



FINANCE FOR ENERGY MARKET RESEARCH CENTRE



Two Algorithms for the Discrete Time Approximation of Markovian Backward Stochastic Differential Equations under Local Conditions

Plamen Turkedjiev

Working Paper
RR-FiME-13-04
September 2013

Two algorithms for the discrete time approximation of Markovian backward stochastic differential equations under local conditions

Plamen Turkedjiev*

Centre de Mathématiques Appliquées
Ecole Polytechnique and CNRS
Route de Saclay
F 91128 Palaiseau cedex, France

September 18, 2013

ABSTRACT: Two discretizations of a novel class of Markovian backward stochastic differential equations (BSDEs) are studied. The first is the classical Euler scheme which approximates a projection of the processes Z , and the second a novel scheme based on Malliavin weights which approximates the marginals of the process Z directly. Extending the representation theorem of Ma and Zhang [MZ02] leads to advanced a priori estimates and stability results for this class of BSDEs. These estimates are then used to obtain competitive convergence rates for both schemes with respect to the number of points in the time-grid. The class of BSDEs considered includes Lipschitz BSDEs with fractionally smooth terminal condition as well as quadratic BSDEs with bounded, Hölder continuous terminal condition.

KEYWORDS: Backward stochastic differential equation, approximation schemes, Malliavin calculus, representation theorem, a priori estimates.

MSC 2010: 60H35, 65C30, 60H07, 60H10.

1 Introduction

► **Framework.** Backward stochastic differential equations play an important role in the theory of mathematical finance, stochastic optimal control, and partial differential equations. In this paper, we study two discrete-time approximations of the for a novel class of Markovian backward stochastic differential equation (BSDE). The first is the well-established Euler scheme for BSDEs, and the second is a novel scheme we call the Malliavin weights scheme for BSDEs. Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}, \mathbb{P})$ a filtered probability space, where $\{\mathcal{F}_t : 0 \leq t \leq T\}$ is the filtration generated by a q -dimensional ($q \geq 1$) Brownian motion W and satisfying the usual

*Currently affiliated to Laboratoire de Finance et des Marchés de l'Energie (FiME). A significant part of the author's research was done at Humboldt University. Email: turkedjiev@cmap.polytechnique.fr. The author would like to thank in particular Dirk Becherer and Emmanuel Gobet for their feedback and spotting some early errors in the manuscript.

conditions of right-continuity and completeness. We look to approximate the $\mathbb{R} \times (\mathbb{R}^q)^\top$ -valued, predictable process (Y, Z) solving the BSDE

$$Y_t = \Phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s. \quad (1.1)$$

Here, $(\mathbb{R}^q)^\top$ is the space of q -dimensional, real valued row vectors; X is an \mathbb{R}^d -valued ($1 \leq d \leq q$) diffusion; and $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ and $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^q)^\top \rightarrow \mathbb{R}$ are deterministic functions that are termed the *terminal condition* and *driver*, respectively. We focus on the setting in which the terminal condition Φ is in the space of **fractionally smooth** functions $\mathbf{L}_{2,\alpha}$ for parameter $\alpha \in (0, 1]$ - see (\mathbf{A}_Φ) in Section 1.2 for details - and the driver is **locally Lipschitz continuous in (x, y, z)** and **locally bounded at 0** in the sense that there exist constants $\theta_L, \theta_c \in (0, 1]$ and finite $L_f, L_X, C_f \geq 0$ and a function $L_f : [0, T] \rightarrow [0, \infty)$ dominated by $L_X(T-t)^{(\theta_L-1)/2}$ such that, for all $t \in [0, T)$ and $(x, y, z), (x', y', z') \in \mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^q)^\top$,

$$\begin{aligned} |f(t, x, y, z) - f(t, x', y', z')| &\leq L_f \frac{|y - y'| + |z - z'|}{(T-t)^{(1-\theta_L)/2}} + L_X(t)|x - x'|, \\ |f(t, x, 0, 0)| &\leq \frac{C_f}{(T-t)^{1-\theta_c}}. \end{aligned} \quad (1.2)$$

We say that (Y, Z) solves a BSDE **under local conditions**. Furthermore, X solves a time-inhomogeneous stochastic differential equation (SDE) with bounded, continuously differentiable coefficients whose derivatives are also bounded, and the volatility is uniformly elliptic; see $(\mathbf{A}_{\mathbf{b},\sigma})$ in Section 1.2. The existence and uniqueness of this class of BSDEs is given in Section 2.3. As we will see below, this class of BSDEs contains a large section of the important class of quadratic BSDEs and can also be used to study proxy schemes, so it is of great interest to find good discrete-time approximations. We note that fully implementable algorithms – admitting the full generality considered in this paper – based on the Euler and Malliavin weights schemes have been studied in detail in [GT13b][GT13a] respectively, but, to the best of our knowledge, this is the first paper considering the discretization error under the full generality of the local conditions.

► **Summary of results.** In the spirit of [GM10], we make use of non-uniform time-grids $\{\pi_N^{(\beta)} := \{0 = t_0^{(N)} < \dots < t_N^{(N)} = T\} : N \geq 1\}$ whose parameter $\beta \in (0, 1]$ determines the time-points $t_i^{(N)} := T - T(1 - i/N)^{1/\beta}$.

The first approximation, studied in Section 3, is the so-called Euler scheme for BSDEs:

$$\begin{aligned} Y_N^{(N)} &= \Phi(X_T), \quad Z_i^{(N)} = \frac{1}{t_{i+1}^{(N)} - t_i^{(N)}} \mathbb{E}[Y_{i+1}^{(N)} (W_{t_{i+1}^{(N)}} - W_{t_i^{(N)}})^\top | \mathcal{F}_{t_i^{(N)}}], \\ Y_i^{(N)} &= \mathbb{E}[Y_{i+1}^{(N)} + f(t_i^{(N)}, X_{t_i^{(N)}}, Y_{i+1}^{(N)}, Z_i^{(N)})(t_{i+1}^{(N)} - t_i^{(N)}) | \mathcal{F}_{t_i}] \end{aligned} \quad (1.3)$$

for each $i \in \{0, \dots, N-1\}$. The random variable $Z_i^{(N)}$ is a discretization of the projection $(t_{i+1}^{(N)} - t_i^{(N)}) \tilde{Z}_{t_i} := \mathbb{E}[\int_{t_i^{(N)}}^{t_{i+1}^{(N)}} Z_s ds | \mathcal{F}_{t_i^{(N)}}]$. This approximation has been frequently studied: [Zha04][BT04][GL07] among others, in the setting where the terminal condition Φ and the driver are uniformly Lipschitz continuous (i.e. $\theta_L = 1$); [GM10] in the setting of the fractionally smooth Φ but uniformly Lipschitz continuous driver; [IDR10][Ric11] in the setting of bounded Lipschitz (resp. Hölder) continuous Φ and quadratic driver; and [Ric12] in the setting of possibly unbounded (locally) Lipschitz continuous Φ and (super-)quadratic driver. Typically, the discretization error of the Euler scheme is measured by

$$\mathcal{E}(N) := \max_{0 \leq i < N} \mathbb{E}[|Y_{t_i^{(N)}} - Y_i^{(N)}|^2] + \sum_{i=0}^{N-1} \int_{t_i^{(N)}}^{t_{i+1}^{(N)}} \mathbb{E}[|Z_t - Z_i^{(N)}|^2] dt \quad (1.4)$$

First, we show in Theorem 3.3 that if $\beta < (2\gamma) \wedge \alpha$,

$$\mathcal{E}(N) \leq CN^{-1}\mathbf{1}_{[1,2]}(\alpha + \theta_L) + CN^{-2\gamma}\mathbf{1}_{(0,1)}(\alpha + \theta_L)$$

where $\gamma = (\frac{\alpha}{2} \wedge \theta_c + \frac{\theta_L}{2}) \wedge \theta_c$. The optimal rate of convergence $O(N^{-1})$ is obtained if $\alpha + \theta_L \geq 1$. This rate is optimal in the sense that it is the same as the rate of convergence obtained in [GM10, Theorem 3.2] in the uniformly Lipschitz driver setting ($\theta_L = 1$). This result can be complimented under the additional assumption that the terminal condition Φ is θ_Φ -Hölder continuous: in Theorem 3.6, we show that if $\beta < (2\gamma) \wedge \alpha \wedge \theta_L$, then

$$\mathcal{E}(N) \leq CN^{-1}\mathbf{1}_{[1,4]}(\theta_\Phi + \beta + 2\gamma) + CN^{-2\gamma}\mathbf{1}_{(0,1)}(\theta_\Phi + \beta + 2\gamma).$$

Now $\theta_\Phi + \beta + 2\gamma \geq 1$ is sufficient to obtain the optimal convergence rate $O(N^{-1})$. Although the complex relationship between θ_Φ , α and γ make it difficult to compare the two results in full generality, the latter is clearly superior if $\theta_c \geq 1/2$.

The second approximation, studied in Section 4, is the so-called Malliavin weights scheme. Rather than approximating the projections of the process Z , this algorithm is used to approximate the version of Z given by Theorem 2.15 at the points of the time grid directly: for each $N \geq 1$, set

$$\begin{aligned} \bar{Y}_N^{(N)} &= \Phi(X_T), \quad \bar{Y}_i^{(N)} := \mathbb{E}[\Phi(X_T) + \sum_{j=i}^{N-1} f(t_j^{(N)}, X_{t_j^{(N)}}, \bar{Y}_{j+1}^{(N)}, \bar{Z}_j^{(N)})(t_{j+1}^{(N)} - t_j^{(N)}) | \mathcal{F}_{t_i^{(N)}}], \\ \bar{Z}_i^{(N)} &:= \mathbb{E}[\Phi(X_T)H_N^i + \sum_{j=i+1}^{N-1} f(t_j^{(N)}, X_{t_j^{(N)}}, \bar{Y}_{j+1}^{(N)}, \bar{Z}_j^{(N)})H_j^i(t_{j+1}^{(N)} - t_j^{(N)}) | \mathcal{F}_{t_i^{(N)}}] \end{aligned} \quad (1.5)$$

for $i \in \{0, \dots, N-1\}$, where $(H_j^i)_{i,j}$ is a suitable discrete time approximation of the so-called Malliavin weights $(H_r^s)_{s,r}$ of Section 2.4. Due to the connection between BSDEs and quasilinear partial differential equations (PDEs), see [Ric12][CD12] and references therein, it may be of interest to approximate the marginals of the process Z rather than the projections. Other schemes that make use of Malliavin calculus are available [BL13][HNS11], but this is, to the best of our knowledge, the first scheme of this form. Convergence results are given for fixed time-points in Theorem 4.5. These results are proven under stronger conditions than for the Euler scheme because the use of stronger a priori estimates – Proposition 3.4 – is essential in the proof: one requires that either the terminal condition has exponential moments or that it is Hölder continuous, and unfortunately we have not yet been able to extend these a priori estimates to second moments of the terminal condition. One also requires a greater constraint $\beta \leq \gamma \wedge \theta_L \wedge \alpha$ on the time-grid. The rate of convergence again depends on the parameters $(\alpha, \theta_L, \theta_c, \beta)$: in the general setting, $\beta + 2\gamma \geq 1$ is required for optimality, whereas in the setting of θ_Φ -Hölder continuous terminal condition, $\beta + \theta_\Phi + 2\gamma \geq 1$. It is shown in [GT13a] that, using Monte Carlo least-squares regression to approximate the conditional expectation, one can theoretically gain an order one improvement on the algorithm complexity using the Malliavin weights scheme compared to the multi-step forward implementation of the Euler scheme [GT13b], so it is of substantial interest to study this scheme.

The main tools in this paper are stability results for approximating sequences of BSDEs combined with dynamical estimates of Malliavin derivatives – see Section 2.3 – and a priori estimates – see Section 2.4. A key result used to obtain these estimates is a representation theorem on the process Z which does not require Φ and f to be differentiable. This theorem is proved in Section 2.5. This theorem gives us important stability estimates, see Proposition 2.11. These stability estimates lead in turn to a priori estimates of the form

$$|Z_t| \leq C\sqrt{(T-t)^{-1}\mathbb{E}_t[\Phi(X_T) - \mathbb{E}_t[\Phi(X_T)]]^2} + C(T-t)^{\theta_c-1/2} + \mathbb{E}_t[\Phi(X_T)^2]^{1/2}(T-t)^{\theta_L/2}$$

for all $t \in [0, T)$ almost surely. Such estimates are quite novel and allow us to study the impact of the regularity of the terminal condition: see Proposition 2.12. Usually, one considers bounds of the form $C(1 + |X_t|^r)$ [MZ02][Ric12] for (locally) Lipschitz continuous Φ only.

► **Contributions to quadratic BSDEs and proxy methods.** We consider the setting where Φ is a bounded, θ_Φ -Hölder continuous function. To make the contributions of the numerical results in this paper clearer, we consider two important examples. Note that these examples have also been given some attention in [GT13b, Section 2].

Quadratic BSDEs have powerful applications in financial mathematics, for example to solve utility optimization problems in incomplete markets [REK00][HIM05]. Let $q = d$ and the measurable function $F : [0, T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy

$$\begin{aligned} |F(t, x, y, z)| &\leq c(1 + |y| + |z|^2), \\ |F(t, x, y, z) - F(t, x, y', z')| &\leq c(1 + |z| + |z'|)(|y - y'| + |z - z'|). \end{aligned}$$

It is known [DG06] that the solution (Y, Z) of the BSDE with terminal condition Φ and driver $F(t, x, y, z)$ exists and is unique and that there is a constant $\theta \in (0, 1]$ and finite $C_u > 0$ such that $|Z_t| \leq C_u(T - t)^{(\theta-1)/2}$ for all $t \in [0, T)$ almost surely. This implies that (Y, Z) also solves the BSDE under local conditions with terminal condition Φ and driver $f(t, x, y, z) := F(t, x, y, \mathcal{T}_{C_u(T-t)^{(\theta-1)/2}}(z))$, where $\mathcal{T}_L(z) := (-L \vee z_1 \wedge L, \dots, -L \vee z_q \wedge L)$. Indeed, $C_f = c$, $\theta_c = 1$, $L_f = c(T^{(1-\theta)/2} + 2\sqrt{d}C_u)$, and $\theta_L = \theta$. The terminal condition is fractionally smooth with parameter α at least as large as θ_Φ - see Remark 1.3. It is shown in Corollary 2.12 that $|Z_t| \leq C(T - t)^{(\theta_\Phi-1)/2}$, so θ_L is at least as large as θ_Φ . Therefore, the error $\mathcal{E}(N)$ of the Euler scheme is bounded above by $C_\beta N^{-1} \mathbf{1}_{[1,4]}(3\theta_\Phi + \beta) + CN^{-2\theta_\Phi} \mathbf{1}_{(0,1)}(3\theta_\Phi + \beta)$ for any $\beta < \theta_\Phi$. In [Ric11], the Euler scheme for bounded, Hölder continuous is also considered, but with a different non-uniform time-grid and a transformation of the terminal condition. The author obtains a rate of convergence $C_\eta N^{\eta-\theta_\Phi}$ for any $\eta > 0$, so we have obtained an improvement in this work. This improvement is likely due to the use of the time-grids $\pi_N^{(\beta)}$ in our scheme - indeed, [GM10] show a rate of convergence $O(N^{-\alpha})$ in the uniformly Lipschitz continuous driver setting if only a uniform time-grid is used.

Next we consider a particular instance of the proxy method. Let $f(t, x, y, z)$ satisfy (1.2), and (Y, Z) satisfy the BSDE with terminal condition Φ and driver $f(t, x, y, z)$. For functions b and σ - the drift and volatility of X - satisfying the conditions of $(\mathbf{A}_{b,\sigma})$ in Section 1.2, the linear parabolic PDE given by

$$\left. \begin{aligned} 0 &= \partial_t u + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^\top)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i} u, \\ u(T, x) &= \Phi(x). \end{aligned} \right\} \quad (1.6)$$

has a unique strong solution $u(t, x)$, and the k -th order ($k \leq 3$) partial derivatives in x of u are bounded by $C_u(T - t)^{(\theta_\Phi-k)/2}$, see Lemma 2.8. It is well known that $(\mathcal{Y}, \mathcal{Z}) = (Y_t - u(t, X_t), Z_t - \nabla_x u(t, X_t)\sigma(t, X_t))_t$ solves a BSDE under local conditions with term with zero terminal condition and driver $F(t, x, y, z) := f(t, x, u(t, x) + y, \nabla_x u(t, x)\sigma(t, x) + z)$, $\theta_c = \theta_\Phi$ and $L_F(t) = L_f C_u(T - t)^{(\theta_\Phi+\theta_L-3)/2}$. Throughout this paper, we actually compute the discretization error of $(\mathcal{Y}, \mathcal{Z})$ in order to obtain that of (Y, Z) , although it is technically not in the scope of (1.2). For $\theta_L = 1$, it is known that this error has a better rate of convergence [GM10], but it is not necessarily so for $\theta_L < 1$. On the other hand, it is shown in the proof of Proposition 3.4 - equation (3.12) - that $|Z_t| \leq C(T - t)^{(\theta_\Phi+(2\theta_c)\wedge\theta_L-1)/2}$, whereas $|Z_t| \leq C(T - t)^{((2\theta_c)\wedge\theta_\Phi-1)/2}$. In fact, smaller almost sure bounds yield improved rates of convergence for numerical methods [GT13a][GT13b], so it is numerically beneficial to simulate $(\mathcal{Y}, \mathcal{Z})$ rather than (Y, Z) .

► **Remarks on extensions.** In this paper, we work with one of the simplest time-inhomogeneous SDE models with stochastic volatility, which, in particular, allows us to make use of powerful results

from the theory of parabolic PDEs [Fri64] – see Lemma 2.8. The representation of Z in Theorem 2.15 also makes use of the uniform ellipticity condition. Our application to quadratic BSDEs also requires these constraints, and additionally that Φ is Hölder continuous and bounded, because we make use of the results of [DG06] to introduce local Lipschitz continuity. In fact, the use of fractionally smooth terminal conditions combined when driver is inevitably introduces a host of technical problems due to time dependency, see also [GG04][GH07] [GM10][GG11][GGG12], and it is the goal of this paper to extend this literature and provide solutions that may be stepping stones to further extension. Another interesting aspect of our general results is that we require neither BMO results nor (local)-Lipschitz continuity of Φ . Combined with the connection to quadratic BSDEs already discussed here, this suggests there may be potential to apply the results of this paper to obtain novel representation theorems, a priori estimates, existence and uniqueness results for (super-)quadratic BSDEs with possibly unbounded and discontinuous terminal conditions. This would be a nice complement to the recent work of [Ric12]. There are already several directions that may help us to avoid the uniformly elliptic condition. The results of [Kus03] [CD12] [Nee11], offer suitable PDE results under UFG conditions. Also, a representation theorem beyond the uniformly elliptic setting has been found by [Zha05] and [GM⁺05] (although only for the zero driver case in the latter). Unfortunately, all of these interesting extensions are beyond the scope of this paper.

1.1 Notation and conventions

► **Time-grids.** Since each result is given for a fixed number of time-points N , we denote the points $\{t_i^{(N)}\}$ of the time-grid simply by $\{t_i\}$. Let $\Delta_i := t_{i+1} - t_i$ and $\Delta W_i := W_{t_{i+1}} - W_{t_i}$. We also suppress the superscript (N) in the Euler and Malliavin weights scheme.

► **Conditional expectations.** The conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$ is denoted by $\mathbb{E}_t[\cdot]$, and $\mathbb{E}_{t_i}[\cdot]$ is denoted $\mathbb{E}_i[\cdot]$. The norm $\sqrt{\mathbb{E}[|\cdot|^2]}$ by $\|\cdot\|_2$. We will make use of a conditional version of Fubini's theorem, stated in Lemma 5.1. In order to simplify notation throughout this paper, we write $\int_0^T \mathbb{E}_t[f_s] ds := \int_0^T F_t(\cdot, s) ds$, where F_t is the process defined in Lemma 5.1.

► **Lebesgue measure** For any Euclidean space E , $\mathcal{B}(E)$ denotes the Borel measurable sets in E , and the Lebesgue measure on the measurable space $(E, \mathcal{B}(E))$ is denoted by m .

► **Processes and spaces.** For two processes X and Y in $\mathbf{L}_0([0, T] \times \Omega; \mathbb{R}^k)$, Y is said to be a version of X if $X = Y$ $m \times \mathbb{P}$ -a.e. $\mathcal{P} \subset \mathcal{B}([0, T]) \otimes \mathcal{F}_T$ is the predictable σ -algebra, generated by the continuous, adapted processes, and \mathcal{H}^2 is the subspace of $\mathbf{L}_2([0, T] \times \Omega)$ containing only predictable processes. For $p \geq 2$, \mathcal{S}^p is the subspace of \mathcal{H}^2 of continuous processes Y such that $\|Y\|_{\mathcal{S}^p} := \mathbb{E}[\sup_{0 \leq s \leq T} |Y_s|^p]^{\frac{1}{p}}$ is finite for all $Y \in \mathcal{S}^p$; $\|\cdot\|_{\mathcal{S}^p}$ is a norm for this space.

► **Linear algebra** We identify the space of $k \times n$ dimensional, real valued matrices with $\mathbb{R}^{k \times n}$. $^\top$ denotes the transpose operator. For any $A \in \mathbb{R}^{k \times n}$, let A_j denote the j -th column vector of A . For any vector $x \in \mathbb{R}^n$, $|x|$ is the vector 2-norm, defined by $(\sum_{i=1}^n |x_i|^2)^{1/2}$, and for any matrix A , $|A|$ is the matrix 2-norm, defined by $\max_{|x|=1} |Ax|$, where $|Ax|$ is the vector 2-norm of the vector Ax .

► **Functions and regularity.** Let $\gamma \in (0, 1]$ and $A(\cdot)$ be a function in the domain $[0, T] \times \mathbb{R}^l$ taking values in $\mathbb{R}^{k \times n}$ (resp. \mathbb{R}^k). We say that $A(t, \cdot)$ is γ -Hölder continuous uniformly in t with Hölder constant L_A if, for all $(x, y) \in (\mathbb{R}^l)^2$ and $t \in [0, T]$, $|A(t, x) - A(t, y)| \leq L_A |x - y|^\gamma$; in the case that $\gamma = 1$, we say that $A(t, \cdot)$ is Lipschitz continuous uniformly in t with Lipschitz constant L_A . Likewise, we say that $A(\cdot, x)$ is γ -Hölder continuous uniformly in x with Hölder constant L_A if, for every $(t_1, t_2) \in [0, T]^2$ and $x \in \mathbb{R}^l$, $|A(t_1, x) - A(t_2, x)| \leq L_A |t_1 - t_2|^\gamma$. For a given multi-index $\alpha = (i_1, \dots, i_{|\alpha|})$ with no zero entries, we define by $\partial_x^\alpha A(t, \cdot)$ the multiple derivative $\partial_{x_{i_1}} \dots \partial_{x_{i_{|\alpha|}}} A(t, \cdot)$. If $A(t, \cdot)$ takes values in \mathbb{R}^k and is differentiable, we define by $\nabla_x A(t, \cdot)$ the $\mathbb{R}^{k \times l}$ valued function whose (u, v) -th component is $\partial_{x_v} A_u(t, \cdot)$. If $A(t, \cdot)$ takes values in $(\mathbb{R}^k)^\top$

and is differentiable, we define by $\nabla_x A(t, \cdot)$ the $\mathbb{R}^{l \times k}$ -valued function whose (u, v) -th component is $\partial_{x_u} A_v(t, \cdot)$. Define by $\|A\|_\infty$ the infinity norm

$$\max_{u,v} \sup_{(t,x) \in [0,T] \times \mathbb{R}^l} |A_{u,v}(t, x)| \quad (\text{resp. } \max_u \sup_{(t,x) \in [0,T] \times \mathbb{R}^l} |A_u(t, x)|).$$

► **Mollifiers.** The following definitions will come in handy.

