FiNANCE FOR ENERGY
MARKET RESEARCH CENTRE

A Principal-Agent Problem for Emissions' Reduction

Giuseppe Benedetti

Working Paper
RR-FiME-12-06
June 2012

A Principal-Agent Problem for Emissions' Reduction*

Giuseppe Benedetti†

June 21, 2012

Abstract

We analyze a principal-agent problem between the state (principal) and a firm (agent) which produces carbon emissions. In particular, the aim of the state is to motivate the firm to reduce those emissions as much as possible by structuring an appropriate incentive policy. We allow for two different kinds of incentives: a "negative" one, typically represented by a fee to pay at a given time T if emissions are too high; and a "positive" one, in the form of continuous-time payments to the agent. Given an incentive policy, we solve the agent's problem using the stochastic maximum principle to derive alternative representations of the optimal effort in terms of Backward Stochastic Differential Equations (BSDEs), and showing uniqueness of the optimal effort up a certain class of policies. This also allows to prove that the agent's utility is (strictly) increasing with respect to incentives and to discuss the sensitivity of the optimal effort when risk aversion or emissions' volatility change. Under some regularity hypotheses, the problem boils down to solving a certain nonlinear PDE, for which we give a suitable discretization scheme. We then perform some numerical experiments to show how the agent behaves in two particular cases: in line with intuition, the optimal effort is bell-shaped in the case of a forfeitary fee while it exhibits a monotone behavior in presence of a linearly increasing fee. The last section is devoted to the problem of the principal, who needs to propose an incentive plan by taking into account the subsequent agent's behaviour. We show that under proper assumptions the problem is similar to the agent's one and can therefore be solved using the same techniques. Again we provide a numerical example to illustrate in particular the optimal choice of continuous-time incentives.

Key-words: Principal-Agent Problems, BSDEs, Emissions markets.
JEL Classification: H23
AMS Classification (2010): 91B70

1 Introduction

The reduction of greenhouse gases emissions has been a subject of crucial importance in the recent years. In particular the introduction of the Kyoto Protocol in 1997 has stimulated an intense debate over the optimal incentives and/or taxation schemes that better motivate the firms to reduce pollution. The academic literature has also followed this growing interest, by focusing especially on the newly created financial market for emission allowances, on its price formation mechanisms and its possible effects on the firms' production decisions (see, for example, [1], [3], [4] and references therein).
However, given that the firms' emission reducing policies are typically only partially observable by the regulator, these kinds of issues seem to be also closely related to the classical economic concept of moral hazard (in the form of the so-called *principal-agent problems*), whose literature has by now quite a long history (see [7] for one of the first mathematical treatments. A partial account of the following developments is given in [6]).
Principal-agent problems in continuous time have been studied recently in particular by [11], [13] and [6]. The usual setting is that of private contracts, where an employer (principal) needs to design a contract in such a way that the agent will i) accept it, and ii) behave afterwards according to the principal's interest. The most general approach to these problems is probably the one in [6], where the authors derive FBSDE systems to characterize the optimal contract under very general assumptions. Similar results in a slightly different context are shown in [13]. In [11], instead, the particular structure

*This research is supported by CREST (Finance/Insurance Laboratory) and the "Laboratoire de Finance des Marchés de l'Energie" (FiME). The author is also particularly grateful to Luciano Campi, Romuald Elie, René Aid and all the participants to the CREST Phd Seminar for fruitful discussions.
†CREST (Finance/Insurance Laboratory) and Université Paris-Dauphine. E-mail: giuseppe.benedetti@ensae.fr.

of the model and the infinite horizon of the optimization problem allow to characterize the optimal contract with an ODE, which can be solved numerically and which permits a deeper study of the properties of the optimal players' behavior.
This paper is the first attempt (to the best of our knowledge) to establish a link between these two recent fields of research. The idea is essentially to apply and develop some of the ideas and techniques of the principal-agent literature to a particular toy economic model where the principal is not a private company but the state, who aims at minimizing the social cost of carbon emissions X_t by imposing an appropriate incentive structure, made up by a continuous incentive process s_t and a final penalty $p(X_T)$ at maturity (which typically intervenes when emissions are too high). The emissions process is modeled by the stochastic process

$$dX_t = X_t l(k_t)dt + X_t \sigma dW_t^k, \tag{1.1}$$

where W^k is a Brownian Motion, $X_0 = x$ and $\sigma > 0$. This is just a Geometric Brownian Motion whose drift is controlled by the agent through his effort $k_t \geq 0$, which can be interpreted as a measure of the efficiency of emission-reducing policies put in place at time t (a higher value of k_t stands for more effort, thus more efficiency). The function $l : [0, +\infty) \mapsto \mathbb{R}$ models the impact of effort on the emissions evolution and is therefore assumed to be strictly decreasing. Other assumptions will be made in order to ensure some good properties of the optimization problem and to be able to apply the measure change techniques.
The principal (state) is assumed to observe the process X, that we call "emissions process" but which may also be interpreted, in line with [1], as a market perception of the cumulative emissions produced by the firm (which become completely known only at maturity T). What the principal does not observe is the agent's effort k, i.e. he observes the left-hand side of (1.1) but he is not able to recover the decomposition on the right-hand side. In particular, he does not observe the Brownian Motion W^k, where the superscript is common in the recent principal-agent literature and indicates that once the process k is also known then the Brownian Motion becomes observable. For technical reasons, we will not define directly the evolution in (1.1), but we will first introduce a reference filtration which is independent of the agent's effort and we will then get to the same representation through a suitably defined measure change. This is called *weak formulation* in the literature (see [6] and [13]): it will prove to be quite powerful to treat the agent's utility maximization problem but we will also apply it to the principal's one, supposing him to know the optimal reactions of the agent to the incentive policy that he puts in place.
As mentioned above, we consider incentive policies made up by two different components: continuous time incentives s and a final penalty p. The incentive process s is assumed to depend on X (not necessarily in a Markovian way) but not on k, which the principal does not observe. The final penalty takes the form $p(X_T)$ where p is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$* (more details will be given later), that typically operates if emissions exceed a certain level $\Lambda > 0$ (see [3] for a related discussion).
We remark that production is not present in our model: the optimal agent's choices concerning his effort plan will only take into account the final regulatory fee and continuous time incentives that he might gain with his effort. However, another interpretation might lead to thinking of $s(t, (X)_{0 \leq s \leq t})$ as a continuous penalty/reward caused by the effects on production of the effort-reducing policy. Finally, we do not model any financial market, hence we do not allow for the possibility of exchanging financial contracts on emissions before maturity (see [1], [3], [4] for a discussion in this direction).
In the first part of the paper we introduce the agent's problem and we give an existence and uniqueness result for the optimal agent's effort. The measure change techniques used in this part are similar to the ones developed in [6] (although, due to the great generality, their hypotheses are not always easy to interpret nor to verify). The particular model that we adopted then allows us to study some of the properties of the optimal effort: for example, we show under which conditions it will be increasing in the agent's risk aversion or in the emissions' volatility. We are also able (under some regularity assumptions) to solve the problem numerically and to visualize the structure of the optimal effort in some special cases: for example, we find a bell shape (in the emissions variable)

*For the subsequent results concerning BSDEs, one could also consider penalty functions depending on the whole process X with minor modifications, and we will use this fact when dealing with the principal's problem. Here we do not look for the greatest generality.

in the case of a forfeitary fee and a monotone behavior with a linearly increasing fee (see Section 4.1 for an interpretation of these results). In the last part of this work we deal with the problem of the principal, who needs to optimally choose an incentive plan (which is considered fixed in the first part). Again we are able to get optimality conditions and to propose a numerical example which shows how continuous-time incentives are optimally chosen in a special case. In particular, we will see that continuous incentives are not necessarily decreasing with respect to emissions.
The paper is structured as follows. In Section 2 we fix an incentive structure and we introduce the optimization problem of the firm (agent), whose solution is characterized in terms of a (F)BSDE. A uniqueness result is also given in a particular set of effort policies. In Section 3 we give alternative BSDE representations of the agent's optimal effort and we find some comparison results. In particular, we look at how the agent's effort and expected utility are affected by a change in the incentive policy. Section 4 is devoted to deriving a (nonlinear) PDE representation for the optimal effort under certain conditions, along with a numerical scheme to solve it. Finally in Section 5 we deal with the principal's problem by giving some necessary and sufficient conditions for optimality. We finally show that in a particular case the problem is quite similar to the agent's one and can be solved with analogous techniques.

2 The agent's problem

Let W^0 be a Brownian Motion on a reference filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is assumed to be the one generated by W^0 augmented by all the P-null sets in $\mathcal{F}$. We define the market perception of the cumulative emissions, or simply *emissions process* X, evolving as a driftless GMB:

$$dX_t = X_t \sigma dW_t^0. \tag{2.1}$$

Note that $(\mathcal{F}_t)_{t\in[0,T]}$ is also generated by X. In order to model the impact of the agent's action to the emissions process we introduce the function $l : [0, +\infty) \mapsto \mathbb{R}$ which verifies the following Assumption.

Assumption 2.1 *The function $l : [0, +\infty) \mapsto \mathbb{R}$ is C^3, strictly decreasing, convex, bounded and with bounded first derivative.*

We then define the change of measure $\Gamma_t^k = \mathcal{E}_t(l/\sigma \cdot W^0)$ associated to the control k (note that it is well defined, as l is bounded), with dynamics $d\Gamma_t^k = \Gamma_t^k l(k_t)/\sigma dW^0$. In this way $dW_t^k := dW_t^0 - \frac{l(k_t)}{\sigma}dt$ is a BM for the measure (induced by) Γ^k. Under this weak formulation† we imagine the agent as regarding the process (2.1) through the probability change Γ^k, which he knows once he decides a technology plan k (the first introduction of this approach to optimization problems dates back to [2]). We will denote E^k the expectation operator under the change of measure induced by k.
Before stating the agent's problem we introduce a utility function $u : \mathbb{R}_+ \mapsto \mathbb{R}$ and a cost function $c : \mathbb{R}_+ \mapsto \mathbb{R}$ with the following regularity properties:

Assumption 2.2 *The following properties hold:*

(i) *u is a C^3 utility function (i.e. strictly increasing and concave) satisfying the Inada conditions, i.e. $u'(0) = +\infty$ and $u'(\infty) = 0$. Moreover, $\frac{u''}{u'}(x) \to -\infty$ as $x \to 0$. We set $u(x) = -\infty$ for $x < 0$.*

(ii) *c is C^3, positive, strictly increasing and convex with $c'(0) = 0$ and $c''(0) > 0$.*

†In another formulation, called 'strong', one starts by fixing a BM, say W, and then works directly with the controlled process

$$dX_t^k = l(k_t)X_t^k dt + \sigma X_t^k dW_t,$$

which may seem more natural at first sight, and we would not need all of the assumptions on l that ensure the well posedness of the change of measure. Apart from this, we preferred the weak formulation for many reasons: for example, it allows to consider a wider class of continuous time incentive policies (i.e. those depending on the history of X) and it requires no stringent conditions on the penalty function, such as differentiability and convexity. As reported in [6], the two methods have sometimes been used together in the literature even if the connection between the two is not always clear and it may hide some subtle measurability issues. We will give some more details in the sequel.

The utility function u models the continuous-time part of the agent's total utility, the one that accounts for the effort plan and continuous-time incentives. Remark that the last condition in its characterization is satified in the most common cases, i.e. for power and logarithmic utilities.
The function c captures the monetary cost associated to effort at each date. It is increasing to reflect the fact that more effort is more costly, while the other technical assumptions (which are satisfied by the common quadratic cost function) will be needed to derive our BSDE representation. Remark that we only model variable costs connected to effort plans, there are no fixed costs linked to increasing or decreasing effort.

Definition 2.1 An admissible incentive policy $(s_t)_{0\leq t\leq T}$ is a positive $\mathcal{F}_t$-adapted stochastic process such that $c(0) < m \leq s_t \leq M$ for some $0 < m < M$.
A penalty function p is called admissible if $p(X_T) \in L^{2+\alpha}(\Omega, P)$ for some $\alpha > 0$.

In this section we will consider an admissible incentive structure (s,p) to be fixed, and we will be concerned with the optimal agent's reaction. To this aim, we define his admissible effort strategies.

Definition 2.2 An admissible effort policy $(k_t)_{0\leq t\leq T}$ is a positive $\mathcal{F}_t$-adapted stochastic process such that

$$E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right] < \infty$$

for some $\alpha > 0$. It is *strongly* admissible if $k_t \leq c^{-1}(s_t - \epsilon)$ a.s. $\forall t \in [0,T]$ for some $\epsilon > 0$.

We will also use the expression "ϵ-admissible" with the obvious meaning.

Remark 2.1 It is clear that a strongly admissible effort policy is admissible. Moreover, since s is bounded, an admissible k must also be bounded. If $u(0)$ is finite, then admissibility of k is equivalent to simply requiring $k_t \leq c^{-1}(s_t)$ a.s. $\forall t \in [0,T]$.

