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Clémence Alasseur, Ivar Ekeland, Romuald Élie,
Nicolás Hernández Santibáñez and Dylan Possamaï

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An adverse selection approach to power tarification*

Clémence ALASSEUR † Ivar EKELAND ‡ Romuald ÉLIE §
Nicolás HERNÁNDEZ SANTIBÁÑEZ ¶ Dylan POSSAMAÏ ‖

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Abstract

We study the optimal design of electricity contracts among a population of consumers with different needs. This question is tackled within the framework of Principal-Agent problem in presence of adverse selection. The particular features of electricity induce an unusual structure on the production cost, with no decreasing return to scale. We are nevertheless able to provide an explicit solution for the problem at hand. The optimal contracts are either linear or polynomial with respect to the consumption. Whenever the outside options offered by competitors are not uniform among the different type of consumers, we exhibit situations where the electricity provider should contract with consumers with either low or high appetite for electricity.

Key words: electricity pricing, adverse selection, power management, contract theory, u-convexity, calculus of variations.

AMS 2000 subject classifications: 91B08; 91B69; 49L20.

1 Introduction

Electricity is non-storable, except marginally: any quantity which is consumed now must be produced now, and conversely. This means that the installed capacity must be sufficient to supply electricity when demand is maximal. As a consequence, part of this installed capacity will stay idle when demand is lower. This is the overcapacity problem, which is compounded by the fact that electricity consumption is far from stable, and little substitutable. People turn the heating on when it is cold, and the air conditioning on when it is warm, they turn the light on as soon as it gets dark and the home appliances when they come back from work; unless they own a generator, there is no other source of energy they can turn to. So the collective demand for electricity must be satisfied, and satisfied right now. In reality, the capacity constraint is not binding: more electricity can always be found, by putting in service less efficient

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†EDF R&D and FIME, Laboratoire de Finance des Marchés de l'Énergie, clemence.alasseur@edf.fr.

‡Université Paris–Dauphine, PSL Research University, CNRS, CEREMADE and Institut de Finance, 75016 Paris, France, ekeland@math.ubc.ca.

§Université Paris-Est Marne-la-Vallée, romuald.elie@univ-mlv.fr .

¶Université Paris–Dauphine, PSL Research University, CNRS, CEREMADE, 75016 Paris, France, and CMM, Universidad de Chile, nhernandez@dim.uchile.cl.

‖Université Paris–Dauphine, PSL Research University, CNRS, CEREMADE, 75016 Paris, France, possamai@ceremade.dauphine.fr.

production units, or by resorting to the spot market. It only becomes extremely expensive when the limits are pushed. So, instead of putting an upper bound on production, we will introduce a production function with steeply increasing marginal cost, as discussed for example in [8]. This production function can be understood either as reflecting the actual cost of producing electricity from primary energies, in which case the power company is a producer, or the financial cost of buying electricity on the open market, in which case the power company is a retailer, or a combination of both.

This paper focuses on the problem of finding an appropriate tariff: for a given installed capacity, or rather, for a given production function, how should a power company price electricity in order to maximize its profit? The company faces a variety of consumers, industrial users and domestic users, some of them are efficient, others less so. Some of them, for instance, live in insulated homes and need less electricity to achieve comfortable temperatures than others. The tariff the producer offers will be time-dependent and consumption-based. It will act in several ways:

- it will make electricity more expensive in peak hours, so that consumers who are able to do so will switch their demand to off-peak periods,
- it will penalize higher consumptions, thereby avoiding the overly expensive parts of the production function,
- it will effectively exclude some of the users, who will find the proposed tariff too expensive, and who will look for better alternatives elsewhere, by swapping for a competitor for instance, or to alternative sources of energy, or by dropping out of business altogether.

The empirical effective response of individual agents to hourly pricing contracts is studied in [3] or [10] and is heterogeneous among the electricity consumers, as pointed out in [18] or [21]. Besides, more resilient measures of high electricity prices appear to be more efficient in practice for incentivizing the reduction of electricity consumption, see [20] or [35]. In any case, the critical importance of the feedback information signals provided to the Agent has been highlighted in the empirical literature [11, 12, 13]. In particular, the development of smart metering systems allows for the implementation of decentralized home automation systems, who partially regulate power consumption, as presented e.g. in [1] or [25]. The adoption of dynamic electricity tariffs is also proven to be heterogeneous among the population [23, 26, 16]. This feature is encompassed in our model through the consideration of Agents with different reservation utilities, potentially reflecting their cost of switching to hourly pricing contracts. The heterogeneity of fallback options also stresses out the effect of competitors on the energy retailer market, whose impact is discussed in [22] or [32].