Definition 1.1. Let n be a non-zero integer. A mollifier is a smooth function $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ with compact support on $\{x \in \mathbb{R}^n : |x| \leq 1\}$ such that $\int_{\mathbb{R}^n} \phi(x) dx = 1$ and $\lim_{R \rightarrow \infty} R^n \phi(Rx) = \delta(x)$ for all $x \in \mathbb{R}^n$, where $\delta(x)$ is the Dirac delta function. For $R > 0$, define the function $\phi_R : \mathbb{R}^n \rightarrow [0, \infty)$ be the function $x \mapsto R^n \phi(Rx)$.

An example of a mollifier is $\phi(x) = e^{-1/(1-|x|)} \mathbf{1}_{|x| < 1}$. The following lemma, which is standard, shows how a mollifier can be used to generate a smooth function from a continuous one.

Lemma 1.2. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous, and define the function $F_R(x) := \int_{\mathbb{R}^n} F(x-y) \phi_R(y) dy$. Then the function $F_R(x)$ is smooth and $\lim_{R \rightarrow \infty} F_R(x) = F(x)$ for all $x \in \mathbb{R}^n$.

1.2 Assumptions

The following assumptions will hold throughout this paper.

(**A_{b,σ}**) X is a solution to the stochastic differential equation (SDE)

$$X_0 = x_0, \quad X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \quad s > 0, \quad (1.7)$$

where $x_0 \in \mathbb{R}^d$ is fixed and b and σ satisfy

- (a) $(t, x) \in [0, T] \times \mathbb{R}^d \mapsto b(t, x)$ is \mathbb{R}^d -valued, measurable and uniformly bounded. Moreover, $b(t, \cdot)$ is twice continuously differentiable with uniformly bounded derivatives and Hölder continuous second derivative, and $b(\cdot, x)$ is 1/2-Hölder continuous uniformly in x .
- (b) $(t, x) \in [0, T] \times \mathbb{R}^d \mapsto \sigma(t, x)$ is $\mathbb{R}^{d \times q}$ -valued, measurable and uniformly bounded. Moreover, $\sigma(t, \cdot)$ is twice continuously differentiable with uniformly bounded derivatives and Hölder continuous second derivative, and $\sigma(\cdot, x)$ is 1/2-Hölder continuous uniformly in x .
- (c) $\sigma(\cdot)$ satisfies a uniformly elliptic condition: there exists some finite $\bar{\beta} > 0$ such that, for any $\zeta \in \mathbb{R}^d$, $\zeta^\top \sigma(t, x) \sigma(t, x)^\top \zeta \geq \bar{\beta} |\zeta|^2$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

(**A_Φ**) The terminal condition $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function and there exists a constant $\alpha \in (0, 1]$ such that $K^\alpha(\Phi) < \infty$, where

$$\left. \begin{aligned} K^\alpha(\Phi)^2 &:= \mathbb{E}[|\Phi(X_T)|^2] + \sup_{0 \leq t < T} \frac{V_{t,T}(\Phi)^2}{(T-t)^\alpha} \\ \text{for } V_{t,T}(\Phi)^2 &:= \mathbb{E}[|\Phi(X_T) - \mathbb{E}_t[\Phi(X_T)]|^2]. \end{aligned} \right\} \quad (1.8)$$

We say that Φ is fractionally smooth, and that it belongs to the space $\mathbf{L}_{2,\alpha}$. We refer to [GM10] for further discussion of and references for the space $\mathbf{L}_{2,\alpha}$.

(**A_f**) The driver $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^q)^\top \rightarrow \mathbb{R}$ satisfies (1.2).

The following assumptions will be needed for interim results only hold when specifically stated.

(**A_{∂f}**) The driver $(t, x, y, z) \mapsto f(t, x, y, z)$ is continuously differentiable with respect (x, y, z) for all $t \in [0, T]$, and the partial derivatives are bounded by $L_f(T - t)^{(\theta_L - 1)/2}$.

(**A_{f_t}**) The driver $f(t, x, y, z)$ is $\frac{1}{2}$ -Hölder continuous in its t uniformly in (x, y, z) with Hölder constant L_f .

(**A_{bΦ}**) The function Φ is uniformly bounded: $\|\Phi\|_\infty < \infty$.

(**A_{expΦ}**) The terminal condition has exponential bounds in the sense that there is a finite $C_\xi > 0$ such that $\mathbb{E}[e^{|\Phi(X_T)|}] \leq C_\xi$.

(**A_{hΦ}**) The function Φ is Hölder continuous: there exists a finite constants K_Φ and $\theta_\Phi \in (0, 1]$ such that $|\Phi(x_1) - \Phi(x_2)| \leq K_\Phi|x_1 - x_2|^{\theta_\Phi}$ for any $x_1, x_2 \in \mathbb{R}^d$.

Remark 1.3. Due to (**A_{b,σ}**), (**A_{hΦ}**) implies (**A_{expΦ}**) and (**A_Φ**). Note that it is possible that $\theta_\Phi < \alpha$: see [GGG12, page 2086, e.g. (i)].

In the proofs below, it will be necessary to compute a right-inverse to the matrix $\sigma(\cdot)$, i.e., for every $(t, x) \in [0, T] \times \mathbb{R}^d$, it will be necessary to find a (q, d) -dimensional matrix $\sigma^{-1}(t, x)$ such that $\sigma(t, x)\sigma^{-1}(t, x) = I_d$. In the case where the dimensions d and q are equal, this is uniquely defined by usual matrix inverse of $\sigma(t, x)$, whose existence is guaranteed by the uniform ellipticity condition (**A_{u.e.}**). If the dimensions d and q are not equal, $\sigma^{-1}(t, x)$ is defined by the pseudoinverse $\sigma(t, x)^\top (\sigma(t, x)\sigma(t, x)^\top)^{-1}$; this is well defined because the uniform ellipticity condition (**A_{u.e.}**) guarantees the existence of the inverse of $\sigma\sigma^\top$. In both cases, the right-inverse is uniformly Lipschitz continuous.

Lemma 1.4. *The right inverse matrix $\sigma(t, \cdot)^{-1}$ is Lipschitz continuous uniformly in t and $\sigma^{-1}(\cdot, x)$ is $1/2$ -Hölder continuous uniformly in x . Its Lipschitz (resp. Hölder) constant depends $\|\sigma\|_\infty$, $\|\nabla_x \sigma\|_\infty$ and $\bar{\beta}$ only, but not on (t, x) . Moreover, $\|\sigma^{-1}\|_\infty \leq \|\sigma\|_\infty / \bar{\beta}$.*

2 Key preliminary results

2.1 Malliavin calculus

We recall briefly some properties and definitions of Malliavin calculus. For details, we refer the reader to [Nua06].

For any $m \geq 1$, define $C_p^\infty(\mathbb{R}^m)$ to be the space of functions taking values in \mathbb{R} which are infinitely differentiable such that all partial derivatives have at most polynomial growth, and denote by $W(h) := \int_0^T h_t dW_t$ the Itô integral of the $(\mathbb{R}^q)^\top$ -valued, deterministic function $h \in \mathbf{L}_2([0, T]; (\mathbb{R}^q)^\top)$. Let $\mathcal{R} \subset \mathbf{L}_2(\mathcal{F}_T)$ be the subspace containing all random variables F of the form $f(W(h_1), \dots, W(h_m))$ for $h_i \in \mathbf{L}_2([0, T]; \mathbb{R}^q)$ and any finite m . Define the derivative operator $D : \mathcal{R} \mapsto \mathbf{L}_2([0, T] \times \Omega)$ by $D_t F := \sum_{i=1}^m \partial_i f(W(h_1), \dots, W(h_m)) h_i(t)$. The derivative operator is extended to $\mathbb{D}^{1,2} \subset \mathbf{L}_2(\mathcal{F}_T)$, the closure of \mathcal{R} in $\mathbf{L}_2(\mathcal{F}_T)$ under the norm $\|F\|_{1,2}^2 := \|F\|_2^2 + \mathbb{E}[\int_0^T |D_t F|^2 dt]$. Define by $\mathbb{D}^{1,2}(\mathbb{R}^k)$ (resp. $\mathbb{D}^{1,2}((\mathbb{R}^k)^\top)$) by the space of random variables $F = (F_1, \dots, F_k)^\top$ (resp. $F = (F_1, \dots, F_k)$) such that $F_i \in \mathbb{D}^{1,2}$ for each $i \in \{0, \dots, k\}$. The Malliavin derivative DF is denoted by the $\mathbb{R}^{k \times q}$ - (resp. $\mathbb{R}^{q \times k}$ -) valued process whose i -th row (resp. column) is DF_i (resp. $(DF_i)^\top$).

The following lemma, termed the *chain rule* of Malliavin calculus, is proved in [Nua06, Proposition 1.2.3].

Lemma 2.1 (Chain rule). *Let $(F_1, \dots, F_m) \in (\mathbb{D}^{1,2})^m$. For any continuously differentiable function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ with bounded partial derivatives, and $F = f(F_1, \dots, F_m) \in \mathbb{D}^{1,2}$, the random variable $f(F) \in \mathbb{D}^{1,2}$ and $Df(F) = \sum_{i=1}^m \partial_i f(F) DF_i = \nabla_x f(F) DF$.*

Remark. In the case that F takes values in $(\mathbb{R}^m)^\top$, the result of Lemma 2.1 hold with $Df(F) = \nabla_x f(F)(DF)^\top$. In the case that f takes values in $(\mathbb{R}^k)^\top$, applying Lemma 2.1 component-wise yields that $f(F)$ is in $\mathbb{D}^{1,2}((\mathbb{R}^k)^\top)$ and $Df(F) = (DF)^\top \nabla_x f(F)$.

For the space

$$\text{dom}(\delta) := \{u \in \mathbf{L}_2([0, T] \times \Omega; (\mathbb{R}^q)^\top) : \exists c \in \mathbb{R} \text{ s.t. } \forall F \in \mathbb{D}^{1,2} \ |\mathbb{E}[\int_0^T (u_s \cdot D_s F) ds]| \leq c \|F\|_2^2 \}$$

define the Skorohod integral operator $\delta : \text{dom}(\delta) \rightarrow \mathbf{L}_2(\Omega)$ as the dual operator to the Malliavin derivative in the sense that $\mathbb{E}[\int_0^T (u_s \cdot D_s F) ds] = \mathbb{E}[F \delta(u)]$. Below are the key properties of the Skorohod integral used in this paper.

Lemma 2.2 (Integration-by-parts). *Suppose that $u \in \text{dom}(\delta)$ and $F \in \mathbb{D}^{1,2}$ are such that $\mathbb{E}[F^2 \int_0^T |u_s|^2 ds] < \infty$. Then, the integration by parts formula holds: $\int_0^T (u_s \cdot D_s F) ds = F \delta(u) - \delta(Fu)$.*

Remark 2.3. Suppose that the process u takes values in $\mathbb{R}^{q \times k}$ is such that u_i^\top is in $\text{dom}(\delta)$ for each $i \in \{0, \dots, k\}$, where u_i is the i -th column of u . The Skorohod integral of u , denoted by $\delta(u)$, is defined by

$$\delta(u) := (\delta(u_1^\top)^\top, \dots, \delta(u_k^\top)^\top). \quad (2.1)$$

The integration by parts formula, Lemma 2.2, is applied column-wise in the case of matrix valued u . where $D_s F u_s$ is understood as a matrix-matrix multiplication, and the Skorohod integrals are defined in the multidimensional sense of equation (2.1).

2.2 SDEs and Malliavin calculus

Fix $t \in [0, T)$ and $x \in \mathbb{R}^d$. We recall some standard properties on the Malliavin calculus applied to SDEs $X^{(t,x)}$ of the form

$$X_s^{(t,x)} = x + \int_t^s b(r, X_r^{(t,x)}) \mathbf{1}_{(t,T]}(r) dr + \int_t^s \sigma(r, X_r^{(t,x)}) \mathbf{1}_{(t,T]}(r) dW_r. \quad (2.2)$$

Observe that the SDE X defined in (1.7) is equal to $X^{(0,x_0)}$. First, we recall the flow $\nabla X^{(t,x)}$ and its inverse $\nabla X^{(t,x,-1)}$, which are respectively defined by the SDEs

$$\begin{aligned} \nabla X_r^{(t,x)} &= I_d + \int_t^r b'_u \nabla X_u^{(t,x)} du + \sum_{j=1}^q \int_0^r \sigma'_{j,u} \nabla X_u^{(t,x)} dW_{j,u}, \\ \nabla X_r^{(t,x,-1)} &= I_d + \int_t^r \nabla X_r^{(t,x,-1)} \left(\sum_{j=1}^q (\sigma'_{j,u})^2 - b'_u \right) dr - \sum_{j=1}^q \int_t^r \nabla X_r^{(t,x,-1)} \sigma'_{j,u} dW_{j,r}, \end{aligned}$$

where b'_u denotes $\nabla_x \sigma_j(u, X_u^{(t,x)})$ and $\sigma'_{j,u}$ denotes $\nabla_x \sigma_j(r, X_r^{(t,x)})$. These processes are linear SDEs, and we list some standard properties used throughout this paper in the following Lemma.

Lemma 2.4. *For every $p > 1$, $\nabla X^{(t,x)}$ and $\nabla X^{(t,x,-1)}$ are in \mathcal{S}^p , and there is a constant C_p depending only on $\|\sigma\|_\infty$, $\|\nabla_x b\|_\infty$, $\|\nabla_x \sigma_j\|_\infty$, T and p such that $\|\nabla X^{(t,x)}\|_{\mathcal{S}^p} + \|\nabla X^{(t,x,-1)}\|_{\mathcal{S}^p} \leq C_p$. Moreover, $\|\nabla X_r^{(t,x)} - \nabla X_s^{(t,x)}\|_2^2 + \|\nabla X_r^{(t,x,-1)} - \nabla X_s^{(t,x,-1)}\|_2^2 \leq C_2 |r-s|$ for all $(t, s) \in [0, T]^2$, and $\nabla X_r^{(t,x)} \nabla X_r^{(t,x,-1)} = I_d$ for all r a.s., and, for any $r < u < s$, $\mathbb{E}_r[\|\nabla X_s^{(t,x)} \nabla X_r^{(t,x,-1)} - \nabla X_u^{(t,x)} \nabla X_r^{(t,x,-1)}\|_2^2] \leq C_2 (s-u)$ a.s.*

The Malliavin derivative of the marginals of $X^{(t,x)}$ is strongly related to the flow and its inverse, as shown in the following Lemma. The proof of the estimates follows directly from Lemma 2.4.

Lemma 2.5. *For all $r \in [0, T]$, $X_r^{(t,x)}$ is in $\mathbb{D}^{1,2}(\mathbb{R}^d)$. Moreover, for all $0 \leq s, r \leq T$,*

$$D_s X_r^{(t,x)} = \nabla X_r^{(t,x)} \nabla X_s^{(t,x,-1)} \sigma(s, X_s^{(t,x)}) \mathbf{1}_{[s,T]}(r) \mathbf{1}_{[t,T]}(s) \quad a.s.$$

whence there exists a constant C_p depending only on $\|\sigma\|_\infty, \|\nabla_x b\|_\infty, \|\nabla_x \sigma_j\|_\infty, T$ and p such that $\mathbb{E}_s[|D_s X_r^{(t,x)}|^p] \leq C_p$, and $\sup_s \mathbb{E}[\sup_{s \leq r \leq T} |D_s X_r^{(t,x)}|^2]^{1/2} \leq C_2$; moreover, for any $x_1, x_2 \in \mathbb{R}^d$, $\mathbb{E}[|D_s X_r^{(t,x_1)} - D_s X_r^{(t,x_2)}|^p] \leq C_p |x_1 - x_2|^p$ and, for any $r < u < s$, $\mathbb{E}_r[|D_r X_s^{(t,x)} - D_r X_u^{(t,x)}|^2] \leq C_2(s - u)$.

2.3 Existence, uniqueness, approximation and decomposition of the BSDE

Since the class of BSDEs under local conditions has, to the best of our knowledge, not been studied in full generality, we now include a proof of the existence and uniqueness of solutions.

Theorem 2.6. *There exists a unique pair of process (Y, Z) in $\mathcal{S}^2 \times \mathcal{H}^2$ solving the BSDE (1.1) with terminal condition $\Phi(X_T) \in \mathbf{L}_2(\mathcal{F}_T)$ and driver f satisfying the locally Lipschitz continuous and boundedness of (1.2).*

Proof. Let (ϕ, ψ) be in $\mathcal{H}^2 \times \mathcal{H}^2$, and define the random function $f(r, y, z) = f(r) := f(r, X_r, \phi_r, \psi_r)$. The function f is predictably measurable and satisfies assumptions (H1)-(H5) of [BDH⁺03, Section 4]. Since f takes no argument in (y, z) , it is only necessary to check (H1), which follows readily the local Lipschitz continuity and local boundedness of the driver (1.2). Thanks to [BDH⁺03, Theorem 4.2], there exists a unique solution $(Y^{(\phi, \psi)}, Z^{(\phi, \psi)})$ to the BSDE

$$Y_t^{(\phi, \psi)} = \Phi(X_T) + \int_t^T f(r) dr - \sum_{j=1}^q \int_t^T Z_{j,r}^{(\phi, \psi)} dW_{j,r}.$$

in $\mathcal{H}^2 \times \mathcal{H}^2$. The function $\Xi : \mathcal{H}^2 \times \mathcal{H}^2 \rightarrow \mathcal{H}^2 \times \mathcal{H}^2$ mapping (ϕ, ψ) to $(Y^{(\phi, \psi)}, Z^{(\phi, \psi)})$ is well defined. In fact, $Y^{(\phi, \psi)}$ is in \mathcal{S}^2 .

As in the proof of [EKPQ97, Theorem 2.1], we prove that Ξ is a contraction. For $k \in \{1, 2\}$, let $(\phi_k, \psi_k) \in \mathcal{H}^2 \times \mathcal{H}^2$ and define the BSDE $(Y_k, Z_k) := \Xi(\phi_k, \psi_k)$. Define the differences $\delta Y = Y_1 - Y_2$, $\delta Z = Z_1 - Z_2$, $\delta \phi = \phi_1 - \phi_2$ and $\delta \psi = \psi_1 - \psi_2$. It then follows from Hölder's inequality that

$$\begin{aligned} \|\delta Y_t\|_2^2 + \int_t^T \|\delta Z_r\|_2 dr &\leq \left\| \int_t^T |f(r, X_r, \phi_{1,r}, \psi_{1,r}) - f(r, X_r, \phi_{2,r}, \psi_{2,r})| dr \right\|_2^2 \\ &\leq L_f^2 (T - t)^{\theta_L} \int_t^T \{\|\delta \phi_r\|_2^2 + \|\delta \psi_r\|_2^2\} dr \end{aligned}$$

for all $t \in [0, T]$. Setting $t_0 = (T - 1/(4L_f^2)^{1/\theta_L} \wedge 1) \vee 0$ ensures, on the one hand, that $L_f^2 (T - t_0)^{\theta_L} \leq 1/4$, and, on the other hand, that $T - t_0 \leq 1$. Integrating the above inequality on the interval $t \in [t_0, T]$ then yields $4 \int_{t_0}^T \{\|\delta Y_r\|_2^2 + \|\delta Z_r\|_2^2\} dr \leq \int_{t_0}^T \{\|\delta \phi_r\|_2^2 + \|\delta \psi_r\|_2^2\} dr$ and $4\|\delta Y_t\|_2 \leq \int_{t_0}^T \{\|\delta \phi_r\|_2^2 + \|\delta \psi_r\|_2^2\} dr$ for all $t \in [t_0, T]$. On the interval $[0, t_0]$, the function $f(t, x, \cdot)$ is Lipschitz continuous with a uniform Lipschitz constant for all (t, x) , so we proceed as in the proof of Theorem [EKPQ97, Theorem 2.1] to show that, for sufficiently large $\eta > 0$,

$$\int_0^{t_0} e^{\eta r} \{\|\delta Y_r\|_2^2 + \|\delta Z_r\|_2^2\} dr \leq e^{\eta t_0} \|\delta Y_{t_0}\|_2 + \frac{1}{2} \int_0^{t_0} e^{\eta r} \{\|\delta \phi_r\|_2^2 + \|\delta \psi_r\|_2^2\} dr$$

Combining this with the above estimates on $\int_{t_0}^T \{\|\delta Y_r\|_2^2 + \|\delta Z_r\|_2^2\} dr$ and $\|\delta Y_{t_0}\|_2$ then yields

$$\int_0^T e^{\eta r} \{\|\delta Y_r\|_2^2 + \|\delta Z_r\|_2^2\} dr \leq \frac{1}{2} \int_0^T e^{\eta r} \{\|\delta \phi_r\|_2^2 + \|\delta \psi_r\|_2^2\} dr$$

where $\eta_r = \eta(r \wedge t_0)$. This is sufficient to prove that Ξ is a contraction. \square

We now introduce an approximation procedure that will be used repeatedly in this paper.

Definition 2.7. Let $\varepsilon \in [0, T)$ and define $f^{(\varepsilon)}(t, x, y, z) := f(t, x, y, z) \mathbf{1}_{[0, T-\varepsilon)}(t)$. Let $(Y^{(\varepsilon)}, Z^{(\varepsilon)})$ be the solution of the BSDE

$$Y_t^{(\varepsilon)} = \Phi(X_T) + \int_t^T f^{(\varepsilon)}(s, X_s, Y_s^{(\varepsilon)}, Z_s^{(\varepsilon)}) ds - \int_t^T Z_s^{(\varepsilon)} dW_s. \quad (2.3)$$

Additionally, let (y, z) be the solution of the BSDE with zero driver $y_t = \Phi(X_T) - \int_t^T z_s dW_s$ and $(y^{(\varepsilon)}, z^{(\varepsilon)})$ the solution of the BSDE with zero terminal condition

$$y_t^{(\varepsilon)} = \int_t^T f^{(\varepsilon)}(s, X_s, y_s + y_s^{(\varepsilon)}, z_s + z_s^{(\varepsilon)}) ds - \int_t^T z_s^{(\varepsilon)} dW_s. \quad (2.4)$$

Since $f^{(\varepsilon)}(t, x, y, z)$ is Lipschitz continuous uniformly in t with Lipschitz constant $L_f \varepsilon^{(\theta_L - 1)/2}$, the solutions of the BSDEs in Definition 2.7 exists in $\mathcal{S}^2 \times \mathcal{H}^2$ and are unique for all $\varepsilon \in [0, T)$ [EKPQ97, Theorem 2.1]. The introduction of the BSDEs in Definition 2.7 is localization technique used extensively in [GM10], and we shall frequently take advantage of it throughout this work. We shall also make use of the decomposition $(Y^{(\varepsilon)}, Z^{(\varepsilon)}) = (y + y^{(\varepsilon)}, z + z^{(\varepsilon)})$, which is standard in BSDE literature [GM10].

We first treat the linear BSDE (y, z) . The following Lemma relates the linear BSDE (y, z) to the PDE in (1.6) and gives some boundedness properties for the function u and its derivatives; these bounds will be used throughout this paper.

Lemma 2.8. Let $(\mathbf{A}_{\mathbf{b}\Phi})$ be in force. Then $u(t, x) = \mathbb{E}[\Phi(X_T) | X_t = x]$, the so-called Feynman-Kac representation, is a classical solution of the PDE (1.6), and the derivatives $\partial_x^\alpha u$ ($|\alpha| \leq 3$), $\partial_t u$, $\partial_t \nabla_x u$ exist and are continuous. There is a constant C depending only on the bound on b and it's derivatives, the bound on σ and it's derivatives, and $\bar{\beta}$ such that $\|\partial_x^\alpha u(t, \cdot)\|_\infty \leq C \|\Phi\|_\infty (T-t)^{|\alpha|/2}$ for all $(t, x) \in [0, T) \times \mathbb{R}^d$. Moreover, $(u(t, X_t), (\nabla_x u(t, X_t) \sigma(t, X_t))^\top)^\top$ is the solution to the linear BSDE (y, z) . For any $x_1, x_2 \in \mathbb{R}^d$, $t \in [0, T)$ recall from (2.2) the SDEs $X^{(t, x_1)}$ and $X^{(t, x_2)}$, and for $\alpha, \beta \in [0, 1]$ define $\bar{X} := \alpha X^{(t, x_1)} + \beta X^{(t, x_2)}$; then $|\nabla_x u(r, \bar{X}_r)|^2 \leq C \mathbb{E}_r [|\Phi(\bar{X}_T) - \mathbb{E}_r[\Phi(\bar{X}_T)]|^2] (T-r)^{-1}$ and $|\nabla_x^2 u(r, \bar{X}_r)|^2 \leq \mathbb{E}_r [|\Phi(\bar{X}_T) - \mathbb{E}_r[\Phi(\bar{X}_T)]|^2] (T-r)^{-2}$ for all $r \in [0, T)$.

