2\sigma_{t-i-2,n}^2(\theta_n)}{{}^i\mathrm{R}_t(\theta_0)+\beta_n(e_k)\prod_{0\leq h\leq i}\beta_n(s_{t-h})\sigma_{t-i-2,n}^2(\theta_n)}\mathbb{1}_{s_{t-i-1}=e_k} \\
&\leq \eta_{t-1}^2 + \sum_{i=0}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_{n_0}(s_{t-h})\eta_{t-i-2}^2\sigma_{t-i-2,n_0}^2(\theta_{n_0})}{{}^i\mathrm{R}_t(\theta_0)+\beta_0(e_k)\prod_{0\leq h\leq i}\beta_{n_0}(s_{t-h})\sigma_{t-i-2,n_0}^2(\theta_{n_0})}\mathbb{1}_{s_{t-i-1}=e_k} := Y_\alpha.
\end{aligned} \tag{C.24}$$

We detail the computation to convince the reader but the arguments used are clearly the same. Similarly, turning to derivative with respect to $\omega(e_k)$ for $k \in \{1,\ldots,d\}$

such that $\omega_0(e_k)=0$, in view of $\theta_0\leq\theta_n$ together with inequality (C.2):

$$\begin{aligned}
&\frac{1}{\sigma^2_{t,n}}\frac{\partial\sigma^2_{t,n}}{\partial\omega(e_k)}(\theta_n)\\
\leq\ &\frac{\mathbb{1}_{S_t=e_k}+\beta_n(s_t)\,\mathbb{1}_{S_{t-1}=e_k}}{\sigma^2_{t,n}(\theta_n)}+\sum_{i=1}^{\infty}\frac{\beta_n(s_t)\ldots\beta_n(s_{t-i})}{\sigma^2_{i,t}(\theta_n,\theta_0)}\,\mathbb{1}_{s_{t-i-1}=e_k}\\
\leq\ &\frac{\mathbb{1}_{S_t=e_k}+\beta_{n_0}(s_t)\,\mathbb{1}_{S_{t-1}=e_k}}{\sigma^2_t(\theta_0)}+\sum_{i=1}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_n(s_{t-h})}{R_{i,t}(\theta_0)+\prod_{0\leq h\leq i}\beta_n(s_{t-h})\Omega_{t-i-1}(\theta_0)}\,\mathbb{1}_{S_{t-i-1}=e_k}\\
\leq\ &\frac{\mathbb{1}_{S_t=e_k}+\beta_{n_0}(s_t)\,\mathbb{1}_{S_{t-1}=e_k}}{\sigma^2_t(\theta_0)}\\
+\ &\sum_{i=1}^{\infty}\frac{\prod_{0\leq h\leq i}\beta_{n_0}(s_{t-h})}{R_{i,t}(\theta_0)+\prod_{0\leq h\leq i}\beta_{n_0}(s_{t-h})\Omega_{t-i-1}(\theta_0)}\,\mathbb{1}_{S_{t-i-1}=e_k}:=Y_\omega. \qquad (C.25)
\end{aligned}$$

Similar upper bounds hold for $\frac{1}{\sigma^2_{t,n}}\frac{\partial\sigma^2_{t,n}}{\partial\omega(e_k)}$, for $k\in\{1,\ldots,d\}$ such that $\omega_0(e_k)\neq 0$ and $\frac{1}{\sigma^2_{t,n}}\frac{\partial\sigma^2_{t,n}}{\partial\alpha(e_k)}$ $k\in\{1,\ldots,d\}$ such that $\alpha_0(e_k)=0$. For any $n\geq n_0$, using (C.4) and remarking that $\frac{\partial\sigma^2_{t,n}}{\partial\theta}\geq\frac{\partial\sigma^2_t}{\partial\theta}$, $\quad\sigma^2_{t,n}\leq\sigma^2_{t,n_0}$ componentwise, we have :

$$\frac{1}{\sigma^2_{t,n}}\frac{\partial\sigma^2_{t,n}}{\partial\theta}\ \geq\ \frac{1}{\sigma^2_{t,n_0}}\frac{\partial\sigma^2_t}{\partial\theta}.$$

Note that the right hand-side term of (C.23), (respectively (C.24) and (C.25)) converges a.s. to its left hand-side term when n_0 tends to ∞. In view of (C.23), (C.24), (C.25), it follows that :

$$Y_t^{(1)}(n_0)\leq\frac{1}{\sigma^2_{t,n}}\frac{\partial\sigma^2_{t,n}}{\partial\theta}\leq Y_t^{(2)}(n_0) \qquad (C.26)$$

for some $(\mathbb{R}^{3d})$-valued, processes $Y_t^{(1)}(n_0)$ and $Y_t^{(2)}(n_0)$. Because the vectors involved in the last inequality have positive components, it follows that:

$$Y_t^{(1)}(n_0)Y_t^{(1)}(n_0)'\leq\frac{1}{\sigma^4_{t,n}}\frac{\partial\sigma^2_{t,n}}{\partial\theta}\frac{\partial\sigma^2_{t,n}}{\partial\theta'}\leq Y_t^{(2)}(n_0)Y_t^{(2)}(n_0)' \qquad (C.27)$$

componentwise. Note that the lower and upper-bounds are independent of n, whenever $n\geq n_0$ and both converge to $\frac{1}{\sigma^4_t}\frac{\partial\sigma^2_t}{\partial\theta}\frac{\partial\sigma^2_t}{\partial\theta'}$. Lemma 1 in Francq and Gautier (2004a) applies to $n^{-1}\sum_{t=1}^n Y_t^{(i)}(n_0)Y_t^{(i)}(n_0)'$ $(i=1,2)$ provided the expectation of $EY_{S,t}^{(2)}(n_0)Y_{S,t}^{(2)}(n_0)$ is finite (with $Y_{S,t}^{(i)}(n_0)(i=1,2)$ defined by replacing (s_t) by (S_t) in the definition of $Y_t^{(i)}(n_0)(i=1,2)$). We thus need to prove $EY_{S,t}^{(2)}(n_0)Y_{S,t}^{(2)}(n_0)<\infty$. We will show it only for components of expression (C.24), (C.23) and (C.25) because these are the terms that need the most careful attention. The cases related to the derivatives with respect to $\omega(e_k)$ (respectively $\alpha(e_k)$) for k such that $\omega_0(e_k)\neq 0$(resp.

$\alpha_0(e_k) \neq 0$) may be treated in a similar way. Let us define $Y_{S,\omega}$(respectively $Y_{S,\alpha}$ and $Y_{S,\beta}$) the right-hand-side term of (C.25), (respectively (C.24) and (C.23)) after replacing s_t by S_t. $Y_{S,\omega}$ being very similar to the right-hand-side term of (7.18), proceeding as in the proof of (7.19) leads to :

$$\begin{aligned}
\|Y_{S,\omega}\|_4 &\leq \left\| \frac{1+\beta_{n_0}(S_t)}{\sigma^2_{S,t}(\theta_0)} \right\|_4 \\
&+ \sum_{i=1}^{\infty} \left\| \frac{1}{\Omega^*_{S,t-i-1}(\theta_0)} \frac{\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\Omega^*_{S,t-i-1}(\theta_0)}{c_{S,t}(\theta_0)+\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\Omega^*_{S,t-i-1}(\theta_0)} \right\|_4 \\
&\leq K + \sum_{i=1}^{\infty} \left\| \frac{1}{\Omega^*_{S,t-i-1}(\theta_0)} \left\{ \frac{\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\Omega^*_{S,t-i-1}(\theta_0)}{c_{S,t}(\theta_0)} \right\}^{v/16} \right\|_4 \\
&\leq K + \sum_{i=1}^{\infty} \left\| \frac{1}{\Omega^{*(1-\frac{v}{16})}_{S,t-i-1}(\theta_0)} \right\|_8 \left\| \beta^{\frac{v}{16}}_{n_0}(S_t)\dots\beta^{\frac{v}{16}}_{n_0}(S_{t-i}) \right\|_{16} \left\| \frac{1}{c^{v/16}_{S,t}(\theta_0)} \right\|_{16} \\
&\leq K + K\sum_{i=1}^{\infty} \rho^{*i/16} < \infty. \qquad\qquad (C.28)
\end{aligned}$$

Indeed, to get this result we have to add two arguments to those used in the proof of (7.19). First, the first inequality relies on $R_{i,S,t}(\theta_0) \geq c_{S,t}(\theta_0)$ coming from (C.3) together with the obvious inequality $\Omega_{S,t}(\theta) \geq \Omega^*_{S,t}(\theta)$. Second, the fourth inequality relies on inequality (C.5). We turn now to $Y_{S,\alpha}$. $Y_{S,\alpha}$ being very similar to the right-hand-side term of (7.20), proceeding as in the proof of (7.22) leads to:

$$\begin{aligned}
&\|Y_{S,\alpha}\|_4 \\
\leq\ & \left\|\eta^2_{t-1}\right\|_4 + K\sum_{i=0}^{\infty} \left\|\eta^2_{t-i-2}\right\|_4 \left\| \left\{ \frac{\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\sigma^2_{S,t-i-2,n_0}(\theta_{n_0})}{d_{S,t}(\theta_0)} \right\}^{v/16} \right\|_4 \\
\leq\ & K + K\sum_{i=0}^{\infty} \left\{ E\left\{\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\right\}^v E\left\{\sigma^2_{S,t-i-2,n_0}(\theta_{n_0})\right\}^v \right\}^{1/16} \left\| \frac{1}{d^{v/16}_{S,t}(\theta_0)} \right\|_8 \\
\leq\ & K + K\left\{E\sigma^{2v}_{S,t,n_0}(\theta_{n_0})\right\}^{1/16} \sum_{i=0}^{\infty} \left\{ E\left\{\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\right\}^v \right\}^{1/16} \\
\leq\ & K(1+\sum_{i=0}^{\infty} \rho^{*i/16}) < \infty. \qquad\qquad (C.29)
\end{aligned}$$

Indeed, to get this result we have to add three arguments to those used in the proof of (7.22). First, the first inequality relies on inequality ${}^{i}\mathrm{R}_{\mathrm{S,t}}(\theta_0) \geq d_{S,t}(\theta_0)$ coming from (C.3). Second, for the third inequality, we need to use the stationarity of (ϵ_{S,t,n_0}). Third, to retrieve the fourth inequality, we need to use (C.5) and (C.6). Turning to $Y_{S,\beta}$ using the inequality ${}_{i}\mathrm{R}_{\mathrm{S,t}}(\theta_0) \geq c_{S,t}(\theta_0)$ coming from (C.3) and proceeding as in the proof of (7.23):

$$
\begin{aligned}
& \|Y_{S,\beta}\|_4 \\
\leq\ & \frac{1}{\beta^*}\sum_{i=0}^{\infty}\left\|\frac{(i+1)\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})c_{S,t-i-1,n_0}(\theta_{n_0})}{c_{S,t}(\theta_0)+\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})c_{S,t-i-1,n_0}(\theta_{n_0})}\right\|_4 \\
\leq\ & K\sum_{i=0}^{\infty}(i+1)\left\|\{\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\}^{v/16}\right\|_{16}\left\|c_{S,t-i-1,n_0}(\theta_{n_0})^{v/16}\right\|_{16} \\
& \left\|c_{S,t}(\theta_0)^{-v/16}\right\|_8 \\
\leq\ & K\sum_{i=0}^{\infty}(i+1)\{E\left\{\beta_{n_0}(S_t)\dots\beta_{n_0}(S_{t-i})\right\}^v\}^{1/4M} \\
\leq\ & K\sum_{i=0}^{\infty}(i+1)\rho^{*i/16}<\infty. \qquad\qquad \text{(C.30)}
\end{aligned}
$$

The second inequality comes from the arguments used in the proof of (7.23). The third one comes from the stationarity of (c_{S,t,n_0}) and (C.6). The fourth one relies on (C.5).