We will frame this problem in a Principal-Agent model, where the Principal is the power company, who proposes a contract, that is, a tariff, and the Agents are the consumers, who either turn down the contract and drop out, or accept it, and adjust their consumption accordingly. In contrast with the usual Principal-Agent models, the profit of the Principal is not the sum of the profits he makes from each agent, because of the nonlinearity of the production function $K(c)$: the cost $K(c_1+c_2)$ of producing c_1+c_2 is higher than $K(c_1)+K(c_2)$. This is a particular difficulty of this problem, which to our knowledge has not been treated previously, and which has deep consequences on the tariff. There are actually earlier works in a bilevel optimization setting, but where linear pricing only is considered, see for example [19] where the question is to provide electricity and to sponsor at the same time saving measures, or [2] where consumers can reduce their consumption at a price of inconvenience. As a related study, we mention as well the Principal-Agent modeling discussed in [14] for reducing energy consumption in a landlord-tenant relationship.

We are able to solve the problem at hand in the present paper explicitly, in some polar cases. The tariff we find is quite natural. It consists in the sum of three components:

- A fixed component, independent of the consumption, which is a subscription to the service.
- A linear component, $p(t)c(t)$, which consists simply of pricing the current consumption at the current price $p(t)$. Recall that this price depends on time in order to discourage peak-hour consumption.
- A non-linear component, which depends on the current consumption $c(t)$ in a non-linear way. This part appears only when $c(t)$ is high enough. Its purpose is to make the high consumers pay for making the Principal produce at very high marginal cost. Indeed, since the production function has increasing marginal cost, a high consumer costs more per unit produced than a low consumer. In addition, since the cost is global, high consumers increase the marginal cost, and hence the price, for everybody

To understand the last point, let us assume that the producer uses a linear pricing rule, charging pc for a consumption of c, and that the producer thereby breaks even: total consumption is C, and profit is $pC - K(C) = 0$. Suppose a new consumer comes in. The total cost becomes $K(C + c)$, and since K is convex (increasing marginal cost) and $K(0) = 0$, we have

$$\frac{K(C+c)}{C+c} > \frac{K(c)}{c} = p,$$

and the profit becomes $p(C + c) - K(C + c) < 0$. To break even, the producer has to increase the price p for everybody. Of course, the larger c is with respect to C, the higher the increase, so high consumers are more costly than low ones. Using a non-linear pricing rule will somehow isolate the latter from the former.

In Principal-Agent models, one is used to the fact that the optimal contract for the Principal will exclude the low end of the market, that is, the least efficient Agents. Indeed, such *shutdown* contracts are quite commonplace in the adverse selection literature, see for instance the seminal papers of Guesnerie and Laffont [15] or Rochet and Choné [28] in discrete-time (see also the monograph by Laffont and Martimort for more details [24]), or the more recent contributions of Cvitanić et al. [6] and Hernández Santibáñez et al. [17] in continuous-time. A remarkable feature of our problem is that, in certain circumstances, the optimal contract excludes the high end of the market: the most efficient agents, those who need less electricity to achieve the same degree of welfare than others, go elsewhere, and only the least efficient ones remain. This corresponds to situations of Cournot competition, where the welfare gain associated with increase in electricity consumption is small beyond a certain point (as in the case of domestic users: one does not need to heat or cool too much), so that the welfare is sensitive to price and not to consumption. In that case it becomes worthwhile not to take part in that competition, which concerns the most efficient Agents, and to concentrate on the least efficient ones, who are still a long way from achieving saturation, and who are more sensitive to changes in consumption.

We now proceed to describe the main features of our model. The Principal's cost, as mentioned above, is a convex function of aggregate production. He offers a contract to the Agents, who may accept or decline it. If they accept, they commit for a period $T > 0$. They decline if the total utility they derive from the contract is less than some reservation utility, which may depend on the Agent, and which expresses the fact that she has alternative options, such as turning to an alternate provider, or substituting another energy source to electricity.