For an admissible k we can now write down the expected agent's utility as

$$\begin{aligned} V(k) = V^{(s,p)}(k) &= E^k\left[\int_0^T u(s_t - c(k_t))dt - p(X_T)\right] \\ &= E\left[\int_0^T \Gamma_t^k u(s_t - c(k_t))dt - \Gamma_T^k p(X_T)\right] \end{aligned} \tag{2.2}$$

where X evolves according to (2.1). The implicit assumption in (2.2) is that the agent's utility separates into two components: a continuous-time part which is captured by u and a lump part at maturity T which is described by p. Therefore the function p is to be interpreted as the (dis)utility the agent gets from the final fee payment, and not as a penalty function *tout court* (unless risk neutrality is assumed). We do not state it explicitly at this stage but we mainly think of p as a nondecreasing function.
For a given admissible incentive structure (s,p), the agent needs to optimally choose his effort plan, that is he must solve the optimization problem

$$v^{(s,p)} := \sup_k V^{(s,p)}(k) \tag{2.3}$$

where the sup is taken over admissible effort policies k.

2.1 Necessary conditions

In order to characterize the solution in terms of a BSDE we will apply the stochastic maximum principle (hereafter SMP), as stated in Theorem 3.2 in [12]. A crucial step in the application of this kind of results is the choice of the state variable(s), as different choices generally lead to different conditions. The peculiarity of the weak formulation lies in the fact that we can take Γ^k as the state variable in the optimization, while X_T is considered as a fixed element of the reference probability space (it does not depend on the control under this formulation). This is quite an advantage as it allows to work under no regularity assumptions on the penalty function p nor on the incentive process s. We remark in particular that no convexity requirements are imposed on p. Choosing Γ^k as state variable the Hamiltonian of the problem can be expressed as

$$H(t, k_t, s_t, Y_t, Z_t) = \Gamma_t^k [Z_t l(k_t)/\sigma + u(s_t - c(k_t))], \tag{2.4}$$

where (Y, Z) are the adjoint variables which follow the BSDE

$$\begin{cases} dY_t = [-Z_t l(k_t)/\sigma - u(s_t - c(k_t))]dt + Z_t dW_t^0 \\ Y_T = -p(X_T). \end{cases} \tag{2.5}$$

We say that a control $(k_t^*)_{0 \leq t \leq T}$ is optimal if it reaches the supremum in the definition of $v^{(s,p)}$. The next result gives some necessary conditions for optimality.

Proposition 2.1 *Let k^* be an optimal strongly admissible control. Then there exist adapted processes (Y, Z) satisfying (2.5) with $k = k^*$ and the optimal control k^* satisfies*

$$\begin{cases} \sigma u'(s_t - c(k_t^*))c'(k_t^*) = Z_t l'(k_t^*) \ \ on \ \{k_t^* > 0\} \\ \sigma u'(s_t - c(k_t^*))c'(k_t^*) \geq Z_t l'(k_t^*) \ \ on \ \{k_t^* = 0\} \end{cases} \tag{2.6}$$

Proof. We will apply the SMP. The non-smoothness of p does not represent a problem since we use Γ^k as state variable. In the notation of [12], we have $h(\Gamma_T^k) = -\Gamma_T^k p(X_T)$, where X_T is independent of the control. Hence $h'(\Gamma_T^k) = -p(X_T)$ and $h''(\Gamma_T^k) = 0$. However, in order to work as it is, the SMP requires that $(k, \Gamma) \mapsto \Gamma u(s - c(k))$ be Lipschitz in both variables. The problem is partially solved by assuming s to be bounded, but there still remains an issue when k is close to $c^{-1}(s)$, and this is where strong admissibility gets in. Let $\epsilon_n \to 0$ and define a sequence of Lipschitz functions $\tilde{u}^n(x)$ which coincide with $u(x)$ for $x \geq \epsilon_n$. Also define $V_n^{(s,p)}(k)$ by replacing u with $\tilde{u}^n$ in the definition of the problem. By definition k^* is ϵ-admissible for some $\epsilon > 0$, therefore we have that $V_n^{(s,p)}(k^*) = V^{(s,p)}(k^*)$ for $n \geq n_0$. Take $0 < \epsilon_0 < \epsilon$ and an ϵ_0-admissible k, then $V^{(s,p)}(k) \leq V_n^{(s,p)}(k^*) = V^{(s,p)}(k^*)$ for $n \geq n_0$. We also have $V_n^{(s,p)}(k) = V^{(s,p)}(k)$ for $n \geq n_1$ and so $V_n^{(s,p)}(k) \leq V_n^{(s,p)}(k^*)$ for $n \geq n_1$. It follows that k^* maximizes V_n over all ϵ_0-admissible k, when $n \geq n_1$. Therefore the SMP can be applied to this new problem, implying that k^* satisfies (2.6) replacing u with $\tilde{u}^n$ and adding a supplementary condition when $k = c^{-1}(s - \epsilon_n)$. However since k^* is ϵ-admissible the replacement is irrelevant and the supplementary condition is never satisfied, so we directly have (2.6). $\square$

Remark 2.2 The requirement of strong admissibilty in the previous result can also be found in the related literature (see [6]) under different (and usually more complex) forms. In our context it could be replaced by simple admissibility if, given the optimal admissible policy k^*, we could find a sequence k^n converging uniformly to k^* and such that k^n is optimal when we only consider controls such that $k_t \leq c^{-1}(s_t - \epsilon_n)$, with $\epsilon_n \downarrow 0$. Indeed for each ϵ_n we can define a function $\tilde{u}^n$ as in the previous proof, so that $\tilde{u}^n \to u$ pointwise. Then the necessary conditions hold for (Y^n, Z^n) and k^n which are the analogous variables in the problem where u_n replaces u. By standard properties of BSDEs (see [8] Theorem 4.4) we have

$$\begin{aligned} &E\left[\sup_{0 \leq t \leq T} |Y_t^m - Y_t^n|^2 + \int_0^T |Z_t^m - Z_t^n|^2 dt\right] \\ &\leq CE\left[\int_0^T [(Z_t^m)^2 |l(k_t^n) - l(k_t^m)|^2 + |u_n(s_t - c(k_t^n)) - u_m(s_t - c(k_t^m))|^2] dt\right] \to 0, \end{aligned}$$

recall that l is bounded, hence (Y^n, Z^n) converge to some (Y, Z) which satisfy (2.5) and the Hamiltonians also converge.

Proposition 2.1 leads to the representation of the target volatility process as

$$\begin{cases} \widehat{Z}_t(s,p,k) = \sigma \frac{u'(s_t - c(k_t))c'(k_t)}{l'(k_t)} \leq 0 \text{ on } \{k_t > 0\} \\ \widehat{Z}_t(s,p,k) \geq 0 \text{ on } \{k_t = 0\} \end{cases} \tag{2.7}$$

It is called *target* (see [13]) because, if the principal wants to induce a strongly admissible technology plan k, it is necessary to act on the incentives s and/or on the fee p in such a way that the volatility process in (2.5) satisfies (2.7).

Definition 2.3 A policy (s,p,k) is said to be

- *promise-keeping* if (s,p,k) imply a solution Y to the BSDE (2.5) with volatility process Z satisfying (2.7).
- *implementable* if, given (s,p), the agent optimally chooses the recommended actions k.

The term promise-keeping (taken from [13]) expresses the idea that under this condition the volatility process Z "keeps the promise" of being equal to its target level.

2.2 Sufficient conditions

In Proposition 2.1 we proved that if (s,p,k) is implementable and k is strongly admissible, then it is promise-keeping. We now aim at proving a converse implication, for which we need a preliminary technical discussion.
For a given admissible effort process k we can rewrite (2.1)-(2.5) as

$$\begin{cases} dX_t = l(k_t)X_t dt + \sigma X_t dW_t^k \\ dY_t = -u(s_t - c(k_t))dt + Z_t dW_t^k \\ X_0 = x, Y_T = -p(X_T) \end{cases} \tag{2.8}$$

This is a simple kind of Forward-Backward SDE, as the link between the two equations only lies in the terminal condition. If we could prove that

$$E^k \left[\int_t^T Z_r dW_r^k \mid \mathcal{F}_t \right] = 0 \tag{2.9}$$

then we would obtain that

$$Y_t = E^k \left[\int_t^T u(s_r - c(k_r))dr - p(X_T) \mid \mathcal{F}_t \right]$$

that is $Y_t = V_t^{(s,p)}(k)$, where $V_t^{(s,p)}(k)$ is naturally defined as the conditional agent's expected utility. We could apply standard existence and uniqueness results directly to (2.8) in order to obtain $E^k \left[\int_0^T Z_r^2 dr \right] < \infty$ which would imply the result. However this can be a little bit tricky, as the system contains the process k which is $\mathcal{F}_t$-adapted, while existence results in this context would hold on the filtration $\mathcal{F}^{W^k}$, which in general differs from $\mathcal{F} = \mathcal{F}^{W^0}$ (though we clearly have $\mathcal{F}_t^{W^k} \subseteq \mathcal{F}_t^{W^0}$). Lemma 7.2 in the Appendix addresses this issue by constructing, for any admissible k, an $\mathcal{F}_t$-measurable solution to (2.8) which exhibits the usual integrability properties of the standard BSDE theory with respect to any measure associated to an effort policy (boundedness of l will play an important role via Lemma 7.1).
The following result completes Proposition 2.1 by providing necessary and sufficient conditions for optimality. Remark that it is stated for strongly admissible policies even if the sufficient part holds with simple admissibility.

Proposition 2.2 *A strongly admissible policy (s,p,k) is implementable if and only if it is promise-keeping. In other words, a strongly admissible effort policy k is optimal for the agent given the incentive structure (s,p) if and only if the (unique) process Z defined as in* (2.8) *with (s,p,k) satisfies condition* (2.7).

Proof. The necessary condition is just Proposition 2.1. To show sufficiency, consider an admissible effort plan k^* and assume that $\widehat{Z} = \widehat{Z}(s,p,k^*)$ defined as in (2.8) with (s,p,k^*) satisfies condition (2.7). The agent's Hamiltonian is (we omit Γ_t^k as it is positive and does not affect the maximization)

$$H^*(t,k_t,s_t) = \widehat{Z}_t(s,p,k^*)l(k_t)/\sigma + u(s_t - c(k_t)).$$

We would like to show that

$$k_t^* = \text{argmax}_{0\le k<c^{-1}(s_t)}[\widehat{Z}_t(s,p,k^*)l(k)/\sigma + u(s_t - c(k))]. \tag{2.10}$$

By calling $b(k)$ the function in the argument we can compute its first and second derivatives

$$b'(k) = \widehat{Z}_t l'(k)/\sigma - u'(s_t - c(k))c'(k)$$

$$b''(k) = \widehat{Z}_t l''(k)/\sigma + u''(s_t - c(k))(c'(k))^2 - u'(s_t - c(k))c''(k)$$

If $\widehat{Z}_t < 0$ then $b''(k) \le 0$ for all $k \in [0, c^{-1}(s_t))$ and $b'(k_t^*) = 0$ since $\widehat{Z}_t$ verifies (2.7), hence (2.10) is true. If $\widehat{Z}_t \ge 0$ then $b'(k) \le 0$ for all $k \in [0, c^{-1}(s_t))$, meaning that $k = 0$ is optimal and again (2.10) is true. Therefore k^* always reaches the maximum in H^*.
It remains to verify that the agent will choose k^* when he faces incentives s and fee p. By Lemma 7.2, the agent's expected utility following k^* is $V^{(s,p)}(k^*) = Y_0^*$, where (Y^*, Z^*) is the solution of (2.5) with k^* replacing k. Then for any admissible k we have

$$\begin{aligned}
V^{(s,p)}(k) - V^{(s,p)}(k^*) =& E^k\left[\int_0^T u(s_t - c(k_t))dt - p(X_T)\right] - E^k\left[Y_0^*\right] \\
=& E^k\left[\int_0^T [u(s_t - c(k_t)) - u(s_t - c(k_t^*))]dt + \int_0^T Z_t^* dW_t^{k^*}\right] \\
=& E^k\left[\int_0^T [u(s_t - c(k_t)) - u(s_t - c(k_t^*))]dt + \int_0^T Z_t^* dW_t^{k}\right] \\
&+ E^k\left[\int_0^T Z_t^*[l(k_t) - l(k_t^*)]\sigma^{-1}dt\right] \\
=& E^k\left[\int_0^T [H^*(t,k_t,s_t) - H^*(t,k_t^*,s_t)]dt\right] \le 0,
\end{aligned}$$

which implies the claim since Z^* satisfies (2.7) by assumption. Here we used the fact that, by definition, we have that $dW_t^{k^*} = dW_t^k + [l(k_t) - l(k_t^*)]\sigma^{-1}dt$. Moreover, Lemma 7.2 ensures that the expected value of the stochastic integral in the third equality is zero. □

For later use we denote g the function appearing in the first line of the optimality condition (2.7):

$$g(s,k) = \sigma\frac{u'(s - c(k))c'(k)}{l'(k)}. \tag{2.11}$$

In economic terms, this can be interpreted as the elasticity of the agent's utility with respect to a change in the emissions' growth rate over a little lapse of time. The following calculation may help grasping this intuition.

$$\frac{E^{k+\delta}\left[\int_t^{t+\epsilon} dV_r^{(s,p)}(k+\delta) \mid \mathcal{F}_t\right] - E^k\left[\int_t^{t+\epsilon} dV_r^{(s,p)}(k) \mid \mathcal{F}_t\right]}{E^{k+\delta}\left[\frac{1}{X_t}\int_t^{t+\epsilon} dX_r \mid \mathcal{F}_t\right] - E^k\left[\frac{1}{X_t}\int_t^{t+\epsilon} dX_r \mid \mathcal{F}_t\right]}$$
$$= \frac{E^{k+\delta}\left[-\int_t^{t+\epsilon} u(s_r - c(k_r+\delta))dr \mid \mathcal{F}_t\right] - E^k\left[-\int_t^{t+\epsilon} u(s_r - c(k_r))dr \mid \mathcal{F}_t\right]}{E^{k+\delta}\left[e^{\int_t^{t+\epsilon} l(k_r+\delta)dr} \mid \mathcal{F}_t\right] - E^k\left[e^{\int_t^{t+\epsilon} l(k_r)dr} \mid \mathcal{F}_t\right]}$$
$$\approx \frac{-u(s_t - c(k_t+\delta))\epsilon - [-u(s_t - c(k_t))]\epsilon}{e^{l(k_t+\delta)\epsilon} - e^{l(k_t)\epsilon}} \approx \frac{u'(s_t - c(k_t))c'(k_t)\delta\epsilon}{e^{l(k_t)\epsilon}l'(k_t)\delta\epsilon} \approx g(s_t, k_t)/\sigma.$$

Some of the sensitivity results in Section 3 will make reference to this quantity.