(C.28)(C.29) and (C.30) allow to get that each component of the vector $Y = (Y_\omega, Y_\alpha, Y_\beta)$ possesses moment at order 4. Thus by the Cauchy-Schwarz inequality, $EY'Y<\infty$. Y is a sub vector of $Y_{S,t}^{(2)}(n_0)$ and the others components of $Y_{S,t}^{(2)}(n_0)$ can be treated in the same way to show that $Y_{S,t}^{(2)}(n_0)$ possesses a fourth-order moment. We omit the proof for brevity. It follows that

$$n^{-1}\sum_{t=1}^{n}Y_t^{(2)}(n_0)Y_t^{(2)}(n_0)'\xrightarrow{\mathbb{P}}EY_{S,t}^{(2)}(n_0)Y_{S,t}^{(2)}(n_0)'.$$

By the Lebesgue Theorem, this expectation converges to J when $n_0\to\infty$. Similarly, $n^{-1}\sum_{t=1}^{n}Y_t^{(1)}(n_0)Y_t^{(1)}(n_0)'\xrightarrow{\mathbb{P}}J$, when n and n_0 tend to $+\infty$. In view of (C.27) we can conclude, when $n\to\infty$, that

$$n^{-1}\sum_{t=1}^{n}\frac{1}{\sigma_{t,n}^4}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\frac{\partial\sigma_{t,n}^2}{\partial\theta'}\xrightarrow{\mathbb{P}}J,$$

which entails (C.21). Let us turn to prove (C.22). Firstly remark that $Y_{S,t}^{(2)}(n_0)$ possesses moment at order 4. Thus $E\{\lambda'Y_{S,t}^{(2)}(n_0)Y_{S,t}^{(2)}(n_0)'\lambda\}^2<\infty$ and using a law of large number

$$\frac{1}{n}\sum_{t=1}^{n}E\{\lambda'Y_t^{(2)}(n_0)Y_t^{(2)}(n_0)'\lambda\}^2\to E\{\lambda'Y_{S,t}^{(2)}(n_0)Y_{S,t}^{(2)}(n_0)'\lambda\}^2<\infty.$$

Then using Markov inequality, $\forall \epsilon > 0$:

$$
\begin{aligned}
& \frac{1}{n}\sum_{t=1}^{n} E(x_{t,n}^2\, \mathbb{1}_{|x_{t,n}^2|\geq n^{1/2}\epsilon}) \\
\leq\ & \frac{1}{\epsilon^2 n^2}\sum_{t=1}^{n} E x_{t,n}^4 \leq \frac{1}{\epsilon^2 n^2}\sum_{t=1}^{n}(\kappa_\eta - 1)E\left\{\lambda' \frac{1}{\sigma_{t,n}^4}\frac{\partial \sigma_{t,n}^2}{\partial \theta}\frac{\partial \sigma_{t,n}^2}{\partial \theta}\lambda\right\}^2 \\
\leq\ & \frac{K}{n^2}\sum_{t=1}^{n} E\{\lambda' Y_t^{(2)}(n_0) Y_t^{(2)}(n_0)'\lambda\}^2 \to 0,
\end{aligned}
$$

showing (C.22). The proof of **b)** is complete.

Proof of **c)**: The first and second derivatives of $\ell_{S,t,n}(\theta) = \frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2(\theta)} + \log \sigma_{S,t,n}^2(\theta)$ are given by :

$$
\begin{aligned}
\frac{\partial \ell_{S,t,n}}{\partial \theta} &= \left\{1 - \frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2}\right\}\left\{\frac{1}{\sigma_{S,t,n}^2}\frac{\partial \sigma_{S,t,n}^2}{\partial \theta}\right\}, \\
\frac{\partial^2 \ell_{S,t,n}}{\partial \theta \partial \theta'} &= \left\{1 - \frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2}\right\}\left\{\frac{1}{\sigma_{S,t,n}^2}\frac{\partial^2 \sigma_{S,t,n}^2}{\partial \theta \partial \theta'}\right\} \\
&+ \left\{2\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2} - 1\right\}\left\{\frac{1}{\sigma_{S,t,n}^2}\frac{\partial \sigma_{S,t,n}^2}{\partial \theta}\right\}\left\{\frac{1}{\sigma_{S,t,n}^2}\frac{\partial \sigma_{S,t,n}^2}{\partial \theta'}\right\}. \qquad \text{(C.31)}
\end{aligned}
$$

To prove **c)**, we need to prove following the scheme of proof of Lemma 7.10, that

$$
\sup_{n\geq n_0}\ \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\sigma_{S,t,n}^2}\frac{\partial \sigma_{S,t,n}^2}{\partial \theta}, \quad \sup_{n\geq n_0}\ \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta}\left\{1 - \frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2}\right\}
$$

$$
\text{and} \quad \sup_{n\geq n_0}\ \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta}\left\{\frac{1}{\sigma_{S,t,n}^2}\frac{\partial^2 \sigma_{S,t,n}^2}{\partial \theta \partial \theta'}\right\}
$$

possess respectively a third (respectively third and second) order moment. Actually, we will show that they all possess a fourth order moment. This result will be used in the proof of **f)**.

As in **b)**, we will only treat the first order derivatives with respect to $\omega(e_k)$ for $k \in \{1, \ldots, d\}$ such that $\omega_0(e_k) = 0$, $\alpha(e_k)$ for $k \in \{1, \ldots, d\}$ such that $\alpha_0(e_k) = 0$ and $\beta(e_k)$. First, we will show that a formula similar to (C.26) holds in some neighborhood $\mathcal{V}(\theta_0)$ of θ_0. Let us define $\mathcal{V}(\theta_0)$ according to Lemma 7.6 with $m \geq 4$. Let us denote $\underline{\theta} = \inf_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \theta$ the infimum being understood componentwise. Defining Θ as an hypercube allows to get that $\underline{\theta} \in \Theta$, which is what we will assume here. Let n_0 be large enough so that $\theta_{n_0} \in \mathcal{V}(\theta_0)$. Then similarly to (C.23), for

$n \geq n_0$,

$$\begin{aligned}
&\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\sigma^2_{S,t,n}}\frac{\partial\sigma^2_{S,t,n}}{\partial\beta(e_k)}(\theta)\\
\leq\;& \frac{1}{\beta_0(e_k)}\sum_{i=0}^{\infty}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\prod_{0\leq h\leq i}\beta(S_{t-h})c_{S,t-i-1,n}(\theta)}{{}_i\mathrm{R}_{S,t}(\underline{\theta})+\prod_{0\leq h\leq i}\beta(S_{t-h})c_{S,t-i-1,n}(\theta)}\sum_{j=0}^{i}\mathbb{1}_{S_{t-j}=e_k}\\
\leq\;& \frac{1}{\beta_0(e_k)}\sum_{i=0}^{\infty}(i+1)\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\prod_{0\leq h\leq i}\beta(S_{t-h})c_{S,t-i-1,n_0}(\theta)}{c_{S,t}(\underline{\theta})+\prod_{0\leq h\leq i}\beta(S_{t-h})c_{S,t-i-1,n_0}(\theta)}.
\end{aligned}$$

The first inequality follows from (7.5) and that for $\theta\in\mathcal{V}(\theta_0)\cap\Theta$ ${}_i\mathrm{R}_{S,t}(\underline{\theta})\leq_i \mathrm{R}_{S,t}(\theta)$. The second one follows from (C.3) and that $x/(a+x)$ is an increasing function of x when $a\geq 0$ (here $a=c_{S,t}(\underline{\theta})\geq 0$) together with $\prod_{0\leq h\leq i}\beta(S_{t-h})c_{S,t-i-1,n}(\theta)\leq \prod_{0\leq h\leq i}\beta(S_{t-h})c_{S,t-i-1,n_0}(\theta)$. by (C.4). Remarking that $\frac{1}{c_{S,t}(\underline{\theta})}\leq\sup_{\theta\in\Theta}\frac{1}{c_{S,t}(\theta)}$ because $\underline{\theta}\in\Theta$, it possesses thus a moment of order v in view of (7.3). Combining this result with arguments from the proofs of (C.30) and (7.23), it can be shown that the right hand-side term possesses a moment at any order $m\in\mathbb{N}$. Thus remarking that this right-hand-side term does not depend on n, we get that the supremum on $n\geq n_0$ of the left hand-side term possesses moment at any order $m\in\mathbb{N}$.