Proof. The Feynman-Kac representation of the solution is well known, see [GM⁺05] among many others. Moreover, X is a Markov process with transition density $p(t, x; \xi, s)$ and, under $(\mathbf{A}_{\mathbf{b}, \sigma})$, the derivative of the transition density $\partial_x^\alpha p(t, x; \xi, s)$ exist for all α such that $|\alpha| \leq 3$ and are bounded [Fri64, Chapter 9, Theorem 7]. The bounds on the derivatives of $u(t, \cdot)$ follow from Lebesgue's differentiation theorem and direct computation.

To show the bound on $|\nabla_x u(r, \bar{X}_r)|$, let us recall first that the result in the case $\alpha = 1$ and $\beta = 0$ is given in [GM10, Lemma 1.1]. The authors use the tools of [GM⁺05, Lemma 2.9] to show that, for every $r \in [0, T)$ and $x \in \mathbb{R}^d$, there is a random variable $H_{r,x}$ such that $\nabla_x u(r, X_r^{(t, x)}) = \mathbb{E}_r[\Phi(X_T) H_{r,x}]$. $H_{r,x}$ satisfies $\mathbb{E}_r[H_{r,x} = 0]$ and $\mathbb{E}_r[|H_{r,x}|^2] \leq C(T-r)^{-1}$, whence their result follows. In fact, one can follow the proof method of [GM⁺05, Lemma 2.9], using additionally the linearity of the Malliavin derivative, to show that, for $\bar{H}_r := \alpha H_{r, x_1} + \beta H_{r, x_2}$, $\nabla_x u(r, \bar{X}_r) = \mathbb{E}_r[\Phi(\bar{X}_T) \bar{H}_r]$, whence the result follows. The proof for the bound on $|\nabla_x^2 u(r, \bar{X}_r)|$ is similar. \square

We move onto the non-linear BSDE $(y^{(\varepsilon)}, z^{(\varepsilon)})$. The following representations and a priori estimates will be critical throughout this paper.

Lemma 2.9. *Let $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$ hold. Define $\Theta_r = (r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})$, and set*

$$\begin{aligned} a_r^{(\varepsilon)} &:= \nabla_x f^{(\varepsilon)}(\Theta_r) + \nabla_y f^{(\varepsilon)}(\Theta_r) \nabla_x u(r, X_r) + \nabla_z f^{(\varepsilon)}(\Theta_r) U(r, X_r)^\top, \\ b_r^{(\varepsilon)} &:= \nabla_y f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}), \quad c_r^{(\varepsilon)} := \nabla_z f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}). \end{aligned} \quad (2.5)$$

where $U(r, x)$ is defined by

$$U(t, x) := \nabla_x^2 u(t, x) \sigma(t, x) + \sum_{j=1}^d (\nabla_x u)_j(t, x) \nabla_x \sigma_j^\top(t, x), \quad (2.6)$$

Then there a finite constant C depending only on $T, d, K^\alpha(\Phi)$, the bound on σ and it's derivatives, the bound on b and it's derivatives, L_f , and θ_L such that

$$\|a_r^{(\varepsilon)}\|_2 \leq C \mathbf{1}_{[0, T-\varepsilon]}(r) (T-r)^{(\alpha+\theta_L-3)/2} \quad (2.7)$$

There exists a unique solution $(U^{(\varepsilon)}, V^{(\varepsilon)}) \in \mathcal{S}^2 \times \mathcal{H}^2$ of the BSDE

$$\begin{aligned} U_t^{(\varepsilon)} &= \int_t^T a_r^{(\varepsilon)} + U_r^{(\varepsilon)} (b_r^{(\varepsilon)} I_d + \nabla_x b(r, X_r) + \sum_{j=1}^q c_{j,r}^{(\varepsilon)} \nabla_x \sigma_j(r, X_r)) dr \\ &\quad + \int_t^T \sum_{j=1}^q (V_{j,r}^{(\varepsilon)})^\top (c_{j,r}^{(\varepsilon)} I_d + \nabla_x \sigma_j(r, X_r)) dr - \sum_{j=1}^q \int_t^T (V_{j,r}^{(\varepsilon)})^\top dW_{j,r} \end{aligned} \quad (2.8)$$

where $\sigma_j(\cdot)$ is the j -th column of $\sigma(\cdot)$, $c_{r,j}^{(\varepsilon)}$ are the j -th component of $c_r^{(\varepsilon)}$, and $V_{j,r}^{(\varepsilon)}$ is the j -th column of $V_r^{(\varepsilon)}$. There is a (possibly different) constant C such that, for any $0 \leq t < T$ and $\varepsilon > 0$,

$$\mathbb{E} \left[\sup_{t \leq r < T} |U_r^{(\varepsilon)}|^2 \right] + \int_t^T \|V_r^{(\varepsilon)}\|_2^2 dr \leq C \left\| \int_t^{T-\varepsilon} |a_r^{(\varepsilon)}| dr \right\|_2^2 \leq \frac{C}{\varepsilon^{1-(\theta_L+\alpha)\wedge 1}}. \quad (2.9)$$

Let us consider $(\nabla y^{(\varepsilon)}, \nabla z^{(\varepsilon)})$ solving the BSDE

$$\begin{aligned} \nabla y_t^{(\varepsilon)} &= \int_t^T f_x^{(\varepsilon)}(\Theta_r) \nabla X_r + f_y^{(\varepsilon)}(\Theta_r) (u(r, X_r) \nabla X_r + \nabla y_r^{(\varepsilon)}) dr \\ &\quad + \int_t^T f_z^{(\varepsilon)}(\Theta_r) U(r, X_r)^\top \nabla X_r + \sum_{j=1}^q f_{j,z}^{(\varepsilon)}(\Theta_r) (\nabla z_{j,r}^{(\varepsilon)})^\top dr - \sum_{j=1}^q \int_t^T (\nabla z_{j,r}^{(\varepsilon)})^\top dW_r. \end{aligned} \quad (2.10)$$

The processes $z^{(\varepsilon)}$ and $\nabla z^{(\varepsilon)}$ satisfy the representations

$$z_t^{(\varepsilon)} = U_t^{(\varepsilon)} \sigma(t, X_t) \quad m \times \mathbb{P} - a.e. \quad (2.11)$$

$$(V_{j,t}^{(\varepsilon)})^\top = (\nabla z_{j,t}^{(\varepsilon)})^\top \sigma^{-1}(t, X_t) - U_t^{(\varepsilon)} \nabla_x \sigma_j(t, X_t) \quad m \times \mathbb{P} - a.e. \quad (2.12)$$

where $\nabla z_{j,t}^{(\varepsilon)}$ is the j -th column of $\nabla z_t^{(\varepsilon)}$.

Proof. In what follows, C may change from line to line. From [GM10, Lemma 1.1], $\|\nabla_x u(t, X_t)\|_2 \leq C(T-t)^{(\alpha-1)/2}$ and $\|\nabla_x^2 u(t, X_t)\|_2 \leq C(T-t)^{(\alpha-2)/2}$. Therefore, $\|a_r^{(\varepsilon)}\|_2 \leq C(T-r)^{(\theta_L+\alpha-3)/2}$ for all $r \in [0, T-\varepsilon]$, which is the bound (2.7), whence

$$\left(\int_0^{T-\varepsilon} \|a_r^{(\varepsilon)}\|_2 dr \right)^2 < \frac{C}{\varepsilon^{1-(\theta_L+\alpha)\wedge 1}} < \infty.$$

This is the second inequality in (2.9). Additionally, for all $t \in [0, T)$, $|b_t^{(\varepsilon)}| + \max_j |c_{j,t}^{(\varepsilon)}| \leq C(T-t)^{(\theta_L-1)/2}$ almost surely. The first inequality in (2.9) follows. Let (ϕ, ψ) be a $(\mathbb{R}^d)^\top \times \mathbb{R}^{d \times q}$ -valued process in \mathcal{H}^2 , and define the random function

$$g(r, y, z) = g(r) := a_r^{(\varepsilon)} + \phi_r((b_r^{(\varepsilon)} I_d + \nabla_x b(r, X_r) + \sum_{j=1}^q c_{j,r}^{(\varepsilon)} \nabla_x \sigma_j(r, X_r)) + \sum_{j=1}^q (\psi_{j,r})^\top (c_{j,r}^{(\varepsilon)} I_d + \nabla_x \sigma_j(r, X_r)).$$

The function g is progressively measurable and satisfies assumptions (H1)-(H5) of [BDH⁺03, Section 4]. Since f takes no argument in (y, z) , it is only necessary to validate (H1): using the triangle inequality, Jensen's inequality, the Cauchy-Schwarz inequality, and assumptions $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}, \sigma})$, it follows that

$$\begin{aligned} \mathbb{E}[(\int_0^T |g(r)| dr)^2]^{1/2} &\leq \mathbb{E}[(\int_0^T |a_r^{(\varepsilon)}| dr)^2]^{1/2} + C(\int_0^T \mathbb{E}[|\phi_r|^2] dr)^{1/2} (\int_0^T \frac{dr}{(T-r)^{1-\theta}})^{1/2} \\ &\quad + C \sum_{j=1}^q (\int_0^T \mathbb{E}[|\psi_{j,r}|^2] dr)^{1/2} (\int_0^T \frac{dr}{(T-r)^{1-\theta}})^{1/2} < \infty. \end{aligned}$$

Thanks to [BDH⁺03, Theorem 4.2], there exists a unique solution (u, v) to the BSDE

$$u_t = \int_t^T g(r) dr - \sum_{j=1}^q \int_t^T v_{j,r} dW_r^j \quad t \in [0, T).$$

in $\mathcal{S}^2 \times \mathcal{H}^2$. The remainder of the proof of existence and uniqueness follows exactly as the proof of Theorem 2.6. To prove the first inequality in (2.9), observe that the driver $g(r)$ satisfies (5.1) from Proposition 5.2 with $f_r = |a_r^{(\varepsilon)}|$ and $\lambda_r = \mu_r = C(T-r)^{(\theta-1)/2}$.

The proofs of (2.11) and (2.12) are given in [GM10, Theorem 2.1]. The inclusion of the local Lipschitz continuity assumptions (1.2) make no difference, because the driver $f(t, x, y, z) \mathbf{1}_{[0, T-\varepsilon)}(t)$ is Lipschitz continuous uniformly in t in (x, y, z) with Lipschitz coefficient $L_f \varepsilon^{(\theta_L-1)/2}$. \square

2.4 A priori estimates

For $0 \leq s < r \leq T$, we define the Malliavin weights by

$$H_r^s = \frac{1}{r-s} \left(\int_s^r (\sigma^{-1}(t, X_t) D_s X_t)^\top dW_t \right)^\top \quad (2.13)$$

where $D_s X_t$ is the Malliavin derivative of X_t at s defined in Section 2.2. It was shown in Lemma 1.4 that $|\sigma^{-1}(t, x)|$ is uniformly bounded in (t, x) . The following constant appears throughout this paper

$$C_M := \|\sigma^{-1}\|_\infty^2 \sup_{0 \leq s < T} \sup_{s < t \leq T} \mathbb{E}_s[|D_s X_t|^2]. \quad (2.14)$$

It is known from Lemma 2.4 that $\sup_{0 \leq s < T} \sup_{s < t \leq T} \mathbb{E}_s[|D_s X_t|^2]$ is bounded. The following result is used in the proof of [GM10, Lemma 1.1]; we include it here for completeness.

Lemma 2.10. *For any $0 \leq s \leq r \leq T$,*

$$\mathbb{E}_s[|H_r^s|^2] \leq \frac{C_M}{r-s} \quad \text{almost surely.}$$

Moreover, for every $p \geq 2$, there is a finite $C_p \geq 0$ depending only on p , $\|\sigma\|_\infty$, $\|\nabla_x b\|_\infty$, $\max_j \|\nabla_x \sigma_j\|_\infty$, and T such that $\|H_r^s\|_p \leq C_p(r-s)^{-p/2}$.

Proof. Observe, using Lemma 2.5 and the fact that $(s-r)^2|H_r^s|^2 - \int_s^r |\sigma^{-1}(t, X_t)D_s X_t|^2 dt$ is a (local) martingale, that

$$\mathbb{E}_s[|H_r^s|^2] = (r-s)^{-2}\mathbb{E}_s\left[\int_s^r |\sigma^{-1}(t, X_t)D_s X_t|^2 dt\right] \leq \frac{\|\sigma^{-1}\|_\infty}{(r-s)^2}\mathbb{E}_s\left[\int_s^r |D_s X_t|^2 dt\right].$$

One then applies the conditional Fubini's lemma, Lemma 5.1, and the uniform bound on $\mathbb{E}_s[|D_s X_t|^2]$ from Lemma 2.5 to complete the proof. The bound on $\|H_r^s\|_p$ is proved using the Burkholder-Davis-Gundy inequality on the continuous local martingale $(t-s)H_t^s$. \square

We now state and prove a priori results on the solutions of BSDEs with drivers satisfying (1.2).

Proposition 2.11. *Let $x \mapsto \Phi_1, \Phi_2 \in \mathbf{L}_2(\mathcal{F}_T)$ and $(\omega, t, y, z) \mapsto f_1(\omega, t, y, z), f_2(\omega, t, y, z)$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}((\mathbb{R}^q)^\top)$ -measurable functions for which there are constants $(\theta_{1,L}, \theta_{2,L}) \in (0, 1]^2$ and $(L_{f_1}, L_{f_2}) \in (0, \infty)^2$ such that*

$$|f_i(\omega, t, y, z) - f_i(\omega, t, y', z')| \leq \frac{L_{f_i}\{|y-y'| + |z-z'|\}}{(T-t)^{(1-\theta_{i,L})/2}} \quad m \times \mathbb{P} - \text{almost everywhere,}$$

and $f_i(\omega, t, 0, 0) \in \mathcal{H}^2$ for $i \in \{1, 2\}$. Let (Y_i, Z_i) be a solution to the FBSDE with terminal condition Φ_i and driver $f_i(t, y, z)$ ($i = 1, 2$ respectively).

Define

$$\begin{aligned} \Delta Y_t &:= Y_{1,t} - Y_{2,t}, & \Delta Z_t &:= Z_{1,t} - Z_{2,t}, \\ \Delta f_t &:= f_1(t, Y_{1,t}, Z_{1,t}) - f_2(t, Y_{1,t}, Z_{1,t}), & \Delta \Phi &:= \Phi_1 - \Phi_2. \end{aligned}$$

Then there is a finite constant $C \geq 0$ depending only on T, L_{f_2} and $\theta_{2,L}$ such that, for all $s < t < T$,

$$\mathbb{E}_s[\Delta Y_t^2] + \mathbb{E}_s\left[\int_t^T |\Delta Z_s|^2 ds\right] \leq C\mathbb{E}_s[\Delta \Phi^2] + C\left(\int_t^T \mathbb{E}_s[\Delta f_r^2]^{1/2} dr\right)^2 \quad (2.15)$$

Moreover, suppose that $Z_{i,t} := \mathbb{E}_t[\Phi_i(X_T)H_T^t + \int_t^T f_i(r, X_r, Y_{i,r}, Z_{i,r})H_r^t dr]$ for all $t \in [0, T]$ almost surely ($i = 1, 2$). Then there is a (possibly different) finite constant $C \geq 0$ depending only on T, C_M, L_{f_2} , and $\theta_{2,L}$ such that,

$$\mathbb{E}_s[|\Delta Z_t|^2]^{1/2} \leq C \frac{\mathbb{E}_s[(\Delta \Phi - \mathbb{E}_t \Delta \Phi)^2]^{1/2}}{\sqrt{T-t}} + C \int_t^T \frac{\mathbb{E}_s[\Delta f_r^2]^{1/2}}{\sqrt{r-t}} dr + C\mathbb{E}_s[\Delta \Phi^2]^{1/2}(T-t)^{\theta_{2,L}/2} \quad (2.16)$$

for all $t \in [0, T]$ almost surely.

Proof. In what follows, C may change from line to line. We prove the result for $s = 0$; the general case is proved analogously, the only difference is that one must use the conditional version of the Minkowski, Cauchy-Schwarz, and Hölder inequalities and the conditional Fubini's theorem (Lemma 5.1). Using the definition of the BSDE (1.1),

$$\Delta Y_t + \int_t^T \Delta Z_s dW_s = \Delta \Phi + \int_t^T \Delta f_s ds + \int_t^T f_2(s, Y_{1,s}, Z_{1,s}) - f_2(s, Y_{2,s}, Z_{2,s}) ds.$$

Using (1.2) and Hölder's inequality,

$$\begin{aligned}
\|\Delta Y_t\|_2^2 + \int_t^T \|\Delta Z_s\|_2^2 ds &\leq 3\|\Delta\Phi\|_2^2 + 3\left\|\int_t^T \Delta f_s ds\right\|_2^2 + 3\left\|\int_t^T f_2(s, Y_{1,s}, Z_{1,s}) - f_2(s, Y_{2,s}, Z_{2,s}) ds\right\|_2^2 \\
&\leq 3\|\Delta\Phi\|_2^2 + 3\left\|\int_t^T |\Delta f_s| ds\right\|_2^2 + 3L_{f_2}^2 \left\|\int_t^T \frac{|\Delta Y_s| + |\Delta Z_s|}{(T-s)^{(1-\theta_{2,L})/2}} ds\right\|_2^2 \\
&\leq 3\|\Delta\Phi\|_2^2 + 3\left\|\int_t^T |\Delta f_s| ds\right\|_2^2 + 3L_{f_2}^2 \int_t^T \frac{1}{(T-s)^{1-\theta_{2,L}}} ds \int_t^T \{\|\Delta Y_s\|_2^2 + \|\Delta Z_s\|_2^2\} ds \\
&\leq 3\|\Delta\Phi\|_2^2 + 3\left\|\int_t^T |\Delta f_s| ds\right\|_2^2 + 3L_{f_2}^2 (T-t)^{\theta_{2,L}} \int_t^T \{\|\Delta Y_s\|_2^2 + \|\Delta Z_s\|_2^2\} ds \quad (2.17)
\end{aligned}$$

Setting $t_0 = (T - 1/(6L_{f_2}^2)^{1/\theta_{2,L}}) \vee 0$ ensures that $3L_{f_2}^2 (T - t_0)^{\theta_{2,L}} \leq 1/2$, and, on the other hand, that $T - t_0 \leq 1$. Integrating (2.17) over (t_0, T) , we obtain

$$\int_{t_0}^T \|\Delta Y_t\|_2^2 + \|\Delta Z_t\|_2^2 ds \leq 6\|\Delta\Phi\|_2^2 + 6L_{f_2}^2 \left\|\int_{t_0}^T |\Delta f_s| ds\right\|_2^2 \quad (2.18)$$

Substituting (2.18) into (2.17) then yields

$$\sup_{t_0 \leq t < T} \|\Delta Y_t\|_2^2 \leq 6\|\Delta\Phi\|_2^2 + 6\left\|\int_{t_0}^T |\Delta f_s| ds\right\|_2^2$$

and this gives the result in the interval $[t_0, T]$.

In the interval $[0, t_0)$, the function $(y, z) \mapsto f_2(\omega, t, y, z)$ is $m \times \mathbb{P}$ Lipschitz continuous with Lipschitz constant $L = L_f (T - t_0)^{(\theta_{2,L} - 1)/2}$. It then follows from [EKPQ97, Proposition 2.1] that

$$\sup_{0 \leq t < t_0} \|\Delta Y_t\|_2^2 + \int_{t_0}^T \|\Delta Z_s\|_2^2 ds \leq C\|\Delta Y_{t_0}\|_2^2 + C\left\|\int_0^{t_0} |\Delta f_s| ds\right\|_2^2$$

and the proof of (2.15) is complete by substituting the bounds on $\|\Delta Y_{t_0}\|_2^2$ from above.

Next, we prove (2.16). Using the representation $Z_{i,t} = \mathbb{E}_t[\Phi_i H_T^t + \int_t^T f_i(r, Y_{i,r}, Z_{i,r}) H_r^t dr]$, it follows from Minkowski's inequality, the Cauchy-Schwarz inequality, and Lemma 2.10 that

$$\|\Delta Z_t\|_2 \leq \frac{CV_{t,T}(\Delta\Phi)}{\sqrt{T-t}} + C \int_t^T \frac{\|\Delta f_r\|_2}{\sqrt{r-t}} dr + C \int_t^T \frac{\|\Delta Y_r\|_2 + \|\Delta Z_r\|_2}{(T-r)^{(1-\theta_{2,L})/2} \sqrt{r-t}} dr. \quad (2.19)$$

where we define $V_{t,T}(\Delta\Phi)$ by $\mathbb{E}[|\Delta\Phi - \mathbb{E}_t \Delta\Phi|^2]^{1/2}$. Defining $\Theta_r = \|\Delta Y_r\|_2 + \|\Delta Z_r\|_2$ and recalling (2.17), it follows that

$$\Theta_t \leq C\|\Delta\Phi\|_2 + \frac{CV_{t,T}(\Delta\Phi)}{\sqrt{T-t}} + C \int_t^T \frac{\|\Delta f_r\|_2}{\sqrt{r-t}} dr + C \int_t^T \frac{\Theta_r}{(T-r)^{(1-\theta_{2,L})/2} \sqrt{r-t}} dr. \quad (2.20)$$

Applying Lemma 5.6 with $u_t = \Theta_t$ and

$$w_t = C\|\Delta\Phi\|_2 + \frac{CV_{t,T}(\Delta\Phi)}{\sqrt{T-t}} + C \int_t^T \frac{\|\Delta f_r\|_2}{\sqrt{r-t}} dr,$$

it follows that

$$\Theta_r \leq Cw_t + C \int_t^T \frac{w_r}{(T-r)^{(1-\theta_{2,L})/2} \sqrt{r-t}} dr + C \int_t^T \frac{\Theta_r}{(T-r)^{(1-\theta_{2,L})/2}} dr$$

whence it follows from Lemma 5.7 that

$$\int_t^T \frac{\Theta_r}{(T-r)^{(1-\theta_{2,L})/2} \sqrt{r-t}} dr \leq C \int_t^T \frac{w_r}{(T-r)^{(1-\theta_{2,L})/2} \sqrt{r-t}} dr$$

Substituting this into (2.19) and applying Lemma 5.5 leads to

$$\begin{aligned} \|\Delta Z_t\|_2 &\leq \frac{CV_{t,T}(\Delta\Phi)}{\sqrt{T-t}} + C \int_t^T \frac{\|\Delta f_r\|_2}{\sqrt{r-t}} dr + C \int_t^T \frac{w_r}{(T-r)^{(1-\theta_{2,L})/2} \sqrt{r-t}} dr \\ &= \frac{CV_{t,T}(\Delta\Phi)}{\sqrt{T-t}} + C \int_t^T \frac{\|\Delta f_r\|_2}{\sqrt{r-t}} dr + C \int_t^T \frac{V_{r,T}(\Delta\Phi)}{(T-r)^{(2-\theta_{2,L})/2} \sqrt{r-t}} dr \\ &\quad + C \int_t^T \frac{\int_r^T \|\Delta f_s\|_2 (s-r)^{-1/2} ds}{(T-r)^{(1-\theta_{2,L})/2} \sqrt{r-t}} dr + C \|\Delta\Phi\|_2 (T-t)^{\theta_{2,L}/2} \\ &= \frac{CV_{t,T}(\Delta\Phi)}{\sqrt{T-t}} + C \int_t^T \frac{\|\Delta f_r\|_2}{\sqrt{r-t}} dr + C \int_t^T \frac{V_{r,T}(\Delta\Phi)}{(T-r)^{(2-\theta_{2,L})/2} \sqrt{r-t}} dr \\ &\quad + C \int_t^T \|\Delta f_s\|_2 \left\{ \int_r^s (s-r)^{-1+\theta_{2,L}} (r-t)^{-1/2} dr \right\} ds + C \|\Delta\Phi\|_2 (T-t)^{\theta_{2,L}/2}. \end{aligned}$$

The proof is completed by observing that $V_{r,T}(\Delta\Phi)$ is non-increasing in r . \square

The estimates (2.16) allow us to determine a priori estimates on the conditional second moments of the solution of the BSDE (Y, Z) .