Remark 2.3 Standard procedures can be used to look for a candidate solution to (2.8), when the policy (s, p, k) is fixed and Markovian (i.e. s and k depend only on t and X_t). In particular, by assuming that $Y_t = \theta^{(s,p,k)}(t, X_t)$ then θ is solution (supposing it is sufficiently regular) of

$$\begin{cases} \theta_t + \frac{1}{2}\theta_{xx}x^2\sigma^2 + xl(k(t,x))\theta_x + u(s(t,x) - c(k(t,x))) = 0 \\ \theta(T,x) = -p(x) \end{cases} \tag{2.12}$$

and $Z_t = \theta_x^{(s,p,k)}(t, X_t)\sigma X_t$.
In this way the implementability constraints on the volatility process Z given in (2.7) can be re-expressed in terms of the solution of the PDE (2.12)[‡]. We can therefore state that a Markovian policy (s, p, k) is implementable if and only if the solution $\theta^{(s,p,k)}(t, X_t)$ to (2.12) satisfies

$$\begin{cases} \theta_x^{(s,p,k)}(t,x) = \frac{u'(s_t - c(k_t))c'(k_t)}{xl'(k_t)} \leq 0 \text{ if } k_t > 0 \\ \theta_x^{(s,p,k)}(t,x) \geq 0 \text{ if } k_t = 0 \end{cases}$$

This result also has a clear economic intuition. The quantity $xl'(k)$ represents the marginal average emissions' reduction when effort is increased, hence $xl'(k)\theta_x$ is the marginal expected utility benefit from increasing effort. On the other hand, $u'(s-c(k))c'(k)$ is the marginal cost of effort. In equilibrium, the marginal benefit should be equal to the marginal cost. When $\theta_x \geq 0$ it means that the marginal expected utility benefit from increasing effort is negative, a pathological situation that will typically only occur when p is increasing, i.e. it is no longer a penalty but a reward for polluting. In this unrealistic case the optimal effort is going to be zero.
At this stage, however, our conditions are too weak to ensure the existence of a classical solution to (2.12). Other PDE results will be derived in Section 4.

2.3 Existence of the optimal effort

We still consider an admissible incentive structure (s, p) to be fixed. In the following we will investigate the question of whether an optimal effort k^* exists and is unique. The main ingredient to do this is going to be the inversion of the the conditions for optimality stated in (2.7). This is done in the following Lemma.

Lemma 2.1 *Given $0 < m < s \leq M$ and $z \in \mathbb{R}$, there exists a unique $k = F(s, z)$ satisfying*

$$\begin{cases} z = \sigma\frac{u'(s-c(k))c'(k)}{l'(k)} = g(s,k)(\leq 0) \text{ if } k > 0 \\ z \geq 0 \text{ if } k = 0 \end{cases} \tag{2.13}$$

[‡]This result also gives an idea of how one can heuristically recover the optimal effort without solving a nonlinear equation, i.e. one can solve the PDE (2.12) backwards with a standard implicit finite-difference scheme by making sure that the discretized versions of the implementability conditions be satisfied at each point in space and time. This procedure gives rise, at each time step, to a nonlinear equation with a number of unknowns equal to the dimension of the spacial grid, which is usually well handled numerically by Matlab.

The function $F(s,\cdot)$ is nonincreasing, Lipschitz and continuously differentiable on $\mathbb{R}\setminus\{0\}$. If $2l''(k)^2 - l'(k)l^{(3)}(k) \geq 0$ and $c^{(3)}(k) \geq 0$ then $F(s,\cdot)$ is concave on $(-\infty, 0]$.

Proof. See Appendix.

Inverting the conditions in (2.7) through the function F allows us to rewrite (2.5) by incorporating the implementability constraints *inside* the BSDE. In this spirit, to any admissible (s,p) we can associate the system

$$\begin{cases} dX_t = X_t\sigma dW_t^0 \\ dY_t = [-Z_t l(F(s_t,Z_t))/\sigma - u(s_t - c(F(s_t,Z_t)))]dt + Z_t dW_t^0 \\ X_0 = x, Y_T = -p(X_T) \end{cases} \tag{2.14}$$

By Proposition 2.2 the existence of a unique solution to this equation is equivalent to the existence of an optimal effort policy, which is characterized by posing $k_t = F(s_t, Z_t)$. Theorem 2.1 gives such an existence result.

Theorem 2.1 *There exists an admissible optimal effort k^* for the agent's problem* (2.3)*. Uniqueness holds in the class of strictly admissible policies.*

Proof. We want the FBSDE (2.14) to have a unique solution (Y,Z). Since it is decoupled, we can treat it as a simple BSDE as far as existence is concerned. Hence it is enough to check that $f(s,z) := u(s - c(F(s,z))) + zl(F(s,z))/\sigma$ is uniformly Lipschitz continuous in z. We have

$$f_z(s,z) = l(F(s,z))/\sigma + F_z(s,z)[zl'(F(s,z))/\sigma - u'(s - c(F(s,z)))c'(F(s,z))].$$

Recall that $F(s,\cdot)$ is not differentiable in 0 and therefore the previous expression should be interpreted at first as a right/left derivative in zero. Let us focus on the second term: if $z \leq 0$ the term in brackets is zero by definition of F, while if $z > 0$ then $F_z(s,z) = 0$. Therefore f is differentiable and we have $f_z(s,z) = l(F(s,z))/\sigma$, which is bounded by assumption. By admissibility of s and p and Theorem 6.2.1 in [10] equation (2.14) has a unique solution (Y,Z) and therefore $k_t = F(s_t,Z_t)$ is an optimal effort. We need to verify that it is admissible, i.e. that

$$E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right] < \infty.$$

for some $\alpha > 0$. Considering the function $a(s,z) = u(s - c(F(s,z)))$, we have for $z < 0$

$$\begin{aligned} 0 \leq a_z(s,z) &= -u'(s-c(F))c'(F)F_z \\ &= -\frac{u'(s-c(F))c'(F)l'(F)^2}{l'(F)[-u''(s-c(F))c'(F)^2 + u'(s-c(F))c''(F)] - l''(F)[u'(s-c(F))c'(F)]} \\ &= \frac{c'(F)l'(F)^2}{l'(F)[\frac{u''}{u'}(s-c(F))c'(F)^2 - c''(F)] + l''(F)c'(F)}. \end{aligned}$$

Since $\frac{u''}{u'}(x) \to \infty$ as $x \to 0$ by Assumption 2.2 we deduce that $a_z(s,z) \to 0$ as $z \to -\infty$. Moreover, $a_{z-}(s,0)$ is finite and $a_z(s,z) = 0$ for $z > 0$. It follows that $a(s,z)$ has a sublinear growth in z, i.e. we can write $|a(s,z)| \leq K_1 + K_2|z|^{1/\beta}$ for some $\beta > 1$ and constants $K_1, K_2 > 0$ (which can be chosen independently of s taking into account $m \leq s \leq M$). Now if we take $\alpha = 2\beta - 2 > 0$ we obtain

$$E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right] \leq K_1 + K_2 E\left[\int_0^T |Z_t|^2 dt\right] < \infty,$$

Therefore $k_t = F(s_t, Z_t)$ is indeed admissible.
As for uniqueness, the results follows from the fact that the necessary conditions in Proposition 2.1 only hold in the class of strongly admissible policies. ☐

In the next Section we will see that under additional regularity assumptions on the penalty function we can ensure that the optimal effort coming from (2.14) is strongly admissible.

3 The optimal effort and comparison results

Having characterized the agent's value function and optimal effort in terms of the solution of a BSDE, it is now natural to look at how the solution is affected by a change in the parameters (i.e. a change in the incentive structure). As a starting point, the next lemma shows that the agent's value function $v^{(s,p)}$ defined in (2.3) reacts positively to higher incentives or a lower final penalty. This is quite intuitive and it could in part also be deduced directly from definition (2.3) (though the last claim seems to require the comparison theorem for BSDEs).

LEMMA 3.1 *Assume that we have two admissible incentive policies (s,p) and $(\bar{s},\bar{p})$ such that $\bar{s}_t \geq s_t$ a.s. for all t and $\bar{p}(x) \leq p(x)$ for all $x \geq 0$. Then $v^{(\bar{s},\bar{p})} \geq v^{(s,p)}$. Moreover, if $\bar{p}(x) < p(x)$ on a set of strictly positive Lebesgue measure, or if $\bar{s}_t > s_t$ on a set of strictly positive measure $dt \times dP$, then $v^{(\bar{s},\bar{p})} > v^{(s,p)}$.*

Proof. The agent's conditional value function given incentives s follows the BSDE

$$\begin{cases} -dY_t = f(s_t, Z_t)dt - Z_t dW_t^0 \\ Y_T = -p(X_T) \end{cases} \tag{3.1}$$

where $f(s,z) = zl(F(s,z)) + u(s - c(F(s,z)))$. We have that

$$\begin{aligned} f_s(s,z) &= zl'(F)F_s + u'(s-c(F))[1 - c'(F)F_s] \\ &= [zl'(F) - u'(s-c(F))c'(F)]F_s + u'(s-c(F)) \end{aligned}$$

The first term in brackets is zero when $z \leq 0$, while $F_s = 0$ when $z \geq 0$, thus

$$f_s(s,z) = u'(s - c(F)) > 0$$

since u is strictly increasing. This implies that $f(s'_t, z) \geq f(s_t, z)$ for all $z \in \mathbb{R}$ and the claim follows by standard comparison theorems for BSDEs (see [10], Theorem 6.2.2). □

We now turn to the study of the optimal effort. Since it is defined as a function of the Z-part of BSDE (2.14), the starting point must be a better characterization of Z. To our best knowledge, comparison theorems for the Z-part of a BSDE seem to be lacking in the literature, therefore we will directly look for a new BSDE solved by Z. This procedure, however, requires stronger regularity conditions on the incentives and on the final penalty function p. For the rest of the section we will therefore work under the following additional assumption.

ASSUMPTION 3.1 *Continuous time incentives are Markovian, i.e. $s_t = s(t, X_t)$, where (with a slight abuse of notation) $s : [0,T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is $C^{1,2}$, bounded ($m \leq s \leq M$) and with bounded derivatives. Moreover $s_x(\cdot, x) \leq 0$.*
The functions $x \mapsto p(x)$ and $x \mapsto xp'(x)$ defined on $\mathbb{R}_+$ are positive, bounded and C^∞.

Remark that supposing $s_x(\cdot,x) \leq 0$ and $p'(x) \geq 0$ reflects the natural assumption that higher emissions should induce lower incentives and a higher final fee[§].
We are now able to give a BSDE characterization of the optimal effort that results from BSDE (2.14). Recall that it might not be the unique optimal effort, in the sense that there may exist other optimal policies which are not strongly admissible.

[§]Remark that the hypothesis $s_x(\cdot,x) \leq 0$ is indeed quite natural but it might not be always optimal for the principal: in certain cases he may be willing to offer higher incentives when emissions are high, with the aim of inducing more effort and thus reduce the final social cost at maturity. See Section 5.