Turning to derivative with respect to $\alpha(e_k)$ for $k\in\{1,\ldots,d\}$ such that $\alpha_0(e_k)=0$ and similarly to (C.24) for $n\geq n_0$,

$$\begin{aligned}
&\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{\sigma^2_{S,t,n}}\frac{\partial\sigma^2_{S,t,n}}{\partial\alpha(e_k)}(\theta)\\
\leq\;& \sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\epsilon^2_{S,t-1,n}}{\sigma^2_{S,t-1,n}(\theta)}\\
+\;& \sum_{i=0}^{\infty}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\prod_{0\leq h\leq i}\beta(S_{t-h})\eta^2_{t-i-2}\sigma^2_{S,t-i-2,n}(\theta_n)}{{}^i\mathrm{R}_{S,t}(\underline{\theta})+\underline{\beta}(e_k)\prod_{0\leq h\leq i}\beta(S_{t-h})\sigma^2_{S,t-i-2,n}(\theta)}\mathbb{1}_{S_{t-i-1}=e_k}\\
\leq\;& \eta^2_{t-1}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\sigma^2_{S,t-1,n}(\theta_n)}{\sigma^2_{S,t-1,n}(\theta)}+\sum_{i=0}^{\infty}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\left\{\frac{\sigma^2_{S,t-i-2,n}(\theta_n)}{\sigma^2_{S,t-i-2,n}(\theta)}\right\}\\
&\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\prod_{0\leq h\leq i}\beta(S_{t-h})\eta^2_{t-i-2}\sigma^2_{S,t-i-2,n}(\theta)}{d_{S,t}(\underline{\theta})+\underline{\beta}(e_k)\prod_{0\leq h\leq i}\beta(S_{t-h})\sigma^2_{S,t-i-2,n}(\theta)}\\
\leq\;& \eta^2_{t-1}\left\{\sup_{n\geq n_0}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\sigma^2_{S,t-1,n}(\theta_n)}{\sigma^2_{S,t-1,n}(\theta)}\right\}+\sum_{i=0}^{\infty}\eta^2_{t-i-2}\left\{\sup_{n\geq n_0}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\sigma^2_{S,t-i-2,n}(\theta_n)}{\sigma^2_{S,t-i-2,n}(\theta)}\right\}\\
&\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\prod_{0\leq h\leq i}\beta(S_{t-h})\sigma^2_{S,t-i-2,n_0}(\theta)}{d_{S,t}(\underline{\theta})+\underline{\beta}(e_k)\prod_{0\leq h\leq i}\beta(S_{t-h})\sigma^2_{S,t-i-2,n_0}(\theta)}. \qquad \text{(C.32)}
\end{aligned}$$

The first inequality follows from (7.5) and that for $\theta\in\mathcal{V}(\theta_0)\cap\Theta$ ${}^i\mathrm{R}_{S,t}(\underline{\theta})\leq^i \mathrm{R}_{S,t}(\theta)$. The second one follows from (C.3). The third one relies on two arguments. Firstly

$x/(a+bx)$ is an increasing function of x when $a \geq 0, b > 0$ (here $a = d_{S,t}(\underline{\theta}) \geq 0$ and $b = \underline{\beta}(e_k) > 0$) and $\prod_{0 \leq h \leq i} \beta(s_{t-h}) \sigma^2_{S,t-i-2,n}(\theta) \leq \prod_{0 \leq h \leq i} \beta(s_{t-h}) \sigma^2_{S,t-i-2,n_0}(\theta)$. Secondly, we take supremum on $\sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma^2_{S,t,n}(\theta_n)}{\sigma^2_{S,t,n}(\theta)}$ for $n \geq n_0$.

Now, for any $m > 0$ and with w defined in Lemma C.2:

$$
\begin{aligned}
&\left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\prod_{0 \leq h \leq i} \beta(S_{t-h}) \sigma^2_{S,t-i-2,n_0}(\theta)}{d_{S,t}(\underline{\theta}) + \underline{\beta}(e_k) \prod_{0 \leq h \leq i} \beta(S_{t-h}) \sigma^2_{S,t-i-2,n_0}(\theta)} \right\|_m \\
\leq\ & K \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\prod_{0 \leq h \leq i} \beta^{w/4m}(S_{t-h}) \sigma^{w/2m}_{S,t-i-2,n_0}(\theta)}{d^{w/4m}_{S,t}(\underline{\theta})} \right\|_m \\
\leq\ & K \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \prod_{0 \leq h \leq i} \beta^{\frac{w}{4m}}(S_{t-h}) \right\|_{2m} \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \sigma^{\frac{w}{2m}}_{S,t-i-2,n_0}(\theta) \right\|_{4m} \left\| \frac{1}{d^{\frac{w}{4m}}_{S,t}(\underline{\theta})} \right\|_{4m} \\
\leq\ & K \{ E \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \prod_{0 \leq h \leq i} \beta^{\frac{w}{2}}(S_{t-h}) \}^{1/2m} \leq K (\rho^{* \frac{w}{vm}})^{(i+1)}. \qquad \text{(C.33)}
\end{aligned}
$$

The first inequality follows from the elementary inequality $x/(1+x) \leq x^{\frac{w}{4m}}$ with $\frac{w}{4m} < 1$. The second one relies on the Cauchy-Schwarz inequality. Remarking that $\frac{1}{d_{S,t}(\underline{\theta})} \leq \sup_{\theta \in \Theta} \frac{1}{d_{S,t}(\theta)}$ because $\underline{\theta} \in \Theta$ and using (7.3), we get $\left\| \frac{1}{d^{\frac{w}{4m}}_{S,t}(\underline{\theta})} \right\|_{4m} < \infty$. Using (C.7) and the stationarity of (σ^2_{S,t,n_0}), we get the third inequality. The last one follows from the Hôlder inequality and (7.8). Now choosing $M > 0$ such that $\frac{1}{M} + \frac{1}{4+\nu} = \frac{1}{4}$ with ν defined accordingly to Lemma C.2 and taking the supremum on the left hand-side term of (C.32), using the Minkowski inequality :

$$
\begin{aligned}
&\left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^2_{S,t,n}} \frac{\partial \sigma^2_{S,t,n}}{\partial \alpha(e_k)}(\theta) \right\|_4 \\
\leq\ & \left\| \eta^2_{t-1} \right\|_4 \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma^2_{S,t-1,n}(\theta_n)}{\sigma^2_{S,t-1,n}(\theta)} \right\|_4 \\
+\ & \sum_{i=0}^{\infty} \left\| \eta^2_{t-i-2} \right\|_4 \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma^2_{S,t-i-2,n}(\theta_n)}{\sigma^2_{S,t-i-2,n}(\theta)} \right\|_{4+\nu} \\
& \left\| \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\prod_{0 \leq h \leq i} \beta(S_{t-h}) \sigma^2_{S,t-i-2,n_0}(\theta)}{d_{S,t}(\underline{\theta}) + \underline{\beta}(e_k) \prod_{0 \leq h \leq i} \beta(S_{t-h}) \sigma^2_{S,t-i-2,n_0}(\theta)} \right\|_M \\
\leq\ & K + K \sum_{i=0}^{\infty} (\rho^{* \frac{w}{vM}})^{(i+1)} < \infty.
\end{aligned}
$$

The first inequality relies on two arguments. Firstly it follows from the independence between η^2_{t-i-2} and the other term inside the sum. Secondly, it follows from the Hôlder inequality. **A7"**, the stationarity of (η^2_t) and $\left(\frac{\sigma^2_{S,t,n}(\theta_n)}{\sigma^2_{S,t,n}(\theta)} \right)$, (C.7) and (C.33) lead to the second one. The result follows from $\rho^* < 1$.

Now we turn to the first-order derivative with respect to $\omega(e_k)$ for $k \in \{1, \ldots, d\}$ such that $\omega_0(e_k) = 0$. Noting that $\frac{\partial \sigma^2_{S,t,n}}{\partial \omega(e_k)}(\theta) = \frac{\partial \sigma^2_{S,t}}{\partial \omega(e_k)}(\theta)$, we get that

$$\sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^2_{S,t,n}} \frac{\partial \sigma^2_{S,t,n}}{\partial \omega(e_k)}(\theta) \leq \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^2_{S,t}} \frac{\partial \sigma^2_{S,t}}{\partial \omega(e_k)}(\theta)$$

It was shown in (7.19), that the right hand-side term of this inequality possesses under the assumptions of Theorem 3.2 a moment at order 3. Using the same type of proof, it possesses a moment at order 4 under assumption **A7"** and **A9'**. Thus we have established that

$$0 \leq \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^2_{S,t,n}} \frac{\partial \sigma^2_{S,t,n}}{\partial \theta}(\theta) \leq Y^{(3)}_{S,t}(n_0) \tag{C.34}$$

for some vector $Y^{(3)}_t(n_0)$ admitting moments at order 4. Similar arguments show that for $i, j \in \{1, \ldots, 3d\}$,

$$0 \leq \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma^2_{S,t,n}} \frac{\partial^2 \sigma^2_{S,t,n}}{\partial \theta_i \partial \theta_j}(\theta) \leq Y^{(4)}_{i,j,S,t}(n_0) \tag{C.35}$$

for some variables $Y^{(4)}_{i,j,S,t}(n_0)$ admitting moments at order 4. To handle terms of (C.31) involving $\frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)}$, remark that, using (C.7) and the assumption **A7"**,

$$\left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} \right\|_4 = \{E \eta_t^8\}^{1/4} \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\sigma^2_{S,t,n}(\theta_n)}{\sigma^2_{S,t,n}(\theta)} \right\|_4 < \infty. \tag{C.36}$$

The proof of **c)** is thus achieved.

Proof of **d)**: Following the lines of the proof of Theorem 7 *iv)* in RZ, we have almost surely, for t sufficiently large with $\sup_{\theta \in \Theta} \prod_{j=1}^d \beta^{\pi_j}(e_j) < \beta_* < 1$:

$$\sup_{\theta \in \Theta} \left\| \frac{\partial \sigma^2_{t,n}}{\partial \theta} - \frac{\partial \tilde{\sigma}^2_{t,n}}{\partial \theta} \right\| < K \beta_*^t, \quad \sup_{\theta \in \Theta} \left\| \frac{\partial^2 \sigma^2_{t,n}}{\partial \theta \partial \theta} - \frac{\partial^2 \tilde{\sigma}^2_{t,n}}{\partial \theta \partial \theta'} \right\| < K \beta_*^t. \tag{C.37}$$

In view of (7.6) and $\tilde{\sigma}^2_{t,n}(\theta) \geq c_{t,n}(\theta)$ we have,

$$\left| \frac{1}{\sigma^2_t(\theta)} - \frac{1}{\tilde{\sigma}^2_{t,n}(\theta)} \right| = \left| \frac{\sigma^2_{t,n}(\theta) - \tilde{\sigma}^2_{t,n}(\theta)}{\sigma^2_{t,n}(\theta) \tilde{\sigma}^2_{t,n}(\theta)} \right| \leq \frac{K \beta_*^t}{\sigma^2_{t,n}(\theta) c_{t,n}(\theta)}, \quad \frac{\sigma^2_{t,n}(\theta)}{\tilde{\sigma}^2_{t,n}(\theta)} \leq 1 + K \frac{\beta_*^t}{c_{t,n}(\theta)}. \tag{C.38}$$