Corollary 2.12. *Assume that $Z_t = \mathbb{E}_t[\Phi(X_T)H_T^t + \int_t^T f(r, X_r, Y_r, Z_r)H_r^t dr]$ for all $t \in [0, T]$ almost surely. Then there is a constant C depending only on $L_f, \theta_L, C_f, \theta_c, K^\alpha(\Phi)$ and T such that, for all $t \in [0, T]$ and $s \in [0, t]$, we have*

$$\begin{aligned} \sup_{s \leq t \leq T} \mathbb{E}_s[|Y_t|^2]^{1/2} &\leq C(1 + \mathbb{E}_s[|\Phi(X_T) - \mathbb{E}_s[\Phi(X_T)]|^2]^{1/2}), \\ \mathbb{E}_s[|Z_t|^2]^{1/2} &\leq \frac{C\mathbb{E}_s[|\Phi(X_T) - \mathbb{E}_s[\Phi(X_T)]|^2]^{1/2}}{\sqrt{T-t}} + \frac{C}{(T-t)^{(1-2\theta_c)/2}} + C\mathbb{E}_s[|\Phi(X_T)|^2]^{1/2}(T-t)^{\theta_L/2}. \end{aligned}$$

In particular, $\|Y_t\|_2 \leq C$ and $\|Z_t\|_2 \leq C(T-s)^{((2\theta_c) \wedge \alpha - 1)/2}$ for all $t \in [0, T]$, and

$$\|f(s, X_s, Y_s, Z_s)\|_2 \leq \frac{C}{(T-s)^{1-((2\theta_c) \wedge \alpha + \theta_L)/2}} + \frac{C}{(T-s)^{1-\theta_c}}. \quad (2.21)$$

If $(\mathbf{A}_{\mathbf{h}\Phi})$ is in force, we have additionally that $|Z_t| \leq CK_\Phi(T-s)^{((2\theta_c) \wedge \theta_\Phi - 1)/2}$ for all $t \in [0, T]$ almost surely.

Proof. In what follows, C may change from line to line. As in Proposition 2.11, we only prove the result for $s = 0$. Apply (2.16) from Proposition 2.11 with $(Y_1, Z_1) = (0, 0)$ and $(Y_2, Z_2) = (Y, Z)$ to obtain (for all $t \in [0, T]$)

$$\begin{aligned} \|Z_t\|_2 &\leq C \frac{V_{t,T}(\Phi)}{\sqrt{T-t}} + C \int_t^T \frac{\|f(r, X_r, 0, 0)\|_2}{\sqrt{r-t}} dr + C \|\Phi\|_2 (T-t)^{\theta_L/2} \\ &\leq \frac{C}{(T-t)^{(1-\alpha)/2}} + C \int_t^T \frac{dr}{(T-r)^{1-\theta_c} \sqrt{r-t}} + C(T-t)^{\theta_L/2}. \end{aligned}$$

Combining the local Lipschitz continuity and boundedness of f in (1.2) leads to the required bound on the conditional second moments of Z_t . The estimate on the conditional moments of Y_t is obtained similarly starting from (2.16). The remaining bounds are obtained by taking into account (1.2) and the regularity of the terminal condition $((\mathbf{A}_\Phi)$ or $(\mathbf{A}_{\mathbf{h}\Phi}))$. \square

Recall $(Y^{(\varepsilon)}, Z^{(\varepsilon)})$ from Definition 2.7 in Section 2.3, the BSDE with terminal condition Φ and driver $f^{(\varepsilon)}(t, x, y, z) := f(t, x, y, z)\mathbf{1}_{[0, T-\varepsilon)}(t)$. The following corollary of Proposition 2.11 will be used extensively throughout this paper; it provides a stability results between the BSDEs (Y, Z) and $(Y^{(\varepsilon)}, Z^{(\varepsilon)})$ that are controlled by ε .

Corollary 2.13. *Let $\gamma := (\theta_c \wedge \frac{\alpha}{2} + \frac{\theta_L}{2}) \wedge \theta_c$. Suppose that $Z = \mathbb{E}_t[\Phi(X_T)H_T^t + \int_t^T f(s, X_s, Y_s, Z_s)H_s^t ds]$ and $Z_t^{(\varepsilon)} = \mathbb{E}_t[\Phi(X_T)H_T^t + \int_t^T f^{(\varepsilon)}(s, X_s, Y_s^{(\varepsilon)}, Z_s^{(\varepsilon)})H_s^t ds]$ for all $t \in [0, T)$ almost surely. Then there is a constant C such that*

$$\sup_{0 \leq t \leq T} \|Y_t - Y_t^{(\varepsilon)}\|_2^2 + \int_0^T \|Z_t - Z_t^{(\varepsilon)}\|_2^2 dt \leq C\varepsilon^{2\gamma}, \quad (2.22)$$

$$\|Z_t - Z_t^{(\varepsilon)}\|_2 \leq C \int_{t \vee (T-\varepsilon)}^T \frac{ds}{(T-s)^{1-\gamma} \sqrt{s-t}} \quad (2.23)$$

for all $t \in [0, T)$. In particular, $(Y^{(\varepsilon)}, Z^{(\varepsilon)}) \rightarrow (Y, Z)$ as $\varepsilon \rightarrow 0$ in $\mathcal{S}^2 \times \mathcal{H}^2$.

Proof. In what follows, C may change from line to line.

It follows from (2.15) in Proposition 2.11 that

$$\sup_{0 \leq t \leq T} \|Y_t - Y_t^{(\varepsilon)}\|_2^2 + \int_0^T \|Z_s - Z_s^{(\varepsilon)}\|_2^2 ds \leq C \left(\int_{T-\varepsilon}^T \|f(s, X_s, Y_s, Z_s)\|_2 ds \right)^2. \quad (2.24)$$

Substituting (2.21) into (2.24) combined with $\left(\int_{T-\varepsilon}^T \frac{ds}{(T-s)^{(1-\gamma)}} \right)^2 \leq C\varepsilon^{2\gamma}$ completes the proof of (2.22). Next, it follows from (2.16) that

$$\|Z_t - Z_t^{(\varepsilon)}\|_2 \leq C \int_{t \vee (T-\varepsilon)}^T \frac{\|f(s, X_s, Y_s, Z_s)\|_2}{\sqrt{s-t}} ds \quad \text{for all } t \in [0, T).$$

Substituting (2.21) above proves (2.23). \square

To end this section, we present a mollification procedure that will be used frequently to allow us to extend results under the assumptions $(\mathbf{A}_{\partial\mathbf{f}})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$ to the same results without these assumptions. The following corollaries are a trivial consequence of Proposition 2.11 and the properties of mollifiers.

Corollary 2.14. *Let $M > 0$ be finite, and $M \mapsto R(M) \geq 1$ be increasing w.r.t. M . Define $\Phi_M(x) := -M \vee \Phi(x) \wedge M$ and, recalling the mollifier ϕ of Definition 1.1,*

$$f_M(t, x, y, z) := \int_{\mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^q)^\top} f(t, x - x', y - y', z - z') \phi_{R(M)}(x', y', z') d(x', y', z'),$$

Let (Y_M, Z_M) be the solution of the BSDE with terminal condition Φ_M and driver $f_M(t, x, y, z)$. Then Φ_M satisfies $(\mathbf{A}_{\mathbf{b}\Phi})$, f_M satisfies $(\mathbf{A}_{\partial\mathbf{f}})$, and $(Y_M, Z_M) \rightarrow (Y, Z)$ as $M \rightarrow \infty$ in $\mathcal{S}^2 \times \mathcal{H}^2$.

2.5 Representation theorem

Theorem 2.15. *Suppose that $\Phi \in L_{2,\alpha}$ and $(t, x, y, z) \mapsto f(t, x, y, z)$ satisfies (1.2). Then, there is a predictable version \tilde{Z} of Z which satisfies*

$$Z_t = \mathbb{E}_t[\Phi(X_T)H_T^t + \int_t^T f(s, X_s, Y_s, Z_s)H_s^t ds] \quad \text{for all } t \in [0, T) \quad \mathbb{P} - a.s. \quad (2.25)$$

where H_s^t are the Malliavin weights given in (2.13).

Proof. In the following, we C is a constant whose value may change from line to line.

To start with, assume $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$. Recall the BSDEs $(Y^{(\varepsilon)}, Z^{(\varepsilon)})$, (y, z) and $(y^{(\varepsilon)}, z^{(\varepsilon)})$ from Section 2.3, and furthermore the decomposition $(Y^{(\varepsilon)}, Z^{(\varepsilon)}) = (y + y^{(\varepsilon)}, z + z^{(\varepsilon)})$. We obtain the representation theorem for the BSDEs $(y^{(\varepsilon)}, z^{(\varepsilon)})$ and use the decomposition to obtain representation theorem for $(Y^{(\varepsilon)}, Z^{(\varepsilon)})$; the representation theorem for (y, z) is well known, see the proof of [GM10, Lemma 1.1].

Firstly, observe from Lemma 2.8 that $(y^{(\varepsilon)}, z^{(\varepsilon)})$ solves a BSDE with terminal condition 0 and driver $F(t, x, y, z) := f^{(\varepsilon)}(t, x, u(t, x) + y, (\nabla_x u(t, x)\sigma(t, x))^\top + z)$ on the time interval $[0, T - \varepsilon]$; on this interval, the driver F is uniformly Lipschitz continuous ($\theta_{L,F} = 1$) in (x, y, z) and uniformly bounded at $(x, y, z) = (0, 0)$ ($\theta_{C,F} = 1$), due to the bounds on u and it's derivatives, and continuous in t . Therefore, [MZ02, Theorem 4.2] applies to the BSDE in the interval $[0, T - \varepsilon]$, i.e. $z_t^{(\varepsilon)} = \mathbb{E}_t[\int_t^{T-\varepsilon} F(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})H_r^t dr]$ for all $t \in [0, T - \varepsilon]$ almost surely. On the other hand, $z_t^{(\varepsilon)}$ and $F(t, x, y, z)$ are 0 for all $t \in (T - \varepsilon, T]$ almost surely, so the representation holds trivially in the latter interval. Therefore, including the representation of (y, z) , it follows that

$$Z_t^{(\varepsilon)} = \mathbb{E}_t[\Phi(X_T)H_T^t + \int_t^T f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})H_r^t dr] \quad \text{for all } t \in [0, T) \quad \mathbb{P} - a.s. \quad (2.26)$$

Define by \mathcal{Z} the predictable projection [JS03, Theorem 2.28] of the process $(\mathcal{X}_t := \Phi(X_T)H_T^t + \int_t^T f(r, X_r, Y_r, Z_r)H_r^t dr)_{t \in [0, T]}$. In what follows, we show that $\|Z_t^{(\varepsilon)} - \mathcal{Z}_t\|_2 \rightarrow 0$ as $\varepsilon \rightarrow 0$ for almost all $t \in [0, T)$. This implies, by the dominated convergence theorem, that $Z^{(\varepsilon)} \rightarrow \mathcal{Z}$ in \mathcal{H}^2 . Since $Z^{(\varepsilon)} \rightarrow Z$ in \mathcal{H}^2 was determined in Corollary 2.13, this implies that $Z_t = \mathcal{Z}_t$ $m \times \mathbb{P} - a.e.$, which completes the proof under the assumptions $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$.

We first need some intermediate bounds. Analogously to Corollary 2.12, we have that

$$\|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})\|_2 \leq \frac{C}{(T-r)^{1-\gamma}} \quad \text{for all } r \in [0, T). \quad (2.27)$$

Fix $t \in [0, T)$ and $\eta > 0$. Using the representation formula (2.26), it follows from Minkowski's inequality, the Cauchy-Schwarz inequality, and Lemma 2.10 that

$$\begin{aligned} \|Z_t^{(\varepsilon)} - \mathcal{Z}_t\|_2 &= \|\mathbb{E}_t[\int_t^T (f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r, Z_r))H_r^t dr]\|_2 \\ &\leq \|\mathbb{E}_t[\int_t^T (f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}))H_r^t dr]\|_2 \\ &\quad + C_M^{1/2} \int_t^T \frac{\|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r, Z_r)\|_2}{\sqrt{r-t}} dr. \end{aligned} \quad (2.28)$$

Taking $\varepsilon < (T-t)/2$ and using (2.27), it follows that

$$\begin{aligned} \|\mathbb{E}_t[\int_t^T (f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}))H_r^t dr]\|_2 &\leq C_M^{1/2} \int_{T-\varepsilon}^T \frac{\|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})\|_2}{\sqrt{r-t}} dr \\ &\leq \frac{C_M^{1/2}}{\sqrt{T-t-\varepsilon}} \int_{T-\varepsilon}^T \|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})\|_2 dr \leq \frac{C}{\sqrt{T-t}} \int_{T-\varepsilon}^T \frac{dr}{(T-r)^{1-\gamma}} = \frac{C\varepsilon^\gamma}{\sqrt{T-t}}. \end{aligned}$$

Taking $\varepsilon < \eta^{1/\gamma}(T-t)^{1/(2\gamma)}/C$, where C is the last constant in the inequality above, is sufficient

to bound the above term by η . On the other hand, letting $\delta < (T - t)/2$,

$$\begin{aligned} & \int_t^T \frac{\|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r, Z_r)\|_2}{\sqrt{r-t}} dr \\ & \leq C_M^{1/2} \frac{\int_{t+\delta}^T \|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r, Z_r)\|_2 dr}{\sqrt{\delta}} \\ & \quad + C_M^{1/2} \int_t^{t+\delta} \frac{\|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r, Z_r)\|_2}{\sqrt{r-t}} dr \end{aligned} \quad (2.29)$$

To bound the first integral term above, apply Hölder's inequality and the Lipschitz continuity of $f(t, \cdot)$ to obtain

$$\begin{aligned} & C_M^{1/2} \int_{t+\delta}^T \|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r, Z_r)\|_2 dr \\ & \leq C_M^{1/2} L_f \left(\int_0^T \frac{dr}{(T-r)^{1-\theta_L}} \right)^{1/2} \left(\sup_{0 \leq s \leq T} \|Y_s - Y_s^{(\varepsilon)}\|_2^2 + \int_0^T \|Z_r - Z_r^{(\varepsilon)}\|_2^2 dr \right)^{1/2} \end{aligned}$$

Using that $(Y^{(\varepsilon)}, Z^{(\varepsilon)}) \rightarrow (Y, Z)$ in $\mathcal{S} \times \mathcal{H}^2$ as $\varepsilon \rightarrow 0$ (Corollary 2.13), set ε sufficiently small so that the above is bounded above by $\sqrt{\delta}\eta$. To bound the second integral term in (2.29), use (2.21) and (2.27) combined with the triangle inequality to show that

$$\begin{aligned} & C_M^{1/2} \int_t^{t+\delta} \frac{\|f(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)}) - f(r, X_r, Y_r, Z_r)\|_2}{\sqrt{r-t}} dr \\ & \leq \frac{C}{(T-t-\delta)^{1-\gamma}} \int_t^{t+\delta} \frac{dr}{\sqrt{r-t}} \leq \frac{C\sqrt{\delta}}{(T-t)^{1-\gamma}} \end{aligned}$$

and set δ sufficiently small so that the above is bounded above by η . Therefore, we have shown that for almost every $t \in [0, T)$ and every $\eta > 0$, there is a sufficiently small ε such that $\|Z_t^{(\varepsilon)} - Z_t\|_2 < 3\eta$. In other words, $\mathbb{E}[\|Z_t^{(\varepsilon)} - Z_t\|_2^2] \rightarrow 0$ as $\varepsilon \rightarrow 0$ for every t , as required.

To prove the result without $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$, recall the mollified BSDE (Y_M, Z_M) from Corollary 2.14. Since Φ_M satisfies $(\mathbf{A}_{\mathbf{b}\Phi})$ and f_M satisfies $(\mathbf{A}_{\partial f})$, there is a predictable version \mathcal{Z}_M of Z_M satisfying $\mathcal{Z}_{M,t} = \mathbb{E}_t[\Phi_M(X_T)H_T^t + \int_t^T f_M(r, X_r, Y_{M,r}, Z_{M,r})H_r^t dr]$ for all $t \in [0, T)$ almost surely. Now, we can use analogous to the above arguments, together with the point-wise convergence of f_M to f and Φ_M to Φ , and the convergence of (Y_M, Z_M) to (Y, Z) in $\mathcal{S}^2 \times \mathcal{H}^2$ from Corollary 2.14, to complete the proof in the general case. \square

3 Convergence rate of the Euler scheme for BSDEs

Throughout this section, the assumption $(\mathbf{A}_{\mathbf{f}})$ is in force. The following theorem serves as the basis of our estimates in this section; the proof is analogous to the proof of [GL06, Theorem 1], one must only use the result $\Delta_k/(T-t_k)^{1-\theta_L} \leq T^{\theta_L}(\beta N)^{-1}$ for $\beta \leq \theta_L$ (see Lemma 5.3) in order to compensate for the local Lipschitz constant of the driver.

Proposition 3.1. *Let $\beta \leq \theta_L$. For the Euler scheme for BSDEs defined on the time-grids $\{\pi_N^{(\beta)} : N \geq 1\}$, there is a constant C depending only on L_f , θ_L , β , and T , but not on N , such that, for all $N \geq 1$,*

$$\mathcal{E}_1(N) \leq CN^{-1} + C \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_t - \tilde{Z}_{t_i}\|_2^2 dt$$

where $\tilde{Z}_{t_i} := \frac{1}{\Delta_i} \mathbb{E}_i[\int_{t_i}^{t_{i+1}} Z_t dt]$.

Following on from Section 2.5, we work with the version of Z given by (2.25) in Theorem 2.15. Since $(\tilde{Z}_{t_i})_i$ is the projection of Z onto the space of adapted discrete processes with nodes on π under the scalar product $(u, v) = \mathbb{E} \int_0^T (u_s \cdot v_s) ds$, it follows that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_t - \tilde{Z}_{t_i}\|_2^2 dt \leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_t - Z_{t_i}\|_2^2 dt. \quad (3.1)$$

To bound $\mathcal{E}_1(N)$, it follows from Proposition 3.1 that it is sufficient to bound the term on the right-hand side of (3.1). To do so, we use an approximation procedure. Recall the BSDE $(Y^{(\varepsilon)}, Z^{(\varepsilon)})$ from Definition 2.7 in Section 2.3. In what follows, we work with the version of $Z^{(\varepsilon)}$ given by Theorem 2.15. The following lemma decomposes the \mathbf{L}_2 -regularity of Z into the \mathbf{L}_2 -regularity of $Z^{(\varepsilon)}$ and terms controlled by ε .

Lemma 3.2. *Let $\beta \in (0, 1]$. Then there is a constant C depending only on $L_f, C_M, \theta_L, \theta_c, \beta, C_f, K^\alpha(\Phi)$, and T , such that for all $N \geq 1$*

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s - \tilde{Z}_{t_i}\|_2^2 ds \leq C \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s^{(\varepsilon)} - \tilde{Z}_{t_i}^{(\varepsilon)}\|_2^2 ds + C\varepsilon^{2\gamma}.$$

Proof. In what follows, C may change in value from line to line. Using the Cauchy inequality and the orthogonality of the projections, $\frac{1}{2} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s - \tilde{Z}_{t_i}\|_2^2 ds \leq \int_0^T \|Z_s - Z_s^{(\varepsilon)}\|_2^2 ds + \sum_{i=0}^{N-1} \|\tilde{Z}_{t_i} - \tilde{Z}_{t_i}^{(\varepsilon)}\|_2^2 \Delta_i + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s^{(\varepsilon)} - \tilde{Z}_{t_i}^{(\varepsilon)}\|_2^2 ds$. Recall from Corollary 2.13 that $\int_0^T \|Z_s - Z_s^{(\varepsilon)}\|_2^2 ds \leq C\varepsilon^{2\gamma}$. Moreover, using Jensen's inequality,

$$\sum_{i=0}^{N-1} \|\tilde{Z}_{t_i} - \tilde{Z}_{t_i}^{(\varepsilon)}\|_2^2 \Delta_i = \sum_{i=0}^{N-1} \left\| \frac{1}{\Delta_i} \mathbb{E}_i \left[\int_{t_i}^{t_{i+1}} (Z_t - Z_t^{(\varepsilon)}) dt \right] \right\|_2^2 \Delta_i \leq \int_0^T \|Z_s - Z_s^{(\varepsilon)}\|_2^2 ds \leq C\varepsilon^{2\gamma}$$

and this completes the proof. \square

We now come to our first and most general estimate on the $\mathcal{E}(N)$. Later, in Theorem 3.6, we augment this result under stronger assumptions.

Theorem 3.3. *Let $0 < \beta < (2\gamma) \wedge \alpha$. There is a constant C depending only on $L_f, C_M, \theta_L, \theta_c, \beta, C_f, K^\alpha(\Phi)$, and T , but not on N , such that for all $N \geq 1$,*

$$\mathcal{E}(N) \leq CN^{-1} \mathbf{1}_{[1,2]}(\alpha + \theta_L) + CN^{-2\gamma} \mathbf{1}_{(0,1)}(\alpha + \theta_L)$$

Proof. From Proposition 3.1, it is sufficient to bound $\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_t - \tilde{Z}_{t_i}\|_2^2 dt$. In what follows, C may change in value from line to line. To start with, assume $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$. Recall the BSDEs $(y^{(\varepsilon)}, z^{(\varepsilon)})$ from Definition 2.7 and $(U^{(\varepsilon)}, V^{(\varepsilon)})$ from (2.8) in Section 2.3. In the proof of [GM10, Theorem 3.1], the authors show that for any i and $s \in [t_i, t_{i+1})$,

$$\|z_s^{(\varepsilon)} - z_{t_i}^{(\varepsilon)}\|_2 \leq C \int_{t_i}^s \|a_r^{(\varepsilon)}\|_2 dr + C \int_{t_i}^s \|V_r^{(\varepsilon)}\|_2 dr + C\Delta_i^{1/2}. \quad (3.2)$$

Using $(\int_0^T \|a_r^{(\varepsilon)}\|_2 dr)^2 + \int_0^T \|V_r^{(\varepsilon)}\|_2^2 dr \leq C\varepsilon^{-1+(\theta_L+\alpha)\wedge 1}$ from (2.9) in Lemma 2.9, and (3.1), it follows from Jensen's inequality that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|z_s^{(\varepsilon)} - \tilde{z}_{t_i}^{(\varepsilon)}\|_2^2 ds \leq \frac{C}{N} + \frac{C \max_{0 \leq i \leq N-1} \Delta_i}{\varepsilon^{1-(\theta_L+\alpha)\wedge 1}} \leq CN^{-1}(1 + \varepsilon^{(\theta_L+\alpha)\wedge 1-1})$$

where $\max_i \Delta_i \leq CN^{-1}$ follows from (5.2) in Lemma 5.3. Combining this estimate with $\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|z_s - z_{t_i}\|_2^2 ds \leq CN^{-1}$, shown in [GM10, Theorem 1.3], $Z^{(\varepsilon)} = z + z^{(\varepsilon)}$, and the results of Lemma 3.2, (3.1) it follows that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s - \tilde{Z}_{t_i}\|_2^2 ds \leq CN^{-2\gamma/\beta} + C\varepsilon^{2\gamma} + CN^{-1}(1 + \varepsilon^{(\theta_L + \alpha)\wedge 1-1}). \quad (3.3)$$

Let $\varepsilon = N^{-\delta}$. To complete the proof under $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$, we take $\delta = 1/(2\gamma)$ if $\alpha + \theta_L \geq 1$ and $\delta = 1$ otherwise, and notice that $2\gamma \leq \alpha + \theta_L$.