Proposition 3.1 *Under Assumption 3.1, if the optimal effort is strongly admissible then it follows the BSDE*

$$\begin{cases} -dk_t = \left[G(t,X_t,k_t)\Theta_t^2 + D(t,X_t,k_t)\Theta_t + C(t,X_t,k_t)\right]dt - \Theta_t dW_t^0 \\ k_T = F(s(T,X_T), -\sigma X_T p'(X_T)) \end{cases} \tag{3.2}$$

where

$$\begin{aligned} G(t,x,k) &= \frac{1}{2}\frac{g_{kk}}{g_k} \\ D(t,x,k) &= \frac{l(k)}{\sigma} + \frac{s_x\sigma x}{g_k^2}\left(g_{kk}g_s + g_{sk}g_k\right) \\ C(t,x,k) &= \frac{g_s}{g_k}\frac{\partial}{\partial t}s + \frac{u'(s-c(k))}{g_k}s_x\sigma x + \frac{1}{2}\frac{\sigma^2x^2}{g_k}\left(g_{ss}s_x^2 + g_s s_{xx} + \frac{g_{kk}g_s^2}{g_k^2}s_x^2\right) \end{aligned}$$

(we omit the argument (s,k) from g and its derivatives, and (t,x) from s and its derivatives for the sake of clarity).

Proof. We start from the dynamics of the optimal agent's expected utility Y

$$\begin{cases} dX_t = \sigma X_t dW_t^0 \\ dY_t = -f(s(t,X_t),Z_t)dt + Z_t dW_t^0 \\ X_0 = x \\ Y_T = -p(X_T) \end{cases} \tag{3.3}$$

where $f(s,z) = zl(F(s,z))/\sigma + u(s - c(F(s,z)))$.
We want to recover the dynamics of Z_t starting from (3.3). Recall first that f is continuously differentiable in both variables with $f_z(s,z) = l(F(s,z))/\sigma$ and $f_s(s,z) = u'(s - c(F(s,z)))$: the first is bounded by assumption while the second can also be considered bounded since we assume a strongly admissible effort policy. Therefore we can assume that $Y_t = L(t,X_t)$ where L is $C^{1,2}$ (see Chapter 4, Theorem 2.3 in [8]). As a consequence we can write $Z_t = L_x(t,X_t)\sigma X_t$. We also have $\nabla Y_t = L_x(t,X_t)\nabla X_t$, which implies

$$Z_t = \sigma X_t(\nabla X_t)^{-1}\nabla Y_t,$$

where ∇ denotes the derivative of the process with respect to x. The dynamics of the tangent processes are given by

$$\begin{cases} d\nabla Y_t = -[f_z(s_t,Z_t)\nabla Z_t + f_s(s_t,Z_t)s_x(t,X_t)\nabla X_t]dt + \nabla Z_t dW_t^0 \\ d\nabla X_t = \sigma\nabla X_t dW_t^0 \\ d(\nabla X_t)^{-1} = -\sigma(\nabla X_t)^{-1}dW_t^0 + \sigma^2(\nabla X_t)^{-1}dt. \end{cases}$$

We have therefore $d[X_t(\nabla X_t)^{-1}] = 0$, so that $X_t(\nabla X_t)^{-1} = x$ and $dZ_t = \sigma x d\nabla Y_t$, hence Z follows the BSDE

$$\begin{cases} dZ_t = -[l(F(s_t,Z_t))/\sigma N_t + u'(s_t - c(F(s_t,Z_t)))s_x(t,X_t)\sigma X_t]dt + N_t dW_t^0 \\ Z_T = -\sigma X_T p'(X_T) \end{cases} \tag{3.4}$$

where we have identified N_t with $\sigma x\nabla Z_t$.
Now the optimal effort is given by $k_t = F(s_t,Z_t) = F(s(t,X_t),Z_t)$, therefore by Itô's rule we get

$$\begin{aligned} dk_t =& F_s(s_t,Z_t)s_x(t,X_t)dX_t + \frac{1}{2}\sigma^2X_t^2[F_{ss}(s_t,Z_t)s_x(t,X_t)^2 + F_s(s_t,Z_t)s_{xx}(t,X_t)]dt \\ &+ F_z(s_t,Z_t)dZ_t + \frac{1}{2}F_{zz}(s_t,Z_t)N_t^2dt + F_{sz}(s_t,Z_t)s_x(t,X_t)N_tX_t\sigma dt + F_s(s_t,Z_t)\frac{\partial}{\partial t}s(t,X_t)dt \\ =& [-G(t,X_t,k_t)\Theta_t^2 - D(t,X_t,k_t)\Theta_t - C(t,X_t,k_t)]dt + \Theta_t dW_t^0 \end{aligned}$$

where we identified $\Theta_t = F_z(s, Z_t)N_t + F_s(s, Z_t)s_x(t, X_t)\sigma X_t$. In the previous computation we used the fact that

$$F_z(s, z) = \frac{1}{g_k(s, F(s, z))},$$

therefore

$$F_{zz}(s, z) = \frac{-g_{kk}(s, F(s, z))F_z(s, z)}{[g_k(s, F(s, z))]^2},$$

and finally

$$\frac{F_{zz}}{F_z^2}(s, z) = -\frac{g_{kk}(s, F(s, z))}{g_k(s, F(s, z))}.$$

Similarly we have that $F_s = -\frac{g_s}{g_k}$, $F_{ss} = 2\frac{g_{sk}g_s}{g_k^2} - \frac{g_{ss}}{g_k}$, $F_{sz} = -\frac{g_{sk}}{g_k^2}$.

One last thing to be remarked is that we replaced $f_s(s_t, Z_t)$ with $f_s(s_t, g(s_t, k_t))$, and this is only justified when Z in (3.4) is negative. To prove this, note first that the term $f_s(s, z)s_x(t, x) = u'(s - c(F(s, z)))s_x(t, x)$ in the generator of Z in (3.4) is negative by Assumption 3.1. Since we also have $Z_T \leq 0$ by Assumption 3.1, the comparison theorem gives that $Z_t \leq 0$.

The dynamics of the optimal effort $k_t = F(s_t, Z_t)$ is therefore given by (3.2) as claimed. □

Remark 3.1 Using a strong formulation of the problem would lead to a state/adjoint system of the type

$$\begin{cases} dX_t = l(F(s_t, \tilde{Y}_t))X_t dt + \sigma X_t dW_t \\ d\tilde{Y}_t = -\sigma u'(s_t - c(F(s_t, \tilde{Y}_t)))s_x(t, X_t)X_t dt + \tilde{Z}_t dW_t \\ \tilde{Y}_T = -\sigma p'(X_T)X_T, \end{cases} \tag{3.5}$$

which is very similar to (3.4). Hence under this formulation the adjoint variable $\tilde{Y}$ would play the role of Z in the weak formulation, which would require stronger regularity assumptions on the penalty function from the beginning. Moreover, the drift component $l(F)$ moves from the backward to the forward part of the system, thus making (3.5) a *coupled* FBSDE (whose solvability is in general harder to prove).

3.1 Constant incentives

We now suppose that s is constant (that is, it no longer represents incentives but a constant agent's revenue). BSDE (3.2) for the agent's effort simplifies significantly to

$$\begin{cases} -dk_t = \left[G(s, k_t)\Theta_t^2 + \frac{l(k_t)}{\sigma}\Theta_t\right] dt - \Theta_t dW_t^0 \\ k_T = F(s, -\sigma X_T p'(X_T)) \end{cases} \tag{3.6}$$

where $G(s, k) = \frac{1}{2}\frac{g_{kk}(s,k)}{g_k(s,k)}$ ¶, while Z solves

$$\begin{cases} dZ_t = -\frac{l(F(s,Z_t))}{\sigma}N_t dt + N_t dW_t^0 \\ Z_T = -\sigma X_T p'(X_T) \end{cases} \tag{3.7}$$

Remark 3.2 Since we assume Z_T to be bounded, the comparison theorem gives us that Z_t is uniformly bounded, therefore the optimal effort is strongly admissible and (3.6) holds automatically without assuming strong admissibility as we did in Proposition 3.1.

We can also prove that $E^k\left[\int_t^T N_r dW_r^k \mid \mathcal{F}_t\right] = 0$, where $k = F(s, Z_t)$ is the optimal effort policy. Indeed we have that $dZ_t = N_t dW_t^k$, and so Z is a uniformly bounded local martingale under the measure Γ^k, hence a true martingale. This gives the representation

$$Z_t = -E^k\left[\sigma X_T p'(X_T) \mid \mathcal{F}_t\right].$$

¶The definition of G has been slightly changed and adapted to this particular case. We believe this is pretty natural and hope it will cause no confusion.

We now aim at studying the effects of risk aversion on the optimal effort. In order to do so we consider the power utility function $u(x) = u^\gamma/\gamma$, parametrized by $\gamma < 1$. The next result gives some sufficient conditions for the optimal effort to be increasing with respect to risk aversion.

PROPOSITION 3.2 *In the power utility case, if*

- $k_T \leq c^{-1}(s-1)$, *or equivalently*

$$p'(x)x \leq \frac{c'}{|l'|}\left(c^{-1}(s-1)\right) \tag{3.8}$$

  for all $x \in \mathbb{R}_+$,

- $c^{(3)}(k) \leq 0$, $l^{(3)}(k) \leq 0$,

then the optimal effort is decreasing in γ, *therefore increasing in the risk aversion coefficient* $1-\gamma$.

Proof. Remark that now the functions F and G also depend on γ, so we are allowed to differentiate them with respect to this variable. We want to apply the comparison theorem in [5] to the quadratic BSDE (3.6), therefore we need to study the reaction of its generator and terminal condition to a change in γ. As for the terminal condition, remark that $F_\gamma(s, Z_T) = -\frac{g_\gamma(s,k_T)}{g_k(s,k_T)}$, where we recall that in this case $g(s,k) = \sigma\frac{u'(s-c(k))c'(k)}{l'(k)} = \sigma\frac{\gamma(s-c(k))^{\gamma-1}c'(k)}{l'(k)}$. Hence we have that $F_\gamma(s, Z_T) \leq 0$ if $g_\gamma(s,k_T) \leq 0$, or equivalently $k_T \leq c^{-1}(s-1)$. This gives the first condition of the Proposition (by replacing $k_T = F(s, -\sigma X_T p'(X_T))$).
We now turn to the generator of (3.6): we compute

$$\begin{aligned}
G_\gamma(s,k) = &- \Big\{c'(k)l'(k)\left[(\gamma-1)^2c'(k)^4l'(k) + 3(s-c(k))^2l'(k)c''(k)^2\right.\\
&+(2\gamma-3)(s-c(k))c'(k)^3l''(k) - (s-c(k))^2c'(k)\left(3c''(k)l''(k) + l'(k)c^{(3)}(k)\right)\\
&\left.+(s-c(k))c'(k)^2\left((3-2\gamma)l'(k)c''(k) + (s-c(k))l^{(3)}(k)\right)\right]\Big\}\\
&\times\left((s-c(k))\left((\gamma-1)c'(k)^2l'(k) + (-s+c(k))l'(k)c''(k) + (s-c(k))c'(k)l''(k)\right)^2\right)^{-1}
\end{aligned}$$

and we remark that $G_\gamma(s,k) \leq 0$ if $c^{(3)}(k) \leq 0$ and $l^{(3)}(k) \leq 0$, which gives the second condition of the Proposition.
In order to conclude with the the comparison theorem stated in [5], Theorem 2.6, we need to ensure that the coefficient of the quadratic term in the generator stays bounded, which is guaranteed by the following observation: since Z_t is uniformly bounded we deduce that k_t is bounded away from $c^{-1}(s)-\epsilon$ for some $\epsilon > 0$. Now take a bounded function $\tilde{G}(s,k)$ which coincides with G when $k \leq c^{-1}(s)-\epsilon$: we deduce that the optimal effort still solves (3.6) with G replaced by $\tilde{G}$. □

The first condition in the previous result imposes that the optimal effort towards maturity must not be too high in relation to the agent's revenue, otherwise a higher risk aversion might induce the agent to decrease effort and save some money. The second condition is probably less intuitive and is related to the rate of increase of costs and benefits when effort is increased. It is going to be satisfied in the model that we consider in Section 4, i.e. by a quadratic cost function and $l(k) = \frac{1-k}{1+k}$.
The reaction of the optimal agent's expected utility to a change in risk aversion is less clear to investigate. To get an intuition of why this is so, notice that for example under the conditions of Proposition 3.2 a higher value of γ reduces the optimal effort: this increases the continuous-time part of the agent's utility but will also in general increase final emissions, thus reducing the expected utility.
We now examine the effects of volatility on the effort.

PROPOSITION 3.3 *Suppose the process* N *solution of* (3.4) *is negative for any* $\sigma > 0$. *Then if* $l(k) \leq 0$ *(resp* $l(k) \geq 0$*) the optimal effort is increasing (resp. decreasing) in* σ.

Proof. We use (3.6), and we remark that

$$\frac{\partial}{\partial\sigma}F(s,-\sigma xp'(x))=0$$

(since F also depends on σ through g) and that G does not depend on σ. If $N_t \le 0$ then $\Theta_t \ge 0$ by the proof of Proposition 3.1, and the claim follows by the comparison theorem for quadratic BSDEs in [5], using the function $\tilde{G}$ as explained in the proof of Proposition 3.2. □

The previous result requires the knowledge of N, which can be computed by solving the nonlinear PDE (4.1) that will be presented in the next section.
As for the dependence of the optimal effort on revenues s, the analysis is more complex: if we consider the terminal condition in (3.2), we see that $F_s(s,z) = -F_z g_s(s,F) \ge 0$, but the reaction of the generator to a change in s is harder to examine.