We have, similarly as in Lemma 7.11, using (C.38), the first inequality in (C.37) and some computations detailed in the lines of the proof of Theorem 7 *iv)* in RZ, that for n_0 sufficiently large and $n \geq n_0$,

$$\begin{aligned} \left| \frac{\partial \tilde{\ell}_{t,n}}{\partial \theta_i}(\theta_n) - \frac{\partial \ell_{t,n}}{\partial \theta_i}(\theta_n) \right| &\leq K \frac{\beta_*^t}{c_{t,n}(\theta_n)} (1 + \eta_t^2) \left| 1 + \left\{ \frac{1}{\sigma^2_{t,n}(\theta_n)} \frac{\partial \sigma^2_{t,n}(\theta_n)}{\partial \theta_i} \right\} \right| \\ &\leq K \frac{\beta_*^t}{c_t(\theta_0)} (1 + \eta_t^2) \left| 1 + \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ \frac{1}{\sigma^2_{t,n}(\theta)} \frac{\partial \sigma^2_{t,n}(\theta)}{\partial \theta_i} \right\} \right|. \end{aligned}$$

Two arguments are needed to retrieve the second inequality. Firstly, $c_t(\theta_0) \leq c_{t,n}(\theta_n)$. Secondly, noting that $\theta_n \in \mathcal{V}(\theta_0) \cap \Theta$ for n_0 sufficiently large, we are allowed to upperbound $\left\{\frac{1}{\sigma_{t,n}^2(\theta_n)} \frac{\partial \sigma_{t,n}^2(\theta_n)}{\partial \theta_i}\right\}$ by the supremum on $\mathcal{V}(\theta_0) \cap \Theta$ of the function $\frac{1}{\sigma_{t,n}^2} \frac{\partial \sigma_{t,n}^2}{\partial \theta_i}$, which is naturally bounded by its supremum for $n \geq n_0$. Using the elementary inequality $(x+y)^u \leq x^u + y^u$, for $x, y \geq 0$ and $0 < u < 1$, it follows that for any $0 < u < 1$ and $n \geq n_0$,

$$\begin{aligned} & \left| n^{-1/2} \sum_{t=1}^{n} \left\{ \frac{\partial \ell_{t,n}(\theta_n)}{\partial \theta_i} - \frac{\partial \tilde{\ell}_{t,n}(\theta_n)}{\partial \theta_i} \right\} \right|^u \\ \leq \;& K n^{-\frac{u}{2}} \sum_{t=1}^{n} K \frac{\beta_*^{ut}}{c_t^u(\theta_0)} (1+\eta_t^2)^u \left| 1 + \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ \frac{1}{\sigma_{t,n}^2(\theta)} \frac{\partial \sigma_{t,n}^2(\theta)}{\partial \theta_i} \right\} \right|^u . \end{aligned} \tag{C.39}$$

Let us choose $u < \min(r/2, 1/2)$. From **c)**, $\sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ \frac{1}{\sigma_{S,t,n}^2(\theta)} \frac{\partial \sigma_{S,t,n}^2(\theta)}{\partial \theta_i} \right\}$ possesses a moment of order 1, thus a moment of order 2u. We get thus using (7.4) that:

$$E \frac{1}{c_{S,t}^u(\theta_0)} < \infty, \quad E \frac{1}{c_{S,t}^{2u}(\theta_0)} < \infty, \quad E \left| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma_{S,t,n}^2(\theta)} \frac{\partial \sigma_{S,t,n}^2(\theta)}{\partial \theta_i} \right|^{2u} < \infty, \tag{C.40}$$

for $u > 0$ sufficiently small. Using the independence between η_t and $(\sigma_{S,t}^2(\theta_0), c_{S,t}(\theta_0))$, the stationarity of $(\sigma_{S,t,n}^2)$ and $\left(\frac{1}{\sigma_{S,t,n}^2(\theta)} \frac{\partial \sigma_{S,t,n}^2(\theta)}{\partial \theta_i}\right)$ (for $n \geq n_0$):

$$\begin{aligned} & E \left(\sum_{t=1}^{n} \frac{\beta_*^{ut}}{c_{S,t}^u(\theta_0)} (1+\eta_t^2)^u \left| 1 + \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma_{S,t,n}^2(\theta)} \frac{\partial \sigma_{S,t,n}^2(\theta)}{\partial \theta_i} \right|^u \right) \\ \leq \;& E((1+\eta_t^2)^u) \left(E \frac{1}{c_{S,t}^u(\theta_0)} + E \frac{1}{c_{S,t}^u(\theta_0)} \left| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma_{S,t,n}^2(\theta)} \frac{\partial \sigma_{S,t,n}^2(\theta)}{\partial \theta_i} \right|^u \right) \sum_{t=1}^{n} \beta_*^{ut/2} \\ \leq \;& K \sum_{t=1}^{\infty} \beta_*^{ut} < \infty. \end{aligned}$$

The second inequality follows from the Cauchy-Schwarz inequality, the existence of moment of order u for η_t^2 and (C.40). The result follows from $\beta_*^u < 1$. This result shows that $\sum_{t=1}^{n} \frac{\beta_*^{ut}(1+\eta_t^2)^u}{c_{S,t}^u(\theta_0)} \left| 1 + \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{1}{\sigma_{S,t,n}^2(\theta)} \frac{\partial \sigma_{S,t,n}^2(\theta)}{\partial \theta_i} \right|^u$ converges a.s.. Thus, by a straightforward extension of Lemma 1 in Francq and Gautier (2004a), the right hand-side term of (C.39) converges to 0, a.s. showing (C.19).

We turn now to prove (C.20). For any $0 < u < 1$, using the inequalities (C.37) and

(C.38) and following some computations detailed in lines of Proof of Theorem 7 *iv)* in RZ, it can be shown that for $n \geq n_0$,

$$\sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \left| \frac{1}{n} \sum_{t=1}^{n} \left\{ \frac{\partial^2 \ell_{t,n}(\theta)}{\partial\theta_i \partial\theta_j} - \frac{\partial^2 \tilde{\ell}_{t,n}(\theta)}{\partial\theta_i \partial\theta_j} \right\} \right|^u \leq Kn^{-1} \sum_{t=1}^{n} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{\beta_*^{ut}}{c_t^u(\theta)} \mathcal{Y}_t^u, \tag{C.41}$$

where

$$\mathcal{Y}_t = \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \left\{ 1 + \frac{\epsilon_{t,n}^2}{\sigma_{t,n}^2} \right\} \left\{ 1 + \frac{1}{\sigma_{t,n}^2} \frac{\partial^2 \sigma_{t,n}^2}{\partial\theta_i \partial\theta_j} + \frac{1}{\sigma_{t,n}^4} \frac{\partial \sigma_{t,n}^2}{\partial\theta_i} \frac{\partial \sigma_{t,n}^2}{\partial\theta_j} \right\}.$$

Let us define:

$$\mathcal{Y}_{S,t} = \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \left\{ 1 + \frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2} \right\} \left\{ 1 + \frac{1}{\sigma_{S,t,n}^2} \frac{\partial^2 \sigma_{S,t,n}^2}{\partial\theta_i \partial\theta_j} + \frac{1}{\sigma_{S,t,n}^4} \frac{\partial \sigma_{S,t,n}^2}{\partial\theta_i} \frac{\partial \sigma_{S,t,n}^2}{\partial\theta_j} \right\}.$$

Note that $\mathcal{Y}_{S,t}$ involves only some nonnegative terms, thus is nonnegative. The Cauchy-Schwarz and Minkowski inequalities entail that for any $u > 0$:

$$\begin{aligned} \|\mathcal{Y}_{S,t}\|_u \leq & \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \left\{ 1 + \frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2} \right\} \right\|_{2u} \\ & \left\{ 1 + \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\sigma_{S,t,n}^2} \frac{\partial^2 \sigma_{S,t,n}^2}{\partial\theta_i \partial\theta_j} \right\|_{2u} + \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{1}{\sigma_{S,t}^4} \frac{\partial \sigma_{S,t,n}^2}{\partial\theta_i} \frac{\partial \sigma_{S,t,n}^2}{\partial\theta_j} \right\|_{2u} \right\}. \end{aligned}$$

Using the lines of the proof of lemma 7.10, the terms involving the first and second-order derivatives of $\sigma_{S,t}^2$ and $\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2}$ possess moments of order u for $u > 0$ sufficiently small. Let choose $0 < u < r/2$ such that $E\mathcal{Y}_{S,t}^{2u} < \infty$. Now, using the stationarity of $(c_{S,t})$ and $(\mathcal{Y}_{S,t})$:

$$\begin{aligned} E \sum_{t=1}^{n} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{\beta_*^{ut}}{c_{S,t}^u(\theta)} \mathcal{Y}_{S,t}^u & \leq (\sum_{t=1}^{\infty} \beta_*^{ut}) E \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{1}{c_{S,t}^u(\theta)} \mathcal{Y}_{S,t}^u \\ & \leq KE \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{1}{c_{S,t}^{2u}(\theta)} E\mathcal{Y}_{S,t}^{2u} < \infty. \end{aligned}$$

The second inequality follows from the Cauchy-Schwarz inequality and $\beta_*^u < 1$. The result relies on $E\mathcal{Y}_{S,t}^{2u} < \infty$ and (7.4). Thus $\sum_{t=1}^{n} \sup_{\theta \in \mathcal{V}(\theta_0)\cap\Theta} \frac{\beta_*^{ut}}{c_{S,t}^u(\theta)} \mathcal{Y}_{S,t}^u$ converges a.s , and thus $\sum_{t=1}^{n} \frac{\beta_*^{ut}}{c_t^u(\theta)} \mathcal{Y}_t^u$ converges a.s. by a straightforward extension of Lemma 1 in Francq and Gautier (2004a). The right hand-side term of (C.41) thus converges to 0 a.s. which shows (C.20). The proof of **d)** is complete.