In order to prove the general result, recall the BSDE (Y_M, Z_M) from Corollary 2.14; its terminal condition satisfies $(\mathbf{A}_{\mathbf{b}\Phi})$ and its driver satisfies $(\mathbf{A}_{\partial f})$. Moreover, [GM10, Lemma 3.1] yields $K^\alpha(\Phi_M) \leq K^\alpha(\Phi)$. Therefore, working with the version of Z_M given by Theorem 2.15, it follows from the triangle inequality and the results obtained above that

$$\mathcal{E}(N) \leq 2 \int_0^T \|Z_s - Z_{M,s}\|_2^2 ds + 2 \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_{M,s} - \tilde{Z}_{M,t_i}\|_2^2 ds$$

and letting $M \rightarrow \infty$ with Corollary 2.14 yields the result. \square

In the remainder of this section, we give a complementary result to Theorem 3.3 under $(\mathbf{A}_{\text{exp}\Phi})$. This is achieved using the an additional a priori estimates on $\|V_t^{(\varepsilon)}\|_2$, given in the proposition below. This a priori estimate will be very important in Section 4.

Proposition 3.4. *Suppose that $(\mathbf{A}_{\partial f})$ is in force and $\Phi(x)$ is not zero everywhere in \mathbb{R}^d . If $(\mathbf{A}_{\mathbf{b}\Phi})$ is in force, there exists version of $V^{(\varepsilon)}$ and a finite constant C depending only on L_f , the bounds on b and σ and their partial derivatives, $\bar{\beta}$, C_M , θ_L , θ_c , C_f , and T such that for any $\varepsilon \in (0, T]$ and every $t \in [0, T)$, $\|V_t^{(\varepsilon)}\|_2 \leq C\phi(t, \varepsilon, \theta_L)$, where*

$$\phi(t, \varepsilon, \theta_L) := \|\Phi\|_\infty \int_t^{T-\varepsilon} \frac{dr}{(T-r)^{(3-\theta_L)/2} \sqrt{r-t}}. \quad (3.4)$$

If $(\mathbf{A}_{\mathbf{h}\Phi})$ is in force, there exists a version of $V^{(\varepsilon)}$, such that for any $\varepsilon \in (0, T]$ and every $t \in [0, T)$, $\|V_t^{(\varepsilon)}\|_2 \leq C\phi(t, \varepsilon, \theta_L, \theta_\Phi)$, where

$$\phi(t, \varepsilon, \theta_L, \theta_\Phi) := K_\Phi \int_t^{T-\varepsilon} \frac{dr}{(T-r)^{(3-\theta_L-\theta_\Phi)/2} \sqrt{r-t}}. \quad (3.5)$$

Remark. The integrals in (3.4,3.5) exists and are bounded by $C\varepsilon^{-(1-\theta_L)/2}(T-t)^{(\alpha-1)/2}$.

Proof. In what follows, C may change from line to line.

For all $(t, x) \in [0, T) \times \mathbb{R}^d$, define the FBSDE

$$y_s^{(\varepsilon, t, x)} = \int_s^T F(r, X_r^{(t, x)}, y_r^{(\varepsilon, t, x)}, z_r^{(\varepsilon, t, x)}) dr - \int_s^T z_r^{(\varepsilon, t, x)} dW_r, \quad s \in [t, T), \quad (3.6)$$

where $F(t, x, y, z) = f^{(\varepsilon)}(t, x, u(t, x) + y, (\nabla_x u(t, x)\sigma(t, x))^\top + z)$ and $X^{(t, x)}$ is the solution of the SDE (2.2). Note that the BSDE $(y^{(\varepsilon)}, z^{(\varepsilon)})$ from Section 2.3 is equal to $(y^{(\varepsilon, 0, x_0)}, z^{(\varepsilon, 0, x_0)})$ because, thanks to Lemma 2.8, (y, z) is equal to $(u(\cdot, X), \nabla_x u(\cdot, X)\sigma(\cdot, X))$ and X is equal to $X^{(0, x_0)}$. Since $f^{(\varepsilon)}(t, \cdot)$ is Lipschitz continuous for all $t \in [0, T]$, $F(t, \cdot)$ is also Lipschitz continuous, with Lipschitz constant $C\mathbf{1}_{[0, T-\varepsilon)}(t)\varepsilon^{(\theta_L-3)/2}$, for all $t \in [0, T)$. Now, letting $H_r^{(t, x, s)} := \frac{\mathbf{1}_{(t, T]}(s)}{r-s} (\int_s^r \sigma^{-1}(r, X_r^{(t, x)}) D_s X_r^{(t, x)})^\top dW_r$ where $D_s X^{(t, x)}$ is the Malliavin derivative of $X_s^{(t, x)}$,

it follows from [MZ02, Theorem 4.2], because the terminal condition of the BSDE satisfied by $(y^{(\varepsilon,t,x)}, z^{(\varepsilon,t,x)})$ is zero, that $z_r^{(\varepsilon,t,x)}$ is equal to $z^{(\varepsilon)}(r, X_r^{(t,x)})$ $m \times \mathbb{P}$ -almost everywhere, where $z^{(\varepsilon)} : [0, T] \times \mathbb{R}^d \rightarrow (\mathbb{R}^q)^\top$ is a continuous, deterministic function given by

$$z^{(\varepsilon)}(t, x) = \mathbb{E}\left[\int_t^T F(r, X_r^{(t,x)}, y_r^{(\varepsilon,t,x)}, z_r^{(\varepsilon,t,x)}) H_r^{(t,x,t)} dr\right]; \quad (3.7)$$

we work with this version of $z^{(\varepsilon,t,x)}$ from hereon. It will be sufficient to show that $z^{(\varepsilon)}(t, \cdot)$ is Lipschitz continuous for all $t \in [0, T]$ with Lipschitz constant $C\phi(t, \cdot)$ for ϕ relevant to $(\mathbf{A}_{\mathbf{b}\Phi})$ or $(\mathbf{A}_{\mathbf{h}\Phi})$. Indeed, for a mollified version of $z^{(\varepsilon)}(t, \cdot)$, $|\nabla_x z_r^{(\varepsilon)}(t, X_t)| \leq C\phi(t, \cdot)$. The chain-rule of Malliavin calculus – Lemma 2.1 – yields $D_s z^{(\varepsilon)}(t, X_t)$ equals $(D_s X_t)^\top \nabla_x z^{(\varepsilon)}(t, X_t)$. Moreover, $(\sigma^{-1}(s, X_s) \nabla X_s)^\top D_s z^{(\varepsilon)}(t, X_t)$ is equal to $(\nabla X_s)^\top \nabla_x z^{(\varepsilon)}(t, X_t)$ due to Lemma 2.5, and the Cauchy-Schwarz inequality and Lemma 2.5 then imply that

$$\|(\sigma^{-1}(s, X_s) \nabla X_s)^\top D_s z^{(\varepsilon)}(t, X_t)\|_2 \leq C\phi(t, \cdot) \quad \text{for all } s.$$

Since the bound on $\|\nabla_x z_r^{(\varepsilon)}(t, X_t)\|_4$ is independent of the mollification, it follows that $z^{(\varepsilon)}(t, X_t) \in \mathbb{D}^{1,2}$ for all t , and the above estimate holds. Using the version of $z_t^{(\varepsilon)}$ given by $z^{(\varepsilon)}(t, X_t)$, and $\nabla z_t^{(\varepsilon)} = (\sigma^{-1}(s, X_s) \nabla X_s)^\top D_s z_t^{(\varepsilon)}$ from [MZ02, Lemma 2.4] it follows that $\|\nabla z_t^{(\varepsilon)}\|_2 \leq C\phi(t, \cdot)$. Now, using Lemma 2.9,

$$\left\| \sup_{s \leq r < T} U_r^{(\varepsilon)} \right\|_2 \leq C \int_s^{T-\varepsilon} \|a_r^{(\varepsilon)}\|_2 dr \leq C \int_s^{T-\varepsilon} \frac{dr}{(T-r)^{(3-\alpha)/2}} \leq C\phi(s, \cdot),$$

and $(V_{j,t}^{(\varepsilon)})^\top = (\nabla z_{j,t}^{(\varepsilon)})^\top \sigma^{-1}(t, X_t) - U_t^{(\varepsilon)} \nabla_x \sigma_j(t, X_t)$, therefore we conclude that $\|V_t^{(\varepsilon)}\|_2 \leq C\phi(s, \cdot)$ as required.

Let us now prove that $z^{(\varepsilon)}(t, \cdot)$ is Lipschitz continuous. Fix $s \in [t, T]$. Using the representation (3.7) of $z^{(\varepsilon,t,x)}$, it follows that

$$\begin{aligned} \|z_s^{(\varepsilon,t,x_1)} - z_s^{(\varepsilon,t,x_2)}\|_2 &\leq \|\mathbb{E}_s\left[\int_s^T F(r, X_r^{(t,x_1)}, y_r^{(\varepsilon,t,x_1)}, z_r^{(\varepsilon,t,x_1)}) H_r^{(t,x_1,s)} dr\right] \\ &\quad - \mathbb{E}_s\left[\int_s^T F(r, X_r^{(t,x_2)}, y_r^{(\varepsilon,t,x_2)}, z_r^{(\varepsilon,t,x_2)}) H_r^{(t,x_1,s)} dr\right]\|_2 \\ &\quad + \|\mathbb{E}_s\left[\int_s^T F(r, X_r^{(t,x_2)}, y_r^{(\varepsilon,t,x_2)}, z_r^{(\varepsilon,t,x_2)}) (H_r^{(t,x_1,s)} - H_r^{(t,x_2,s)}) dr\right]\|_2 \\ &=: \mathcal{A}_1 + \mathcal{A}_2. \end{aligned}$$

We start with an estimate for \mathcal{A}_2 . Using the Cauchy-Schwarz inequality, it follows that

$$\mathcal{A}_2 \leq \int_s^T \|F(r, X_r^{(t,x_2)}, y_r^{(\varepsilon,t,x_2)}, z_r^{(\varepsilon,t,x_2)})\|_4 \|H_r^{(t,x_1,s)} - H_r^{(t,x_2,s)}\|_4 dr \quad (3.8)$$

Using the same techniques as in the proof of Lemma 2.10, one shows that

$$\begin{aligned} \|H_r^{(t,x_1,s)} - H_r^{(t,x_2,s)}\|_4 &\leq C_4 \frac{\|\sigma^{-1}(s, X_s^{(t,x_1)}) - \sigma^{-1}(s, X_s^{(t,x_2)})\|_{\mathcal{S}^8} \mathbb{E}[\sup_{s \leq u \leq T} |D_s X_u^{(t,x_1)}|^8]^{1/8}}{\sqrt{r-s}} \\ &\quad + C_4 \frac{\|\sigma^{-1}\|_\infty \mathbb{E}[\sup_{s \leq u \leq T} |D_s X_u^{(t,x_1)} - D_s X_u^{(t,x_2)}|^8]^{1/8}}{\sqrt{r-t}}. \end{aligned} \quad (3.9)$$

where C_4 is the constant coming from the BDG inequality. Thanks to [RY99, Theorem IX.2.4], we have that

$$\|X_s^{(t,x_1)} - X_s^{(t,x_2)}\|_{\mathcal{S}^8} \leq C|x_1 - x_2| \quad (3.10)$$

The function $\sigma^{-1}(t, \cdot)$ is Lipschitz continuous uniformly in t with Lipschitz constant as given in Lemma 1.4. for all $s \in [t, T)$. Moreover, Lemma 2.5 gives that

$$\mathbb{E}[\sup_{s \leq u \leq T} |D_s X_u^{(t, x_1)}|^8]^{1/8} \leq C \quad \text{and} \quad \mathbb{E}[\sup_{s \leq u \leq T} |D_s X_u^{(t, x_1)} - D_s X_u^{(t, x_2)}|^8]^{1/8} \leq C|x_1 - x_2|.$$

Combining these estimates, it follows that $\|H_r^{(t, x_1, t)} - H_r^{(t, x_2, t)}\|_4 \leq C|x_1 - x_2|/\sqrt{r-t}$.

In order to find a bound for $\|F(r, X_r^{(t, x_2)}, y_r^{(\varepsilon, t, x_2)}, z_r^{(\varepsilon, t, x_2)})\|_4$, we show that $y_r^{(\varepsilon, t, x_2)}$ and $z_r^{(\varepsilon, t, x_2)}$ are in $\mathbf{L}_4(\mathcal{F}_r)$. Once this has been shown, we take advantage of the local Lipschitz continuity and boundedness (1.2) of f , and the uniform bounds on u and its partial derivatives from Lemma 2.8, in order to show that

$$\begin{aligned} |F(r, X_r^{(t, x_2)}, 0, 0)| &\leq |f(r, X_r^{(t, x_2)}, 0, 0)| + L_f \frac{|u(r, X_r^{(t, x_2)})| + \|\sigma\|_\infty |\nabla_x u(r, X_r^{(t, x_2)})|}{(T-r)^{(1-\theta_L)/2}} \\ &\leq \frac{C_f}{(T-r)^{1-\theta_c}} + \frac{CB_r(\Phi)}{(T-r)^{1-\theta_L/2}} \leq \frac{CB_r(\Phi)}{(T-r)^{1-\theta_c \wedge \frac{\theta_L}{2}}} \end{aligned} \quad (3.11)$$

where $B_r(\Phi) := \mathbb{E}_r[|\Phi(X_T^{(t, x_2)}) - \mathbb{E}_r[\Phi(X_T^{(t, x_2)})]|^2]^{1/2}$. It follow from the triangle inequality, the local Lipschitz continuity (1.2) and the inequality (3.11) that

$$|F(r, X_r^{(t, x_2)}, y_r^{(\varepsilon, t, x_2)}, z_r^{(\varepsilon, t, x_2)})| \leq |F(r, X_r^{(t, x_2)}, 0, 0)| + L_f \frac{|y_r^{(\varepsilon, t, x_2)}| + |z_r^{(\varepsilon, t, x_2)}|}{(T-r)^{(1-\theta_L)/2}}$$

But $y_r^{(\varepsilon, t, x_2)}$ and $z_r^{(\varepsilon, t, x_2)}$ are bounded: apply Proposition 2.11 with $(Y_1, Z_1) = (0, 0)$ and $(Y_2, Z_2) = (y^{(\varepsilon, t, x_2)}, z^{(\varepsilon, t, x_2)})$, combined with inequality (3.11) and Lemma 5.5 to obtain that

$$\begin{aligned} |y_r^{(\varepsilon, t, x_2)}| &\leq C \int_r^{T-\varepsilon} \mathbb{E}_r[|F(u, X_u^{(t, x_2)}, 0, 0)|^2]^{1/2} du \leq CB_r(\Phi)(T-r)^{\theta_c \wedge \theta_L/2}, \\ |z_r^{(\varepsilon, t, x_2)}| &\leq C \int_r^{T-\varepsilon} \mathbb{E}_r[|F(u, X_u^{(t, x_2)}, 0, 0)|^2]^{1/2} (u-r)^{-1/2} du \leq CB_r(\Phi)(T-r)^{\theta_c \wedge \frac{\theta_L}{2} - \frac{1}{2}} \end{aligned} \quad (3.12)$$

for all $r \in [t, T)$. Therefore, $\|F(r, X_r^{(t, x_2)}, y_r^{(\varepsilon, t, x_2)}, z_r^{(\varepsilon, t, x_2)})\|_4$ is bounded above by $C\|B_r(\Phi)\|_4(T-r)^{\theta_c \wedge \frac{\theta_L}{2} - 1}$. Under either $(\mathbf{A}_{\mathbf{b}\Phi})$ or $(\mathbf{A}_{\mathbf{h}\Phi})$, $\|B_r(\Phi)\|_4$ is bounded uniformly in r by, say, $B_4(\Phi)$. Substituting this and the bound on $\|H_r^{(t, x_1, t)} - H_r^{(t, x_2, t)}\|_4$ into (3.8)

$$\mathcal{A}_2 \leq CB_4(\Phi)|x_1 - x_2| \int_s^T \frac{dr}{(T-r)^{1-\theta_c \wedge \frac{\theta_L}{2}} \sqrt{r-s}} \leq \frac{CB_4(\Phi)|x_1 - x_2|}{(T-s)^{\frac{1}{2} - \theta_c \wedge \frac{\theta_L}{2}}}$$

Now, we estimate \mathcal{A}_1 . Using the conditional Fubini's theorem, Lemma 5.1, and the Cauchy-Schwarz inequality,

$$\mathcal{A}_1 \leq \left\| \int_s^{T-\varepsilon} \mathbb{E}_s[|F(r, X_r^{(t, x_1)}, y_r^{(\varepsilon, t, x_1)}, z_r^{(\varepsilon, t, x_1)}) - F(r, X_r^{(t, x_2)}, y_r^{(\varepsilon, t, x_2)}, z_r^{(\varepsilon, t, x_2)})|^2]^{1/2} \mathbb{E}_s[|H_r^{(t, x_1, s)}|^2]^{1/2} dr \right\|_2$$

Analogously to Lemma 2.10, $\mathbb{E}_s[|H_r^{(t, x_1, s)}|^2]^{1/2} \leq C_M(r-t)^{-1/2}$, therefore Minkowski's inequality implies

$$\mathcal{A}_1 \leq C \int_s^{T-\varepsilon} \frac{\|F(r, X_r^{(t, x_1)}, y_r^{(\varepsilon, t, x_1)}, z_r^{(\varepsilon, t, x_1)}) - F(r, X_r^{(t, x_2)}, y_r^{(\varepsilon, t, x_2)}, z_r^{(\varepsilon, t, x_2)})\|_2}{\sqrt{r-s}} dr.$$

By applying the Lipschitz continuity of $f^{(\varepsilon)}(r, \cdot)$, \mathcal{A}_1 is bounded by

$$\begin{aligned} & C \int_s^{T-\varepsilon} \frac{\|X_r^{(t,x_1)} - X_r^{(t,x_2)}\|_2 + \|\sigma\|_\infty \|u(r, X_r^{(t,x_1)}) - u(r, X_r^{(t,x_2)})\|_2}{(T-r)^{(1-\theta_L)/2} \sqrt{r-t}} dr \\ & + C \int_s^{T-\varepsilon} \frac{\|\nabla_x \sigma\|_\infty \|\nabla_x u(r, X_r^{(t,x_1)}) - \nabla_x u(r, X_r^{(t,x_2)})\|_2}{(T-r)^{(1-\theta_L)/2} \sqrt{r-t}} dr \\ & + C \int_s^{T-\varepsilon} \frac{\|y_r^{(\varepsilon,t,x_1)} - y_r^{(\varepsilon,t,x_2)}\|_2 + \|z_r^{(\varepsilon,t,x_1)} - z_r^{(\varepsilon,t,x_2)}\|_2}{(T-r)^{(1-\theta_L)/2} \sqrt{r-t}} dr \end{aligned}$$

Using the differentiability of $u(s, \cdot)$, it follows that

$$\begin{aligned} \|u(r, X_r^{(t,x_1)}) - u(r, X_r^{(t,x_2)})\|_2 &\leq \|\mathcal{R}(u, r, X_r^{(t,x_1)}, X_r^{(t,x_2)})\|_2, \\ \|\nabla_x u(r, X_r^{(t,x_1)}) - \nabla_x u(r, X_r^{(t,x_2)})\|_2 &\leq \|\mathcal{R}(\nabla_x u, r, X_s^{(t,x_1)}, X_s^{(t,x_2)})\|_2 \end{aligned}$$

for all $r \in [t, T)$, where, for a differentiable function g , $\mathcal{R}(g, r, x, x')$ is the remainder from the first order Taylor expansion of $g(r, x) - g(r, x')$: in the case of g taking values in \mathbb{R} , this is equal to

$$\mathcal{R}(g, r, x, x') = \left\{ \int_0^1 \nabla_x g(\delta x + (1-\delta)x') d\delta \right\} (x - x'); \quad (3.13)$$

in the multidimensional case, the expansion (3.13) is defined component-wise. Denote by $\mathcal{R}(r)$ the sum of the normed residuals $\|\mathcal{R}(u, r, X_r^{(t,x_1)}, X_r^{(t,x_2)})\|_2 + \|\mathcal{R}(\nabla_x u, r, X_s^{(t,x_1)}, X_s^{(t,x_2)})\|_2$. Therefore, denoting by Θ_r the normed sum $\|y_r^{(\varepsilon,t,x_1)} - y_r^{(\varepsilon,t,x_2)}\|_2 + \|z_r^{(\varepsilon,t,x_1)} - z_r^{(\varepsilon,t,x_2)}\|_2$, the final bound on \mathcal{A}_1 is

$$\mathcal{A}_1 \leq C \int_s^{T-\varepsilon} \frac{\mathcal{R}(r) dr}{(T-r)^{(1-\theta_L)/2} \sqrt{r-t}} + C \int_s^{T-\varepsilon} \frac{\Theta_r}{(T-r)^{(1-\theta_L)/2} \sqrt{r-t}} dr.$$

It follows from Lemma 2.8 that

$$\mathcal{R}(r) \leq \begin{cases} C \|\Phi\|_\infty \|X_r^{(t,x_1)} - X_r^{(t,x_2)}\|_2 (T-r)^{-1} C \|\Phi\|_\infty |x_1 - x_2| (T-r)^{-1} \leq & \text{under } (\mathbf{A}_{\mathbf{b}\Phi}), \\ CK_\Phi \|X_r^{(t,x_1)} - X_r^{(t,x_2)}\|_2 (T-r)^{\frac{\theta_\Phi}{2}-1} \leq CK_\Phi |x_1 - x_2| (T-r)^{\frac{\theta_\Phi}{2}-1} & \text{under } (\mathbf{A}_{\mathbf{h}\Phi}), \end{cases}$$

where the bound $\|X_r^{(t,x_1)} - X_r^{(t,x_2)}\|_2 \leq C|x_1 - x_2|$ is obtained from [RY99, Theorem IX.2.4]. It is clear from the bounds above that the integral in $\mathcal{R}(r)$ in the bound of \mathcal{A}_1 dominates the upper bound on \mathcal{A}_2 . Therefore,

$$\|z_s^{(\varepsilon,t,x_1)} - z_s^{(\varepsilon,t,x_2)}\|_2 \leq C \int_s^{T-\varepsilon} \frac{\mathcal{R}(r) dr}{(T-r)^{(1-\theta_L)/2} \sqrt{r-t}} + C \int_s^{T-\varepsilon} \frac{\Theta_r}{(T-r)^{(1-\theta_L)/2} \sqrt{r-t}} dr.$$

Since $y_s^{(\varepsilon,t,x)} = \mathbb{E}_s[\int_s^T f^{(\varepsilon)}(r, X_r^{(t,x)}, Y_r^{(\varepsilon,t,x)}, Z_r^{(\varepsilon,t,x)}) dr]$, similar estimates yield

$$\Theta_s \leq C \int_s^{T-\varepsilon} \frac{\mathcal{R}(r) dr}{(T-r)^{(3-\theta_L)/2} \sqrt{r-s}} + C \int_s^{T-\varepsilon} \frac{\Theta_r}{(T-r)^{(1-\theta_L)/2} \sqrt{r-s}} dr$$

for all $s \in [t, T-\varepsilon)$. Let $(\mathbf{A}_{\mathbf{b}\Phi})$ be in force. Applying Lemma 5.6 with $w_r = C\|\Phi\|_\infty|x_1 - x_2| \int_r^{T-\varepsilon} (T-u)^{(\theta_L-3)/2} (u-r)^{-1/2} du$ and $u_r = \Theta_r$, it follows that

$$\begin{aligned} \Theta_s &\leq C \|\Phi\|_\infty |x_1 - x_2| \int_s^{T-\varepsilon} \frac{dr}{(T-r)^{(3-\theta_L)/2} \sqrt{r-s}} \\ &+ C \|\Phi\|_\infty |x_1 - x_2| \int_s^{T-\varepsilon} \frac{\int_s^u (u-r)^{\theta_L/2-1} (r-s)^{-1/2} dr}{(T-u)^{(3-\theta_L)/2}} du + C \int_s^{T-\varepsilon} \frac{\Theta_r}{(T-r)^{(1-\theta_L)/2}} dr \\ &\leq C \|\Phi\|_\infty |x_1 - x_2| \int_s^{T-\varepsilon} \frac{dr}{(T-r)^{(3-\theta_L)/2} \sqrt{r-s}} + C \int_s^{T-\varepsilon} \frac{\Theta_r}{(T-r)^{(1-\theta_L)/2}} dr, \end{aligned}$$

where we have used Lemma 5.3 to bound the integral $\int_s^u (u-r)^{\theta_L/2-1}(r-s)^{-1/2}dr$. Then, applying Lemma 5.7 to bound the integral $\int_s^{T-\varepsilon} \Theta_r(T-r)^{(\theta_L-1)/2}dr$, final bound on $\|z_s^{(\varepsilon,t,x_1)} - z_s^{(\varepsilon,t,x_2)}\|_2$ for all $t \in [0, T)$, $(x_1, x_2) \in (\mathbb{R}^d)^2$, and $s \in [t, T)$ is

$$\begin{aligned} \|z_s^{(\varepsilon,t,x_1)} - z_s^{(\varepsilon,t,x_2)}\|_2 &\leq C\|\Phi\|_\infty|x_1 - x_2| \int_s^{T-\varepsilon} \frac{dr}{(T-r)^{(3-\theta_L)/2}\sqrt{r-s}} \\ &\quad + C\|\Phi\|_\infty|x_1 - x_2| \int_s^{T-\varepsilon} \frac{\int_r^{T-\varepsilon} (T-u)^{(\theta_L-3)/2}(u-r)^{-1/2}du}{(T-r)^{(1-\theta_L)/2}} dr \end{aligned}$$

and application of Lemma 5.5 yields the final upper bound $C\phi(t, \varepsilon, \theta_L)|x_1 - x_2|$. Therefore, setting $t = s$ yields that the function $z^{(\varepsilon)}(t, \cdot)$ is Lipschitz continuous, as required, with Lipschitz constant $C\phi(t, \varepsilon, \theta_L)$. The proof under $(\mathbf{A}_{\mathbf{h}\Phi})$ is analogous. \square

In order to make use of Proposition 3.4, it is necessary to approximate Z by an intermediate process Z_M which satisfies the hypotheses of Proposition 3.4.

