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A note on super-hedging for investor-producers

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Abstract

We study the situation of an investor-producer who can trade on a financial market in continuous time and can transform some assets into others by means of a discrete time production system, in order to price and hedge derivatives on produced goods. This general framework covers the interesting case of an electricity producer who wants to hedge a financial position and can trade commodities which are also inputs for his system. This extends the framework of [1] to continuous time for concave and bounded production functions. We introduce the flexible concept of *conditional sure profit* along the idea of the *no sure profit* condition of Rasonyi [15] and show that it allows one to provide a closedness property for the set of super-hedgeable claims in a very general setting. Using standard separation arguments, we then deduce a dual characterization of the latter.

Key words : markets with proportional transaction costs, non-linear returns, no-arbitrage, super replication theorem, electricity markets, energy derivatives.

1 Introduction

The recent deregulation of electricity markets in many countries has opened a new range of applications for financial techniques in order to hedge energy risks. However, the non-storability of electricity forbids any possible trading strategy based on the spot price and the standard mathematical toolbox cannot be exploited to hedge and price derivative products upon this asset. The challenge must still be taken up for electricity producers who are endowed with such claims or for financial agents endowed with a power plant.

This is typically the situation of an agent who can trade on a market (especially cash and raw material such as fuel) but also who has the possibility to transform some assets into others by the mean of a production system, e.g., to produce electricity out of fuel and sell it on the market. In the framework of purely financial portfolios, the Arbitrage Pricing Theory ensures by an economical assumption, the no-arbitrage condition, properties on the set of attainable terminal wealth for self financing portfolios.

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This often allows one to find a linear pricing rule and to use the powerful martingale approach to price contingent claims. In our particular framework, if the financial market runs as usual, production is not bound up with any particular economical condition : it is an idiosyncratic action of the agent, so that we might allow him to make sure profits, at least in a reasonable way.
This specific situation has been explored in a discrete time framework for markets with proportional transaction costs in [1]. In the latter, it has been proposed to extend the no-arbitrage of second kind condition of Rasonyi [15] to portfolios augmented by the possibility to produce with a linear production system. A condition for general production functions, the *no marginal arbitrage for high production regime* condition, has then been introduced using the extended condition above in order to allow marginal arbitrages for reasonable levels of production. The fundamental closedness property of the set of attainable terminal positions follows from this condition and allows one to propose super-replication results and to show the existence of portfolio optimization problems.
This present note intends to push forward this study by proposing an alternative condition which has a close economical interpretation. We also focus on investors-producers with specific means of production and financial possibilities corresponding to a very large class of market models. We study production possibilities in discrete time as in [1] but we also assume concavity and boundedness of the production function. In counterpart, the financial possibilities can be modelized by a continuous time market with or without frictions. The main result of this note is that if we forbid arbitrage possibilities on the financial market, and the Fatou-closedness property of the set of terminal positions of financial portfolios as a corollary, then this closedness property can easily be extended to portfolios with production possibilities under an additional economical condition, as long as we keep suitable convexity and admissibility properties. By showing the closedness property of the set of attainable wealth for investors-producers, we can propose a super-replication theorem for contingent claims these typical agents can face.
Section 2 is dedicated to the introduction of that framework. Section 3 presents the economical condition we impose, the *conditional sure profit* condition, and the properties it induces. Finally, we illustrate in section 4 this situation in the context of an electricity provider endowed with a future contract. Proofs are collected in section 5.

General notations: throughout this note, $x \in \mathbb{R}^d$ will be viewed as a column vector with entries x^i, $i \le d$. The transpose of a vector x will be denoted x', so that $x'y$ stands for the scalar product. As usual, $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ stand for the positive and negative orthans of $\mathbb{R}^d$ respectively, i.e., $[0,+\infty)^d$ and $(-\infty,0]^d$. For a given probability space $(\Omega,\mathcal{G},\mathbb{P})$ and a $\mathcal{G}$-measurable random set E, $L^0(E,\mathcal{G})$ will denote the set of $\mathcal{G}$-measurable random variables taking values in E $\mathbb{P}$-almost surely, $L^1(E,\mathcal{G})$ the set of $\mathbb{P}$-integrable random variables takings values in E $\mathbb{P}$-almost surely and $L^\infty(E,\mathcal{G})$ the set of random variables taking values $\mathbb{P}$-almost surely in a bounded $\mathcal{G}$-measurable subset of E. The notation $\text{conv}(E)$ will denote the closed convex set generated by convex combinations of elements of E, and $\text{cone}(E)$ the closed convex cone generated by $\text{conv}(E)$. All the inclusions or inequalities are to be understood in the almost sure sense unless otherwise specified.

2 The framework

We first introduce the financial possibilities of the agent. We do so by considering an abstract setting, mainly inspired by [7], and the technical assumptions which are illustrated by two very distinct examples: the frictionless continuous time case and a discrete time financial market with convex transaction costs. We then introduce the production possibilities.

2.1 The set of financial positions

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a continuous-time filtered stochastic basis on a finite time interval $[0,T]$ satisfying the usual conditions. We assume without loss of generality that $\mathcal{F}_0$ is trivial and $\mathcal{F}_{T^-} = \mathcal{F}_T$. For any $0 \le t \le T$, let $\mathcal{T}_{[t,T]}$ denote the family of stopping times taking values in $[t,T]$ $\mathbb{P}$-almost surely.

On $[0,T]$, the agent has the possibility to trade on a financial market with a finite number d of assets. For $\rho \in \mathcal{T}_{[0,T]}$, we then consider the set of portfolio processes corresponding to self financing strategies between ρ and T, starting from 0 at time ρ, and we denote it $\mathfrak{X}^0_\rho$. The superscript 0 stands for "no production", or "pure financial". Every process $\xi \in \mathfrak{X}^0_\rho$ is a d-dimensional $\mathbb{F}$-adapted process and verifies $\xi_{\rho^-} = 0$. The null process is in $\mathfrak{X}^0_0$. This multidimensional setting allows us to consider portfolios labelled in physical units of assets, which is of great interest when facing transaction costs. We then define the set of terminal positions at time T: $\mathfrak{X}^0_\rho(T) := \{\xi_T \ : \ \xi \in \mathfrak{X}^0_\rho\}$.

In order to compare the financial positions, we need to introduce a partial order on $\mathbb{R}^d$. To this end, we consider a set-valued $\mathbb{F}$-adapted process $\widehat{K}$ such that for almost every $(t,\omega) \in [0,T] \times \Omega$, $\widehat{K}_t(\omega)$ is a proper convex closed cone of $\mathbb{R}^d$ and containing $\mathbb{R}^d_+$. This allows one to define a partial order on $\mathbb{R}^d$: $\xi \succeq_s -\kappa$ for $(\xi,\kappa) \in L^0(\mathbb{R}^{2d}, \mathcal{F}_s)$ if and only if $\xi + \kappa \in L^0(\widehat{K}_s, \mathcal{F}_s)$. This can be extended to stopping times in $\mathcal{T}_{[0,T]}$. In the literature on markets with transaction costs, $\widehat{K}_t$ usually stands for the solvency region at time t, and $-\widehat{K}_t$ for the set of possible trades at time t, see [11] and the reference therein. We also suppose, for any $\rho \in \mathcal{T}_{[0,T]}$, that $\mathfrak{X}^0_\rho(T)$ is a convex subset of $L^0(\mathbb{R}^d, \mathcal{F}_T)$ and

$$\mathfrak{X}^0_\rho(T) - L^\infty(\widehat{K}_s, \mathcal{F}_s) \subseteq \mathfrak{X}^0_\rho(T), \quad \forall s \in [\rho, T] \ \mathbb{P} - \text{a.s.} \tag{2.1}$$

This convexity property holds in most of market models, see sections 2.5 and 4. Relation (2.1) means that whatever the financial position of the agent is, it is always possible for him to throw away a non-negative quantity of assets at any time, or to do an arbitrarily large transfer of assets allowed by the cone $-\widehat{K}_s$. Finally, we assume the following concatenation property, which also holds in most of market models and often reveals their Markovian behaviour:

$$\mathfrak{X}^0_\rho(T) = \left\{\xi_\sigma + \zeta_T \ : \ (\xi,\zeta) \in \mathfrak{X}^0_\rho \times \mathfrak{X}^0_\sigma, \ \text{for any } \sigma \in \mathcal{T}_{[0,T]} \text{ s.t. } \sigma \ge \rho\right\} \ . \tag{2.2}$$

Note that, since the null process is in $\mathfrak{X}^0_0$, relation (2.2) implies that $0 \in \mathfrak{X}^0_\rho(T) \subset \mathfrak{X}^0_0(T)$ for any $\rho \in \mathcal{T}_{[0,T]}$.