Proof of **e)**: Similar to (C.31) the second order derivatives of $\ell_{t,n}$ is :

$$\begin{aligned}\frac{\partial^2 \ell_{t,n}}{\partial\theta\partial\theta'} &= \left\{1-\frac{\epsilon_{t,n}^2}{\sigma_{t,n}^2}\right\}\left\{\frac{1}{\sigma_{t,n}^2}\frac{\partial^2\sigma_{t,n}^2}{\partial\theta\partial\theta'}\right\} \\ &+ \left\{2\frac{\epsilon_{t,n}^2}{\sigma_{t,n}^2}-1\right\}\left\{\frac{1}{\sigma_{t,n}^2}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\right\}\left\{\frac{1}{\sigma_{t,n}^2}\frac{\partial\sigma_{t,n}^2}{\partial\theta'}\right\}. \qquad \text{(C.42)}\end{aligned}$$

First consider the second group of terms at the value θ_n. In view in (C.27), we have for $n \geq n_0$,

$$\begin{aligned}& n^{-1}\sum_{t=1}^{n}(2\eta_t^2-1)\frac{1}{\sigma_{t,n}^4}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\frac{\partial\sigma_{t,n}^2}{\partial\theta'} \\ \leq\ & n^{-1}\sum_{t=1}^{n}(2\eta_t^2-1)\,\mathbb{1}_{2\eta_t^2\geq 1}\,Y_t^{(2)}(n_0)Y_t^{(2)}(n_0)' \\ +\ & n^{-1}\sum_{t=1}^{n}(2\eta_t^2-1)\,\mathbb{1}_{2\eta_t^2< 1}\,Y_t^{(1)}(n_0)Y_t^{(1)}(n_0)'.\end{aligned}$$

Lemma 1 in Francq and Gautier (2004a) applied to sums of the right hand-side term, yields, a.s.

$$\begin{aligned}& \limsup_{n\to\infty} n^{-1}\sum_{t=1}^{n}(2\eta_t^2-1)\frac{1}{\sigma_{t,n}^4}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\frac{\partial\sigma_{t,n}^2}{\partial\theta'} \\ \leq\ & E\{(2\eta_t^2-1)\,\mathbb{1}_{2\eta_t^2\geq 1}\}E\{Y_{S,t}^{(2)}(n_0)Y_{S,t}^{(2)}(n_0)'\} \\ +\ & E\{(2\eta_t^2-1)\,\mathbb{1}_{2\eta_t^2< 1}\}E\{Y_{S,t}^{(1)}(n_0)Y_{S,t}^{(1)}(n_0)'\},\end{aligned}$$

from the independence between η_t and the variables $Y_{S,t}^{(i)}(n_0)$. We have already seen that $E\{Y_{S,t}^{(i)}(n_0)Y_{S,t}^{(i)}(n_0)'\} \to J$ for $i=1,2$, as $n_0\to\infty$. It follows that, a.s.

$$\limsup_{n\to\infty} n^{-1}\sum_{t=1}^{n}(2\eta_t^2-1)\frac{1}{\sigma_{t,n}^4}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\frac{\partial\sigma_{t,n}^2}{\partial\theta'} \leq E\{(2\eta_t^2-1)(\mathbb{1}_{2\eta_t^2\geq 1}+\mathbb{1}_{2\eta_t^2<1})\}J = J.$$

Similarly we have

$$\begin{aligned}& \liminf_{n\to\infty} n^{-1}\sum_{t=1}^{n}(2\eta_t^2-1)\frac{1}{\sigma_{t,n}^4}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\frac{\partial\sigma_{t,n}^2}{\partial\theta'} \\ \geq\ & E\{(2\eta_t^2-1)\,\mathbb{1}_{2\eta_t^2\geq 1}\}E\{Y_{S,t}^{(1)}(n_0)Y_{S,t}^{(2)}(n_0)'\} \\ +\ & E\{(2\eta_t^2-1)\,\mathbb{1}_{2\eta_t^2< 1}\}E\{Y_{S,t}^{(2)}(n_0)Y_{S,t}^{(1)}(n_0)'\},\end{aligned}$$

which converges to J as $n_0\to\infty$. Thus we have that

$$\lim_{n\to\infty} n^{-1}\sum_{t=1}^{n}(2\eta_t^2-1)\frac{1}{\sigma_{t,n}^4}\frac{\partial\sigma_{t,n}^2}{\partial\theta}\frac{\partial\sigma_{t,n}^2}{\partial\theta'} = J.$$

The first group of terms in the right hand-side of (C.42) can be treated analogously, using some appropriate lower and upper-bounds of $\frac{1}{\sigma_{t,n}^2}\frac{\partial^2\sigma_{t,n}^2}{\partial\theta\partial\theta'}$. Therefore we have a.s., $\lim_{n\to\infty} n^{-1}\sum_{t=1}^n(1-\eta_t^2)\frac{1}{\sigma_{t,n}^2}\frac{\partial^2\sigma_{t,n}^2}{\partial\theta\partial\theta'}=0$. The convergence in **e)** follows.

Proof of **f)**: Some similar arguments to those used in the proof of **c)** show that for $i,j,k\in\{1,\dots,3d\}$ and for $n_0>0$ sufficiently large:

$$0\le \sup_{n\ge n_0}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{1}{\sigma_{S,t,n}^2}\frac{\partial^2\sigma_{S,t,n}^2}{\partial\theta_i\partial\theta_j\partial\theta_k}(\theta)\le Y_{i,j,k,S,t}^{(5)}(n_0) \tag{C.43}$$

for some variables $Y_{i,j,k,S,t}^{(5)}(n_0)$ admitting moments at order 4. Differentiating (C.31), we get :

$$\begin{aligned}
&\frac{\partial^3\ell_{S,t,n}(\theta)}{\partial\theta_i\partial\theta_j\partial\theta_k}\\
=\;&\left\{1-\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2(\theta)}\right\}\left\{\frac{1}{\sigma_{S,t,n}^2(\theta)}\frac{\partial^3\sigma_{S,t}^2(\theta)}{\partial\theta_i\partial\theta_j\partial\theta_k}\right\}\\
+\;&\left\{2\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2(\theta)}-1\right\}\left\{\frac{1}{\sigma_{S,t,n}^2(\theta)}\frac{\partial\sigma_{S,t,n}^2(\theta)}{\partial\theta_i}\right\}\left\{\frac{1}{\sigma_{S,t,n}^2}\frac{\partial^2\sigma_{S,t,n}^2(\theta)}{\partial\theta_j\partial\theta_k}\right\}\\
+\;&\left\{2\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2(\theta)}-1\right\}\left\{\frac{1}{\sigma_{S,t,n}^2(\theta)}\frac{\partial\sigma_{S,t,n}^2(\theta)}{\partial\theta_j}\right\}\left\{\frac{1}{\sigma_{S,t,n}^2}\frac{\partial^2\sigma_{S,t,n}^2(\theta)}{\partial\theta_i\partial\theta_k}\right\}\\
+\;&\left\{2\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2(\theta)}-1\right\}\frac{1}{\sigma_{S,t,n}^4(\theta)}\left\{\frac{\partial\sigma_{S,t,n}^2(\theta)}{\partial\theta_k}\frac{\partial^2\sigma_{S,t,n}^2(\theta)}{\partial\theta_i\partial\theta_j}\right\}\\
+\;&\left\{2-6\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2(\theta)}\right\}\frac{1}{\sigma_{S,t,n}^6(\theta)}\left\{\frac{\partial\sigma_{S,t,n}^2(\theta)}{\partial\theta_i}\frac{\partial\sigma_{S,t,n}^2(\theta)}{\partial\theta_j}\frac{\partial\sigma_{S,t,n}^2(\theta)}{\partial\theta_k}\right\}.
\end{aligned}$$

Thus taking the supremum firstly on $\mathcal{V}(\theta_0)\cap\Theta$ then on $n\ge n_0$, for $n_0>0$ sufficiently large

$$\begin{aligned}
&\sup_{n\ge n_0}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\partial^3\ell_{S,t,n}(\theta)}{\partial\theta_i\partial\theta_j\partial\theta_k}\\
\le\;&\left\{6\sup_{n\ge n_0}\sup_{\theta\in\mathcal{V}(\theta_0)\cap\Theta}\frac{\epsilon_{S,t,n}^2}{\sigma_{S,t,n}^2(\theta)}+2\right\}\\
&\{Y_{i,j,k,S,t}^{(5)}(n_0)+Y_{S,t,i}^{(3)}(n_0)Y_{S,t,j}^{(3)}(n_0)Y_{S,t,k}^{(3)}(n_0)+Y_{S,t,i}^{(3)}(n_0)Y_{j,k,S,t}^{(4)}(n_0)\\
&+Y_{S,t,j}^{(3)}(n_0)Y_{i,k,S,t}^{(4)}(n_0)+Y_{S,t,k}^{(3)}(n_0)Y_{i,j,S,t}^{(4)}(n_0)\}.
\end{aligned}\tag{C.44}$$

Now using the Hôlder and Minkowski inequalities, for $n_0 > 0$ sufficiently large

$$\begin{aligned} & E \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 6 \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} + 2 \right\} Y^{(3)}_{S,t,i}(n_0) Y^{(3)}_{S,t,j}(n_0) Y^{(3)}_{S,t,k}(n_0) \\ \leq & \ \{6 \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} \right\|_4 + 2\} \left\| Y^{(3)}_{S,t,i}(n_0) \right\|_4 \left\| Y^{(3)}_{S,t,j}(n_0) \right\|_4 \left\| Y^{(3)}_{S,t,k}(n_0) \right\|_4 < \infty. \end{aligned}$$

The result follows from (C.36) together with (C.34). Similarly combining (C.36), (C.34) and (C.35) leads for $n_0 > 0$ sufficiently large to

$$\begin{aligned} & E \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 6 \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} + 2 \right\} Y^{(3)}_{S,t,i}(n_0) Y^{(4)}_{j,k,S,t}(n_0) \\ \leq & \ \{6 \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} \right\|_2 + 2\} \left\| Y^{(3)}_{S,t,i}(n_0) \right\|_4 \left\| Y^{(4)}_{j,k,S,t}(n_0) \right\|_4 < \infty. \end{aligned}$$

Combining (C.36) and (C.43) leads for $n_0 > 0$ sufficiently large to

$$\begin{aligned} & E \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \left\{ 6 \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} + 2 \right\} Y^{(5)}_{i,j,k,S,t}(n_0) \\ \leq & \ \{6 \left\| \sup_{n \geq n_0} \sup_{\theta \in \mathcal{V}(\theta_0) \cap \Theta} \frac{\epsilon^2_{S,t,n}}{\sigma^2_{S,t,n}(\theta)} \right\|_2 + 2\} \left\| Y^{(5)}_{i,j,k,S,t}(n_0) \right\|_2 < \infty. \end{aligned}$$

Thus the right hand-side term of (C.44) possesses a moment of order 1 for $n_0 > 0$ sufficiently large, which leads to the proof of **f)**.

C.3 Asymptotic distribution of $\hat{\theta}_n$

This part of the proof relies on the identical arguments existing in the proof of Theorem 3 in Francq and Zakoïan (2009) once provided (C.17) and (C.18).