Lemma 3.5. *Assume that $(\mathbf{A}_{\mathbf{exp}\Phi})$ is in force. Recall the BSDE (Y_M, Z_M) defined in Corollary 2.14. Take the version of Z_M given by Theorem 2.15. For $M = (3\ln(N))^{1/4}$ and $R(M)$ equal to $3L_f e^{M^2/2}$, there is a finite constant C depending only on $L_f, C_M, \theta_L, C_\xi$ and T , but not on N , such that for all $N \geq 1$*

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s - \tilde{Z}_{t_i}\|_2^2 ds \leq C \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_{M,s} - \tilde{Z}_{M,t_i}\|_2^2 ds + CN^{-1}.$$

Proof. In what follows, C may change from line to line. Using Cauchy's inequality and the orthogonality of the projections,

$$\frac{1}{2} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s - \tilde{Z}_{t_i}\|_2^2 ds \leq \int_0^T \|Z_s - Z_{M,s}\|_2^2 ds + \sum_{i=0}^{N-1} \|\tilde{Z}_{t_i} - \tilde{Z}_{M,t_i}\|_2^2 \Delta_i + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_{M,s} - \tilde{Z}_{M,t_i}\|_2^2 ds.$$

From Jensen's inequality, it follows that $\sum_{i=0}^{N-1} \|\tilde{Z}_{t_i} - \tilde{Z}_{M,t_i}\|_2^2 \Delta_i \leq \int_0^T \|Z_s - \bar{Z}_{M,s}\|_2^2 ds$. Proposition 2.11 with $(Y_1, Z_1) = (Y_M, Z_M)$ and $(Y_2, Z_2) = (Y, Z)$ yields that, for any $s \in [0, T)$,

$$\int_0^T \|Z_t - Z_{M,t}\|_2^2 dt \leq C\|\Phi(X_T) - \Phi_M(X_T)\|_2^2 + C \int_0^T \|f(t, X_t, Y_{M,t}, Z_{M,t}) - f_M(t, X_t, Y_{M,t}, Z_{M,t})\|_2^2 dt, \quad (3.14)$$

It follows from Markov's exponential inequality and $(\mathbf{A}_{\mathbf{exp}\Phi})$ that

$$\|\Phi(X_T) - \Phi_M(X_T)\|_2^2 = \int_{M^2}^\infty \mathbb{P}(|\Phi(X_T)|^2 \geq x) dx \leq C_\xi \int_{M^2}^\infty e^{-\sqrt{x}} dx = 2C_\xi(1+M^4)e^{-M^4} \leq CN^{-2}. \quad (3.15)$$

The last inequality is obtained by substituting the value of M . On the other hand, the basic properties of the mollifier in Definition 1.1 yields

$$\begin{aligned} &|f(t, X_t, Y_{M,t}, Z_{M,t}) - f_M(t, X_t, Y_{M,t}, Z_{M,t})| \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^q)^\top} |f(t, X_t, Y_{M,t}, Z_{M,t}) - f(t, X_t - x, Y_{M,t} - y, Z_{M,t} - z)| \phi_{R(M)}(x, y, z) d(x, y, z) \\ &\leq \frac{L_f}{(T-t)^{(1-\theta_L)/2}} \int_{\{|x|^2 + |y|^2 + |z|^2 \leq R(M)^{-2}\}} (|x| + |y| + |z|) \phi_{R(M)}(x, y, z) d(x, y, z) \leq \frac{3L_f}{(T-t)^{(1-\theta_L)/2} R(M)} \end{aligned} \quad (3.16)$$

Substituting the value of $R(M)$ then gives $\|f(t, X_t, \bar{Y}_{M,t}, \bar{Z}_{M,t}) - f_M(t, X_t, \bar{Y}_{M,t}, \bar{Z}_{M,t})\|_2 \leq (T-t)^{(\theta_L-1)/2} N^{-1/2}$ for all $t \in [0, T]$. Substituting (3.15) and (3.16) into (3.14) Lemma 5.5 then yields

$$\int_0^T \|Z_s - Z_{M,s}\|_2^2 ds \leq CN^{-1} + CN^{-2} \sum_{i=0}^{N-1} \frac{\Delta_i}{T-t_i},$$

The sum on the right hand side above is bounded by $1 + \int_0^{t_{N-1}} (T-t)^{-1} dt = 1 + C \ln(N)$, whence the proof is complete. \square

We now provide an extension to Theorem 3.3 under $(\mathbf{A}_{\mathbf{h}\Phi})$ with the aid of Proposition 3.4.

Theorem 3.6. *Let $(\mathbf{A}_{\mathbf{h}\Phi})$ be in force and $0 < \beta < (2\gamma) \wedge (\alpha \wedge \theta_L)$. There is a constant C depending only on $L_f, C_M, \theta_L, \theta_c, \beta, C_f, K_\Phi$ and T , but not on N , such that for all $N \geq 1$,*

$$\mathcal{E}(N) \leq CN^{-1} \mathbf{1}_{[1,3]}(\theta_\Phi + \beta + 2\gamma) + CN^{-2\gamma} \mathbf{1}_{(0,1)}(\theta_\Phi + \beta + 2\gamma). \quad (3.17)$$

Proof. In what follows, C may change from line to line. To start with, we assume that $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$ are in force. Recall (3.2). From the bounds $\|a_r^{(\varepsilon)}\|_2 \leq C(T-r)^{(\alpha+\theta_L-3)/2}$ in the proof on Lemma 2.9 the first sum $\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (\int_{t_i}^t \|a_r^{(\varepsilon)}\|_2 dr)^2 dt$ is bounded above by

$$\begin{aligned} & C \sum_{i=0}^{N-2} \int_{t_i}^{t_{i+1}} \frac{\left(\int_{t_i}^t \frac{dr}{(t-r)^{(1-\theta_L)/2}}\right)^2}{(T-t)^{2-\alpha}} dt + C \int_{t_{N-1}}^T \frac{\left(\int_{t_{N-1}}^t \frac{dr}{(t-r)^{1-\theta_L/2}}\right)^2}{(T-t)^{1-\alpha}} dt \\ & \leq C \sum_{i=0}^{N-2} \int_{t_i}^{t_{i+1}} \frac{(T-t)^{\beta-1} (t-t_i)^{1+\theta_L}}{(T-t)^{1+\beta-\alpha}} dt + C \int_{t_{N-1}}^T \frac{(t-t_{N-1})^{\theta_L}}{(T-t)^{1-\alpha}} dt \\ & \leq C \sum_{i=0}^{N-2} \frac{\Delta_i^{1+\theta_L}}{(T-t_{i+1})^{1-\beta}} \int_{t_i}^{t_{i+1}} \frac{dt}{(T-t)^{1+\beta-\alpha}} + C \Delta_{N-1}^{\theta_L+\alpha}. \end{aligned} \quad (3.18)$$

Using (5.3) from Lemma 5.3, $\Delta_i \leq C \Delta_{i+1}$ for $i < N-1$, which, combined with (5.2), yields

$$\max_{0 \leq i \leq N-2} \frac{\Delta_i^{1+\theta_L}}{(T-t_{i+1})^{1-\beta}} \leq C \max_{0 \leq i \leq N-1} \frac{\Delta_i^{1+\theta_L}}{(T-t_i)^{1-\beta}} < CN^{-1-\theta_L}.$$

Additionally, $\beta < \alpha$ implies that $\Delta_{N-1}^{\alpha+\theta_L} = CN^{-(\alpha+\theta_L)/\beta} \leq CN^{-1}$. Substituting these results into (3.18) gives

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^t \|a_r^{(\varepsilon)}\|_2 dr\right)^2 dt \leq CN^{-1}. \quad (3.19)$$

The refined estimates $\|V_r^{(\varepsilon)}\|_2 \leq C\phi(r, \varepsilon, \theta_L, \theta_\Phi) dr - a.e.$ from Proposition 3.4 are used to bound $\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (\int_{t_i}^t \|V_r^{(\varepsilon)}\|_2 dr)^2 dt$. Using Lemma 5.5 and Jensen's inequality,

$$\begin{aligned} & \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^t \phi(r, \varepsilon, \theta_L, \theta_\Phi) dr\right)^2 dt = K_\Phi^2 \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^t \left\{ \int_r^{T-\varepsilon} \frac{(T-u)^{(\beta-1)/2} du}{(T-u)^{\frac{1+\beta-\theta_\Phi-\theta_L}{2}} \sqrt{u-r}} \right\} dr\right)^2 dt \\ & \leq \frac{K_\Phi^2}{\varepsilon^{1-\theta_\Phi-\beta}} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^t \left\{ \int_r^{T-\varepsilon} \frac{du}{(T-u)^{1-(\theta_L-\beta)/2} \sqrt{u-r}} \right\} dr\right)^2 dt \\ & \leq \frac{C}{\varepsilon^{1-\theta_\Phi-\beta}} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^t \frac{dr}{(T-r)^{(1-(\theta_L-\beta))/2}}\right)^2 dt \\ & \leq \frac{C(\max_{0 \leq i \leq N-1} \Delta_i)^2}{\varepsilon^{1-\theta_\Phi-\beta}} \int_0^T \frac{dr}{(T-t)^{1-(\theta_L-\beta)}} dt \leq \frac{CN^{-2}}{\varepsilon^{1-\theta_\Phi-\beta}} \end{aligned} \quad (3.20)$$

where we have used (5.2) in Lemma 5.3 for the bound $\max_{0 \leq i \leq N-1} \Delta_i \leq CN^{-1}$. Substituting (3.19) and (3.20) into (3.2) finally yields

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|z_s^{(\varepsilon)} - z_{t_i}^{(\varepsilon)}\|_2^2 ds \leq CN^{-1} + \frac{CN^{-2}}{\varepsilon^{1-\theta_\Phi-\beta}}. \quad (3.21)$$

Then, using $Z^{(\varepsilon)} = z^{(\varepsilon)} + z$, Lemma 3.2, and $\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|z_s - z_{t_i}\|_2^2 ds \leq CN^{-1}$, shown in [GM10, Theorem 1.3], it follows that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s - Z_{t_i}\|_2^2 ds \leq CN^{-1} + CN^{-2} \varepsilon^{\theta_\Phi+\beta-1} + CN^{-2\gamma/\beta} + C\varepsilon^{2\gamma}. \quad (3.22)$$

Let $\delta \in [0, 1]$ be sufficiently large so that $N^{-\delta} \ln(N) \leq 1$; for $N \in \{1, 2\}$, take $\delta = 0$, and for $N > 2$, $\delta = \ln \ln(N) / \ln(N)$. We have ascertained the value of δ to be δ_N given in the Theorem statement. Set $\varepsilon = N^{-(1+\delta)/(2\gamma)}$. Recalling further that $2\gamma < \beta$, this implies that, under $(\mathbf{A}_{\mathbf{b}\Phi})$ and $(\mathbf{A}_{\partial\mathbf{f}})$,

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|Z_s - Z_{t_i}\|_2^2 ds \leq CN^{-1} + CN^{-2-(\theta_\Phi+\beta-1)(1+\delta_N)/(2\gamma)}.$$

To obtain the general result, recall the BSDE (Y_M, Z_M) from Corollary 2.14. The driver of (Y_M, Z_M) satisfies assumptions $(\mathbf{A}_{\partial\mathbf{f}})$. The proof is complete by taking M equal to $(3 \ln(N))^{1/4}$, $R(M)$ equal to $3L_f e^{M^4/2}$, and applying Lemma 3.5. \square

4 Convergence rate of the Malliavin weights scheme

Recall the Malliavin derivative of the the marginals of the process X in Section 2.2. In the definition of the Malliavin weights scheme (1.5), we use the following discrete-time approximation of the Malliavin weights (2.13):

$$H_j^i := \frac{1}{t_j - t_i} \left(\sum_{k=i}^{j-1} (\sigma^{-1}(t_k, X_{t_k}) D_{t_i} X_{t_k} \sigma(t_i, X_{t_i}))^\top \Delta W_k \right)^\top \quad (4.1)$$

Notice that H_j^i satisfies $\mathbb{E}_i[H_j^i] = 0$ and $\mathbb{E}_i[|H_j^i|^2] \leq C_M(t_j - t_i)^{-1}$; the latter property is proved exactly like Lemma 2.10. If the marginals of X and $D_{t_i} X$ are not known explicitly, one can use an SDE scheme to provide approximations, but this is beyond the scope of this work; see Section 3 and after of [GM⁺05] for some results in the zero driver case. In what follows, we use the version of Z given by Theorem 2.15. However, the first term in this decomposition

We start with some preliminary results.

Lemma 4.1. *There is a constant C depending only on the bound on b and it's derivatives, the bound on σ and it's derivatives, $\bar{\beta}$, L_f , θ_L , C_f , θ_c , $K^\alpha(\Phi)$ and T such that, for any $0 \leq i < j \leq N$,*

$$\begin{aligned} \|\mathbb{E}_i[\Phi(X_T)(H_{t_j}^{t_i} - H_j^i)]\|_2 &\leq \frac{CN^{-1/2}}{(T - t_i)^{(1-\alpha)/2}}, \\ \|\mathbb{E}_i[f_j(X_{t_j}, Y_{t_{j+1}}, Z_{t_j})(H_{t_j}^{t_i} - H_j^i)]\|_2 &\leq \frac{CN^{-1/2}}{(T - t_j)^{1-\gamma} \sqrt{t_j - t_i}} \end{aligned}$$

Proof. For any $j > i$ and $t \geq t_j$, define $N_t^{t_i} := \sigma^{-1}(t, X_t) D_{t_i} X_t \sigma(t_i, X_{t_i})$. Observe the decompositions into telescopic sums

$$\begin{aligned} N_t^{t_i} - N_{t_j}^{t_i} &= \sigma^{-1}(t, X_t) (D_{t_i} X_t - D_{t_i} X_{t_j}) \sigma(t_i, X_{t_i}) \\ &\quad + (\sigma^{-1}(t, X_t) - \sigma^{-1}(t_j, X_{t_j})) D_{t_i} X_{t_j} \sigma(t_i, X_{t_i}), \end{aligned}$$

it follows from the boundedness and Lipschitz continuity of σ and σ^{-1} (Lemma 1.4) that for any $j > i$ and $t \in [t_j, t_{j+1}]$,

$$\mathbb{E}_i[|H_{t_j}^{t_i} - H_j^i|^2] = \frac{\sum_{k=i}^{j-1} \int_{t_k}^{t_{k+1}} \mathbb{E}_i[|N_t^{t_i} - N_{t_k}^{t_i}|^2] dt}{(t_j - t_i)^2} \leq \frac{C \sum_{k=i}^{j-1} \int_{t_k}^{t_{k+1}} \mathbb{E}_i[|D_{t_i} X_t - D_{t_i} X_{t_k}|^2 + |X_t - X_{t_k}|^2] dt}{(t_j - t_i)^2}.$$

It now follows from Lemma 2.5 the usual bound $\mathbb{E}_i[|X_t - X_{t_j}|^2] \leq C(t - t_j)$, which can be easily shown from $(\mathbf{A}_{\mathbf{b}, \sigma})$, and Lemma 5.3 that $\mathbb{E}_i[|H_{t_j}^{t_i} - H_j^i|^2] \leq C \max_k \Delta_k (t_j - t_i)^{-1} \leq CN^{-1}(t_j - t_i)^{-1}$. The bound on $\|\mathbb{E}_i[\Phi(X_T)(H_{t_j}^{t_i} - H_j^i)]\|_2$ now follows from the Cauchy-Schwarz inequality and $V_{t,T}(\Phi) \leq K^\alpha(\Phi)(T - t)^{\alpha/2}$. The bound on $\|\mathbb{E}_i[f_j(X_{t_j}, Y_{t_{j+1}}, Z_{t_j})(H_{t_j}^{t_i} - H_j^i)]\|_2$ follows from the Cauchy-Schwarz inequality the fact that, similarly to (2.21), $\|f_j(X_{t_j}, Y_{t_{j+1}}, Z_{t_j})\|_2 \leq C(T - t_j)^{\gamma-1}$. \square

Lemma 4.2. *For all $t_i, t_j \in \pi$ such that $t_i \leq t_j$ and $r \in [t_j, T]$,*

$$\mathbb{E}_i[f(r, X_r, Y_r, Z_r)H_r^{t_i}] = \mathbb{E}_i[f(r, X_r, Y_r, Z_r)H_{t_j}^{t_i}]. \quad (4.2)$$

Moreover,

$$\begin{aligned} \mathbb{E}_i\left[\int_{t_i}^T f(r, X_r, Y_r, Z_r)H_r^{t_i} dr\right] &= \mathbb{E}_i\left[\sum_{j=i+1}^{N-1} f_j(X_{t_j}, Y_{t_{j+1}}, Z_{t_j})H_{t_j}^{t_i} \Delta_j\right] + \mathbb{E}_i\left[\int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r)H_r^{t_i} dr\right] \\ &\quad + \mathbb{E}_i\left[\sum_{j=i+1}^{N-1} \int_{t_j}^{t_{j+1}} (f(r, X_r, Y_r, Z_r) - f_j(X_{t_j}, Y_{t_{j+1}}, Z_{t_j}))H_r^{t_i} dr\right]. \end{aligned} \quad (4.3)$$

Proof. First let $(\mathbf{A}_{\partial f})$ be in force and recall, as argued in the proof of Theorem 2.15, that the BSDE solved by $(y^{(\varepsilon)}, z^{(\varepsilon)})$ in Definition 2.7 satisfies the conditions of [MZ02, Theorem 4.2]. A key element of the proof of that Theorem is to show that, for almost all $v \in [0, r]$,

$$\begin{aligned} D_v f^{(\varepsilon)}(\Theta_r) \sigma^{-1}(v, X_v) \nabla X_v &= f_x^{(\varepsilon)}(\Theta_r) \nabla X_r + f_y^{(\varepsilon)}(\Theta_r) (u(r, X_r) \nabla X_r + \nabla y_r^{(\varepsilon)}) \\ &\quad + f_z^{(\varepsilon)}(\Theta_r) (U(r, X_r) \nabla X_r + \nabla z_r^{(\varepsilon)}) \end{aligned} \quad m \times \mathbb{P} - a.e;$$

where $U(r, x)$ is defined in (2.6); see the equality just above equation (4.19) in [MZ02]. Integrating with respect to v over $v \in [t_i, t_j]$, on the one hand, and between $v \in [t_i, r]$, on the other, which yields

$$\frac{1}{t_j - t_i} \int_{t_i}^{t_j} D_v f^{(\varepsilon)}(\Theta_r) \sigma^{-1}(v, X_v) \nabla X_v dv = \frac{1}{r - t_i} \int_{t_i}^r D_v f^{(\varepsilon)}(\Theta_r) \sigma^{-1}(v, X_v) \nabla X_v dv.$$

One then follows the proof of [MZ02, Theorem 4.2], which essentially uses integration-by-parts for Malliavin calculus – Lemma 2.2 – to show that $\mathbb{E}_i[f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})H_r^{t_i}] = \mathbb{E}_i[f^{(\varepsilon)}(r, X_r, Y_r^{(\varepsilon)}, Z_r^{(\varepsilon)})H_{t_j}^{t_i}]$. One extends to the general case (4.2) by convergence arguments as in the proof of Theorem 2.15. The relation (4.3) is now straightforward to obtain from (4.2). \square

Lemma 4.3. *There is a finite constant C depending only on the bound on b and its derivatives,*

the bound on σ and its derivatives, L_f , θ_L , C_f , θ_c , and T such that, for all $i \in \{0, \dots, N-1\}$,

$$\|\mathbb{E}_{t_i}[\int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r) H_r^{t_i} dr]\|_2 \leq C \int_{t_i}^{t_{i+1}} \frac{dr}{(T-r)^{1-\gamma} \sqrt{r-t_i}}, \quad (4.4)$$

$$\begin{aligned} & \left\| \sum_{j=i+1}^{N-1} \mathbb{E}_{t_i}[(f_j(Y_{t_{j+1}}, Z_{t_j}) - f_j(\bar{Y}_{j+1}, \bar{Z}_j)) H_{t_j}^{t_i}] \Delta_j \right\|_2 \\ & \leq C \sum_{j=i+1}^{N-1} \frac{\|Y_{t_{j+1}} - \bar{Y}_{j+1}\|_2 + \|Z_{t_j} - \bar{Z}_j\|_2}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \Delta_j, \end{aligned} \quad (4.5)$$

$$\begin{aligned} & \left\| \sum_{j=i+1}^{N-1} \mathbb{E}_{t_i}[\int_{t_j}^{t_{j+1}} (f(r, X_r, Y_r, Z_r) - f(t_j, X_{t_j}, Y_{t_{j+1}}, Z_{t_j})) H_{t_j}^{t_i} dr]\|_2 \\ & \leq CN^{-1/2} + C \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \{\|Y_r - Y_{t_{j+1}}\|_2 + \|Z_r - Z_{t_j}\|_2\} dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}}. \end{aligned} \quad (4.6)$$

Proof. In what follows, C may change from line to line. Using Lemma 2.10, Minkowski's inequality, and the moment bound (2.21) of Corollary 2.12,

$$\|\mathbb{E}_{t_i}[\int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r) H_r^{t_i} dr]\|_2 \leq C \int_{t_i}^{t_{i+1}} \frac{dr}{(T-r)^{1-\gamma} \sqrt{r-t_i}}.$$

Using the Lipschitz continuity of f , Minkowski's inequality, and Lemma 2.10,

$$\left\| \sum_{j=i+1}^{N-1} \mathbb{E}_{t_i}[(f_j(Y_{t_{j+1}}, Z_{t_j}) - f_j(\bar{Y}_{j+1}, \bar{Z}_j)) H_{t_j}^{t_i}] \Delta_{j-1} \right\|_2 \leq C \sum_{j=i+1}^{N-1} \frac{\|Y_{t_{j+1}} - \bar{Y}_{j+1}\|_2 + \|Z_{t_j} - \bar{Z}_j\|_2}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \Delta_j$$

For (4.6), the t -Hölder continuity of f in $(\mathbf{A}_{\mathbf{t}_t})$, the Cauchy-Schwarz inequality, Minkowski's inequality, and Hölder's inequality are needed:

$$\begin{aligned} & \left\| \sum_{j=i+1}^{N-1} \mathbb{E}_{t_i}[\int_{t_j}^{t_{j+1}} (f(r, X_r, Y_r, Z_r) - f_j(X_{t_j}, Y_{t_{j+1}}, Z_{t_j})) H_{t_j}^{t_i} dr]\|_2 \\ & \leq C \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \|f(r, X_r, Y_r, Z_r) - f_j(X_r, Y_r, Z_r)\|_2 dr}{\sqrt{t_j-t_i}} + C \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \|f_j(X_r, Y_r, Z_r) - f_j(X_{t_j}, Y_{t_{j+1}}, Z_{t_j})\|_2 dr}{\sqrt{t_j-t_i}} \\ & \leq C \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \sqrt{r-t_j} dr}{\sqrt{t_j-t_i}} + C \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \{\|X_r - X_{t_j}\|_2 + \|Y_r - Y_{t_{j+1}}\|_2 + \|Z_r - Z_{t_j}\|_2\} dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \end{aligned}$$

The usual bound $\|X_r - X_{t_j}\|_2 \leq C\sqrt{r-t_j}$ is easily shown from $(\mathbf{A}_{\mathbf{b},\sigma})$. Note that $3 \int_{t_j}^{t_{j+1}} \sqrt{r-t_j} dr = 2\Delta_j^{3/2} \leq CN^{-1/2} \Delta_j$ follows from Lemma 5.3. Applying Lemma 5.5 to bound the remaining sums by C is then sufficient to complete the proof. \square

In the following proposition, we obtain a bound for the error terms on the right hand side of (4.6); these error terms are intrinsically related to the discretization error of the Malliavin weights scheme. Proposition 3.4 will be essential in the proof of this result.