3.2 Impatience rate

Sometimes an *impatience rate* $\delta \ge 0$ is incorporated in principal-agent models (see [11]) in order to account for the time preferences of the agent, in the sense that he gives a lower weight to cash flows that are far away in the future. This can be easily done in our framework by reformulating the agent's expected utility in this way:

$$V(k)=E^k\left[\int_0^T e^{-\delta t}u(s_t-c(k_t))dt-e^{-\delta T}p(X_T)\right].$$

All the results above can be readily adapted with minor modifications. In particular the agent's conditional value function given constant incentives s now follows the BSDE

$$\begin{cases}-dY_t = f^\delta(t,s,Z_t)dt - Z_t dW_t^0\\ Y_T=-e^{-\delta T}p(X_T)\end{cases}$$

where $f^\delta(t,s,z)=zl(F^\delta(t,s,z))+e^{-\delta t}u(s-c(F^\delta(t,s,z)))$ and $F^\delta(t,s,z)$ is now the inverse (in k) of

$$g^\delta(t,s,k)=e^{-\delta t}\sigma\frac{u'(s-c(k))c'(k)}{l'(k)}. \tag{3.9}$$

In the same way as before we obtain the following BSDE for Z:

$$\begin{cases}dZ_t = -\frac{l(F^\delta(t,s,Z_t))}{\sigma}N_t dt + N_t dW_t^0\\ Z_T=-e^{-\delta T}\sigma X_T p'(X_T),\end{cases} \tag{3.10}$$

and the optimal effort therefore solves

$$\begin{cases}-dk_t = \left[-\delta\frac{g(t,s,k_t)}{g_k(t,s,k_t)}+G(t,s,k_t)\Theta_t^2+\frac{l(k_t)}{\sigma}\Theta_t\right]dt - \Theta_t dW_t^0\\ k_T=F^\delta(t,s,-e^{-\delta T}\sigma X_T p'(X_T))=F(s,-\sigma X_T p'(X_T))\end{cases} \tag{3.11}$$

where $G(t,s,k)=\frac{1}{2}\frac{g^\delta_{kk}(t,s,k)}{g^\delta_k(t,s,k)}=\frac{1}{2}\frac{g_{kk}(t,s,k)}{g_k(t,s,k)}$. Here we used the fact that $\frac{\partial}{\partial t}F^\delta(t,s,z)=\delta\frac{g^\delta(t,s,F^\delta)}{g^\delta_k(t,s,F^\delta)}=\delta\frac{g(t,s,F^\delta)}{g_k(t,s,F^\delta)}$. Now since $\frac{g^\delta}{g^\delta_k}\ge 0$ and G does not depend on δ, we deduce the following result.

Proposition 3.4 *The optimal effort is decreasing in the impatience rate δ.*

Since the terminal condition in (3.11) does not depend on δ, we see that the changes will be more relevant as the time to maturity increases. Remark that this result also holds when incentives s are not necessarily constant, though we presented it in this simpler case.

4 Numerical computation of the optimal contract

We still assume that continuous-time incentives are constant, or at least space independent (i.e. $s = s(t)$).
From (3.7) and Assumption 3.1 we can write $Z_t = \phi(t, X_t)$ where ϕ solves (in the classical sense, see [8], Chapter 9, Section 2.1)

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 x^2 \phi_{xx} + l(F(s,\phi))x\phi_x = 0 \\ \phi(T,x) = -\sigma x p'(x) \end{cases} \tag{4.1}$$

which is usually easier to treat than (3.6). The idea is therefore to approximate Z first and then recover k. We set $y = \log x$ and $\theta(t,y) = \phi(T-t,x)$ from which we get

$$\begin{cases} \theta_t - \frac{1}{2}\sigma^2 \theta_{yy} - b(\theta)\theta_y = 0 \\ \theta(0,y) = -\sigma e^y p'(e^y) \end{cases} \tag{4.2}$$

where $b(\theta) = l(F(s,\theta)) - \frac{1}{2}\sigma^2$. The solution to (4.2) can be approximated numerically using a standard scheme that we briefly recall and adapt to our case (see [8], Chapter 9 for details).
We set the space and time steps $h > 0$, $\Delta t > 0$. We let $y_i = ih$, $i = 0, \pm 1, ..., \pm i_0$, and $t^j = j\Delta t$, $j = 0, 1, ..., N$. We denote $h_i^j = h(t^k, y_i)$ the grid value of the function h, and $h^j = h(t^j, \cdot)$. We define for each j the approximate solution w^j by the following recursive steps:

(i) *Step 0*: Set $w_i^0 = -\sigma e^{y_i} p'(e^{y_i})$, $i = 0, \pm 1, ..., \pm i_0$; use linear interpolation to obtain a function $w^0(y)$ defined on $y \in \mathbb{R}$.

(ii) *Step j*: Suppose that $w^{j-1}(y)$ is defined for $y \in \mathbb{R}$ and set

$$\begin{cases} b_i^j = b(w_i^{j-1}) \\ \bar{y}_i^j = y_i - b_i^j \Delta t, \quad \bar{w}_i^{j-1} = w^{j-1}(\bar{y}_i^j) \\ \delta^2(w)_i^j = h^{-2}[w_{i+1}^j - 2w_i^j + w_{i-1}^j] \end{cases}$$

Obtain the grid values for the j-th step approximate solution by solving

$$\frac{w_i^j - \bar{w}_i^{j-1}}{\Delta t} = \frac{1}{2}\sigma^2 \delta^2(w)_i^j.$$

Use again linear interpolation to extend the grid values to all $y \in \mathbb{R}$.

Define the error function on the grid by $\zeta_i^j = \theta_i^j - w_i^j$, where θ_i^j represent grid values for the true solution. One can prove that

$$\sup_{j,i} |\zeta_i^j| = \mathcal{O}(h + \Delta t).$$

The approximation for the optimal effort is then recovered by setting $k_i^j = F(s, w_i^j)$. Since F has bounded derivatives, the same $\mathcal{O}(h + \Delta t)$ rate of convergence holds for the approximation of the optimal effort.
We just mention for completeness (without discussing regularity and convergence of the numerical schemes) the two other possible ways to compute the optimal effort. In the first we use (3.6) and supposing $k_t = \varphi(t, X_t)$ we recover φ as the solution to

$$\begin{cases} \varphi_t + \frac{1}{2}\sigma^2 x^2 \left[\varphi_{xx} + G(s,\varphi)(\varphi_x)^2\right] + l(\varphi)x\varphi_x = 0 \\ \varphi(T,x) = F(s, -\sigma x p'(x)). \end{cases} \tag{4.3}$$

Another idea is to write the analogue of the PDE (2.12) that takes into account the implementability constraints by forcing $k_t = F(s, Z_t) = F(s, \theta_x \sigma x)$:

$$\begin{cases} \theta_t + \frac{1}{2}\theta_{xx}\sigma^2 x^2 + x\theta_x l(F(s,\theta_x\sigma x)) + u(s - c(F(s,\theta_x\sigma x))) = 0, \\ \theta(T,x) = -p(x). \end{cases} \tag{4.4}$$

4.1 Interpretation of results

For our numerical experiments we take the functions $c(k) = k^2/2$, $u(x) = 2\sqrt{x}$ and $l(k) = \frac{1-k}{1+k}$ (the first two are quite standard, the third is just a decreasing bounded function on $[0, \infty)$ with bounded first derivative).

Figure 1 shows the numerical approximation of the optimal effort by choosing (a proper regularization of) a penalty function of the type $p(x) = \lambda \mathbf{1}_{[\Lambda,\infty)}(x)$ (i.e. a fixed amount is charged when a certain level of emissions is exceeded). The economic interpretation is straightforward: since s is time and space independent, it can be more naturally considered as an income flow, and not as a real incentive policy. Therefore in this example we have in a sense isolated the effects on effort provided by the final fee to pay at maturity T. At every date the optimal effort is bell-shaped: loosely speaking, when emissions are too high the firm has little hope to reduce them and finds no reason to bear the cost of trying (the fee being fixed); on the other hand, when emissions are sufficiently small the agent can be reasonably sure that they will end up below Λ even without any positive effort. As maturity approaches, the short time left to act makes it optimal to take on some effort only when emissions are close to Λ.

The situation changes if we choose a penalty function of the type $p(x) = \lambda(x - \Lambda)^+$, corresponding to a situation where the agent is charged proportionally for each unit of emissions that exceeds a certain threshold Λ at maturity. This case is shown in Figure 2: we see that it is no more optimal to stop putting effort when emissions are high, since there is always an opportunity to reduce the final payment.

In Figure 3 we plotted some simulated paths by using (1.1) and the optimal effort dynamics of Figure 2: we observe a natural tendency for emissions to be driven close to the threshold level at maturity.

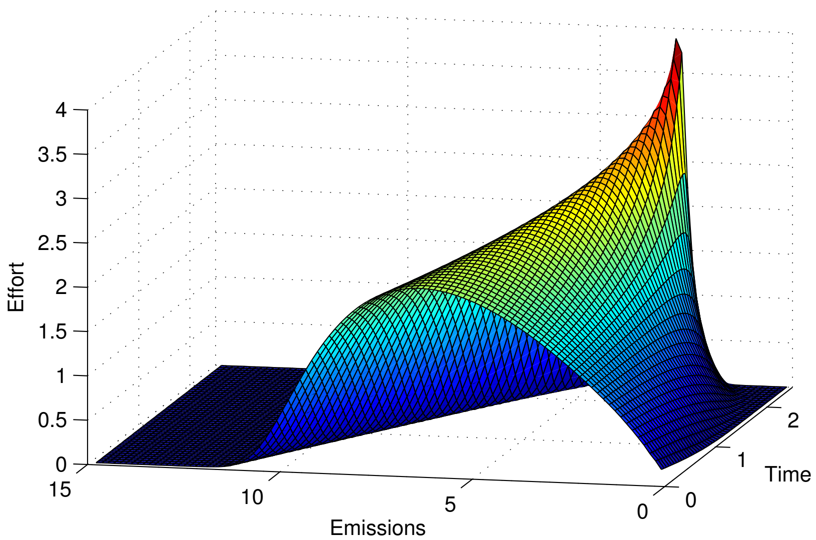

Figure 1: Optimal effort dynamics with a constant incentive policy $s_t = 10$ and fee $p(x) = 4\mathbf{1}_{[3,\infty)}(x)$. Parameter values: $\sigma = 0.22$, $\gamma = 0.5$, $T = 2.5$.

The effort dynamics of the previous examples can be considered as a benchmark situation when there are no continuous-time incentives (or they are trivially constant). The principal can further modify this basic effort structures by properly acting on s, according for example to some social cost function (as we will see the next section).

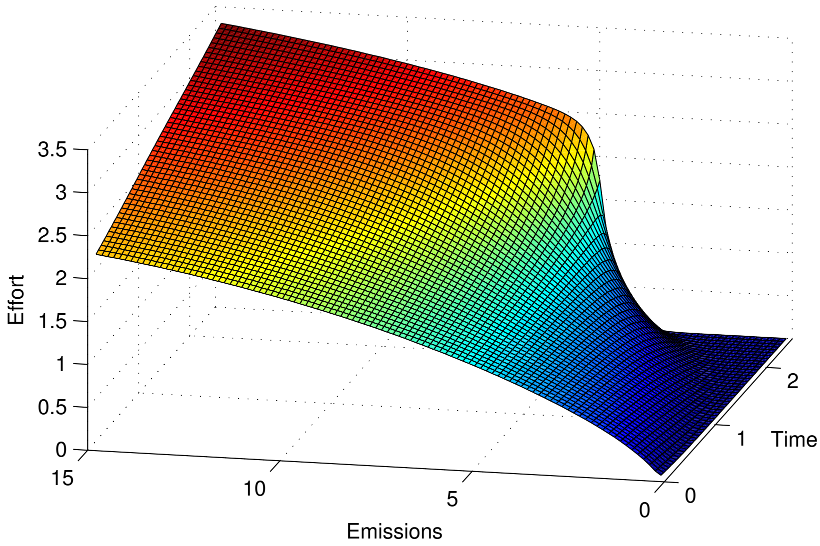

Figure 2: Optimal effort dynamics with a constant incentive policy $s_t = 10$ and $p(x) = (x - 5)^+$

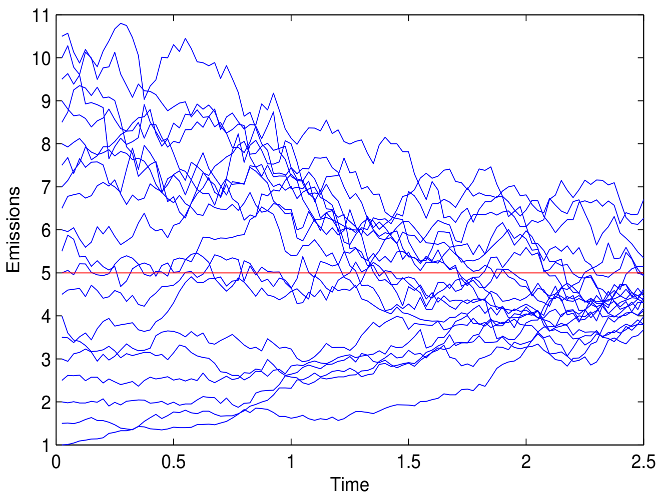

Figure 3: Simulation of typical emissions paths following the agent's optimal effort with a constant incentive policy $s_t = 10$ and different starting values. The red line shows $\Lambda = 5$.