2.2 Absence of arbitrage on the financial market

The possibilities of the investor-producer on the financial market shall be the same as every financial agent, i.e., it is not possible to make financial arbitrage. We elaborate this condition below by relying on the core result of Arbitrage Pricing Theory, which resides in the following fact. Formally, when the financial market prices are represented by a process S, the no-arbitrage property for this market holds if and only if there exists a stochastic deflator, i.e., a strictly positive martingale ρ such that the process $Z := \rho S$ is a martingale. The process Z can then be seen as the market price of assets that forbid arbitrage opportunities. In order to express a no-arbitrage condition for the market in the general framework, we will suppose that such a process Z exists. This is undertaken by defining a *dual set* $\mathcal{M}$ and assuming that it contains at least one element. For this purpose, we introduce the set-valued process $\widehat{K}^*$ defined by

$$\widehat{K}^*_t(\omega) := \left\{ y \in \mathbb{R}^d_+ \ : \ xy \geq 0, \ \forall x \in \widehat{K}_t(\omega) \right\} . \tag{2.3}$$

Since $\widehat{K}_t(\omega)$ is proper, $\widehat{K}^*_t(\omega)$ is not reduced to 0, for almost every $(t, \omega) \in [0,T] \times \Omega$.

Definition 2.1. *Let $\mathcal{M}$ be the set of $\mathbb{F}$-adapted martingales Z on $[0,T]$ taking values in $\widehat{K}^*$, with strictly positive components, such that*

$$\sup \left\{ \mathbb{E}\left[Z'_T \xi_T\right] \ : \ \xi \in \mathfrak{X}^0_0 \text{ such that } \xi_\tau \succeq_\tau -\kappa \text{ for all } \tau \in \mathcal{T}_{[0;T]}, \text{ for some } \kappa \in \mathbb{R}^d_+ \right\} < +\infty \ .$$

We will furthermore assume that at least one of these elements exists, i.e., $\mathcal{M} \neq \emptyset$.

This definition needs some comments. Following the common interpretation in the literature, $\mathcal{M}$ shall express the set of linear pricing measures for financial positions and strategies. When the set $\mathfrak{X}^0_0(T)$ is a cone, which is the case of frictionless markets and markets with proportional transaction costs, absence of arbitrage should reasonably implies that the above formula is non positive for $Z \in \mathcal{M}$. Recall that $0 \in \mathfrak{X}^0_0(T)$. By the concatenation property (2.2) and the martingale property of Z, $Z'\xi$ is a supermartingale for any $Z \in \mathcal{M}$ and $\xi \in \mathfrak{X}^0_0$ verifying the above condition in the set. This condition on ξ is a specific definition of admissibility, which is quickly discussed in the next section. The real process $Z'\xi$ being a supermartingale makes us meet a more common definition of martingale dual processes and absence of arbitrage, see subsection 2.5.1 below and Section 4. It appears in the general non conical case, see subsection 2.5.2 below, that the support function in Definition 2.1 might not be null. This justifies that we take it only finite. Finally, defining Z as above is tailor-made for separation arguments, see the proof of Theorem 3.1 in section 5 below.

2.3 Admissibility of portfolios

If $d = 1$, a position ξ_t is naturally solvable if $\xi_t \geq 0$ $\mathbb{P}-$a.s. Considering a general setting with $d \geq 1$, we decide to define that a solvable position at time t is a vector ξ such that $\xi \succeq_t 0$ $\mathbb{P}-$a.s. In markets with proportional transaction costs, it is natural to define the solvency region by the inverse image of $\mathbb{R}^d_+$ by immediate transfers ℓ_τ allowed at time τ, which is precisely $\widehat{K}_\tau$. With a generally convex structure, there is no problem to compare two "static" positions in the same way. However, if one of the positions implies a transfer ξ_τ at precise time τ, the addition of the transfer ℓ_τ getting a non negative position in every asset shall keep $\xi_\tau + \ell_\tau$ in allowed transfers at this time. This question is particular to a continuous time setting with convex constraints, such as liquidity matters. We avoid to focus on this problem by keeping the partial order introduced in the section 2.1 as in the case of proportional transaction costs. It does not affect the further mathematical developments, even if the issue of admissibility in the non conical setting still remains.

The notion of solvency implies the one of admissibility, which is central to the theory. In our framework, we ought to choose the weakest notion among the plethora of definitions in order to deal with the different cases, see [4, 2, 7, 6] and the illustrations below. Here, we use the definition proposed in [2].

Definition 2.2. *For some constant vector $\kappa \in \mathbb{R}^d_+$, a portfolio $\xi \in \mathfrak{X}^0_0$ is said to be* κ-admissible *if $Z'_\tau \xi_\tau \geq -Z'_\tau \kappa$ for all $\tau \in \mathcal{T}_{[0,T]}$ and all $Z \in \mathcal{M}$, and $\xi_T \succeq_T -\kappa$.*

Given $\mathcal{M} \neq \emptyset$, the concept of admissibility allows to consider a wider class of terminal wealth than those considered in the definition of $\mathcal{M}$. Notice that if a portfolio ξ verifies $\xi_\tau \succeq_\tau -\kappa$ for all $\tau \in \mathcal{T}_{[0,T]}$ and some $\kappa \in \mathbb{R}^d_+$, the portfolio ξ is κ-admissible in the above sense according to Definition 2.1. The reciprocal is not always true, and is the object of the so-called **B** assumption investigated in [7]. We define the set of admissible elements of $\mathfrak{X}^0_t$,

$$\mathfrak{X}^0_{t,adm} := \left\{ \xi \in \mathfrak{X}^0_t, \ \xi \text{ is } \kappa\text{-admissible for some } \kappa \in \mathbb{R}^d_+ \right\} ,$$

and $\mathfrak{X}^0_{t,adm}(T)$ the set of terminal values of such elements.

2.4 Closedness property

Admissibility is a central concept in continuous time because the closedness property is often proved for $\mathfrak{X}^0_{0,adm}(T)$. Definition 2.2 allows us to define a prior closedness property for $\mathfrak{X}^0_{0,adm}(T)$ before adding production possibilities. When it comes to prove that $\mathcal{M}$ is not empty from a no-arbitrage condition, it is often fundamental to have this closedness property in order to use a separation argument. In our context, we will convey this under the following technical and standing assumption:

Assumption 2.1. *For $t \in [0,T]$, let $(\xi^n)_{n\geq 1} \subset \mathfrak{X}^0_{t,adm}$ be a sequence of admissible portfolios such that $\xi^n_T \succeq_T -\kappa$ for some $\kappa \in \mathbb{R}^d_+$ and all $n \geq 1$. Then there exists a sequence $(\zeta^n)_{n\geq 1} \subset \mathfrak{X}^0_{t,adm}$ constructed as a convex combination (with strictly positive weights) of $(\xi^n)_{n\geq 1}$, i.e., $\zeta^n \in \mathrm{conv}(\xi^k)_{k\geq n}$, such that ζ^n_T Fatou-converges to $\zeta^\infty_T \in \mathfrak{X}^0_t(T)$ with n.*

Recall that a sequence of random variable is Fatou-convergent if it is bounded by below by some constant with respect to the considered order, and is convergent in the almost sure sense. We can indifferently replace the above convergence by the almost sure convergence. Note that the convexity of the set $\mathfrak{X}^0_0(T)$ propagates to $\mathfrak{X}^0_{0,adm}(T)$, which ensures that the new sequence lies in the set.

A word of motivation is in order. It appears in the Arbitrage Pricing Theory that the key property of Fatou-closedness of $\mathfrak{X}^0_0(T)$ often relies on a convergence lemma, see section 4. In [17], Schachermayer introduced a version of Komlos Lemma that has been of fundamental use in [4] for frictionless markets, see the example of subsection 2.5.1, and in [2] with Campi for markets with strictly positive proportional transaction costs, see section 4. Assumption 2.1 expresses a synthesis of this result. It implies the Fatou-closedness of $\mathfrak{X}^0_{\tau,adm}(T)$ for any $\tau \in \mathcal{T}_{[0,T]}$. It is also a central tool in the proof of the main theorem of this note.