□

D Proofs of Lemmas 7.1,7.2 and 7.3

In this section, we present the proofs of Lemmas 7.1,7.2 and 7.3. The assumptions **A0'** and **A1** imply that Θ is a compact included in $]0, \infty[^d \times [0, \infty[^{2d}$. Letting for $\gamma > 0$, $P_\gamma = [\gamma, \infty[^d \times [0, \infty[^{2d}$, under **A0'** and **A1**, Θ is thus included in P_γ for some $\gamma > 0$. Letting

$$\Theta_\gamma(I, b) = \{\theta \in [0, \infty[^{3d}, \omega(e_i) \geq \gamma \quad \forall i \notin I, \quad \alpha(e_i) \geq \gamma \quad \forall i \in I, \quad \beta(.) \geq a(I)^{1/b}\}.$$

$$\Theta_\gamma(I) = \{\theta \in [0,\infty[^{3d}, \omega(e_i) \geq \gamma \quad \forall i \notin I, \quad \alpha(e_i) \geq \gamma \quad \forall i \in I, \quad \beta(.) \geq \gamma\}.$$

As Θ is a compact and $0 \leq a(.) < 1$, **A0"'** implies

$$\Theta \subseteq \Gamma_\gamma = P_\gamma \cup \{\cup_{I \in \mathcal{E}} \Theta_\gamma(I)\} \tag{D.1}$$

for some $\gamma > 0$ and the assumptions **A9** or **A9'** imply

$$\Theta \subseteq \Gamma_{g,\gamma} = P_\gamma \cup \{\cup_{I \in \mathcal{E}} \Theta_\gamma(I, g)\} \tag{D.2}$$

with $g = 3 + \gamma$ under **A9** or $g = 4 + \gamma$ under **A9'**, for some $\gamma > 0$.

We will need the following Lemma :

Lemma D.1 *Let $\omega(.)$ a function with values from E to $\mathbb{R}^+$, $E_0 = \{e_i \in E \quad \omega(e_i) = 0\}$. and $0 < \rho < 1$ and $\Omega_t := \Omega_t(\rho) = \sum_{i=0}^{\infty} \rho^i \omega(S_{t-i})$. Let us denote for all $t \geq 0$, $\mathcal{E}_t = \{S_j \in E_0, \quad \forall j \in \{1, \dots, t\}\}$. If $(\mathbb{P}(\mathcal{E}_t)^{1/t})$ converges towards a constant $a \in [0, 1)$ such that :*

$$\mathbb{P}(\mathcal{E}_t) \overset{t \to \infty}{=} O(a^t) \tag{D.3}$$

then for all $\rho > 0$,

$$E\left\{\frac{1}{\Omega_t(\rho)}\right\}^u < \infty, \quad E \log^- \Omega_t(\rho) < \infty. \tag{D.4}$$

for some $u > 0$. More over for all $m > 0$, if $\rho > a^{1/m}$ then

$$E\left\{\frac{1}{\Omega_t}\right\}^m < \infty \tag{D.5}$$

.

Proof of Lemma D.1 For any $E_0 \subseteq E$, without loss of generality, prooving the lemma for $\omega(x) = \mathbb{1}_{x \notin E_0}$ allows to proove it to any function $\omega_1(.)$ such that $\{e_i \in E \quad \omega_1(e_i) = 0\} = E_0$. Defining $\omega_{1,*} = \min_{x \in E, x \notin E_0} \omega_1(x) > 0$ and $\omega_1^* = \min_{x \in E, x \notin E_0} \omega_1(x) > 0$, we have then $\omega_1(.) \geq \omega_{1,*}\omega(.)$. Thus, viewing Ω as a function of $\omega(.)$ in lemma D.1, if (D.4) is true for $\omega(.)$, it will be true for $\omega_1(.)$. In the following, we will treat the case $\omega(x) = \mathbb{1}_{x \notin E_0}$. Firstly, noting that $a < 1$:

$$\mathbb{P}(\Omega_t = 0) = \lim_{J \to \infty} \mathbb{P}(S_{t-j} \in E_0, \quad \forall j \in \{1, \dots, J\}) = \lim_{J \to \infty} \mathbb{P}(\mathcal{E}_J) = \lim_{J \to \infty} a^J = 0 \tag{D.6}$$

Secondly, let us define $j \geq 0$. It is obvious that $\mathcal{E}_{j-1} \subseteq \left\{\Omega_1 \leq \frac{\rho^j}{1-\rho}\right\}$. Let us define for all $0 < \rho < 1$, K the biggest $k \in \mathbb{N}$ such that $\frac{\rho^k}{1-\rho} \geq 1$. It can be easily shown that $K := K(\rho) = \left[\frac{\log(1-\rho)}{\log \rho}\right]$. Now, for $j > K$ if $i \le j$ is thus that $\frac{\rho^j}{1-\rho} \ge \rho^i$ then $i \ge j - K$. Thus $\left\{\Omega_1 \le \frac{\rho^j}{1-\rho}\right\} \subseteq \mathcal{E}_{j-K-1}$. We can thus conclude that

$$\mathcal{E}_{j-1} \subseteq \left\{\Omega_1 \le \frac{\rho^j}{1-\rho}\right\} \subseteq \mathcal{E}_{j-K-1}. \tag{D.7}$$

Let us prove the first term of (D.4). By (D.6), remarking that $\Omega_1 \le \frac{1}{1-\rho}$ and $u_n := \frac{\rho^n}{1-\rho}$ decreases to 0 when $n \to \infty$:

$$E\left\{\frac{1}{\Omega_1^u}\right\} = \sum_{n=0}^{\infty} E\left\{\frac{1}{\Omega_1^u}\mathbb{1}_{\frac{\rho^{n+1}}{1-\rho}<\Omega_1\le\frac{\rho^n}{1-\rho}}\right\}. \tag{D.8}$$

In view of (D.7), it can be shown that $\mathbb{P}(\rho^{n+1} < (1-\rho)\Omega_1 \le \rho^n) \le \mathbb{P}(\mathcal{E}_{n-K-1})$. This inequality leads to :

$$\begin{aligned} E\left\{\frac{1}{\Omega_1^u}\mathbb{1}_{\frac{\rho^{n+1}}{1-\rho}<\Omega_1\le\frac{\rho^n}{1-\rho}}\right\} &\le \frac{(1-\rho)^u}{\rho^{u(n+1)}}\mathbb{P}\Big(\frac{\rho^{n+1}}{1-\rho} < \Omega_1 \le \frac{\rho^n}{1-\rho}\Big) \\ &\le \frac{(1-\rho)^u}{\rho^{u(n+1)}}\mathbb{P}(\mathcal{E}_{n-K-1}) \\ &\overset{n\to\infty}{=} O\Big(\frac{a^{n-K-1}}{\rho^{u(n+1)}}\Big) \overset{n\to\infty}{=} O\Big(\left(\frac{a}{\rho^u}\right)^n\Big). \end{aligned}$$

The first equality follows from (D.3). Now, as $\rho > 0$ and $a < 1$, it is possible to choose $u > 0$ such that $a < \rho^u$. $\frac{a}{\rho^u} < 1$ implies that the sum in the right hand side term of (D.8) converges to a finite limit for this u. The first part of (D.4) straightforwardly follows from the stationarity of the process $(\Omega(S_t))$. When $\rho > a^{1/m}$ by choosing $u = m$ as $\frac{a}{\rho^u} < 1$, the sum in the right hand side term of (D.8) converges to a finite limit for this u. The stationarity of the process $(\Omega(S_t))$ leads to the proof of (D.5).

We turn now to the second part of (D.4). Let us choose $\rho \in [0, 1]$ and define $N \ge 0$ such that for all $i \ge N$, $\frac{\rho^i}{1-\rho} \le 1$. We get similarly to (D.8) that :

$$E\log^-\Omega_1 = \sum_{n=N}^{\infty} E\left\{-\log\Omega_1\,\mathbb{1}_{\frac{\rho^{n+1}}{1-\rho}<\Omega_1\le\frac{\rho^n}{1-\rho}}\right\}.$$

We can deduce :

$$\begin{aligned} E\log^-\Omega_1 &\le \sum_{n=N}^{\infty}\left|\log\frac{\rho^{n+1}}{1-\rho}\right|\mathbb{P}\left(\frac{\rho^{n+1}}{1-\rho} < \Omega_1 \le \frac{\rho^n}{1-\rho}\right) \\ &\le \sum_{n=N}^{\infty}((n+1)|\log\rho| + |\log(1-\rho)|)\mathbb{P}\left(\frac{\rho^{n+1}}{1-\rho} < \Omega_1 \le \frac{\rho^n}{1-\rho}\right) < \infty \end{aligned} \tag{D.9}$$

The sum in the last inequality converges because $\mathbb{P}\left(\frac{\rho^{n+1}}{1-\rho} < \Omega_1 \le \frac{\rho^n}{1-\rho}\right) = O(a^n)$, when $n \to \infty$ with $a < 1$ and we get the second term of (D.4)

D.1 Proof of Lemma 7.1

Now we turn to the proof of Lemma 7.3. We will use using inclusion inequality (D.1) because this Lemma is stated under the assumption **A0"'**. On P_γ, $\omega(.) \geq \gamma$, and thus (7.2) is true when replacing Θ by P_γ, because the three functions are respectively bounded by $1/\gamma^u$, $\log^- \gamma$ and $\log \gamma$ for all $u > 0$.