Proposition 4.4. *Let either $(\mathbf{A}_{\text{exp}\Phi})$ or $(\mathbf{A}_{\mathbf{h}\Phi})$ be in force and suppose that $0 < \beta < (2\gamma) \wedge \alpha \wedge \theta_L$. For $\delta, K > 0$, define $\mathcal{C}(\delta, K) := K(N^{-1/2} \mathbf{1}_{[1,3]}(\delta) + N^{-\gamma} \mathbf{1}_{(0,1)}(\delta))$, and, for $j \in \{0, \dots, N-1\}$, $\Psi_j := \int_{t_j}^{t_{j+1}} \{\|Y_r - Y_{t_{j+1}}\|_2 + \|Z_r - Z_{t_j}\|_2\} dr$. There is a constant C depending only on L_f , θ_L , C_f ,*

$\theta_c, \beta, \bar{\beta}$, the bound on b and its derivatives, the bound on σ and its derivatives, and T , but not on N , such that, for all $N \geq 1$,

$$\begin{aligned} \sum_{j=0}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \Psi_j}{(T-t_j)^{(1-\theta_L)/2}} &\leq C(T-t_i)^{(1+\theta_L-\beta)/2} \mathcal{C}(\beta+2\gamma, \ln(N)^{1/4} \vee 1), \\ \sum_{j=i+1}^{N-1} \frac{\Psi_j dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} &\leq C(T-t_i)^{-(1+\beta-\theta_L)/2} \mathcal{C}(\beta+2\gamma, \ln(N)^{1/4} \vee 1) \end{aligned}$$

in the case of $(\mathbf{A}_{\text{exp}\Phi})$, and

$$\begin{aligned} \sum_{j=0}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \Psi_j}{(T-t_j)^{(1-\theta_L)/2}} &\leq C(T-t_i)^{(1+\theta_L-\beta)/2} \mathcal{C}(\beta+\theta_\Phi+2\gamma, K_\Phi), \\ \sum_{j=i+1}^{N-1} \frac{\Psi_j dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} &\leq C(T-t_i)^{-(1+\beta-\theta_L)/2} \mathcal{C}(\beta+\theta_\Phi+2\gamma, K_\Phi) \end{aligned}$$

in the case of $(\mathbf{A}_{\text{h}\Phi})$.

Proof. We will prove the bounds for

$$\sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \{\|Y_r - Y_{t_{j+1}}\|_2 + \|Z_r - Z_{t_j}\|_2\} dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}}$$

The bounds for the other sum are obtained analogously. Moreover, we will only prove the result for the terms in Z . The bound for the terms in Y are also obtained analogously. In what follows, C may change from line to line. We first prove the result under $(\mathbf{A}_{\partial f})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$, and then obtain the general result by means of mollification. Fix $\varepsilon \leq \Delta_{N-1}$ and recall the BSDE $(Y^{(\varepsilon)}, Z^{(\varepsilon)})$ from Definition 2.7 in Section 2.3. We use the version of $Z^{(\varepsilon)}$ provided by Theorem 2.15. First, apply the triangle inequality to the integrand in order to obtain $\|Z_t - Z_{t_i}\|_2 \leq \|Z_t - Z_t^{(\varepsilon)}\|_2 + \|Z_{t_i} - Z_{t_i}^{(\varepsilon)}\|_2 + \|Z_t^{(\varepsilon)} - Z_{t_i}^{(\varepsilon)}\|_2$.

To bound the terms in $Z - Z^{(\varepsilon)}$, recall the bound (2.23) from Corollary 2.13. For $j \leq N-2$, the bound on $\|Z_t - Z_t^{(\varepsilon)}\|_2$ implies that

$$\int_{t_j}^{t_{j+1}} \|Z_t - Z_t^{(\varepsilon)}\|_2 dt \leq C \int_{t_j}^{t_{j+1}} \frac{\int_{T-\varepsilon}^T (T-r)^{\gamma-1} dr}{\sqrt{t_{N-1}-t}} dt \leq C\varepsilon^\gamma \int_{t_j}^{t_{j+1}} \frac{dt}{\sqrt{t_{N-1}-t}}.$$

Lemma 5.4 yields $\int_{t_j}^{t_{j+1}} (t_{N-1}-t)^{-1/2} dt \leq 2\Delta_j(t_{N-1}-t_j)^{-1/2}$. Therefore, applying Lemma 5.5 implies that

$$\sum_{j=i+1}^{N-2} \frac{\int_{t_j}^{t_{j+1}} \|Z_t - Z_t^{(\varepsilon)}\|_2 dt}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \leq C\varepsilon^\gamma \sum_{j=i+1}^{N-2} \frac{\Delta_j}{(t_{N-1}-t_j)^{1-\theta_L/2} \sqrt{t_j-t_i}} \leq \frac{C\varepsilon^\gamma}{(t_{N-1}-t_i)^{(1-\theta_L)/2}}. \quad (4.7)$$

Then, use $(t_{N-1}-t_i)^{-1/2} \leq 2(T-t_i)^{-1/2}$ on the denominator on the right hand side. For the outstanding term, $j = N-1$, we implement Lemma 5.5 to show that

$$\int_{t_{N-1}}^T \|Z_t - Z_t^{(\varepsilon)}\|_2 dt \leq C \int_{t_{N-1}}^T \left\{ \int_t^T (T-r)^{\gamma-1} (r-t)^{-1/2} dr \right\} dt \leq C\Delta_{N-1}^{1/2+\gamma},$$

whence it follows that

$$\frac{\int_{t_{N-1}}^T \|Z_t - Z_t^{(\varepsilon)}\|_2 dt}{\Delta_{N-1}^{(1-\theta_L)/2} \sqrt{t_{N-1} - t_i}} \leq \frac{C\Delta_{N-1}^{\gamma+\theta_L/2}}{\sqrt{t_{N-1} - t_i}} \leq \frac{C\Delta_{N-1}^{\gamma+\theta_L/2}}{\sqrt{T - t_i}} \quad (4.8)$$

Combining (4.7) and (4.8), it follows that

$$\sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \|Z_t - Z_t^{(\varepsilon)}\|_2 dr}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}} \leq \frac{C\varepsilon^\gamma}{(T - t_i)^{(1-\theta_L)/2}} + \frac{CN^{-1}}{\sqrt{T - t_i}} \quad (4.9)$$

where we have used that $\Delta_{N-1}^{2\gamma+\theta_L} = TN^{(2\gamma+\theta_L)/\beta}$ and $\beta < (2\gamma) \wedge \theta_L$. Analogously, we can also show that

$$\sum_{j=i+1}^{N-1} \frac{\|Z_{t_j} - Z_{t_j}^{(\varepsilon)}\|_2 \Delta_j}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}} \leq \frac{C\varepsilon^\gamma}{(T - t_i)^{(1-\theta_L)/2}} + \frac{CN^{-1}}{\sqrt{T - t_i}} \quad (4.10)$$

Recalling the BSDEs (y, z) and $(y^{(\varepsilon)}, z^{(\varepsilon)})$ from Definition 2.7 and that $Z^{(\varepsilon)} = z + z^{(\varepsilon)}$, the triangle inequality yields $\|Z_t^{(\varepsilon)} - Z_{t_i}^{(\varepsilon)}\|_2 \leq \|z_t - z_{t_i}\|_2 + \|z_t^{(\varepsilon)} - z_{t_i}^{(\varepsilon)}\|_2$. In the proof of [GM10, Theorem 1.1], in bounding the terms E_1 and E_2 , it is shown that, for all $t \in [0, T]$,

$$\|z_t - z_{t_i}\|_2^2 \leq \frac{C(t - t_i)}{(T - t)^{1-\alpha}} + C \int_{t_i}^t \|\nabla_x^2 u(r, X_r)\|_2^2 dr.$$

Lemma 2.8 implies $\int_{t_i}^t \|\nabla_x^2 u(r, X_r)\|_2^2 dr \leq C \int_{t_i}^t (T - r)^{\alpha-2} dr$. Now, applying Jensen's inequality, Lemma 5.4, Lemma 5.5, and the above bound, one obtains

$$\begin{aligned} \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \|z_r - z_{t_j}\|_2 dr}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}} &\leq \sum_{j=i+1}^{N-1} \frac{C\Delta_j \int_{t_j}^{t_{j+1}} (T - r)^{(\alpha-1)/2} dr + C \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^r (T - t)^{\alpha-2} dt \right)^{1/2} dr}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}} \\ &\leq \sum_{j=i+1}^{N-1} \frac{C\Delta_j^2 (T - t_j)^{(\alpha-1)/2} + C\Delta_j^{1/2} \left(\int_{t_j}^{t_{j+1}} (t_{j+1} - t)(T - t)^{\alpha-2} dt \right)^{1/2}}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}}. \end{aligned} \quad (4.11)$$

For $j \leq N - 2$, one can apply Lemma 5.3 and Lemma 5.4 to show that

$$\int_{t_j}^{t_{j+1}} (t_{j+1} - t)(T - t)^{\alpha-2} dt \leq \frac{\Delta_j}{(T - t_{j+1})^{1-\beta}} \int_{t_j}^{t_{j+1}} (T - t)^{\alpha-\beta-1} dt \leq \frac{CN^{-1}\Delta_j}{(T - t_j)^{1-\alpha+\beta}}.$$

On the other hand, for $j = N - 1$, since $\beta < \alpha$,

$$\int_{t_{N-1}}^T (T - t)(T - t)^{\alpha-2} dt = \frac{1}{\alpha} \Delta_{N-1}^\alpha = \frac{T}{\alpha} N^{-\alpha/\beta} \leq \frac{T}{\alpha} N^{-1}.$$

Substituting these bounds into (4.11) and implementing Lemma 5.3 and Lemma 5.5, we obtain

$$\begin{aligned} \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \|z_r - z_{t_j}\|_2 dr}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}} &\leq CN^{-1/2} \sum_{j=i+1}^{N-1} \frac{\Delta_j}{(T - t_j)^{1+(\beta-\alpha-\theta_L)/2} \sqrt{t_j - t_i}} + CN^{-1/2} \frac{\Delta_{N-1}^{\theta_L/2}}{\sqrt{t_{N-1} - t_i}} \\ &\leq \frac{CN^{-1/2}}{(T - t_i)^{(1+\beta-\alpha-\theta_L)/2}} + \frac{CN^{-1}}{\sqrt{T - t_i}}. \end{aligned} \quad (4.12)$$

In the bounds (3.2), we used the inequality

$$\|z_r^{(\varepsilon)} - z_{t_i}^{(\varepsilon)}\|_2 \leq C \int_{t_i}^r \|a_t^{(\varepsilon)}\|_2 dt + C \int_{t_i}^r \|V_t^{(\varepsilon)}\|_2 dt + C\Delta_i^{1/2}. \quad (4.13)$$

Using $\|a_t^{(\varepsilon)}\|_2 \leq C(T-t)^{(\alpha+\theta_L-3)/2}$ as shown Lemma 2.9, Lemma 5.3, Lemma 5.4, and Lemma 5.5, it follows that

$$\begin{aligned}
& \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \{\int_{t_j}^r \|a_t^{(\varepsilon)}\|_2 dt\} dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \leq C \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \{\int_{t_j}^r (T-t)^{(\theta_L+\alpha-3)/2} dt\} dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \\
& \leq C \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} (T-r)^{(\alpha-2)/2} dr \int_{t_i}^{t_{j+1}} (T-t)^{(\theta_L-1)/2} dt}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \leq C \sum_{j=i+1}^{N-1} \frac{\Delta_j^{(3+\theta_L)/2}}{(T-t_j)^{(3-\alpha-\theta_L)/2} \sqrt{t_j-t_i}} \\
& \leq C \left(\max_{0 \leq i \leq N-1} \frac{\Delta_i^{1+\theta_L}}{(T-t_i)^{1-\beta}} \right)^{1/2} \sum_{j=i+1}^{N-1} \frac{\Delta_j}{(T-t_j)^{(2+\beta-\theta_L-\alpha)/2} \sqrt{t_j-t_i}} \leq \frac{CN^{-1/2}}{(T-t_i)^{(1+\beta-\theta_L-\alpha)/2}}
\end{aligned} \tag{4.14}$$

On the other hand, we obtain bounds for $\|V_t^{(\varepsilon)}\|_2$ from Proposition 3.4 under $(\mathbf{A}_{\text{exp}\Phi})$ or $(\mathbf{A}_{\mathbf{h}\Phi})$. Let us work under $(\mathbf{A}_{\text{exp}\Phi})$. It follows from Lemma 5.4 and Lemma 5.5 that, for all j and $r \in [t_j, t_{j+1}]$,

$$\begin{aligned}
\int_{t_j}^r \|V_t^{(\varepsilon)}\|_2 dt & \leq C \|\Phi\|_\infty \int_{t_j}^r \left\{ \int_t^{T-\varepsilon} (T-s)^{(\beta-1)/2} (T-s)^{(\theta_L-\beta-2)/2} (s-t)^{-1/2} ds \right\} dt \\
& \leq C \|\Phi\|_\infty \varepsilon^{(\beta-1)/2} \int_{t_j}^r \left\{ \int_t^T (T-s)^{\theta_L-\beta-1} (s-t)^{-1/2} ds \right\} dt \\
& \leq C \|\Phi\|_\infty \varepsilon^{(\beta-1)/2} \int_{t_j}^r (T-t)^{(\theta_L-\beta-1)/2} dt \leq C \|\Phi\|_\infty \frac{\varepsilon^{(\beta-1)/2} \Delta_j}{(T-t_j)^{(1+\beta-\theta_L)/2}}
\end{aligned}$$

Therefore, using Lemma 5.3, Lemma 5.5 and the above bound,

$$\begin{aligned}
\sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \{\int_{t_j}^r \|V_t^{(\varepsilon)}\|_2 dt\} dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} & \leq C \|\Phi\|_\infty \varepsilon^{(\beta-1)/2} \max_i \Delta_i \sum_{j=i+1}^{N-1} \frac{\Delta_j}{(T-t_j)^{1-(\theta_L-\beta)/2} \sqrt{t_j-t_i}} \\
& \leq \frac{C \|\Phi\|_\infty \varepsilon^{(\beta-1)/2} N^{-1}}{(T-t_i)^{(1+\beta-\theta_L)/2}}.
\end{aligned} \tag{4.15}$$

Now, substituting (4.14) and (4.15) into (4.13), it follows that

$$\sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \|z_r^{(\varepsilon)} - z_{t_j}^{(\varepsilon)}\|_2 dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \leq \frac{CN^{-1/2}}{(T-t_i)^{(1+\beta-\theta_L-\alpha)/2}} + \frac{C \|\Phi\|_\infty \varepsilon^{(\beta-1)/2} N^{-1}}{(T-t_i)^{(1+\beta-\theta_L)/2}} \tag{4.16}$$

Combining (4.9), (4.10), (4.12) and (4.16) yields

$$\begin{aligned}
& \sum_{j=i+1}^{N-1} \frac{\int_{t_j}^{t_{j+1}} \|Z_r - Z_{t_j}\|_2 dr}{(T-t_j)^{(1-\theta_L)/2} \sqrt{t_j-t_i}} \\
& \leq \frac{CN^{-1/2}}{(T-t_i)^{(1+\beta-\theta_L-\alpha)/2}} + \frac{CN^{-1}}{\sqrt{T-t_i}} + \frac{C\varepsilon^\gamma}{(T-t_i)^{(1-\theta_L)/2}} + \frac{C \|\Phi\|_\infty \varepsilon^{(\beta-1)/2} N^{-1}}{(T-t_i)^{(1+\beta-\theta_L)/2}}
\end{aligned}$$

and we take $\varepsilon = N^{-1/(2\gamma)}$ if $1 - \beta - 2\gamma < 0$ and $\varepsilon = N^{-1}$ otherwise to complete the proof under $(\mathbf{A}_{\partial\mathbf{f}})$ and $(\mathbf{A}_{\mathbf{b}\Phi})$. The proof under $(\mathbf{A}_{\partial\mathbf{f}})$ and $(\mathbf{A}_{\mathbf{h}\Phi})$ is analogous.

To prove the result without $(\mathbf{A}_{\partial\mathbf{f}})$ or $(\mathbf{A}_{\mathbf{b}\Phi})$, recall the mollified BSDE (Y_M, Z_M) from Corollary 2.14. Set $M = (3 \ln(N))^{1/4}$ and $R(M)$ equal to $3L_f e^{M^2/2}$. Substituting equations (3.15) and (3.16)

into (3.14), $\|Z_s - Z_{M,s}\|_2 \leq CN^{-1}(T-s)^{-1/2}$ for all $s \in [0, T]$, whence the triangle inequality and Lemma 5.4 imply

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \|Z_r - Z_{t_j}\|_2 dr &\leq \int_{t_j}^{t_{j+1}} \|Z_r - Z_{M,r}\|_2 dr + \|Z_{t_j} - Z_{M,t_j}\|_2 \Delta_j + \int_{t_j}^{t_{j+1}} \|Z_{M,r} - Z_{M,t_j}\|_2 dr \\ &\leq CN^{-1} \Delta_j (T - t_j)^{-1/2} + \int_{t_j}^{t_{j+1}} \|Z_{M,r} - Z_{M,t_j}\|_2 dr. \end{aligned}$$

The proof is then completed with use of Lemma 5.5. \square

We come to the main result of this section, namely the error estimation for the Malliavin weights scheme.

Theorem 4.5. *Let $(\mathbf{A}_{\text{exp}\Phi})$ or $(\mathbf{A}_{\mathbf{h}\Phi})$ be and force and suppose that $0 < \beta < \gamma \wedge \alpha \wedge \theta_L$. For $\delta, K > 0$, define $\mathcal{C}(\delta, K) := KN^{-1/2} \mathbf{1}_{[1,3]}(\delta) + N^{-\gamma} \mathbf{1}_{(0,1)}(\delta)$. There is a constant C depending only on $L_f, \theta_L, C_f, \theta_c, \beta, \bar{\beta}$, the bound on b and its derivatives, the bound on σ and its derivatives, $K^\alpha(\Phi)$ and T , but not on N , such that, for all $N \geq 1$,*

$$\left. \begin{aligned} \|Y_{t_i} - \bar{Y}_i\|_2 &\leq C(T - t_i)^{(1+\theta_L-\beta)/2} \mathcal{C}(\beta + 2\gamma, \ln(N)^{1/4} \vee 1), \\ \|Z_{t_i} - \bar{Z}_i\|_2 &\leq C(T - t_i)^{-(1\beta-\theta_L)/2} \mathcal{C}(\beta + 2\gamma, \ln(N)^{1/4} \vee 1) \end{aligned} \right\} \text{ in the case of } (\mathbf{A}_{\text{exp}\Phi}),$$

$$\left. \begin{aligned} \|Y_{t_i} - \bar{Y}_i\|_2 &\leq C(T - t_i)^{(1+\theta_L-\beta)/2} \mathcal{C}(\beta + \theta_\Phi + 2\gamma, K_\Phi), \\ \|Z_{t_i} - \bar{Z}_i\|_2 &\leq C(T - t_i)^{-(1\beta-\theta_L)/2} \mathcal{C}(\beta + \theta_\Phi + 2\gamma, K_\Phi). \end{aligned} \right\} \text{ in the case of } (\mathbf{A}_{\mathbf{h}\Phi}).$$

Proof In what follows, C may change from line to line. For simplicity, we omit the process X from the driver, so that $f(t, y, z) := f(t, X_t, y, z)$ and $f_j(y, z) := f_j(X_{t_j}, y, z)$. Fix $i \in \{0, \dots, N-1\}$. Using the estimates from Lemma 4.1 and Lemma 4.3, and (4.3) from Lemma 4.2, it follows that

$$\begin{aligned} \|Z_{t_i} - Z_i\|_2 &= \|\mathbb{E}_i[\Phi(X_T)H_T^{t_i} - \Phi(X_T)H_N^i] + \int_{t_i}^T f(t, Y_t, Z_t)H_t^{t_i} dt - \sum_{j=i+1}^{N-1} f_j(\bar{Y}_{j+1}, \bar{Z}_j)H_j^i \Delta_j\|_2 \\ &\leq \|\mathbb{E}_i[\int_{t_i}^{t_{i+1}} f(r, Y_r, Z_r)H_r^{t_i} dr]\|_2 + \|\mathbb{E}_i[\Phi(X_T)(H_T^{t_i} - H_N^i)]\|_2 + \|\sum_{j=i+1}^{N-1} \mathbb{E}_i[f_j(Y_{t_{j+1}}, Z_{t_j})(H_{t_j}^{t_i} - H_j^i)]\Delta_j\|_2 \\ &\quad + \|\sum_{j=i+1}^{N-1} \mathbb{E}_i[(f_j(Y_{t_{j+1}}, Z_{t_j}) - f_j(\bar{Y}_{j+1}, \bar{Z}_j))H_j^i]\Delta_j\|_2 + \|\sum_{j=i+1}^{N-1} \mathbb{E}_i[\int_{t_j}^{t_{j+1}} (f(r, Y_r, Z_r) - f_j(Y_{t_{j+1}}, Z_{t_j}))H_{t_j}^{t_i} dr]\|_2 \\ &\leq \frac{CN^{-1/2}}{(T - t_i)^{(1-\alpha \wedge (2\gamma))/2}} + C \int_{t_i}^{t_{i+1}} \frac{dr}{(T-r)^{1-\gamma} \sqrt{r-t_i}} + C\mathcal{E}(i) + C \sum_{j=i+1}^{N-1} \frac{\Theta_j \Delta_j}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}} \end{aligned} \tag{4.17}$$

where $\Theta_j := \|Y_{t_{j+1}} - \bar{Y}_{j+1}\|_2 + \|Z_{t_j} - \bar{Z}_j\|_2$ and

$$\mathcal{E}(i) := \sum_{j=i+1}^{N-1} \frac{\Psi(j)}{(T - t_j)^{(1-\theta_L)/2} \sqrt{t_j - t_i}}, \quad \Psi(j) := \int_{t_j}^{t_{j+1}} \{\|Y_r - Y_{t_{j+1}}\|_2 + \|Z_r - Z_{t_j}\|_2\} dr.$$

In (4.17), we have estimated $\|\sum_{j=i+1}^{N-1} \mathbb{E}_i[f_j(Y_{t_{j+1}}, Z_{t_j})(H_{t_j}^{t_i} - H_j^i)]\Delta_j\|_2$ by $CN^{-1/2} \sum_{j=i+1}^{N-1} (T - t_j)^{\gamma-1} (T_j - t_i)^{-1/2} \Delta_j$ using Lemma 4.1, and the latter sum by $C(T - t_i)^{\gamma-1/2}$ using Lemma 5.5.