2.5 Illustration of the framework by examples of financial markets

To justify the setting we introduced above, we propose two examples covering very different cases. They are based on [4, 5] and [14] respectively. In section 4, we also apply our results to a continuous time market with càdlàg price processes and proportional transaction costs, as studied in [2].

2.5.1 Example 1: multidimensional frictionless market in continuous time

Consider a filtered stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on $[0,T]$. Let S be a locally bounded $(0,\infty)^d$-valued $\mathbb{F}$-adapted càdlàg semimartingale, representing the price process of d risky assets. We suppose the existence of a non risky asset which is taken constant on $[0,T]$ without loss of generality. Let Θ be the set of $\mathbb{F}$-predictable S-integrable processes, representing the possible financial strategies and Π the set of $\mathbb{F}$-predictable increasing processes on $[0,T]$ which represent possible liquidation or consumption in the portfolio. We define

$$\mathfrak{X}^0_t := \left\{ \xi = (\xi^1, 0, \dots, 0) \ : \ \xi^1_s = \int_t^s \vartheta_u . dS_u - (\ell_s - \ell_{t^-}) \ : \ (\vartheta, \ell) \in \Theta \times \Pi, \ s \in [t,T] \right\} , \quad \forall t \leq T .$$

The starting time t can be changed for arbitrary $\rho \in \mathcal{T}_{[0,T]}$ and $\mathfrak{X}^0_\rho(T)$ is a convex cone of $\mathbb{R} \times \{0\}^{d-1}$ containing 0. It also verifies (2.1) and (2.2).

In this context, Delbaen and Schachermayer introduced the No Free Lunch with Vanishing Risk (NFLVR) condition and proved (Theorem 1.1 in [4]) that it is equivalent to

$$\{\mathbb{Q} \sim \mathbb{P} \text{ such that } S \text{ is a } \mathbb{Q} - \text{local martingale}\} \neq \emptyset \ .$$

To relate the NFLVR condition to Definition 2.1, we define $\mathcal{M}$ as the set of local martingale measure processes $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_\cdot}$ for $\mathbb{Q}$ belonging to the above set. If S is a locally bounded martingale, elements of $\mathfrak{X}^0_0$ are local martingales. We now apply Definition 2.2 of admissibility. We take without ambiguity $\widehat{K} = \mathbb{R}_+ \times \{0\}^{d-1}$, implying that $\widehat{K}^* = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ and that the first component of $Z \in \mathcal{M}$ is strictly positive at any time. As a consequence, a portfolio $\xi \in \mathfrak{X}^0_{0,adm}$ is κ-admissible only if $\xi^1_t \geq -\kappa$ for all $t \in [0,T]$, and we retrieve the definition of admissibility introduced by Delbaen and Schachermayer in [4]. Therefore, any admissible portfolio is a true supermartingale under $\mathbb{Q}$ defined as above.

The same authors proved that the NFLVR condition implies that $\mathfrak{X}^0_{0,adm}(T)$ is Fatou closed (Theorem 4.2 in [4]) by using the following convergence property: they proved that for any 1-admissible sequence $\xi^n \in \mathfrak{X}^0_0$, it is possible to find $\zeta^n \in \text{conv}(\xi^k)_{k \geq n}$ such that ζ^n converges in the semimartingale topology (Lemma 4.10 and 4.11 in [4]), and thus ζ^n_T Fatou converges in $\mathfrak{X}^0_0(T)$. This can be easily extended to $\mathfrak{X}^0_\tau(T)$ for any $\tau \in \mathcal{T}_{[0,T]}$ and for any bound of admissibility. In our case, $\xi^{1,n}_T \geq -\kappa$ for some $\kappa \geq 0$ and all $n \geq 1$ implies that $\xi^{1,n}_\tau \geq -\kappa$ for all $\tau \in \mathcal{T}_{[0,T]}$ because ξ^1 is a $\mathbb{Q}$-supermartingale. We then retrieve the uniform admissibility of [4] and Assumption 2.1.

2.5.2 Example 2 : physical market with convex transaction costs in discrete time

Let $(t_i)_{0 \leq i \leq N} \subset [0,T]$ be an increasing sequence of deterministic times with $t_N = T$. Let us consider the discrete filtration $\mathbb{G} := (\mathcal{F}_{t_i})_{0 \leq i \leq N}$. Here, the market is modelized by a $\mathbb{G}$-adapted sequence $C = (C_{t_i})_{0 \leq i \leq N}$ of closed-valued mappings $C_{t_i} : \Omega \mapsto \mathbb{R}^d$ with $\mathbb{R}^d_- \subset C_{t_i}(\omega)$ and $C_{t_i}(\omega)$ convex for every $0 \leq i \leq N$ and $\omega \in \Omega$. We define the recession cones $C^\infty_t(\omega) = \bigcap_{\alpha > 0} \alpha C_t(\omega)$ and their positive polar cones $C^{\infty,*}_t(\omega) = \left\{ y \in \mathbb{R}^d \ : \ xy \geq 0, \ \forall x \in C^\infty_t(\omega) \right\}$, see also [14] for a freestanding definition

This setting has been introduced in [14] to model markets with convex transaction costs, such as currency markets with illiquidity costs, in discrete time. Every financial position is labelled in physical units of the d assets, and the sets C_{t_i} denote the possible self financing changes of position at time t_i, so that

$$\mathfrak{X}^0_{t_i}(T) := \left\{ \sum_{k=i}^{N} \xi_{t_k} \ : \ \xi_{t_k} \in L^0(C_{t_k}, \mathcal{F}_{t_k}), \ \forall i \le k \le N \right\} \text{ for all } 0 \le i \le N \ .$$

In this context, the convexity assumption, equations (2.1) and (2.2) trivially hold.

If $C_{t_i}(\omega)$ is a cone in $\mathbb{R}^d$ for all $0 \le i \le N$ and $\omega \in \Omega$, i.e., $C = C^\infty$, we retrieve a market with proportional transaction costs as described in [10]. In the latter, Kabanov and al. show that the Fundamental Theorem of Asset Pricing can be expressed with respect to the *robust no-arbitrage* property, see [10] for a definition. This condition is equivalent to the existence of a martingale process Z such that $Z_{t_i} \in L^\infty(\mathrm{ri}(C^{\infty,*}_{t_i}), \mathcal{F}_{t_i})$. It is the super replication theorem, see Lemma 3.3.2 in [11], which allows $\mathcal{M}$ given by Definition 2.1 to be characterized by such elements Z. In that case, the reader can see that C^∞ replaces our conventional cone process $\widehat{K}$.

As mentioned in [14], the case of general convex transaction costs leads to two possible definitions of arbitrage, and one of them is based on the recession cone. Following the terminology of [14], the market represented by C satisfies the *robust no-scalable arbitrage* property if C^∞ satisfies the robust no-arbitrage property. This definition implies that arbitrages might exist, but they are limited for elements of $\mathfrak{X}^0_0(T)$ and even not possible for the recession cone. Pennanen and Penner [14] proved that the set $\mathfrak{X}^0_0(T)$ is closed in probability under this condition, which is stronger than the Fatou closedness property. The convergence result used in this context is a different argument than the one of Assumption 2.1. However, the latter can be applied, see [1] in which Assumption 2.1 has been applied in a very similar context. The notion of admissibility can be avoided in the discrete time case.