5 The principal's problem

We will stick to the weak formulation to treat the principal's optimization problem, in order to avoid measurability issues and inconsistencies that can arise when one switches from the two formulations (as we partially mentioned in the Introduction and as is also reported in [6]).
In the agent's case we were trying to find the best possible effort policy k given an incentive structure made up by a penalty function p and continuous-time payments s. When considering the principal, we therefore look for a criterion to find the couple (s, p) that maximizes a certain utility functional. To do so, we model the principal's expected profit given (s, p, k) as

$$\begin{aligned} I(s,p,k) &= E^k \left[p_1(p(X_T)) - p_2(X_T) - \int_0^T u_1(s_r)dr \right] \\ &= E \left[\Gamma_T^k p_1(p(X_T)) - \Gamma_T^k p_2(X_T) - \int_0^T \Gamma_r^k u_1(s_r)dr \right], \end{aligned} \tag{5.1}$$

where

(i) $p_1 : \mathbb{R} \mapsto \mathbb{R}$ is a concave function, C^2 and with bounded derivatives. It relates the final (dis)utility of the agent to the final utility of the principal.

(ii) $p_2 : \mathbb{R}_+ \to \mathbb{R}_+$ has the role to capture the social costs related to the level of emissions. We assume $p_2(X_T) \in L^{2+\alpha}$ for some $\alpha > 0$.

(iii) $u_1 : \mathbb{R}_+ \mapsto \mathbb{R}$ is a C^2 utility function (i.e. increasing and concave) that takes into account continuous-time payments to the agent.

If we assume, however, that the state can forecast the optimal agent's response given (s, p), it is convenient to simply consider $I(s, p)$, defined by replacing k in (5.1) with the optimal effort policy given (s, p) (forgetting for a moment that this might not be unique). This way of proceeding, though quite natural, is not very easy to pursue as it would involve some form of optimization over a function space to recover p.
Since the state knows the optimal agent's reaction given his choices, a more efficient way to attack the problem is to directly assume that he is able to choose the couple (s, k), provided he then adjusts the final fee structure accordingly. Quite intuitively, however, with no other constraints the problem will easily be ill-posed: in fact, if the state can arbitrarily increase the final fees then his maximal expected profit will diverge to infinity (and the agent's one to minus infinity). To avoid this problem, in the classical literature on private contracts (see [6], [11], [13]) the principal also has to guarantee to the agent a certain initial utility, that has to be coherent with the other opportunities available on the market. We are not in such a situation since the agent is now typically forced to enter the contract, however we still assume that the state chooses to provide the agent with a certain initial utility R (which can be a function of initial emissions). The assumption is quite reasonable considering that the aim of the state is not to ruin the firm but rather to push it to act in some socially convenient way.
Recall that the agent's utility given (s, k, p) follows the BSDE (where we add the superscript "A" for "Agent")

$$\begin{cases} dY_t^A = [-Z_t^A l(k_t)/\sigma - u(s_t - c(k_t))]dt + Z_t^A dW_t^0 \\ Y_T^A = -p(X_T) \end{cases} \tag{5.2}$$

If, however, the principal chooses the initial agent's utility, the terminal condition above is replaced by the initial condition $Y_0^A = R$, thus the backward SDE (5.2) becomes a forward SDE for the principal and the terminal value $Y_T^A =: -C_T$ will now be an output of the initial choice of (s, k), once Z^A is fixed. Clearly C_T will not in general be of the form $p(X_T)$ (see Example 5.1), but we will still write (s, C_T, k) with a slight abuse of notation to refer to a policy with C_T as final penalty.
Now remark that we must also make sure that the resulting triplet (s, C_T, k) is implementable: in

other words, when the agent faces the incentive structure (s, C_T) he must actually optimally choose k. By Proposition 2.2, this can be achieved by setting $Z_t^A = g(s_t, k_t)$ in (5.2), which then becomes

$$Y_t^A = R - \int_0^t [u(s_r - c(k_r)) + g(k_r, s_r)l(k_r)/\sigma]dr + \int_0^t g(k_r, s_r)dW_r^0 \tag{5.3}$$

We define $C_T^{(s,k)} := -Y_T^A$, where Y^A follows (5.3) with (s, k). The next result is now straightforward.

Lemma 5.1 *The triplet $(s, C_T^{(s,k)}, k)$ is the unique implementable policy for any strongly admissible couple (s, k).*

Proof. Since (s, k) is strongly admissible, it follows that $g(s_t, k_t)$ stays bounded and consequently $C_T^{(s,k)}$ is also admissible (it is in $L^{2+\alpha}(\Omega)$ for any $\alpha > 0$). Now remark that, even if Proposition 2.2 is stated for final penalties of the form $p(X_T)$, it still holds for a general claim $\xi \in L^{2+\alpha}(\Omega, \mathcal{F}_T)$. Theorem 2.1 then gives uniqueness. □

Example 5.1 Suppose the state aims at inducing a constant level of effort $k_t = \bar{k}$ over time. Assume continuous time incentives are kept constant at some $s_t = \bar{s}$ such that $\bar{k} < c^{-1}(\bar{s})$ (strong admissibility). Denote $\bar{g} = g(\bar{s}, \bar{k})$, $\bar{u} = u(\bar{s} - c(\bar{k}))$ and $\bar{l} = l(\bar{k})$. The final penalty that has to be proposed in this case is then

$$C_T^{(s,k)} = -R + [\bar{u} + \bar{g}\bar{l}/\sigma]T + |\bar{g}|W_T^0.$$

Since $X_T = x\exp\{\sigma W_T^0 - \sigma^2 T/2\}$ we can write

$$C_T^{(s,k)} = -R + \left[\bar{u} + \bar{g}\frac{\bar{l} + \sigma^2/2}{\sigma}\right]T + \frac{|\bar{g}|}{\sigma}\log\frac{X_T}{x}.$$

We see that, even in this simple example, the final penalty is not of the form $p(X_T)$, since the initial value x of the emissions process appears in the formula. Indeed this kind of final fee penalizes the proportional increase in the emissions' level from the beginning of the period. Remark also that if the proportional reduction is sufficiently large then $C_T^{(s,k)}$ can become negative and therefore represents a reward more than a fee.
Conversely, once continuous time incentives are constant and fixed at $\bar{s}$, any final penalty of the form $K + BW_T^0$ for some $K \in \mathbb{R}$ and $B \in \mathbb{R}_+$ induces a constant optimal effort, which can be recovered by solving for $\bar{k}$ the equation $|g(\bar{s}, \bar{k})| = B$. The corresponding agent's initial utility is then given by $R = \left[\bar{u} + \bar{g}\frac{\bar{l}+\sigma^2/2}{\sigma}\right]T - K$.

We then naturally redefine the expected profit of the principal as

$$J(s, k) = E\left[\Gamma_T^k p_1(C_T^{(s,k)}) - \Gamma_T^k p_2(X_T) - \int_0^T \Gamma_r^k u_1(s_r)dr\right]. \tag{5.4}$$

The principal's optimization problem is

$$v_P := \sup_{(s,k)} J(s, k), \tag{5.5}$$

where the sup is taken over all strongly admissible policies. In order to solve it, we now need one additional state equation, so that our state system becomes

$$\begin{cases} \Gamma_t^k = 1 + \int_0^t \Gamma_r^k l(k_r)/\sigma dW_r^0 \\ Y_t^A = R - \int_0^t [u(s_r - c(k_r)) + g(k_r, s_r)l(k_r)/\sigma]dr + \int_0^t g(k_r, s_r)dW_r^0. \end{cases}$$

We will apply again the SMP (Theorem 3.2 in [12]). Define the two adjoint processes

$$\begin{cases} dY_t^P = [-l(k_t)/\sigma Z_t^P + u_1(s_t)]dt + Z_t^P dW_t^0 \\ dY_t^1 = Z_t^1 dW_t^0 \\ Y_T^P = p_1(-Y_T^A) - p_2(X_T) \\ Y_T^1 = -\Gamma_T^k p_1'(-Y_T^A) \end{cases} \tag{5.6}$$

and the Hamiltonian

$$\tilde{H}(\Gamma^k, Z^P, Y^1, Z^1, s, k) = -Y^1[u(s-c(k)) + g(k,s)l(k)/\sigma] + Z^P\Gamma^k l(k)/\sigma + Z^1 g(k,s) - \Gamma^k u_1(s). \quad (5.7)$$

The next result gives necessary conditions for optimality.

PROPOSITION 5.1 *Suppose the strongly admissible contract (s^*, k^*) is optimal for the principal's problem. Then there exist two pairs of processes given by (5.6) such that (dropping *-superscripts for clearness)*

$$\begin{cases} -Y_t^1[u'(s_t - c(k_t)) + g_s(k_t, s_t)l(k_t)/\sigma] + Z_t^1 g_s(k_t, s_t) - \Gamma_t^k u_1'(s_t) = 0 & \text{on } \{m < s_t < M\} \\ -Y_t^1[u'(s_t - c(k_t)) + g_s(k_t, s_t)l(k_t)/\sigma] + Z_t^1 g_s(k_t, s_t) - \Gamma^k u_1'(s_t) \geq 0 & \text{on } \{s_t = M\} \\ -Y_t^1[u'(s_t - c(k_t)) + g_s(k_t, s_t)l(k_t)/\sigma] + Z_t^1 g_s(k_t, s_t) - \Gamma^k u_1'(s_t) \leq 0 & \text{on } \{s_t = m\} \\ -Y_t^1 g_k(k_t, s_t)l(k_t)/\sigma + Z_t^P \Gamma_t^k l'(k_t)/\sigma + Z_t^1 g_k(k_t, s_t) = 0 & \text{on } \{k_t > 0\} \\ -Y_t^1 g_k(k_t, s_t)l(k_t)/\sigma + Z_t^P \Gamma_t^k l'(k_t)/\sigma + Z_t^1 g_k(k_t, s_t) \leq 0 & \text{on } \{k_t = 0\} \end{cases} \quad (5.8)$$

Proof. Similar to Proposition 2.1. Here we have the additional control s, which by admissibility (see Definition 2.1) takes its values in $[m, M]$. □

Sufficient conditions are much harder to derive in this case, with respect to the agent's problem. Indeed, a long but straightforward calculation gives, for any admissible control (s, k), that

$$\begin{aligned} J(s,k) - J(s^*, k^*) &= Y_0^P - Y_0^{P^*} = E^{k^*}[Y_0^P - Y_0^{P^*}] = E[\Gamma_T^{k^*}(Y_0^P - Y_0^{P^*})] \\ &= E\left[\int_0^T [\tilde{H}(\Theta_r^*, Z_r^P, s_r, k_r) - \tilde{H}(\Theta_r^*, Z_r^{P*}, s_r^*, k_r^*) - \tilde{H}_{Z^P}(\Theta_r^*, Z_r^{P*}, s_r^*, k_r^*)(Z^P - Z^{P*})]dr\right] \end{aligned}$$

where we call $\Theta_r^* := (\Gamma_r^{k^*}, Y_r^{1*}, Z_r^{1*})$. In order to conclude that $J(s,k) - J(s^*, k^*) \leq 0$ we need to show that $\tilde{H}$ is jointly concave in (Z^P, s, k), but this is not true because of the term $Z^P\Gamma^k l(k)/\sigma$. We will be able to give some sufficient conditions in the particular case studied the next section.

5.1 The case $p_1(x) = x$

Let $p_1(x) = x$, i.e. the final agent's disutility linked to the payment of the fee corresponds to a principal's utility of the same amount. Then $Y_t^1 = -\Gamma_t^k$ and hence $Z_t^1 = -\Gamma_t^k l(k_t)/\sigma$. The necessary conditions now simply become

$$\begin{cases} u'(s_t - c(k_t)) - u_1'(s_t) = 0 & \text{on } \{m < s_t < M\} \\ u'(s_t - c(k_t)) - u_1'(s_t) \geq 0 & \text{on } \{s_t = M\} \\ u'(s_t - c(k_t)) - u_1'(s_t) \leq 0 & \text{on } \{s_t = m\} \\ l'(k_t)Z_t^P = 0 & \text{on } \{k_t > 0\} \\ l'(k_t)Z_t^P \leq 0 & \text{on } \{k_t = 0\} \end{cases} \quad (5.9)$$

The first condition has a clear economic meaning: the principal will choose continuous-time incentives s_t in such a way that, at any time, the marginal cost $u_1'(s_t)$ of an additional quantity is equal to the marginal benefit $u'(s_t - c(k_t))$[‖].