Let us choose $I \in \mathcal{E}$ and $\theta \in \Theta_\gamma(I)$. Denoting $E_0 = \{e_i \in E, \quad i \in I\}$, let us define: $\Omega_{S,I,t}(\rho) = \sum_{i=0}^{\infty} \rho^i \mathbb{1}_{S_t \notin E_0}$. It is clear that

$$\Omega^*_{S,t}(\theta) \geq \min_{x \notin E_0} \omega(x) \Omega_{S,I,t}(\beta_*(\theta)), \tag{D.10}$$

and thus for all $u > 0$:

$$\frac{1}{\Omega^{*u}_{S,t}(\theta)} \leq \frac{1}{\min_{x \notin E_0} \omega(x)^u} \frac{1}{\Omega^u_{S,0,t}(\beta_*(\theta))}.$$

Using this inequality, noting that for all $\theta \in \Theta_\gamma(I)$,

$$\min_{x \notin E_0} \omega(x)^u \geq \gamma^u \quad \text{and} \quad \beta_*(\theta) \geq \gamma \tag{D.11}$$

and that $\Omega_{S,0,t}$ is an increasing function of ρ, we obtain for all $u > 0$:

$$\sup_{\theta \in \Theta_\gamma(I)} \Omega^{*-u}_{S,t}(\theta) \leq \gamma^{-u} \Omega^{-u}_{S,I,t}(\gamma). \tag{D.12}$$

$\Omega_{S,I,t}(\gamma)$ satisfies all the assumptions needed to apply Lemma D.1 (here, with notations of Lemma D.1, $a = a(I) < 1$ by assumption **A0"** and $\rho = \gamma > 0$). Thus, for all $I \in \mathcal{E}$ there exists $u(I) > 0$ such that $E\Omega^{-u(I)}_{S,I,t}(\gamma) < \infty$. Choosing $u = \min_{I \in \mathcal{E}} u(I)$, by the Hölder inequality $E\Omega^{-u}_{S,I,t}(\gamma) < \infty$ for all $I \in \mathcal{E}$. Taking the expectation on (D.12), we get thus that for this $u > 0$ and for all $I \in \mathcal{E}$,

$$E \sup_{\theta \in \Theta_\gamma(I)} \frac{1}{\Omega^{*u}_{S,t}(\theta)} < \infty.$$

Combining this result with $E \sup_{\theta \in P_\gamma} \frac{1}{\Omega^{*u}_{S,t}(\theta)} < \infty$ and using the elementary inequality $\max(x, y) \leq x + y$ for $x, y \geq 0$,

$$\begin{aligned} E \sup_{\theta \in \Theta} \frac{1}{\Omega^{*u}_{S,t}(\theta)} &= E \max(\max(\sup_{\theta \in \Theta_\gamma(I)} \frac{1}{\Omega^{*u}_{S,t}(\theta)}, I \in \mathcal{E}), \sup_{\theta \in P_\gamma} \frac{1}{\Omega^{*u}_{S,t}(\theta)}) \\ &\leq E \left\{ \sum_{I \in \mathcal{E}} \sup_{\theta \in \Theta_\gamma(I)} \frac{1}{\Omega^{*u}_{S,t}(\theta)} + \sup_{\theta \in P_\gamma} \frac{1}{\Omega^{*u}_{S,t}(\theta)} \right\} < \infty \end{aligned} \tag{D.13}$$

and we get the first term of (7.2). In view of (D.10) and (D.11), remarking that for any $\theta \in \Theta_\gamma(I)$, $\Omega^*_{S,t}(\theta) \geq \gamma\Omega_{S,0,t}(\gamma)$ (as $\Omega_{S,0,t}$ is an increasing function of ρ), and that $\log^-(.)$ is a decreasing function:

$$\log^- \Omega^*_{S,t}(\theta) \leq \log^-(\gamma\Omega_{S,0,t}(\gamma)) \quad \text{for} \quad \theta \in \Theta_\gamma(I). \tag{D.14}$$

Proceeding similarly to (D.13):

$$\begin{aligned}
\sup_{\theta\in\Theta} \log^- \Omega^*_{S,t}(\theta) &= \max(\max(\sup_{\theta\in\Theta_\gamma(I)} \log^- \Omega^*_{S,t}(\theta), I \in \mathcal{E}), \sup_{\theta\in P_\gamma} \log^- \Omega^*_{S,t}(\theta)) \\
&\leq \left\{\sum_{I\in\mathcal{E}} \sup_{\theta\in\Theta_\gamma(I)} \log^- \Omega^*_{S,t}(\theta) + \sup_{\theta\in P_\gamma} \log^- \Omega^*_{S,t}(\theta)\right\} \\
&\leq \left\{\sum_{I\in\mathcal{E}} \log^-(\gamma\Omega_{S,I,t}(\gamma)) + \log^- \gamma\right\} \\
&\leq \sum_{I\in\mathcal{E}} \log^- \gamma + \log^- \gamma + \left\{\sum_{I\in\mathcal{E}} \log^- \Omega_{S,I,t}(\gamma)\right\} < \infty. \qquad \text{(D.15)}
\end{aligned}$$

The second inequality follows from (D.14) together with the obvious inequality $\sup_{\theta\in P_\gamma} \log^- \Omega^*_{S,t}(\theta) < \log^- \gamma$. The third one comes from the elementary inequality $\max(0, a+b) \leq \max(0, a) + \max(0, b)$. Lemma (D.1) applied to $\Omega_{S,0,t}(\gamma)$ leading to $E \log^- \Omega_{S,0,t}(\gamma) < \infty$, taking expectation on (D.15) implies : $E \sup_{\theta\in\Theta} \log^- \Omega^*_{S,t}(\theta) < \infty$. Now

$$\sup_{\theta\in\Theta} E \log^- \Omega^*_{S,t}(\theta) \leq E \sup_{\theta\in\Theta} \log^- \Omega^*_{S,t}(\theta) < \infty$$

and we get the second term of (7.2). Noting that $\theta^* = (\gamma, 0, 0, \gamma, 0, 0, \gamma, \ldots) \in \Theta$ and remarking that $\log \geq -\log^-$ we get

$$\begin{aligned}
&\inf_{\theta\in\Theta} -\log^- \Omega^*_{S,t}(\theta) \leq \inf_{\theta\in\Theta} \log \Omega^*_{S,t}(\theta) \leq \log \Omega^*_{S,t}(\theta^*) \\
&-\sup_{\theta\in\Theta} \log^- \Omega^*_{S,t}(\theta) \leq \inf_{\theta\in\Theta} \log \Omega^*_{S,t}(\theta) \leq \log \gamma. \qquad \text{(D.16)}
\end{aligned}$$

The second inequality comes from $\log \Omega^*_{S,t}(\theta^*) = \log \gamma$ and the obvious definition of the supremum. Taking expectation on (D.16) using $E \sup_{\theta\in\Theta} \log^- \Omega^*_{S,t}(\theta) < \infty$ leads to the third term of (7.2). The lemma is thus proved.

D.2 Proof of Lemma 7.3

Now we turn to the proof of Lemma 7.3. We will indifferently treat the Lemma under the assumption **A9** or **A9'** using the inclusion inequality (D.2). Letting $\xi < \gamma$ and coefficients b and g chosen according to their definition in (D.2) and Lemma 7.3. It is clear that $b < g$. On P_γ, $\omega(.) \geq \gamma$, and thus (7.4) is true when replacing Θ by P_γ, because the three terms are respectively bounded by $1/\gamma^b$, $\log^- \gamma$ and $\log \gamma$. Let us choose $I \in \mathcal{E}$, and $\theta \in \Theta_\gamma(I, g)$. Denoting $E_0 = \{e_i \in E, \quad i \in I\}$, let us define: $\Omega_{S,I,t}(\rho) = \sum_{i=0}^{\infty} \rho^i \, \mathbb{1}_{S_t \notin E_0}$. First, using the elementary inequality :

$$\frac{1}{\Omega_{S,t}^{*b}(\theta)} \leq \frac{1}{\min_{x \notin E_0} \omega(x)^b} \frac{1}{\Omega_{S,I,t}^b(\beta_*(\theta))}$$

and second, noting that for $\theta \in \Theta_\gamma(I, g)$,

$$\min_{x \notin E_0} \omega(x)^b \geq \gamma^b \quad \text{and} \quad \beta_*(\theta) \geq a(I)^{1/g} > a(I)^{1/b},$$

and third noting that $\Omega_{S,0,t}$ is an increasing function of ρ, we obtain :

$$E \sup_{\theta \in \Theta_\gamma(I,g)} \frac{1}{\Omega_{S,t}^{*b}(\theta)} \leq \frac{1}{\gamma^b} E \frac{1}{\Omega_{S,I,t}^b(a(I)^{1/g})} < \infty.$$

The result comes from application of Lemma D.1 to $\Omega_{S,I,t}(a(I)^{1/g})$ with $m = b$ (because $a(I)^{1/g} > a(I)^{1/b}$ in view of $a(I) < 1$ and $b < g$). Combining this result with

$$E \sup_{\theta \in P_\gamma} \frac{1}{\Omega_{S,t}^{*b}(\theta)} < \infty$$

and using the elementary inequality $\max(x, y) \leq x + y$ for $x, y \geq 0$,

$$\begin{aligned} E \sup_{\theta \in \Theta} \frac{1}{\Omega_{S,t}^{*b}(\theta)} &= E \max(\max(\sup_{\theta \in \Theta_\gamma(I,g)} \frac{1}{\Omega_{S,t}^{*b}(\theta)}, I \in \mathcal{E}), \sup_{\theta \in P_\gamma} \frac{1}{\Omega_{S,t}^{*b}(\theta)}) \\ &\leq E \left\{ \sum_{I \in \mathcal{E}} \sup_{\theta \in \Theta_\gamma(I,g)} \frac{1}{\Omega_{S,t}^{*b}(\theta)} + \sup_{\theta \in P_\gamma} \frac{1}{\Omega_{S,t}^{*b}(\theta)} \right\} < \infty \end{aligned}$$

and we get the first term of (7.4).

□

D.3 Proof of Lemma 7.2

We will choose here $r = s$ with s defined in **A8** and according to Lemma 7.1 (these coefficients can be chosen equals without loss of generality). Assumption **A0"** allows to use (D.1). Remarking that first $\sigma_{S,t}^2(\theta) \geq \Omega_{S,t}^*(\theta)$, applying Lemma 7.1 to $\Omega_{S,t}^*$, we get easily

$$\epsilon_{S,t}^{-2r} = \sigma_{S,t}^{-2r}(\theta_0) \eta_t^{-2r} \leq \Omega_{S,t}^{*-r}(\theta_0) \eta_t^{-2r}$$

Using independence between (η_t) and (S_t) in view of **A0**, using **A8** and applying Lemma 7.1 to $\Omega^{*-r}_{S,t}$ we obtain:

$$E\frac{1}{\epsilon^{2r}_{S,t}} \leq E\frac{1}{\Omega^{*r}_{S,t}(\theta_0)}E\frac{1}{\eta^{2r}_t} < \infty. \tag{D.17}$$

Assumption **A0"'** leads to:

$$\sup_{\theta\in\Theta}\frac{1}{c^r_{S,t}} \leq \max\left(\sup_{\theta\in P_\gamma}\frac{1}{(\omega(S_t)+\alpha(S_t)\epsilon^2_{S,t})^r}, \max_{I\in\mathcal{E}}\sup_{\theta\in\Theta_\gamma(I)}\frac{1}{(\omega(S_t)+\alpha(S_t)\epsilon^2_{S,t})^r}\right)$$

$$\leq \sup_{\theta\in P_\gamma}\frac{1}{\omega(S_t)^r}+\sum_{I\in\mathcal{E}}\sup_{\theta\in\Theta_\gamma(I)}\frac{1}{\alpha^r(S_t)\epsilon^{2r}_{S,t}} \leq \frac{1}{\gamma^r}+\sum_{I\in\mathcal{E}}\frac{1}{\gamma^r\epsilon^{2r}_{S,t}} \leq \frac{1}{\gamma}(1+K\frac{1}{\epsilon^{2r}_{S,t}}).$$

The second inequality follows from the elementary inequality $\max(x,y) \leq |x|+|y|$, the third one from $\omega(.) \geq \gamma$ on P_γ together with $\alpha(.) \geq \gamma$ on $\Theta_\gamma(I)$, in view of (D.1). Taking the expectation on this last inequality, we retrieve the first term of (7.3), in view of (D.17). Following same arguments we can show that

$$\sup_{\theta\in\Theta}\frac{1}{d^r_{S,t}} \leq \frac{1}{\gamma}(1+K\frac{1}{\Omega^{*r}_{S,t}(\theta_0)\eta^{2r}_t})$$

and taking expectation in this inequality leads to the proof of the second term of (7.3), using (D.17). The lemma is thus proved.