2.6 Addition of production possibilities

The previous introduction of a financial market comes from the possibility to interpret the available assets on the market as raw material or saleable goods for a producer. Therefore, it seems coherent to modelize the production as a function transforming a consumption of the d assets in a new position in $\mathbb{R}^d$. Another observation that leads to our upcoming setting is that a general production process is subject to physical constraints. It seems realistic not only to consider a discrete time framework as a production calendar, but also to introduce a delay in the production process, as a natural constraint and a generalization. The discrete time framework is common in production optimization, and especially well-suited for generation management, see [13] for a monograph illustrating that concern. The electricity spot price is indeed quoted with an hourly frequency in all deregulated markets, and production decisions are given in this framework. For these reasons, we introduce a deterministic collection of increasing distinct times $(t_i)_{0 \le i \le N} \subset [0, T]$ and the following set:

$$\mathcal{B} := \left\{ (\beta_{t_i})_{0 \le i < N} \ : \ \beta_{t_i} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_i}), \ 0 \le i < N \right\} \ .$$

This set corresponds to the set of controls of the production. As explained above, a control takes values in $\mathbb{R}^d_+$ to represent the non negative consumption of each type of asset put into the production system. We now represent the production as follows. Let $(R_{t_i})_{0 < i \le N}$ be a collection of maps such that, for $0 < i \le N$, R_{t_i} is a $\mathcal{F}_{t_i}$-measurable map from $\mathbb{R}^d_+$ to $\mathbb{R}^d$, in the sense that $R_{t_i}(\beta_{t_{i-1}}) \in L^0(\mathbb{R}^d, \mathcal{F}_{t_i})$ for $\beta_{t_{i-1}} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{i-1}})$. Moreover we will place ourselves in a specific case by assuming the following.

Assumption 2.2. *The production function has the three following properties:*

(i) Concavity*: for all* $0 < i \leq N$*, for all* $(\beta^1, \beta^2) \in L^0(\mathbb{R}^{2d}_+, \mathcal{F}_{t_{i-1}})$ *and* $\lambda \in L^0([0,1], \mathcal{F}_{t_{i-1}})$*,*

$$\lambda R_{t_i}(\beta^1) + (1-\lambda) R_{t_i}(\beta^2) \preceq_{t_i} R_{t_i}(\lambda\beta^1 + (1-\lambda)\beta^2) \ .$$

(ii) Boundedness*: there exists* $\mathfrak{K} \in \mathbb{R}^d_+$ *such that for all* $0 < i \leq N$*,*

$$\mathfrak{K} - |R_{t_i}(\beta) - \beta| \in \mathbb{R}^d_+ \ \mathbb{P} - a.s. \ , \quad \text{for all } \beta \in \mathbb{R}^d_+ \ .$$

(iii) Continuity*: For any* $0 < i \leq N$*, we have that* $\lim_{\beta^n \to \beta^0} R_{t_i}(\beta^n) = R_{t_i}(\beta^0)$ *.*

Just notice that we use the partial order $\succeq$ to define the concavity or R, but that the upper bound $\mathfrak{K}$ is given with respect to $\mathbb{R}^d_+$. This is a useful artefact in the proofs, but also a meaningful expression of a physical bound of production. It is possible to fairly approximate a generation supply curve in this class of functions, see section 4. Condition (ii) does not only ensure the admissibility of investment-production portfolios when we add production: it also provides a realistic framework for physical production systems. Finally, condition (iii) is a technical assumption. Note that according to condition (ii), R_{t_i} is lower semicontinuous and condition (iii) is only needed on the boundary of $\mathbb{R}^d_+$.

Let us explain here how the agent can manage his production system. For $0 \leq i < N$, the agent puts a quantity of assets β_{t_i} at time t_i into the production system. The latter returns a position $R_{t_{i+1}}(\beta_{t_i})$ labelled in assets at time t_{i+1}. At this time, the agent also decides the regime of production $\beta_{t_{i+1}}$ for the next step of time, and so on until time reaches t_N. This allows to write the set of investment-production self financing portfolio processes as

$$\mathfrak{X}^R_t := \left\{ V \ : \ V_s := \xi_s + \sum_{i=1}^N R_{t_i}(\beta_{t_{i-1}} \mathbb{1}_{\{t_{i-1} \geq t\}}) \mathbb{1}_{\{t_i \leq s\}} - \beta_{t_{i-1}} \mathbb{1}_{\{t \leq t_{i-1} \leq s\}}, \ (\xi, \beta) \in \mathfrak{X}^0_{t,adm} \times \mathcal{B} \right\} \ .$$

We can thus define $\mathfrak{X}^R_t(T) := \left\{ V_T \ : \ V \in \mathfrak{X}^R_t \right\}$ and use the concept of admissibility in definition 2.2 without a change for the pure financial part. Notice that it has no mathematical cost to consider separate times of injection and times of production, i.e., a non-decreasing sequence

$$\{t_0, s_0, t_1, s_1, \ldots, t_N, s_N\} \subset [0, T]$$

with $t_i < s_i$, $(t_i)_{0 \leq i < N}$ and $(s_i)_{0 < i \leq N}$ allowing to define $\mathcal{B}$ and R respectively. It is also possible to consider an increasing sequence of stopping times in $\mathcal{T}_{[0,T]}$. Finally, $\mathcal{B}$ can be defined via sequences $(\beta_{t_i})_{0 \leq i < N}$ such that β_{t_i} takes values in a convex closed subset of $\mathbb{R}^d_+$. The proofs in section 5 would be identical.

Remark 2.1. *The generalization to continuous time controls raises mathematical difficulties. When coming to a continuous time control, we have to make a distinction between the continuous and the discontinuous part of the control, i.e., between a regime of production as a rate and an instantaneous consumption of assets put in the production system. This natural distinction has already been observed for liquidity matters in financial markets, see [3]. This implies a separate treatment of consumption in the function R. With a continuous control, the production becomes naturally a linear function* *of that control, which is very restrictive and similar to the polyhedral cone setting of markets with proportional transaction costs. With a discontinuous control, non linearity can appear but we face two difficulties. If the number of discontinuities is bounded, it is easy to see that the set of controls is not convex. On the contrary, if it is not bounded, the set is not closed. This problem typically appears in impulse control problems and is not easy to overpass. We ought to focus on that difficulty in another paper.*

3 The conditional sure profit condition and super-replication theorem

In the situation of our agent, even if we accept no arbitrage on the financial market, there is no economical justification for the interdiction of sure profits coming from the production. This is the reason why the concept of *no marginal arbitrage for high production regime* has been introduced in [1] (**NMA** for short). The **NMA** condition expresses the possibility to make sure profits coming from the production possibilities, but that marginally tend to zero if the production regime β is pushed toward infinity. This condition relied on an affine bound for the production function, introducing then an auxiliary linear production function for which sure profits are forbidden.

We propose here another parametric condition, still based on the idea of possibly making solvable profits for a small regime of production. However we express the new condition directly with the production function R. This condition comes from the following observation. Coming back to the usual case of a financial market, a possible interpretation of a no-arbitrage condition is that there is no strategy which is $\mathbb{P}$-almost surely better than the null strategy (or an equivalent strategy reaching 0 at time T). The idea is to transpose this interpretation to production with a slight modification: we will allow strategies which are better than doing nothing at the condition that the regime of production is bounded. Here, doing nothing implies that the agent is subject to possible fixed costs expressed by $R(0)$.

Definition 3.1. *We say that there are only conditional sure profits for production function R,* **CSP***(R) holds for short, if there exists $C > 0$ such that for all $0 \leq k < N$ and for all $(\xi, \beta) \in \mathfrak{X}^0_{t_k,adm} \times \mathcal{B}$ we have:*

$$\xi_T + \sum_{i=k}^{N-1} R_{t_{i+1}}(\beta_{t_i}) - \beta_{t_i} \succeq_T \sum_{i=k}^{N-1} R_{t_{i+1}}(0)\ \mathbb{P} - a.s. \quad \Longrightarrow \quad \|\beta_{t_i}\| \leq C \text{ for } k \leq i \leq N\ .$$

Let us propose additional explanations for that condition. In our framework, we focus on terminal attainable wealth at a fixed date T, whereas we do not specify portfolio by an initial holding. The reason is that the set of financial or productive possibilities shall not depend in our context of an initial position, which is a common implicit assumption. Consequently, we can focus on portfolios starting at any time before T with any initial holding. The condition **CSP**(R) thus reads as follows. If the agent starts an investment-production strategy at an intermediary date $t \in (t_{k-1}, t_k]$ for some k (whatever his initial position is at t), then he can start his production at index k. We then assess that he can do better than the strategy $(0, 0) \in \mathfrak{X}^0_0 \times \mathcal{B}$ only if the regime of production is bounded. The terminology **CSP**(R) refers to the *no sure profit* property introduced by Rasonyi [15] (which became the *no sure gain in liquidation value* condition in the final version), since it is formulated in a very similar way and expresses the interdiction for sure profit if some condition is not fulfilled. The **CSP** property is indeed very flexible. It is possible to change the condition $\|\beta_{t_i}\| \le C$ by any restriction of the form:

$$\text{There exists a bounded set } C \text{ verifying } 0 \in C \subset \mathbb{R}^d_+ \text{ s.t. } \beta_{t_i} \in C \text{ for all } 0 \le i < N \ .$$

This can convey the condition that the regime of production shall be null or greater than a threshold to allow profits, or observe a more precise condition on its components as long as it also constrains the norm of β. Our first main result shows that the closedness property on the financial market alone, Assumption 2.1 above, transmits to the market with production possibilities, whenever our **CSP**(R) condition stands.