Finding a candidate solution to (5.9) might not be trivial. Here we suggest a possible way to proceed, by working under the following assumption (which includes the definition of the two additional functions I and L).

[‖]The quantity $u'(s_t - c(k_t))$, by (5.3), can be seen as the average increase in the agent's final fee following an increase in s_t (recall that an increase in continuous time incentives in this context reduces the average final fee, as the initial agent's utility is fixed). Since $p_1(x) = x$, the same quantity $u'(s_t - c(k_t))$ is also interpreted as a marginal benefit to the principal.

Assumption 5.1 *Suppose that*

(i) *We are able to invert uniquely the first three conditions in* (5.9) *by constructing a continuous and a.e. differentiable function I such that $s = I(k)$ verifies them.*

(ii) *We can uniquely define a continuous and a.e. differentiable function $L(z)$ that solves (in k) the implicit equation $g(I(k), k) = z$ for all $z \leq 0$. We set $L(z) = 0$ if $z > 0$ (as we did with the function F in the preceding sections).*

Also suppose that $k_t > 0$ a.s. for all $t \in [0, T]$ at the optimum. Then the agent's optimality conditions (2.6) now give that $Z_t^A = g(s_t, k_t) = g(I(k_t), k_t)$, hence $k_t = L(Z_t^A)$. Also (5.9) implies that at the optimum $Z_t^P = 0$, therefore

$$Y_0^P = C_T^{(s,k)} - p_2(X_T) - \int_0^T u_1(s_r)dr,$$

so that $C_T^{(s,k)} = p_2(X_T) + \int_0^T u_1(s_r)dr + c$ for some constant $c \in \mathbb{R}$. This allows to identify the optimal terminal condition to the agent's problem. Now plugging this into the agent's BSDE we get

$$\begin{cases} dY_t^A = [-Z_t^A l(L(Z_t^A))/\sigma - u(I(L(Z_t^A)) - c(L(Z_t^A)))]dt + Z_t^A dW_t^0 \\ Y_T^A = -p_2(X_T) - \int_0^T u_1(I(L(Z_t^A)))dt - c \end{cases} \tag{5.10}$$

where the parameter c is there to ensure that $Y_0^A = R$. We then have the following corollary to the necessary conditions.

Corollary 5.1 *Suppose the strongly admissible contract (s^*, k^*) is optimal for the principal's problem* (5.5) *with $p_1(x) = x$ and verifies $k_t^* > 0$ a.s. for all $t \in [0, T]$. Then, under Assumption 5.1, there exists a solution (Y^A, Z^A) to* (5.10) *and $k_t^* = L(Z_t^A)$, $s_t^* = I(k_t^*) = I(L(Z_t^A))$.*

Conversely, if a solution to (5.10) exists, then it will be a candidate for the optimal solution. In order to derive some sufficient conditions, we introduce the modified Hamiltonian

$$H^M(s, k, z) := -u_1(s) + u(s - c(k)) + zl(k)/\sigma.$$

Proposition 5.2 *Suppose Assumption 5.1 holds and that* (5.10) *admits a solution (Y^{A*}, Z^{A*}). Denote $k_t^* = L(Z_t^{A*})$ and $s_t^* = I(k_t^*)$. Then (s^*, k^*) is optimal for the principal's problem* (5.5) *with $p_1(x) = x$.*

Proof. From $C_T^{(s^*,k^*)} = -Y_T^{A*}$ we get

$$\begin{cases} dY_t^{P*} = [-l(k_t^*)/\sigma Z_t^{P*} + u_1(s_t^*)]dt + Z_t^{P*} dW_t^0 \\ Y_T^{P*} = C_T^{(s^*,k^*)} - p_2(X_T) = \int_0^T u_1(s_t^*)dt + c, \end{cases}$$

therefore $Z_t^{P*} = 0$ a.s. for all $t \in [0, T]$. Hence we have for any strongly admissible couple (s, k)

$$\begin{aligned} J(s,k) - J(s^*, k^*) =& E^k\left[p_1(C_T^{(s,k)}) - p_2(X_T) - \int_0^T u_1(s_r)dr\right] - E^k\left[Y_0^{P*}\right] \\ =& E^k\left[C_T^{(s,k)} - C_T^{(s^*,k^*)} - \int_0^T [u_1(s_r) - u_1(s_r^*)]dr\right] \\ =& E^k\left[-\int_0^T [u_1(s_r) - u_1(s_r^*)]dr\right] \\ &+ E^k\left[\int_0^T \{[u(s_r - c(k_r)) - u(s_r^* - c(k_r^*))]dr + g(k_r^*, s_r^*)[l(k_r) - l(k_r^*)]/\sigma\}dr\right] \\ =& E^k\left[\int_0^T [H^M(s_r, k_r, g(s_t^*, k_t^*)) - H^M(s_r^*, k_r^*, g(s_t^*, k_t^*))]dr\right] \leq 0 \end{aligned}$$

The last line follows from the fact that by definition (s_t^*, k_t^*) is the stationary point of $H^M(\cdot,\cdot,g(s_t^*,k_t^*))$. Indeed, the FOC in s gives that $s = I(k)$, then substituting in $H^M(\cdot,\cdot,g(s_t^*,k_t^*))$ gives

$$H^M(I(k),k,g(s_t^*,k_t^*)) = -u_1(I(k)) + u(I(k)-c(k)) + g(I(k_t^*),k_t^*)l(k)/\sigma.$$

Differentiating in k we obtain

$$\begin{aligned}H_k^M(I(k),k,g(s_t^*,k_t^*)) &= I'(k)[u'(I(k)-c(k)) - u_1'(I(k))] - u'(I(k)-c(k))c'(k) + g(I(k_t^*),k_t^*)l'(k)/\sigma \\ &= -u'(I(k)-c(k))c'(k) + g(I(k_t^*),k_t^*)l'(k)/\sigma\end{aligned}$$

by the FOC in s (remark that $I'(k) = 0$ iff $s = I(k) = m$ or $s = I(k) = M$ since u is strictly increasing, hence the first term in the expression disappears). Equating to zero we obtain $g(I(k),k) = g(I(k_t^*),k_t^*)$, implying $k = k_t^*$ by Assumption 5.1 (ii). We now want to prove that this unique stationary point is a global maximum. It suffices to notice that, since $g(s_t^*,k_t^*) \le 0$, $H_k^M(I(0),0,g(s_t^*,k_t^*)) = g(s_t^*,k_t^*)l'(k)/\sigma \ge 0$ and $H_k^M(I(k),k,g(s_t^*,k_t^*)) \to -\infty$ when $k \to c^{-1}(M)$, which implies the claim. $\square$

Remark that we did not need joint concavity of $H^M(\cdot,\cdot,g(s_t^*,k_t^*))$ with respect to (s,k), as is quite common for this kind of results (see [6]). This would have required verifying that the matrix

$$\begin{bmatrix} u''(s-c(k)) - u_1''(s) & -u''(s-c(k))c'(k) \\ -u''(s-c(k))c'(k) & u''(s-c(k))c'(k)^2 - u'(s-c(k))c''(k) + g(I(k_t^*),k_t^*)l''(k) \end{bmatrix}$$

is negative semi-definite, which can be hard to do in practice and might not be true in general.

5.1.1 A special case

Here we consider the power utility function $u(x) = x^\gamma/\gamma$ and we take $u_1(x) = 2u(x)$ (see Remark 5.2 below). Setting $K_m = c^{-1}(m(1-2^{\frac{1}{\gamma-1}}))$ now the function I defined in Assumption 5.1 (i) takes the form

$$I(k) = \begin{cases} \frac{c(k)}{1-2^{\frac{1}{\gamma-1}}} & \text{if } K_m \le k \le K_M \\ m & \text{if } 0 \le k \le K_m \\ M & \text{if } k \ge K_M. \end{cases}$$

In order to apply Proposition 5.2 we still need to verify that point (ii) in Assumption 5.1 holds, that is we need to show that the function $\tilde g(k) := \sigma\frac{u'(I(k)-c(k))c'(k)}{l'(k)}$ is strictly decreasing (as clearly $\tilde g(0) = 0$ and $\tilde g(k) \to -\infty$ as $k \to c^{-1}(M)$). Computing its derivative gives

$$\tilde g'(k) := \sigma\frac{[u''(I(k)-c(k))(I'(k)-c'(k))c'(k) + u'(I(k)-c(k))c''(k)]l'(k) - u'(I(k)-c(k))c'(k)l''(k)}{l'(k)^2},$$

hence a sufficient condition (remarking that the last term in the espression above is strictly positive except at $k = 0$) is

$$u''(I(k)-c(k))(I'(k)-c'(k))c'(k) + u'(I(k)-c(k))c''(k) \ge 0 \tag{5.11}$$

a.e. on $K_m \le k \le K_M$, which in our power utility case is equivalent to

$$(\gamma-1)c'(k)^2 + c(k)c''(k) \ge 0.$$

With a quadratic cost function this is verified for $1/2 \le \gamma < 1$.
Defining $\tilde Y_t^A = Y_t^A + \int_0^t 2u(I(L(Z_r^A)))dr$ then (5.10) becomes

$$\begin{cases} -d\tilde Y_t^A = \tilde f(Z_t^A)dt - Z_t^A dW_t^0 \\ \tilde Y_T^A = -p_2(X_T) - c \end{cases} \tag{5.12}$$

with $\tilde{f}(z) = zl(L(z))/\sigma + u(I(L(z)) - c(L(z))) - 2u(I(L(z)))$. Using the definitions of I and L we can compute

$$
\begin{aligned}
\tilde{f}'(z) =& l(L(z))/\sigma + zl'(L(z))L'(z)/\sigma \\
&+ u'(I(L(z)) - c(L(z)))[I'(L(z)) - c'(L(z))]L'(z) - 2u'(I(L(z)))I'(L(z))L'(z) \\
=& l(L(z))/\sigma + [zl'(L(z))/\sigma - u'(I(L(z)) - c(L(z)))c'(L(z))]L'(z) \\
&+ [u'(I(L(z)) - c(L(z))) - 2u'(I(L(z)))]I'(L(z))L'(z) \\
=& l(L(z))/\sigma.
\end{aligned}
$$

Remark that, even if I and L are not differentiable at two points the previous equality still holds, by considering I' and L' as a left/right derivatives in the first place.
As in the proof of Proposition 3.1 we can show that, under Assumption 3.1 for p_2, Z^A follows the BSDE

$$
\begin{cases} dZ_t^A = -\frac{l(L(Z_t^A))}{\sigma} N_t dt + N_t dW_t^0 \\ Z_T^A = -\sigma X_T p_2'(X_T). \end{cases} \tag{5.13}
$$

In this way the same numerical method proposed in Section 4 can be applied to (5.13) with minor modifications to recover the optimal effort $k_t = L(Z_t^A)$ and the optimal incentives $s_t = I(k_t)$.

Example 5.2 An example is shown in Figure 4, where we used again the functions $l(k) = \frac{1-k}{1+k}$, $c(k) = k^2/2$ and $u(x) = 2\sqrt{x}$. We chose (a mollified version of) the capped proportional penalty function $p_2(x) = (x-4)^+ - (x-8)^+$ and the minimal (maximal) incentive value is set to $m = 2$ ($M = 10$). Assumption 5.1 holds in this case by the previous discussion, since $\gamma = 1/2$.
The shape of the optimal effort is similar to the one we have already seen in Section 4.1. As for the optimal incentives, they are set most of the time at their minimal level $m = 2$, while they are raised towards the end of the period in order to generate a higher effort in the region where it is more effective (i.e. for emissions values between 4 and 8). Notice that continuous-time incentives are not necessarily a decreasing function of X: higher emissions (notably towards maturity) may induce the principal to increase incentives, in order to generate a higher effort which will reduce the final social cost at maturity.

Remark 5.1 Suppose the principal does not optimize over continuous-time incentives s, so that s is just a constant. The candidate final fee is given by $C_T^{(s,k)} = p_2(X_T) + \beta$ for some constant $\beta \in \mathbb{R}$ and the associated agent's BSDE is now

$$
\begin{cases} dY_t^A = [-Z_t^A l(F(s, Z_t^A))/\sigma - u(s - c((F(s_t, Z_t^A)))]dt + Z_t^A dW_t^0 \\ Y_T^A = -p_2(X_T) - \beta \end{cases} \tag{5.14}
$$

which is equivalent to the agent's formulation with final fee p_2.

Remark 5.2 The choice of $u_1(x) = 2u(x)$ (which readily generalizes to $u_1(x) = \delta u(x)$ with $\delta > 1$) has been done to guarantee a nontrivial solution, in the sense that in this way different levels of incentives will be chosen by the principal. By taking, for example, $u_1(x) = u(x)/2$, we would obtain $I(k) = M$ for all $k \geq 0$ (recall also that $c(0) = 0$ in our example) and the problem would be equivalent to the one where incentives are fixed (see the previous Remark).

6 Conclusions

In this paper we studied a principal-agent problem in the context of emissions-reducing incentive policies. We looked at the problem from the two points of view of the firm (agent) and the state (principal), by deriving optimality conditions, BSDE/PDE representations and sensitivity results. A discretization scheme along with numerical experiments in some particular cases are also provided. It would be interesting to extend our results by allowing for switching costs for changing effort regimes or by including the possibility to trade emissions contracts on a financial market. This is left for future research.