E Proof of Lemma 2.1

We will use the following lemma, which is proved in Berkes, Horváth and Kokoszka (2002).

Lemma E.1 *Let X be a positive random variable such that $E\log X < 0$ and $EX^u < \infty$, for some $u > 0$. Then there exists a number $r > 0$ such that $EX^r < 1$.*

For any $j \in 1,\dots,d$, $E(\alpha_0(e_j)\eta^2_t+\beta_0(e_j)) = \alpha_0(e_j)+\beta_0(e_j) < \infty$ This result and condition (2.8), using Lemma E.1, with $X = \alpha_0(e_i)\eta^2_t+\beta_0(e_j)$ allows to define r_i such that$E(\alpha_0(e_i)\eta^2_t+\beta_0(e_j))^{r_i} < 1$ let us define $r = \min_{i\in\{1,\dots,d\}} r_i$ Using the Hôlder inequality, for any $i \in 1,\dots,d$:

$$E(\alpha_0(e_i)\eta^2_t+\beta_0(e_i))^r \leq \{E(\alpha_0(e_i)\eta^2_t+\beta_0(e_i))^{r_i}\}^{r/r_i} < 1.$$

and it can be shown that r can be choose in $(0,1)$. Now defining

$$X_i = E\left\{a^r_0(S_t,\eta_{t-1})\dots a^r_0(S_{t-i},\eta_{t-i-1})\right\},$$

$$\rho = \max_{i\in\{1,\dots,d\}} E(\alpha_0(e_i)\eta^2_t+\beta_0(e_i))^r,$$

and the filtration $\mathbb{F}_{t_1,t_2} = \sigma(S_{t_1}, S_{t_1-1}, \ldots, \eta_{t_2}^2, \eta_{t_2-1}^2, \ldots)$, then $\rho < 1$ and :

$$\begin{aligned}
E(a_0^r(S_t, \eta_{t-1})|\mathbb{F}_{t-1,t-2}) &= \sum_{j=1}^{d} E(a_0^r(e_j, \eta_{t-1}) \mathbb{1}_{S_t=e_j} |\mathbb{F}_{t-1,t-2}) \\
&= \sum_{j=1}^{d} E(a_0^r(e_j, \eta_{t-1}))P(S_t = e_j|\mathbb{F}_{t-1,t-2}) \\
&\leq \max_{i\in\{1,\ldots,d\}} (E(a_0^r(e_j, \eta_{t-1}))) \leq \rho. \qquad \text{(E.1)}
\end{aligned}$$

The first equality follows from the assumption **A0** that entails the independence between η_{t-1} and $\sigma(S_t, \mathbb{F}_{t-1,t-2})$ and the first inequality follows directly from $\sum_j P(S_t = e_j|\mathbb{F}_{t-1,t-2}) = 1$. Now :

$$\begin{aligned}
X_{i+1} &= E(E(a_0^r(S_t, \eta_{t-1})|\mathbb{F}_{t-1,t-2}) \{a_0^r(S_{t-1}, \eta_{t-2}) \ldots a_0^r(S_{t-i-1}, \eta_{t-i-2})\}) \\
&\leq \rho E \{a_0^r(S_{t-1}, \eta_{t-2}) \ldots a_0^r(S_{t-i-1}, \eta_{t-i-2})\} \\
&\leq \rho E \{a_0^r(S_t, \eta_{t-1}) \ldots a_0^r(S_{t-i}, \eta_{t-i-1})\} \leq \rho X_i \leq X_1 \rho^{i+1}
\end{aligned}$$

The first inequality follows from the inequality (E.1), the second one from the stationarity of $a_0^r(S_t, \eta_{t-1}) \ldots a_0^r(S_{t-i}, \eta_{t-i-1})$ and the last one from a trivial recurrence computation. The lemma is thus proved recalling that $\rho < 1$.

□

F Proof of Lemmas 7.4 and C.1

In view of (2.6), elementary computations show for $i \geq 0$,

$$\begin{aligned}
\sigma_{S,t}^2 &= c_{S,t} + \sum_{i=0}^{\infty} \beta(S_t) \ldots \beta(S_{t-i}) c_{S,t-i-1} \\
= c_{S,t} &+ \sum_{j=0}^{i-1} \prod_{0\leq h\leq j} \beta(S_{t-h}) c_{S,t-j-1} + \prod_{0\leq h\leq i} \beta(S_{t-h}) \Omega_{S,t-i-1} \\
&+ \sum_{j=i}^{\infty} \prod_{0\leq h\leq j} \beta(S_{t-h}) \alpha(S_t) \epsilon_{S,t-j-1}^2 \qquad \text{(F.1)} \\
= c_{S,t} &+ \sum_{j=0}^{i} \prod_{0\leq h\leq j} \beta(S_{t-h}) c_{S,t-j-1} + \prod_{0\leq h\leq i+1} \beta(S_{t-h}) \sigma_{S,t-i-2}^2 \qquad \text{(F.2)} \\
= c_{S,t} &+ \sum_{0\leq j, j\neq i} \prod_{0\leq h\leq j} \beta(S_{t-h}) c_{S,t-j-1} + \prod_{0\leq h\leq i} \beta(S_{t-h}) c_{S,t-i-1} \qquad \text{(F.3)}
\end{aligned}$$

The positivity of θ, $\epsilon^2_{S,t}$ and $c_{S,t}$ together with (F.1) (respectively (F.3) and (F.2)) allow to get easily the first (respectively second and third) inequality of (C.3). In view of the definitions of $({}_i\sigma^2_{S,t})$, $({}^i\sigma^2_{S,t})$ and $(\sigma^2_{i,S,t})$ and (C.3) we get (7.5). Let us denoting $R_{i,S,t,n}$ (respectively ${}_i\mathrm{R}_{S,t,n}$ ${}^i\mathrm{R}_{S,t,n}$) the value of $R_{i,S,t}$ (respectively ${}_i\mathrm{R}_{S,t}$ and ${}^i\mathrm{R}_{S,t}$) after replacing in these functions $\epsilon_{S,t}$ by $\epsilon_{S,t,n}$. We get in view of (F.1), (F.3) and (F.2), for all $\theta \in \Theta$,

$$\sigma^2_{S,t,n}(\theta) = R_{i,S,t,n}(\theta) + \prod_{0\le h\le i} \beta(S_{t-h})\Omega_{S,t-i-1}(\theta),$$
$$\sigma^2_{S,t,n}(\theta) =_i \mathrm{R}_{S,t,n}(\theta) + \prod_{0\le h\le i} \beta(S_{t-h})c_{S,t-i-1,n}(\theta),$$
$$\sigma^2_{S,t,n}(\theta) =^i \mathrm{R}_{S,t,n}(\theta) + \prod_{0\le h\le i+1} \beta(S_{t-h})\sigma^2_{S,t-i-2,n}(\theta).$$

Combining these last equalities with $\Omega(\theta) \ge \Omega(\theta^*)$ (obvious because Ω is an increasing fuction of θ), ${}_i\mathrm{R}_{S,t,n}(\theta) \ge_i \mathrm{R}_{S,t}(\theta^*)$, $R_{i,S,t,n}(\theta) \ge R_{i,S,t}(\theta^*)$ and ${}^i\mathrm{R}_{S,t,n}(\theta) \ge^i \mathrm{R}_{S,t,n}(\theta^*)$ for $\theta \ge \theta^*$ lead to (C.2). The Lemmas are thus proved.

□

G An application to natural gas prices

Regnard and Zakoïan (2011) applied model (2.1) on the Zeebrugge day ahead natural gas spot returns with regimes taken as functions of United Kingdom mean temperature. We refer to their paper for justifications of this approach. They estimated some models with different specifications of the regimes and compare their out of sample forecasting performance and likelihood values to get the best of them. They show significant effect of including these regimes, with respect to a standard GARCH(1,1) model. The best model of their paper has some coefficients that seem to be unsignificant. Denoting by ϵ_t the returns of natural gas spot prices and by T_t the temperature, their best model allows for 7 regimes, but some of them possess regimes with $\alpha_0(e_i)+\beta_0(e_i)$ too close to 0, which is in contradiction with assumption **A0"'**. Despite the forecasting performance, our framework does not allow us to test for the nullity of parameter of this model. To circumvent this problem, we merge some of them and estimate the following model.

$$\sigma_t^2 = \begin{cases} \underset{(0.0009)}{0.0016} + \underset{(0.14)}{0.23}\ \epsilon_{t-1}^2 + \underset{(0.19)}{0.76}\ \sigma_{t-1}^2 & \text{when } T_t \leq 5, \\ \underset{(0.0002)}{0.0005} + \underset{(0.03)}{0.0001}\ \epsilon_{t-1}^2 + \underset{(0.09)}{0.69}\ \sigma_{t-1}^2 & \text{when } 5 < T_t \leq 9, \\ \underset{(0.0002)}{0.0004} + \underset{(0.07)}{0.27}\ \epsilon_{t-1}^2 + \underset{(0.08)}{0.68}\ \sigma_{t-1}^2 & \text{when } 9 < T_t \leq 15.8, \\ \underset{(0.0003)}{0.0004} + \underset{(0.07)}{0.12}\ \epsilon_{t-1}^2 + \underset{(0.15)}{0.73}\ \sigma_{t-1}^2 & \text{when } 15.8 < T_t. \end{cases} \tag{G.1}$$

All coefficients of this model are significant, except $\alpha_0(e_2)$. Table 2 provide result of Wald, QLR and Rao Score tests of the null hypothesis $H_0 : \alpha_0(e_2) = 0$. These three tests accept H_0.

Model (G.1)	Wald test	Rao Score Test	QLR test
P values of test of $H_0 : \alpha_0(e_2) = 0$	> 0.05	> 0.05	> 0.05

Table 2: Tests of nullity for Model (G.1)

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