Proposition 3.1. *The set $\mathfrak{X}^R_0(T)$ is Fatou-closed under* **CSP***(R).*

A corollary of the closedness of $\mathfrak{X}^0_{0,adm}(T)$ is the standard super-replication theorem. This result allows one to characterize random wealth at time T that can be attained by a self financing portfolio. In our case, we retrieve a similar result, which is the main theorem of the paper. For this purpose we introduce the support function for $\mathfrak{X}^R_0(T)$,

$$\alpha^R_0(Z) := \sup \left\{ \mathbb{E}\left[Z'_T V_T\right] \ : \ V_T \in \mathfrak{X}^R_0(T) \right\} \text{ for } Z \in \mathcal{M}$$

so that the following theorem holds as a corollary of Proposition 3.1.

Theorem 3.1. *Let $H \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ be such that $H \succeq_T -\kappa$ for some $\kappa \in \mathbb{R}^d_+$. Then*

$$H \in \mathfrak{X}^R_0(T) \iff \mathbb{E}\left[Z'_T H\right] \le \alpha^R_0(Z), \ \forall Z \in \mathcal{M} \ .$$

4 Application to the pricing of a power future contract

We illustrate the result of Theorem 3.1 by an application to an electricity producer endowed with a generation system converting a raw material, e.g. fuel, into electricity and who has the possibility to trade that asset on a market. We address here the question of a possible price of a term contract a producer can propose when he takes into account that he can use the generation asset. We assume that the financial market is submitted to proportional transaction costs. For this reason, we place ourselves in the financial framework developed by Campi and Schachermayer in [2].

4.1 The financial market

We consider a financial market on $[0, T]$ composed of two assets, cash and fuel, which will be labelled asset 1 and 2 respectively. The market is represented by a so-called bid-ask process, see [2] for a general definition. It is denoted by π, which is a $\mathbb{F}$-adapted càdlàg process taking values in the set of square matrices of dimension 2×2. Here π^{12}_t denotes at time t the quantity of cash necessary to obtain one unit of fuel (in MWh for convenience), and $(\pi^{21}_t)^{-1}$ denotes the quantity of cash that can be obtained by selling one unit of fuel. We assume that $\mathbb{P}(\min(\pi^{12}_t, \pi^{21}_t) > 0 \text{ for all } t \in [0, T]) = 1$.

The terms π_t^{11} and π_t^{22} take the value 1 for all $t \in [0,T]$ $\mathbb{P}$-almost surely. We suppose that the market faces efficient frictions, i.e., strictly positive transaction costs:

$$\pi_t^{12} \times \pi_t^{21} > 1 \text{ for all } t \in [0,T] \ \mathbb{P} - \text{a.s.}$$

The matrix π generates a set-valued random process which defines the solvency region:

$$\widehat{K}_t(\omega) := \text{cone}(e^1, e^2, \pi_t^{12}(\omega)e^1 - e^2, \pi_t^{21}(\omega)e^2 - e^1) \quad \forall (t,\omega) \in [0,T] \times \Omega \, .$$

Here (e^1, e^2) is the canonical base of $\mathbb{R}^2$. The process $\widehat{K}$ is $\mathbb{F}$-adapted and closed convex cone-valued. It allows to define a natural partial order on $\mathbb{R}^2$ at each time t and for almost every $\omega \in \Omega$, see subsection 2.1. In this framework, financial self-financing portfolios are represented by làdlàg $\mathbb{R}^2$-valued $\mathbb{F}$-predictable processes with finite variation. Moreover, every $\xi \in \mathfrak{X}_0^0$ shall verify that for every $(\sigma,\tau) \in \mathcal{T}^2_{[0,T]}$ with $\sigma \le \tau$, we have:

$$(\xi_\tau - \xi_\sigma)(\omega) \in \overline{\text{conv}} \left(\bigcup_{\sigma(\omega) \le u \le \tau(\omega)} -\widehat{K}_u(\omega) \right) ,$$

the bar denoting the closure in $\mathbb{R}^d$. Admissible portfolios are defined via Definition 2.2 and the partial order induced by $\widehat{K}$. Finally, we denote by $\mathcal{M}$ the set of strictly consistent price systems for π, i.e., $\mathbb{R}^2_+$-valued martingales such that $(\pi_t^{21})^{-1} \le Z_t^1/Z_t^2 \le \pi_t^{12}$ $\mathbb{P}-$a.s. and that for all $\sigma \in \mathcal{T}_{[0,T]}$, $(\pi_\sigma^{21})^{-1} < Z_\sigma^1/Z_\sigma^2 < \pi_\sigma^{12}$. If σ is a predictable stopping time, we require that $(\pi_{\sigma^-}^{21})^{-1} < Z_{\sigma^-}^1/Z_{\sigma^-}^2 < \pi_{\sigma^-}^{12}$. For a comprehensive introduction of all these objects, we refer to [2]. Campi and Schachermayer [2] show that, under the assumption that $\mathcal{M} \neq \emptyset$, $Z\xi$ is a supermartingale for all $Z \in \mathcal{M}$ and admissible self-financing portfolio ξ, see Lemma 2.8 in [2], so that $\mathbb{E}\left[Z_T'\xi_T\right] \le 0$. The fact that $\mathcal{M}$ corresponds precisely to Definition 2.1 follows from the construction of $\widehat{K}$ and is a part of the proof of Theorem 4.1 in [2]. Finally, Assumption 2.1 is given by Proposition 3.4 in [2].

4.2 The generation asset

We suppose that the agent possesses a thermal plant allowing to produce electricity on a fixed period of time. The electricity spot price is determined per hour, so that we define the calendar of production as $(t_i)_{0 \le i \le N} \subset [0,T]$, where N represents the number of generation actions for each hour of the fixed period. At time t_i, the agent puts a quantity $\beta_{t_i} = (\beta_{t_i}^1, \beta_{t_i}^2)$ of assets in the plant. The production system transforms at time t_{i+1} the quantity $\beta_{t_i}^2$ of fuel, given a fixed heat rate $q_{i+1} \in \mathbb{R}_+$, into a quantity $q_{i+1}\beta_{t_i}^2$ of electricity (in MWh). The producer has a limited capacity of injection of fuel given by a threshold $\Delta_{i+1} \in L^\infty(\mathbb{R}_+, \mathcal{F}_{t_{i+1}})$. This implies that any additional quantity over Δ_{i+1} of fuel injected in the process will be redirected to storage facilities, i.e., as fuel in the portfolio. The electricity is immediately sold on the market via the hourly spot price. On most of electricity markets, the spot price is legally bounded. It can also happen to be negative. It is thus given by $P_{i+1} \in L^\infty(\mathbb{R}, \mathcal{F}_{t_{i+1}})$. For a given time t_{i+1}, the agent is subject to a fixed cost γ_{i+1} in cash. The agent also faces a cost in fuel in order to maintain the plant activity. This is given by a supposedly non-positive increasing concave function c_{i+1} on $[0, \Delta_{i+1}]$ such that $c'_{i+1}(\Delta_{i+1}) \ge 1$, where c'_{i+1} represents the left derivative. Altogether, we propose to modelize the production possibilities by $R_{t_{i+1}}(\beta_{t_i}) = (R^1_{t_{i+1}}(\beta_{t_i}), R^2_{t_{i+1}}(\beta_{t_i}))$ where

$$R^1_{t_{i+1}}((\beta_{t_i}^1, \beta_{t_i}^2)) = P_{i+1}q_{i+1}\min(\beta_{t_i}^2, \Delta_{i+1}) - \gamma_{i+1} + \beta_{t_i}^1$$

and

$$R^2_{t_{i+1}}((\beta^1_{t_i},\beta^2_{t_i})) = c_{i+1}(\min(\beta^2_{t_i},\Delta_{i+1})) + \max(\beta^2_{t_i}-\Delta_{i+1},0)$$

for $0 \le i < N$. We can constraint $\beta^1_{t_i}$ to be null at every time t_i without any loss of generality. It is easy to show that R verifies Assumption 2.2 (ii). We have indeed

$$|R^1_{t_{i+1}}((\beta^1_{t_i},\beta^2_{t_i})) - \beta^1_{t_i}| \le |P_{i+1}q_{i+1}\Delta_{i+1}| + |\gamma_{i+1}| \in L^\infty(\mathbb{R},\mathcal{F}_{t_{i+1}})$$

and

$$|R^2_{t_{i+1}}((\beta^1_{t_i},\beta^2_{t_i})) - \beta^2_{t_i}| \le \max(|c_{i+1}(0)|,|c_{i+1}(\Delta_{i+1})-\Delta_{i+1}|) \in L^\infty(\mathbb{R}_+,\mathcal{F}_{t_{i+1}}) \ .$$