Using a similar technique, $\|Y_{t_i} - Y_i\|_2$ is bounded above by

$$\begin{aligned} & C \int_{t_i}^{t_{i+1}} \frac{dr}{(T-r)^{1-\gamma}} + C \sum_{j=i}^{N-1} \frac{\Psi(j)}{(T-t_j)^{(1-\theta_L)/2}} + C \sum_{j=i}^{N-1} \frac{\Theta_j \Delta_j}{(T-t_j)^{(1-\theta_L)/2}} \\ & \leq CN^{-1/2} + C \sum_{j=i}^{N-1} \frac{\Psi(j)}{(T-t_j)^{(1-\theta_L)/2}} + C \sum_{j=i}^{N-1} \frac{\Theta_j \Delta_j}{(T-t_j)^{(1-\theta_L)/2}} \end{aligned} \quad (4.18)$$

where we have used Lemma 5.3 and Lemma 5.4 on the first integral to obtain $\int_{t_i}^{t_{i+1}} (T-r)^{\gamma-1} dr \leq C\Delta_j(T-t_j)^{\gamma-1} \leq CN^{-1/2}$. It follows from (4.17) and (4.18) that

$$\Theta_i \leq \frac{CN^{-1/2}}{(T-t_i)^{(1-\alpha\wedge(2\gamma))/2}} + C \int_{t_i}^{t_{i+1}} \frac{dr}{(T-r)^{1-\gamma}\sqrt{r-t_i}} + C\mathcal{E}(i) + C \sum_{j=i+1}^{N-1} \frac{\Theta_j \Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}}$$

Letting $U_i = \Theta_i$ and

$$W_i = N^{-1/2}(T-t_i)^{(\alpha\wedge(2\gamma)-1)/2} + \int_{t_i}^{t_{i+1}} \frac{dr}{(T-r)^{1-\gamma}\sqrt{r-t_i}} + \mathcal{E}(i) =: \Gamma(i) + \Xi(i) + \mathcal{E}(i), \quad (4.19)$$

it follows from Lemma 5.6 that

$$\Theta_i \leq CW_i + C \sum_{j=i+1}^{N-1} \frac{W_j \Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}} + C \sum_{j=i+1}^{N-1} \frac{\Theta_j \Delta_j}{(T-t_j)^{(1-\theta_L)/2}} \quad (4.20)$$

Therefore, using Lemma 5.7 in (4.17) and (4.18),

$$\|Z_{t_i} - Z_i\|_2 \leq W_i + C \sum_{j=i+1}^{N-1} \frac{W_j \Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}}, \quad (4.21)$$

$$\|Y_{t_i} - Y_i\|_2 \leq CN^{-1/2} + C \sum_{j=i}^{N-1} \frac{\Psi(j)}{(T-t_j)^{(1-\theta_L)/2}} + C \sum_{j=i}^{N-1} \frac{W_j \Delta_j}{(T-t_j)^{(1-\theta_L)/2}}. \quad (4.22)$$

Let us consider the sum in the W terms. Firstly, remark that we only need consider the sums for $i < N-1$. Recall the terminology of (4.19). Using Lemma 5.5,

$$\sum_{j=i+1}^{N-1} \frac{\Gamma(j)\Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}} = CN^{-1/2} \sum_{j=i+1}^{N-1} \frac{\Delta_j}{(T-t_j)^{1-\theta_L/2}\sqrt{t_j-t_i}} \leq CN^{-1/2}(T-t_i)^{(\theta_L-1)/2}. \quad (4.23)$$

Using the fact that $\Delta_j \leq \Delta_{j-1}$ to show that $\sqrt{t_{j+1}-t_i}/\sqrt{t_j-t_i} \leq 2$, Lemma 5.3 to show that $\Delta_j/\Delta_{j+1} \leq C$ and $\max_j \Delta_j(T-t_j)^{2\gamma-1} \leq N^{-1}$, one can apply Lemma 5.5 to bound the sum in $\Xi(j)$ as follows:

$$\begin{aligned} & \sum_{j=i+1}^{N-1} \frac{\Xi(j)\Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}} \\ & \leq C \frac{\int_{t_{N-1}}^T (T-r)^{\gamma-1}(r-t_{N-1})^{-1/2} dr \Delta_{N-1}}{\Delta_{N-1}^{(1-\theta_L)/2}\sqrt{t_{N-1}-t_i}} + C \sum_{j=i+1}^{N-2} \frac{\Delta_{j+1}^{3/2}(\Delta_j/\Delta_{j+1})^{3/2}\sqrt{t_{j+1}-t_i}/\sqrt{t_j-t_i}}{(T-t_{j+1})^{(3-\theta_L-2\gamma)/2}\sqrt{t_{j+1}-t_i}} \\ & \leq \frac{C\Delta_{N-1}^{(1+\gamma+\theta_L)/2}\sqrt{\Delta_{N-2}}}{\sqrt{t_{N-1}-t_i}} + C \max_j \sqrt{\frac{\Delta_j}{(T-t_j)^{1-2\gamma}}} \sum_{j=i+1}^{N-2} \frac{\Delta_j}{(T-t_{j+1})^{1-\theta_L/2}\sqrt{t_j-t_i}} \\ & \leq CN^{-3/2} + CN^{-1/2}(T-t_i)^{(\theta_L-1)/2} \end{aligned} \quad (4.24)$$

In order to deal with the sum in $\mathcal{E}(j)$, we change the order of summation and apply Lemma 5.5 to obtain

$$\begin{aligned} \sum_{j=i+1}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}} &= \sum_{j=i+1}^{N-1} \frac{\sum_{k=j+1}^{N-1} \frac{\Psi(k)}{(T-t_k)^{(1-\theta_L)/2}\sqrt{t_k-t_j}} \Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}} \\ &= \sum_{k=i+2}^{N-1} \frac{\sum_{j=i+1}^{k-1} \frac{\Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}} \Psi(k)}{(T-t_k)^{(1-\theta_L)/2}} \\ &\leq C \sum_{j=i+1}^{N-1} \frac{\Psi(j)}{(T-t_j)^{(1-\theta_L)/2}(t_j-t_i)^{(1-\theta_L)/2}} = C\mathcal{E}(i). \end{aligned} \quad (4.25)$$

Combining (4.23) - (4.25), the bound on the sum in W_j is

$$\sum_{j=i+1}^{N-1} \frac{W_j\Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}} \leq CN^{-1/2}(T-t_i)^{(\theta_L-1)/2} + C\mathcal{E}(i). \quad (4.26)$$

Similarly, one can show that

$$\sum_{j=i+1}^{N-1} \frac{W_j\Delta_j}{(T-t_j)^{(1-\theta_L)/2}} \leq CN^{-1/2}(T-t_i)^{(\theta_L+1)/2} + C \sum_{j=i}^{N-1} \frac{\Psi(j)}{(T-t_j)^{(1-\theta_L)/2}}. \quad (4.27)$$

The proof is completed by substituting (4.26) into (4.21), (4.27) into (4.22), and using Proposition 4.4 to bound the remaining terms. \square

5 Appendix

The following *conditional* Fubini's theorem is a consequence of the Monotone Class Theorem.

Lemma 5.1. *Let $f_s \in \mathbf{L}_2([0, T] \times \Omega)$. Then, for all $t \in [0, T]$, there exists a $\mathcal{B}([0, T]) \otimes \mathcal{F}_t$ -measurable processes F_t belonging to $L_2([0, T] \times \Omega)$ such that $(\omega, s) \mapsto F_t(s)$ is a version of $(\omega, s) \mapsto \mathbb{E}_t[f_s]$ and*

$$\mathbb{E}_t\left[\int_0^T f_s ds\right] = \int_0^T F_t(\cdot, s) ds \quad \text{almost surely.}$$

We need the following generalization of the a priori estimates [BDH⁺03, Proposition 3.2]:

Proposition 5.2. *Let k be an integer, and p be an integer greater than or equal to 2. Let $f : \Omega \times [0, T] \times (\mathbb{R}^k)^\top \times \mathbb{R}^{q \times k} \rightarrow (\mathbb{R}^k)^\top$ be $\mathcal{P} \times \mathcal{B}((\mathbb{R}^k)^\top) \otimes \mathcal{B}(\mathbb{R}^{q \times k})$ -measurable, and ξ be an $(\mathbb{R}^k)^\top$ -valued random variable in $\mathbf{L}_p(\mathcal{F}_T)$. Let $(f_t)_{t \in [0, T]}$ be non-negative, predictable process, $\mu \in \mathbf{L}_1([0, T]; m)$ and $\lambda \in \mathbf{L}_2([0, T]; m)$ be \mathbb{R} -valued non-negative. Additionally, assume that $\mathbb{E}[(\int_0^T f_t dt)^p] < \infty$. For any $(y_1, y_2) \in (\mathbb{R}^k)^2$, define the scalar product $(y_1, y_2) := \sum_{j=1}^k y_{1,j} y_{2,j}$ and assume that, for all $(t, y, z) \in [0, T] \times (\mathbb{R}^k)^\top \times \mathbb{R}^{k \times q}$, $(\omega, t, y, z) \mapsto f(\omega, t, y, z)$ satisfies*

$$(|y|^{-1} y \mathbf{1}_{|y|>0}, f(\omega, t, y, z)) \leq f_t(\omega) + \mu_t |y| + \lambda_t |z| \quad \text{almost surely.} \quad (5.1)$$

Let (Y, Z) be a solution to the $((\mathbb{R}^k)^\top, \mathbb{R}^{q \times k})$ -valued BSDE

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr - \sum_{j=1}^q \int_t^T (Z_{j,r})^\top dW_{j,r}.$$

in the space $\mathcal{S}^p \times \mathcal{H}^p$, where \mathcal{H}^p is the space of predictable processes X such that $\mathbb{E}[(\int_0^T |X_s|^2 ds)^{p/2}]$ is finite; Z_j denotes the j -th column of Z .

Then, there exists a constant C_p , depending only on p , such that, for any $\eta_t \geq \mu_t + \lambda_t^2/(p-1)$ in $L_1(\mathbb{R}; dt)$,

$$\mathbb{E}[\sup_t e^{p \int_0^t \eta_r dr} |Y_t|^p + (\int_0^T e^{2 \int_0^t \eta_r dr} |Z_t|^2 dt)^{p/2}] \leq C_p \mathbb{E}[e^{p \int_0^T \eta_r dr} |\xi|^p + (\int_0^T e^{\int_0^t \eta_r dr} f_t dt)^p].$$

Proof. Consider the processes $\tilde{Y}_t = e^{\int_0^t \eta_r dr} Y_t$ and $\tilde{Z}_t = e^{\int_0^t \eta_r dr} Z_t$. Then (\tilde{Y}, \tilde{Z}) satisfies a BSDE with terminal condition $\tilde{\xi} = e^{\int_0^T \eta_r dr} \xi$ and driver $\tilde{f}(t, y, z) = e^{\int_0^t \eta_r dr} f(t, e^{-\int_0^t \eta_r dr} y, e^{-\int_0^t \eta_r dr} z) - \eta_t y$. Moreover, for all $(t, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times q}$, $\tilde{f}(\omega, y, z)$ satisfies

$$(|y|^{-1} y \mathbf{1}_{|y|>0}, \tilde{f}(\omega, t, y, z)) \leq \tilde{f}_t(\omega) + \tilde{\mu}_t |y| + \tilde{\lambda}_t |z| \quad \text{almost surely.}$$

with $\tilde{f}_t = e^{-\int_0^t \eta_r dr} f_t$, $\tilde{\mu}_t = \mu_t - \eta_t$, and $\tilde{\lambda}_t = \lambda_t$. The rest of the proof follows exactly as the proof of [BDH⁺03, Proposition 3.2]. \square

Lemma 5.3. *The time grid $\pi^{(\beta)} = \{0 = t_0 < \dots < t_N = T : t_i = T - T(1 - i/N)^{1/\beta}\}$ with $\beta \in (0, 1]$ satisfies*

$$\max_{0 \leq i < N} \frac{\Delta_k}{(T - t_k)^{1-\theta}} \leq \frac{T^\theta}{\beta} \frac{1}{N^{1 \wedge \frac{\theta}{\beta}}}, \quad (5.2)$$

$$\max_{0 \leq i \leq N-2} \frac{\Delta_k}{\Delta_{k+1}} \leq \frac{1}{\beta} \left(1 \vee \left(\frac{1}{2\beta} \right)^{\frac{1}{\beta}-1} \right), \quad (5.3)$$

for all $\theta \in (0, 1]$.

The following is a trivial result that will come in useful.

Lemma 5.4. *For finite $\delta > 0$, and $s < r \leq R < \infty$, $\int_s^r (R-t)^{\delta-1} dt \leq \frac{1}{\delta} (r-s)(T-s)^{\delta-1}$.*

Proof. Direct computation of the integral term yields

$$\int_s^r \frac{dt}{(R-t)^{1-\delta}} = \frac{1}{\delta} \{(R-s)^\delta - (R-r)^\delta\} \leq \frac{1}{\delta} \left\{ \frac{R-s}{(R-s)^{1-\delta}} - \frac{R-r}{(R-s)^{1-\delta}} \right\} = \frac{r-s}{\delta(R-s)^{1-\delta}}. \quad \square$$

The following three lemmas and their proofs can be found in Section 2.1 of [GT13a]; the results on the integrals are proved exactly as the results on the sums.

Lemma 5.5. *Let $\delta, \rho \in (0, 1]$. Then for $B_{\delta, \rho} := \int_0^1 (1-r)^{\delta-1} r^{\rho-1} dr$, for any $0 \leq s < t \leq T$,*

$$\int_t^s (s-r)^{\delta-1} (r-s)^{\rho-1} dr \leq B_{\delta, \rho} (s-t)^{\delta+\rho-1}.$$

Moreover, on the time-grid $\pi^{(\beta)} = \{0 = t_0 < \dots < t_N = T : t_i = T - T(1 - i/N)^{1/\beta}\}$, for any $0 \leq i < k \leq N$,

$$\sum_{j=i+1}^{k-1} (t_k - t_j)^{\delta-1} (t_j - t_i)^{\rho-1} \Delta_j \leq 2B_{\delta, \rho} (t_k - t_i)^{\delta+\rho-1}.$$

Lemma 5.6. *Let $\delta \in (0, 1/2]$, $\rho > 0$ and $t \in [0, T]$. Suppose that, for a positive constant C_u , the finite positive real functions $u : [t, T] \mapsto [0, \infty)$ and $w : [t, T] \mapsto [0, \infty)$ satisfy*

$$u_t \leq w_t + C_u \int_t^T \frac{u_r dr}{(T-r)^{\frac{1}{2}-\delta} (r-t)^{\frac{1}{2}-\rho}}. \quad (5.4)$$

Then, for constants $\mathcal{C}_{(5.5a)}$ and $\mathcal{C}_{(5.5b)}$ depending only on C_u, T, δ and ρ ,

$$u_t \leq \mathcal{C}_{(5.5a)} w_t + \mathcal{C}_{(5.5a)} \int_t^T \frac{w_r dr}{(T-r)^{\frac{1}{2}-\delta}(r-t)^{\frac{1}{2}-\rho}} + \mathcal{C}_{(5.5b)} \int_t^T \frac{u_r dr}{(T-r)^{\frac{1}{2}-\delta}}. \quad (5.5)$$

Moreover, on the time-grid $\pi^{(\beta)} = \{0 = t_0 < \dots < t_N = T : t_i = T - T(1 - i/N)^{1/\beta}\}$, suppose that the real functions $U : \pi^{(\beta)} \mapsto [0, \infty)$ and $W : \pi^{(\beta)} \mapsto [0, \infty)$ satisfy

$$U_i \leq W_i + C_u \sum_{j=i+1}^{N-1} \frac{U_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\delta}(t_j-t_i)^{\frac{1}{2}-\rho}} \quad (5.6)$$

for all $i \in \{0, \dots, N-1\}$. It follows that

$$U_i \leq 2\mathcal{C}_{(5.5a)} W_i + 2\mathcal{C}_{(5.5a)} \sum_{j=i+1}^{N-1} \frac{W_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\delta}(t_j-t_i)^{\frac{1}{2}-\rho}} + 2\mathcal{C}_{(5.5b)} \sum_{j=i+1}^{N-1} \frac{U_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\delta}}$$

for all $i \in \{0, \dots, N-1\}$.

Lemma 5.7. Let $\delta \in (0, 1/2]$, $\rho > 0$ and $t \in [0, T]$. Suppose that the finite positive real functions $u : [t, T] \mapsto [0, \infty)$ and $w : [t, T] \mapsto [0, \infty)$ satisfy (5.5) for some positive constants $\mathcal{C}_{(5.5a)}$ and $\mathcal{C}_{(5.5b)}$. Then, for $\nu > 0$, there is a positive constant $\mathcal{C}^{(\nu)}$ (depending only on $\mathcal{C}_{(5.5a)}, \mathcal{C}_{(5.5b)}, T, \delta, \rho, \nu$) such that

$$\int_t^T \frac{u_r dr}{(T-r)^{\frac{1}{2}-\delta}(r-t)^{1-\nu}} \leq \mathcal{C}^{(\nu)} \int_t^T \frac{w_r dr}{(T-r)^{\frac{1}{2}-\delta}(r-t)^{1-\nu}} \quad (5.7)$$

Moreover, on the time-grid $\pi^{(\beta)} = \{0 = t_0 < \dots < t_N = T : t_i = T - T(1 - i/N)^{1/\beta}\}$, suppose that the real functions $U : \pi^{(\beta)} \mapsto [0, \infty)$ and $W : \pi^{(\beta)} \mapsto [0, \infty)$ satisfy (5.6) for all $i \in \{0, \dots, N-1\}$. It follows that

$$\sum_{j=i+1}^{N-1} \frac{U_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\delta}(t_j-t_i)^{1-\nu}} \leq 2\mathcal{C}^{(\nu)} \sum_{j=i+1}^{N-1} \frac{W_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\delta}(t_j-t_i)^{1-\nu}}$$

for all $i \in \{0, \dots, N-1\}$.

References

- [BDH⁺03] P. Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica. L^p solutions of backward stochastic differential equations. *Stochastic Processes and their Applications*, 108(1):109–129, 2003.
- [BL13] P. Briand and C. Labart. Simulation of BSDEs by Wiener Chaos Expansion. *To appear in Annals of Applied Probability*, 2013.
- [BT04] B. Bouchard and N. Touzi. Discrete time approximation and Monte Carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111:175–206, 2004.
- [CD12] D. Crisan and F. Delarue. Sharp derivative bounds for solutions of degenerate semi-linear partial differential equations. *Journal of Functional Analysis*, 263(10):3024–3101, 2012.

- [DG06] F. Delarue and G. Guatteri. Weak existence and uniqueness for forward-backward SDEs. *Stochastic Processes and their Applications*, 116(12):1712–1742, 2006.
- [EKPQ97] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance. An International Journal of Mathematics, Statistics and Financial Economics*, 7(1):1–71, 1997.
- [Fri64] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.
- [GG04] C. Geiss and S. Geiss. On approximation of a class of stochastic integrals and interpolation. *Stochastics and Stochastics Reports*, 76(4):339–362, 2004.
- [GG11] S. Geiss and E. Gobet. Fractional smoothness and applications in finance. In *Advanced mathematical methods for finance*, pages 313–331. Springer, Heidelberg, 2011.
- [GGG12] C. Geiss, S. Geiss, and E. Gobet. Generalized fractional smoothness and L_p -variation of BSDEs with non-Lipschitz terminal condition. *Stochastic Processes and their Applications*, 122(5):2078–2116, 2012.
- [GH07] S. Geiss and M. Hujo. Interpolation and approximation in $L_2(\gamma)$. *J. Approx. Theory*, 144(2):213–232, 2007.
- [GL06] E. Gobet and J. P. Lemor. Numerical simulation of BSDEs using empirical regression methods: theory and practice. In *Proceedings of the Fifth Colloquium on BSDEs (29th May - 1st June 2005, Shanghai) - Available on <http://hal.archives-ouvertes.fr/hal-00291199/fr/>*, 2006.
- [GL07] E. Gobet and C. Labart. Error expansion for the discretization of backward stochastic differential equations. *Stochastic Processes and their Applications*, 117(7):803–829, 2007.
- [GM⁺05] E. Gobet, R. Munos, et al. Sensitivity analysis using Itô-malliavin calculus and martingales, and application to stochastic optimal control. *SIAM Journal on control and optimization*, 43(5):1676–1713, 2005.
- [GM10] E. Gobet and A. Makhlouf. \mathbf{L}_2 -time regularity of BSDEs with irregular terminal functions. *Stochastic Processes and their Applications*, 120(7):1105–1132, 2010.
- [GT13a] E. Gobet and P. Turkedjiev. Approximation of backward stochastic differential equations using malliavin weights and least-squares regression. *Available on <http://hal.archives-ouvertes.fr/hal-00802394>*, 2013.
- [GT13b] E. Gobet and P. Turkedjiev. Linear regression mdp scheme for discrete backward stochastic differential equations under general conditions. *Available on <http://hal.archives-ouvertes.fr/hal-00642685>*, 2013.
- [HIM05] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.
- [HNS11] Y. Hu, D. Nualart, and X. Song. Malliavin calculus for backward stochastic differential equations and application to numerical solutions. *Ann. Appl. Probab.*, 21(6):2379–2423, 2011.

- [IDR10] P. Imkeller and G. Dos Reis. Path regularity and explicit convergence rate for BSDE with truncated quadratic growth. *Stochastic Processes and their Applications*, 120(3):348–379, 2010.
- [JS03] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.
- [Kus03] S. Kusuoka. Malliavin calculus revisited. *The University of Tokyo. Journal of Mathematical Sciences*, 10(2):261–277, 2003.
- [MZ02] J. Ma and J. Zhang. Representation theorems for backward stochastic differential equations. *The Annals of Applied Probability*, 12(4):1390–1418, 2002.
- [Nee11] C. Nee. *Sharp Gradient Bounds for the Diffusion Semigroup*. PhD thesis, Imperial College London, 2011.
- [Nua06] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.
- [REK00] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance. An International Journal of Mathematics, Statistics and Financial Economics*, 10(2):259–276, 2000. INFORMS Applied Probability Conference (Ulm, 1999).
- [Ric11] A. Richou. Numerical simulation of BSDEs with drivers of quadratic growth. *The Annals of Applied Probability*, 21(5):1933–1964, 2011.
- [Ric12] A. Richou. Markovian quadratic and superquadratic BSDEs with an unbounded terminal condition. *Stochastic Processes and their Applications*, 122(9):3173–3208, 2012.
- [RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [Zha04] J. Zhang. A numerical scheme for BSDEs. *The Annals of Applied Probability*, 14(1):459–488, 2004.
- [Zha05] J. Zhang. Representation of solutions to BSDEs associated with a degenerate FSDE. *The Annals of Applied Probability*, 15(3):1798–1831, 2005.

Finance for Energy Market Research Centre

Institut de Finance de Dauphine, Université Paris-Dauphine

1 place du Maréchal de Lattre de Tassigny

75775 PARIS Cedex 16

www.fime-lab.org