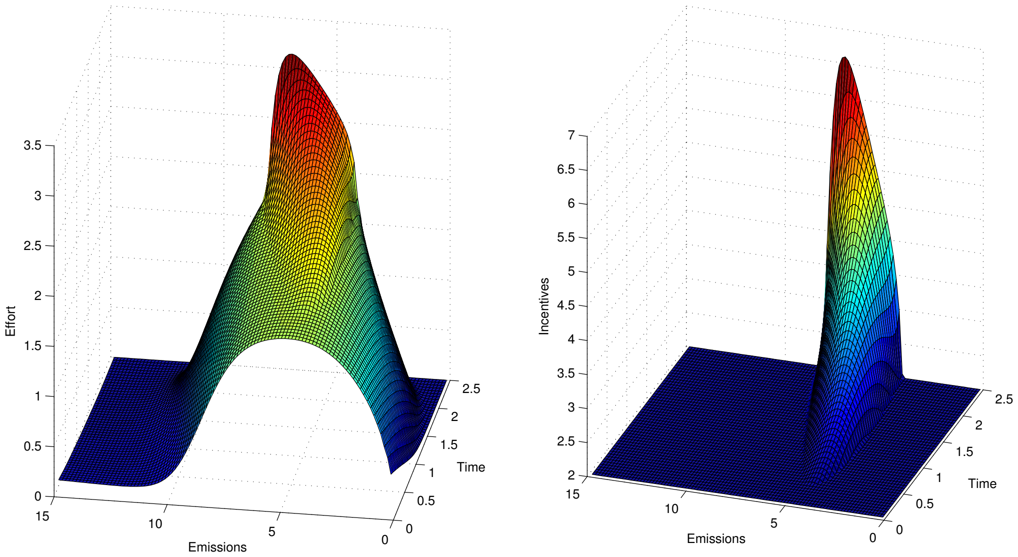

Figure 4: Optimal effort and optimal incentives dynamics with $p_2(x) = (x-4)^+ - (x-8)^+$, $m = 2$, $M = 10$.

References

[1] Belaouar, R., Fahim, A. and Touzi, N. (2011). Optimal Production Policy under the Carbon Emission Market.

[2] Bismut, J.-M. (1978). Duality methods in the control of densities. *SIAM Journal of Control and Optimization.*

[3] Carmona, R., Fehr, M. and Hinz, J. (2009). Optimal Stochastic Control and Carbon Price Formation. *SIAM Journal of Control and Optimization*, 48 (4), 2168-2190.

[4] Carmona, R., Delarue, F., Espinosa, G.-E. and Touzi, N. (2011). Singular Forward-Backward Stochastic Differential Equations and Emission Derivatives. To appear in *Annals of Applied Probability.*

[5] Kobylansky, M. (2000). Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth. *Annals of Applied Probability*, 28 (2) (2000), pp. 558-602.

[6] Cvitanic, J., Wan, X., Zhang, J. (2008). Principal-Agent Problems with Exit Options. *The B.E. Journal in Theoretical Economics*, 8 (1).

[7] Holmstrom, B. (1979). Moral hazard and observability. *Bell J. Econ.*, 10, 74-91.

[8] Ma, J. and Yong, J. (1999). Forward-Backward Stochastic Differential Equations and their Applications. Springer.

[9] Ma, J. and Zhang, J. (2002). Representation Theorems for Backward Stochatic Differential Equations. *Annals of Applied Probability*, 12 (4), 1390-1418.

[10] Pham, H. (2009). Continuous-time Stochastic Control and Optimization with Financial Applications. Springer.

[11] Sannikov, Y. (2008). A Continuous-Time Version of the Principal-Agent Problem. *Review of Economic Studies.*

[12] YONG, J. and ZHOU, X.Y. (1999). Stochastic Controls: Hamiltonian Systems and HJB equations. Springer.

[13] WILLIAMS, N. (2008). On Dynamic Principal-Agent Problems in Continuous Time. Working paper.

7 Appendix

LEMMA 7.1 *The measure changes Γ^k and $(\Gamma^k)^{-1}$ are bounded in L^q for any $q \geq 0$, uniformly on $[0,T]$.*

Proof. Denote $Q_t = (\Gamma_t^k)^q$ for some $q \in \mathbb{R}$. We have in general $dQ_t = Q_t\left[Kl(k_t)dW_t^0 + Cl^2(k_t)dt\right]$, where K, C are constants depending on q. We can choose an increasing sequence T_n of stopping times such that $T_n \to \infty$ and, since l is bounded, we have

$$E[Q_{t\wedge T_n}] = 1 + CE\left[\int_0^{t\wedge T_n} Q_r l^2(k_r)dr\right] \leq 1 + CE\left[\int_0^t Q_{r\wedge T_n}dr\right],$$

Therefore by Gronwall's lemma

$$E[Q_{t\wedge T_n}] \leq e^C,$$

and hence $E[Q_t] \leq e^C$ by Fatou's lemma. Remark that C only depends on q. □

LEMMA 7.2 *Let k be an admissible effort policy. Then system* (2.8) *associated to k admits a unique $\mathcal{F}_t$-measurable solution (Y,Z) which satisfies*

$$E\left[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_t|^2 dt\right] < \infty$$

and also

$$E^{\bar{k}}\left[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_t|^2 dt\right] < \infty.$$

for any admissible $\bar{k}$ (possibly different from k).

Proof. For any admissible k define the system

$$\begin{cases} dX_t = \sigma X_t dW_t^0 \\ d\tilde{Y}_t = -\Gamma_t^k u(s_t - c(k_t))dt + \tilde{Z}_t dW_t^0 \\ X_0 = x, \tilde{Y}_T = \Gamma_T^k \xi \end{cases} \tag{7.1}$$

with $\xi = -p(X_T)$. By Hölder's inequality (with $q = \frac{2+\alpha}{2}$), Lemma 7.1 and admissibility of k we obtain

$$E\left[\int_0^T (\Gamma_t^k)^2 u(s_t - c(k_t))^2 dt\right] \leq E\left[\int_0^T |\Gamma_t^k|^{\frac{2(2+\alpha)}{\alpha}} dt\right]^{\frac{\alpha}{2+\alpha}} E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right]^{\frac{2}{2+\alpha}} < \infty$$

In a similar way we have that $\Gamma_T^k \xi \in L^2$ by Lemma 7.1.
By standard results on BSDEs (see [10], Theorem 6.2.1), or simply by the MRT, there exists a unique solution to (7.1) which satisfies

$$E\left[\sup_{0\leq t\leq T}|\tilde{Y}_t|^2 + \int_0^T |\tilde{Z}_t|^2 dt\right] < \infty.$$

Now define $Y_t = \tilde{Y}_t[\Gamma_t^k]^{-1}$ and $Z_t = [\tilde{Z}_t - l(k_t)/\sigma \tilde{Y}_t][\Gamma_t^k]^{-1}$. We have

$$d\left(\frac{1}{\Gamma_t^k}\right) = -\frac{1}{\Gamma_t^k}\frac{l(k_t)}{\sigma}dW_t^k,$$

recall that $d\Gamma_t^k = \Gamma_t^k \frac{l(k_t)}{\sigma} dW_t^0$, and so

$$\begin{aligned} dY_t =& d\left(\frac{\tilde{Y}_t}{\Gamma_t^k}\right) = \left(\frac{1}{\Gamma_t^k}\right) d\tilde{Y}_t + \tilde{Y}_t d\left(\frac{1}{\Gamma_t^k}\right) - \frac{\tilde{Z}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma}dt = -u(s_t - c(k_t))dt \\ &+ \frac{\tilde{Z}_t}{\Gamma_t^k}dW_t^k + \frac{\tilde{Z}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma}dt - \frac{\tilde{Y}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma}dW_t^k - \frac{\tilde{Z}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma}dt \\ =& -u(s_t - c(k_t))dt + Z_t dW_t^k \end{aligned}$$

hence (Y, Z) solve (2.8) with $Y_T = \xi$. By using (7.1) we get

$$Y_t = E^k\left[\xi + \int_t^T u(s_r - c(k_r))dr \mid \mathcal{F}_t\right]$$

Now define the martingale

$$\widehat{Y}_t = Y_t + \int_0^t u(s_r - c(k_r))dr = E^k\left[\xi + \int_0^T u(s_r - c(k_r))dr \mid \mathcal{F}_t\right].$$

We have by Doob's inequality

$$E^k\left[\left(\sup_{0\le t\le T}|\widehat{Y}_r|\right)^{2+\alpha}\right] \le CE^k\left[\xi^{2+\alpha}\right] + CE^k\left[\int_0^T u(s_r - c(k_r))^{2+\alpha}dr\right] < \infty$$

for $\alpha > 0$ sufficiently small, by admissibility of k. Since $d\widehat{Y}_t = Z_t dW_t^k$ we can also conclude (by BDG) that

$$E^k\left[\left(\int_0^T Z_r^2 dr\right)^{\frac{2+\alpha}{2}}\right] \le CE^k\left[\left(\sup_{0\le t\le T}|\widehat{Y}_r|\right)^{2+\alpha}\right] < \infty$$

By taking another admissible $\bar{k}$ we get

$$E^{\bar{k}}\left[\left(\int_0^T Z_r^2 dr\right)^2\right] = E^k\left[\frac{\Gamma_T^{\bar{k}}}{\Gamma_T^k}\left(\int_0^T Z_r^2 dr\right)^2\right] < \infty$$

by using Hölder's inequality and Lemma 7.1 in a similar way as above. We also have by Doob's inequality

$$E^k\left[\left(\sup_{0\le t\le T}|\widehat{Y}_r|^2\right)^{\beta}\right] \le CE^k\left[\xi^{2\beta}\right] + CE^k\left[\int_0^T u(s_r - c(k_r))^{2\beta}dr\right] < \infty$$

for $\beta > 1$ sufficiently small, hence similarly as above

$$E^{\bar{k}}\left[\sup_{0\le t\le T}|Y_r|^2\right] < \infty.$$

□

Proof of Lemma 2.1. Recall that g is naturally defined for $m < s \le M$ and $0 \le k < c^{-1}(s)$. Assumptions 2.1 and 2.2 ensure that

$$g_k(s,k) = \sigma\frac{l'(k)[-u''(s-c(k))c'(k)^2 + u'(s-c(k))c''(k)] - l''(k)[u'(s-c(k))c'(k)]}{l'(k)^2} \le 0,$$

where g_k is the first derivative of g with respect to the variable k. Moreover, $g(s, 0) = 0$ and $\lim_{k \to c^{-1}(s)} g(s, k) = -\infty$, which implies that for any $s \geq m$ and $z \leq 0$ the equation $g(s, k) = z$ has a unique solution, i.e. $F(s, z)$. Finally we set $F(s, z) = 0$ when $z \geq 0$. Remark that $0 \leq F(s, z) < c^{-1}(s)$, with $\lim_{z \to -\infty} F(s, z) = c^{-1}(s)$. We have

$$F_z(s, z) = \frac{l'(F)^2}{l'(F)[-u''(s - c(F))c'(F)^2 + u'(s - c(F))c''(F)] - l''(F)[u'(s - c(F))c'(F)]}$$

when $z < 0$, and $F_z(s, z) = 0$ when $z > 0$. When $z = 0$ then F has a right derivative $F_{z+}(s, 0) = 0$ and a left derivative

$$F_{z-}(s, 0) = \frac{l'(0)^2}{l'(0)u'(s - c(0))c''(0)}$$

which does not diverge since $s \leq M$ and $c''(0) > 0$. Hence $F(s, \cdot)$ is Lipschitz and continuously differentiable on $\mathbb{R} \setminus \{0\}$.
Finally we can compute

$$\begin{aligned} g_{kk}(s, k) = & -2c'(k) \left(\frac{c''(k)}{l'(k)} - \frac{c'(k)l''(k)}{l'(k)^2} \right) u''(s - c(k)) \\ & + u'(s - c(k)) \left[-\frac{2c''(k)l''(k)}{l'(k)^2} + \frac{c^{(3)}(k)}{l'(k)} + c'(k) \left(\frac{2l''(k)^2}{l'(k)^3} - \frac{l^{(3)}(k)}{l'(k)^2} \right) \right] \\ & + \frac{c'(k) \left(-c''(k)u''(s - c(k)) + c'(k)^2 u^{(3)}(s - c(k)) \right)}{l'(k)} \end{aligned}$$

and the last claim follows by noting that

$$F_{zz}(s, z) = \frac{-g_{kk}(s, F(s, z))F_z(s, z)}{[g_k(s, F(s, z))]^2} = -\frac{g_{kk}(s, F(s, z))}{[g_k(s, F(s, z))]^3}.$$

$\square$

Finance for Energy Market Research Centre

Institut de Finance de Dauphine, Université Paris-Dauphine
1 place du Maréchal de Lattre de Tassigny
75775 PARIS Cedex 16
www.fime-lab.org