Notice that since c_{i+1} is concave with $c'_{i+1}(\Delta_{i+1}) \ge 1$, the function R^2 is concave. The function R is then concave in each component with respect to the usual order, so that Assumption 2.2 (i) holds with the partial order induced by $\widehat{K}$. It is also continuous, so that Assumption 2.2 (iii) holds.

4.3 Conditional sure profit and super replication price

We now fix a condition provided by the agent in order to apply Definition 3.1. For example suppose that the agent knows at time t_i that by producing under a typical regime C and selling the production at the market price, he can refund the quantity of fuel needed to produce. It is a conceivable phenomenon on the electricity spot market. Since the electricity spot price is actually an increasing function of the total amount of electricity produced by the participants, the agent can sell a small quantity of electricity at high price if the total production is elevated. He can then partially or totally recover his fixed cost and even make sure profit. The constant C can depend on external factors of the model, such as the level of aggregated demand of electricity. This is mathematically expressed by the condition

$$R^1_{t_{i+1}}(\beta^2_{t_i}) + \gamma_{i+1} \ge (\pi^{12}_{t_{i+1}})^{-1}(R^2_{t_{i+1}}(\beta^2_{t_i}) - \beta^2_{t_i} - c_{i+1}(0)) \ \mathbb{P}-\text{a.s.} \Longrightarrow \beta^2_{t_i} \le C \tag{4.1}$$

for some $C > 0$. Here, an immediate transfer $\xi_{t_{i+1}}$ of quantity $R^1_{t_{i+1}}(\beta^2_{t_i})$ of asset 1 brought in asset 2 gives $\xi_{t_{i+1}} + R_{t_{i+1}}(\beta_{t_i}) \succeq R_{t_{i+1}}(0)$. The **CSP**(R) condition can then be applied with C. Imposing this condition implies that the set $\mathfrak{X}^R_{0,adm}(T)$ is Fatou-closed, so that we can apply Theorem 3.1. Assuming no arbitrage on the financial market by the existence of $Z \in \mathcal{M}$,

$$\alpha^0_0(Z) := \sup\left\{\mathbb{E}\left[Z'_T V_T\right] \ : \ V \in \mathfrak{X}^0_{0,adm}\right\} = 0, \quad \text{for all } Z \in \mathcal{M}$$

and the support function is given in this case by

$$\alpha^R_0(Z) = \sup_{\beta\in\mathcal{B}} \mathbb{E}\left[\sum_{i=1}^N Z^1_{t_i}\left(P_i q_i \min(\beta^2_{t_{i-1}},\Delta_i) - \gamma_i\right) + Z^2_{t_i}\left(c_i(\min(\beta^2_{t_{i-1}},\Delta_i)) - \min(\beta^2_{t_{i-1}},\Delta_i)\right)\right] \ .$$

Now if the agent wants to sell a future contract on electricity at time $t=0$ of Nx MWh equally shared on the N hours, he has to fix a cash settlement $F(x)$ allowing to hedge the buying (or production) of x MWh per hour paid at the spot price P_i, for $0 < i \le N$. The super-replication price of the forward contract expressed in cash is then given at time 0 by

$$F(x) = \sup_{Z\in\mathcal{M}}\left(\frac{1}{Z^1_0}\mathbb{E}\left[\sum_{i=1}^N Z^1_{t_i} P_i x\right] - \alpha^R_0(Z)\right) \ .$$

According to the Theorem, the agent is capable to find a hedging strategy such that the terminal wealth of the investment-production portfolio constrained to deliver the pay-off of the Future contract is smaller than $F(x)$ almost surely.

5 Proofs

5.1 Proof of Proposition 3.1

We define a collection of sets

$$\widetilde{\mathfrak{X}}^k_t := \left\{ V \ : \ V_s := \xi_s + \sum_{i=1}^{k} R_{t_{N+1-i}}(\beta_{t_{N-i}})\mathbb{1}_{\{t_{N+1-i}\le s\}} - \beta_{t_{N-i}}\mathbb{1}_{\{t_{N-i}\le s\}} \text{ for } s \ge t,\ (\xi,\beta)\in \mathfrak{X}^0_{t,adm}\times \mathcal{B} \right\}$$

and $\widetilde{\mathfrak{X}}^k_t(T) := \left\{ V_T \ : \ V \in \widetilde{\mathfrak{X}}^k_t \right\}$ for $t\in[0,T]$ and $0\le k\le N$, with the convention that

$$\sum_{i=1}^{0} R_{t_{N+1-i}}(\beta_{t_{N-i}}) - \beta_{t_{N-i}} = 0\ .$$

Note thus that $\widetilde{\mathfrak{X}}^0_t(T)$ corresponds precisely to the set $\mathfrak{X}^0_{t,adm}(T)$. We are conducted by the following guideline. According to Assumption 2.1, $\widetilde{\mathfrak{X}}^0_{t_N}(T)$ is Fatou closed. We then proceed by induction in two steps: we first show that $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is closed if $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is closed. Then we prove that $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ is closed if $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is closed.

Proposition 5.1. *For all $0\le k\le N$, the set $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is convex.*

Proof This is a consequence of Assumption 2.2 (i). Indeed take (ξ^1,β^1) and (ξ^2,β^2) in $\mathfrak{X}^0_{t_{N-k},adm}\times\mathcal{B}$ and $\lambda\in[0,1]$. Take $(\kappa^1,\kappa^2)\in\mathbb{R}^{2d}_+$ the respective bounds of admissibility for ξ^1 and ξ^2. Note that $\lambda\xi^1+(1-\lambda)\xi^2$ is clearly $(\lambda\kappa^1+(1-\lambda)\kappa^2)$-admissible since $\widehat{K}$ is a cone-valued process. By Assumption 2.2(i), there exists $(\ell_{t_{N+1-i}})_{1\le i\le k}$ with $\ell_{t_{N+1-i}}\in L^0(-\widehat{K}_{t_{N+1-i}},\mathcal{F}_{t_{N+1-i}})$ such that

$$R_{t_{N+1-i}}(\lambda\beta^1_{t_{N-i}}+(1-\lambda)\beta^2_{t_{N-i}})+\ell_{t_{N+1-i}} = \lambda R_{t_{N+1-i}}(\beta^1_{t_{N-i}})+(1-\lambda)R_{t_{N+1-i}}(\beta^2_{t_{N-i}})\ ,\quad 1\le i\le k\ .$$

Note also that, according to Assumption 2.2 (ii), each $\ell_{t_{N+1-i}}$ is bounded by below by $2\mathfrak{K}$ for $1\le i\le k$. By relation (2.1) and the above fact, $\lambda\xi^1_T+(1-\lambda)\xi^2_T+\sum_{i=1}^k \ell_{t_{N+1-i}}\in\mathfrak{X}^0_{t_{N-k},adm}(T)$. Assembling the parts gives the property. □

Proposition 5.2. *If $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed, then the same holds for $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$.*

Proof Let $(V^n_T)_{n\ge1}\subset\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ be a sequence such that V^n_T Fatou-converges to some V^0_T. Let $(\xi^n)_{n\ge1}\subset\mathfrak{X}^0_{t_{N-(k+1)},adm}$ and $(\beta^n_{t_{N-i}})_{1\le i\le k,n\ge1}$ with $(\beta^n_{t_{N-i}})_{n\ge1}\subset L^0(\mathbb{R}^d_+,\mathcal{F}_{t_{N-i}})$ for $1\le i\le k$, and $\kappa\in\mathbb{R}^d_+$, such that

$$V^n_T = \xi^n_T + \sum_{i=1}^{k} R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} \succeq_T -\kappa \quad \forall n\ge1\ .$$

According to Assumption 2.2 (ii), and since $\mathbb{R}^d_+\subset\widehat{K}_T$, we have that for any $n\ge1$,

$$-k\mathfrak{K}\preceq_T \sum_{i=1}^{k} R_{t_{N+1-i}}(\beta^n_{t_{N-i}})-\beta^n_{t_{N-i}} =: \widehat{V}^n_T\in\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)\ .$$

Due to Assumption 2.2 (ii) also, we have that $\xi^n_T\succeq_T-(\kappa+k\mathfrak{K})$ for all $n\ge1$. According to Assumption 2.1, we can then find a sequence of convex combinations $\widetilde{\xi}^n$ of ξ^n, $\widetilde{\xi}^n\in\mathrm{conv}(\xi^m)_{m\ge n}$, such that $\widetilde{\xi}_T^n$ Fatou-converges to some $\widetilde{\xi}_T^0 \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. The convergence of $\widetilde{\xi}_T^n$ implies, by using the same convex weights, that there exists a sequence $(\widetilde{V}_T^n)_{n\geq 1}$ of convex combinations of $\widehat{V}_T^m$, $m \geq n$, converging $\mathbb{P}-$ a.s. to some $\widetilde{V}_T^0$. By Proposition 5.1 above, the sequence $(\widetilde{V}_T^n)_{n\geq1}$ lies in $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$. Recall that it is also bounded by below. Since $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed, $\widetilde{V}_T^0 \in \widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ and moreover, $\widetilde{V}_T^0$ is of the form $\sum_{i=1}^k R_{t_{N+1-i}}(\beta^0_{t_{N-i}}) - \beta^0_{t_{N-i}} + \ell^0_{t_{N+1-i}}$ for some $\beta^0 \in \mathcal{B}$ and $(\ell^0_{t_{N+1-i}})_{1\leq i\leq k}$ with $\ell^0_{t_{N+1-i}} \in L^\infty(-\widehat{K}_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ for $1 \leq i \leq k$. This is due to Assumption 2.2 (i)-(ii). If we let $(\lambda_m)_{m\geq n}$ be the above convex weights, we can always write for $1 \leq i \leq k$ and $n \geq 1$

$$\sum_{m\geq n} \lambda_m \left(R_{t_{N+1-i}}(\beta^m_{t_{N-i}}) - \beta^m_{t_{N-i}} \right) = R_{t_{N+1-i}}(\sum_{m\geq n} \lambda_m \beta^m_{t_{N-i}}) - \sum_{m\geq n} \lambda_m \beta^m_{t_{N-i}} + \ell^n_{t_{N+1-i}} \,.$$

The sets $L^0(-\widehat{K}_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ and $L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ are closed convex cones for $1 \leq i \leq k$, so that $\ell^n_{t_{N+1-i}}$ and $\sum_{m\geq n} \lambda_m \beta^m_{t_{N-i}}$ and their possible limits stay in those sets respectively. From the boundedness condition of Assumption 2.2 (ii), the vectors $\ell^n_{t_{N+1-i}}$ are uniformly bounded by below by $2\mathfrak{K}$ for any $1 \leq i \leq k$ and $n \geq 1$, and so are $\ell^0_{t_{N+1-i}}$ for $1 \leq i \leq k$. According to (2.1), $\widetilde{\xi}_T^n + \sum_{i=1}^k \ell^0_{t_{N+1-i}} \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. We then have that $\widetilde{\xi}_T^n + \widetilde{V}_T^N$ converges to $\widetilde{\xi}_T^0 + \widetilde{V}_T^0 = V_T^0 \in \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$. □

Proposition 5.3. *If $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is Fatou-closed, then the same holds for $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$.*

Proof Let $(V_T^n)_{n\geq1} \subset \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ such that there exists $\kappa \in \mathbb{R}^d_+$ verifying $V_T^n \succeq_T -\kappa$ for $n \geq 1$, and V_T^n converges $\mathbb{P}-$ a.s. toward $V_T \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ when n goes to infinity. We let $(\bar{V}_T^n, \bar{\beta}^n)_{n\geq1} \subset \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T) \times L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$ be such that $V_T^n = \bar{V}_T^n + R_{t_{N-k}}(\bar{\beta}^n) - \bar{\beta}^n$. Define $\eta^n = |\bar{\beta}^n|$ and the $\mathcal{F}_{t_{N-(k+1)}}$-measurable set $E := \{\limsup_{n\to\infty} \eta^n < +\infty\}$. We consider two cases.

1. First assume that $E = \Omega$. Then $(\bar{\beta}^n)_{n\geq1}$ is $\mathbb{P}-$ a.s. uniformly bounded. According to Lemma 2 in [12], we can find a $\mathcal{F}_{t_{N-(k+1)}}$-measurable random subsequence of $(\bar{\beta}^n)_{n\geq1}$, still indexed by n for sake of clarity, which converges $\mathbb{P}-$ a.s. to some $\bar{\beta}^0 \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$. By Assumption 2.2 (iii), $R_{t_{N-k}}(\bar{\beta}^n)$ converges to $R_{t_{N-k}}(\bar{\beta}^0)$, Recall that $\bar{V}_T^n \succeq -\kappa - \mathfrak{K}$ for $n \geq 1$. Since it is $\mathbb{P}$-almost surely convergent to $V_T - R_{t_{N-k}}(\bar{\beta}^0) + \bar{\beta}^0 =: \bar{V}_T^0$ and that $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is Fatou-closed, the limit $\bar{V}_T^0$ lies in that set. This implies that $V_T \in \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$.

2. Assume now that $\mathbb{P}[E^c] > 0$. Since E^c is $\mathcal{F}_{t_{N-(k+1)}}$-measurable, we argue conditionally to that set and suppose without loss of generality that $E^c = \Omega$. We then know that there exists a $\mathcal{F}_{t_{N-(k+1)}}$-measurable subsequence of $(\eta^n)_{n\geq1}$ converging $\mathbb{P}$-almost surely to infinity with n by an argument similar to the one of Lemma 2 in [12]. We overwrite n by the index of this subsequence. We write V_T^n as follows:

$$V_T^n = \xi_T^n + R_{t_{N-k}}(\beta^n_{t_{N-(k+1)}}) - \beta^n_{t_{N-(k+1)}} + \sum_{i=1}^k R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} \,, \tag{5.1}$$

with $(\xi^n)_{n\geq1} \subset \mathfrak{X}^0_{t_{N-(k+1)},adm}$ and $(\beta^n_{t_{N-i}})_{1\leq i\leq k+1, n\geq1}$ with $(\beta^n_{t_{N-i}})_{n\geq1} \subset L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ for $1 \leq i \leq k+1$, and with the natural convention that for all $n \geq 1$, $\beta^n_{t_{N-(k+1)}} = \bar{\beta}^n$. We then define

$$(\widetilde{V}_T^n, \widetilde{\xi}^n, \widetilde{\beta}^n_{t_{N-(k+1)}}, \ldots, \widetilde{\beta}^n_{t_N}) := \frac{2\|C\|}{1+\eta^n}(V_T^n, \xi^n, \beta^n_{t_{N-(k+1)}}, \ldots, \beta^n_{t_N}) \,. \tag{5.2}$$

Now that $(\widetilde{\beta}^n_{t_{N-(k+1)}})_{n\geq1}$ is a bounded sequence, we can extract a random subsequence, still indexed by n, such that $(\widetilde{\beta}^n_{t_{N-(k+1)}})_{n\geq1}$ converges $\mathbb{P}-$ a.s. to $\beta^0_{t_{N-(k+1)}} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$. Notice for later that $\|\widetilde{\beta}^n_{t_{N-(k+1)}}\|$ converges to $\|\beta^0_{t_{N-(k+1)}}\| = 2\|C\|$. It is clear that Assumption 2.2 (i) allows to write

$$\frac{2\|C\|}{1+\eta^n}\left(R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}}\right) = R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} - \left(1 - \frac{2\|C\|}{1+\eta^n}\right) R_{t_{N+1-i}}(0) + \ell^n_{t_{N+1-i}} , \tag{5.3}$$

with $(\ell^n_{t_{N+1-i}})_{n\geq 1} \subset L^\infty(-\widehat{K}_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ for $1 \leq i \leq k+1$. Note that, according to Assumption 2.2 (iii), the particular case $i = k+1$ gives

$$\lim_{n\uparrow\infty} R_{t_{N-k}}(\widetilde{\beta}^n_{t_{N-(k+1)}}) - \widetilde{\beta}^n_{t_{N-(k+1)}} = R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} . \tag{5.4}$$

The general case $i \leq k$ follows from Assumption 2.2 (ii) applied to equation (5.3): the left hand term converges to 0 and $(1 - \frac{2\|C\|}{1+\eta^n})$ converges to 1, so that

$$\lim_{n\uparrow\infty} R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} + \ell^n_{t_{N+1-i}} = R_{t_{N+1-i}}(0) . \tag{5.5}$$

By construction of the subsequence, the convexity of $\mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$ and the belonging of 0 to that set, $\widetilde{\xi}^n_T \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. By using property of equation (2.1) and since the sequence $(\ell^n_{t_{N+1-i}})_{n\geq 1}$ is uniformly bounded for any $1 \leq i \leq k+1$, see proof of Proposition 5.2 above, we define

$$\widehat{V}^n_T := \widetilde{\xi}^n_T + \ell^n_{t_{N-k}} + \sum_{i=1}^{k} \left(R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} + \ell^n_{t_{N+1-i}}\right) \in \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T) ,$$

which converges by definition and equations (5.4) and (5.5) to $\widehat{V}^0_T$ such that

$$\widehat{V}^0_T + R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \succeq_T \sum_{i=1}^{k+1} R_{t_{N+1-i}}(0) . \tag{5.6}$$

Notice also that by Assumption 2.2 (ii), for all $n \geq 1$

$$\widehat{V}^n_T = \widetilde{V}^n_T - R_{t_{N-k}}(\widetilde{\beta}^n_{t_{N-(k+1)}}) + \widetilde{\beta}^n_{t_{N-(k+1)}} + \sum_{i=1}^{k+1}\left(1 - \frac{2\|C\|}{1+\eta^n}\right) R_{t_{N+1-i}}(0) \succeq_T -(\kappa + (k+1)\mathfrak{K}) .$$

By Fatou-closedness of $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$, we finally obtain that $\widehat{V}^0_T + R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \in \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$. By equation (5.6) and **CSP**(R), $\|\beta^0_{t_{N-(k+1)}}\| \leq C$ but by construction, $\|\beta^0_{t_{N-(k+1)}}\| = 2\|C\|$, so that we fall on a contradiction. The case **2.** is not possible. □

Remark that the flexibility of the **CSP**(R) condition is reflected in the construction in equation (5.2) used in the last lines of the proof of Proposition 5.3. The choice of a good norm for $\widetilde{\beta}$ can indeed vary according to the condition we aim at. Following Propositions 5.2 and 5.3, $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ is Fatou-closed if $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed. Proposition 5.2 is used a last time to pass from the closedness of $\widetilde{\mathfrak{X}}^N_{t_0}(T)$ to the closedness of $\mathfrak{X}^R_0(T)$.

5.2 Proof of Theorem 3.1

Proof The "⇒" sense is obvious. To prove the "⇐" sense, we take $H \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ such that $H \succeq -\kappa$ for some $\kappa \in \mathbb{R}^d_+$ and such that $\mathbb{E}[ZH] \leq \alpha^R_0(Z)$ for all $Z \in \mathcal{M}$ and $H \notin \mathfrak{X}^R_0(T)$, and work toward a contradiction. Let $(H^n)_{n\geq 1}$ be the sequence defined by $H^n := H\mathbb{1}_{\{\|H\|\leq n\}} - \kappa\mathbb{1}_{\{\|H\|>n\}}$.

By Proposition 3.1, $\mathfrak{X}_0^R(T)$ is Fatou-closed, so by Lemma 5.5.2 in [11], $\mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is weak*-closed. Since $H \notin \mathfrak{X}_0^R(T)$, there exists k large enough such that $H^k \notin \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ but, because any $Z \in \mathcal{M}$ has positive components, still satisfies

$$\mathbb{E}\left[Z_T' H^k\right] \leq \alpha_0^R(Z) \quad \text{for all } Z \in \mathcal{M}. \tag{5.7}$$

By Proposition 5.1, the set $\mathfrak{X}_0^R(T)$ is convex, so that we deduce from the Hahn-Banach theorem that we can find $z \in L^1(\mathbb{R}^d, \mathcal{F}_T)$ such that

$$\sup\left\{\mathbb{E}\left[z' V_T\right] \ : \ V_T \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right\} < \mathbb{E}\left[z' H^k\right] < +\infty. \tag{5.8}$$

We define $\widetilde{Z}$ by $\widetilde{Z}_t = \mathbb{E}\left[z|\mathcal{F}_t\right]$. By using the same argument as in Lemma 3.6.22 in [11], we have that $\mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is dense in $\mathfrak{X}_0^R(T)$ and so that the left hand term of equation (5.8) is precisely $\alpha_0^R(\widetilde{Z})$. The process $\widetilde{Z}$ is a non negative martingale and since

$$\left(\mathfrak{X}_0^R(T) - L^\infty(\widehat{K}_t, \mathcal{F}_t)\right) \subset \left(\mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right) \ \forall t \in [0, T]\ ,$$

we have $\widetilde{Z}_t \in L^1(\widehat{K}_t^*, \mathcal{F}_t)$. The contrary would make the left term of equation (5.8) equal to $+\infty$ for a judicious choice of sequences $(\xi^m)_{m \geq 1} \subset \mathfrak{X}_0^R$ (see the proof of Proposition 3.4 in [1]). By using the same arguments as above, and since $\mathfrak{X}_{0,adm}^0(T)$ is Fatou-closed too, we have that $\mathfrak{X}_{0,adm}^0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is dense in $\mathfrak{X}_{0,adm}^0(T)$. This implies that

$$\begin{aligned}
\alpha_0^0(\widetilde{Z}) &:= \sup\left\{\mathbb{E}\left[\widetilde{Z}_T' V_T\right] \ : \ V_T \in \mathfrak{X}_{0,adm}^0(T)\right\} \\
&= \sup\left\{\mathbb{E}\left[\widetilde{Z}_T' V_T\right] \ : \ V_T \in \mathfrak{X}_{0,adm}^0(T) \cap L^\infty(R^d, \mathcal{F}_T)\right\} \\
&\geq \sup\left\{\mathbb{E}\left[\widetilde{Z}_T' V_T\right] \ : \ V \in \mathfrak{X}_0^0 \text{ and } V_\tau \succeq_\tau -\kappa \text{ for all } \tau \in \mathcal{T}_{[0,T]}, \text{ for some } \kappa \in \mathbb{R}_+^d\right\}
\end{aligned}$$

Moreover, according to Assumption 2.2 (ii), $\xi_T + \sum_{i=1}^N R_{t_i}(0) \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ for any $\xi_T \in \mathfrak{X}_{0,ad}^0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$, so that

$$\alpha_0^0(\widetilde{Z}) - N\widetilde{Z}_0'\mathfrak{K} \leq \alpha_0^0(\widetilde{Z}) + \mathbb{E}\left[\widetilde{Z}_T' \sum_{i=1}^N R_{t_i}(0)\right] \leq \sup\left\{\mathbb{E}\left[z' V_T\right] \ : \ V_T \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right\}$$

and then $\alpha_0^0(\widetilde{Z})$ is finite according to equation (5.8). Take $Z \in \mathcal{M}$. Then there exists $\varepsilon > 0$ small enough such that, by taking $\check{Z} = \varepsilon Z + (1-\varepsilon)\widetilde{Z}$,

$$\alpha_0^R(\check{Z}) \leq \varepsilon\alpha_0^R(Z) + (1-\varepsilon)\alpha_0^R(\widetilde{Z}) < \varepsilon\mathbb{E}\left[Z_T' H^k\right] + (1-\varepsilon)\mathbb{E}\left[\widetilde{Z}_T' H^k\right] = \mathbb{E}\left[\check{Z}_T' H^k\right]\ .$$

It is easy to see that $\check{Z} \in \mathcal{M}$, so that the above inequality contradicts (5.7). □

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