The Agents' utilities are separable: the utility which an agent of type x derives from consuming a quantity c of electricity at time t and being charged a (nonlinear) price $p(c)$ is

$$u(t,x,c) - pc,$$

where $u(t,x,\cdot)$ is a concave function of c. This separability assumption is traditional in Principal-Agent problems. In this case, there are additional justifications, as a large part of the Agents are industrial users, who consume electricity in order to produce other goods, so that their utility simply identifies to the profit they derive from this activity. Note also the time-dependence, which reflects the seasonality of consumption.

In the sequel, we will consider CRRA utilities, of the type $\gamma^{-1}c^{\gamma}$, with $\gamma < 1$, and we will provide explicit solutions (except in the case $\gamma = 0$, or $u(c) = \ln c$). The case $\gamma < 0$ reflects the "household" behavior, where electricity fulfills some basic needs, such as lighting or appliances, and 0 consumption is not acceptable while high consumption is not needed. The case $0 < \gamma < 1$ reflects the "industrial" behavior, where high consumption is the norm, subject to decreasing returns to scale. Note, however, that in both cases there is a "fallback" option, a substitute to electricity when it becomes too expensive, for instance an alternative energy source, or simply another provider. This fallback option is expressed by a reservation utility, which may be constant or vary across Agents. Despite the particular structure of the cost function, we are able to solve explicitly the problem at hand. We observe that the optimal contract rewrites as the combination of a fixed cost together with two variable costs, proportional to either the electricity consumption or a power function of it. This tariff structure happens to be quite simple and quite close to the classical tariff structures offered by most electricity providers.

Whenever the fallback option is the same for every Agent, we observe as usual in Principal-Agent problems, that the lower end of the market is not covered: the low types (meaning those households who are less dependent on electricity, or those industry users who are less efficient) will not be offered contracts which they are willing to accept, and will have to fall back on the outside option. More interestingly, we are also able to solve explicitly the case where the fallback option of the Agents depends on their type in a concave manner. In this case, getting more efficient Agents can be too costly, and the electricity provider may concentrate on the less efficient but less expensive consumers.

Finally, a remark on the mathematics. We will be using u-convex analysis, a tool which has been introduced and developed elsewhere, notably by Carlier (see e.g. [5], and which extends classical convex analysis. The economic interpretation of the mathematical formulas is straightforward. Forgetting for ease of presentation about time dependence, assume Agent x has the separable utility $u(x,c) - p(c)$, as described above, and let the Principal set out a price schedule $p(c)$. The maximal utility Agent x derives from this price schedule is

$$\max_c \{u(x,c) - p(c)\}.$$

This number is denoted by $p^{\star}(x)$, and called the indirect utility of Agent x. Note that it depends on the entire price schedule and it can be computed for each Agent x. In this way, we associate with each function $p(c)$ a function $p^{\star}(x)$, which the economist knows as the indirect utility associated with p and which the mathematician knows as the u-conjugate or u-transform of p. Conversely, if the indirect utility $p^{\star}$ is known, the price schedule can be derived by the same formula

$$p(c) = \max_x \{u(x,c) - p^{\star}(x)\}.$$

In the bilinear case, when $u(x,c) = xc$, we get the usual Fenchel formulas of convex analysis.

The first section of the paper sets the model and the main results, i.e. the expression of the tariffs for industrial and residential customers. The two next sections provide the main mathematical results for constant and concave increasing reservation utility; demonstrations are left for appendixes.

2 Main results

The model we are proposing is set up on Principal-Agent relationship where the Principal is an electricity provider and the Agents are electricity consumers. Since the electricity consumption is observed by the Principal, we suppose that there is no moral hazard. On the other hand, adverse selection is in force as the Agent's willingness to pay for electricity is not known by the Principal. This taste for electricity represents how much Agents price a given volume of electricity in term of usefulness. For an industrial Agent, this would represent the benefit he gets by running his industrial process with this given volume of electricity. For a residential Agent, this would represent the comfort he gets by using this given volume of electricity to perform domestic tasks (heating, lighting...). Of course, this depends on the Agent's efficiency of his equipment referred as his type X. As classically assumed in adverse selection setting, even if the Principal does not know the exact type of a particular Agent, he knows the repartition of Agents' type among the population. This hypothesis is realistic as the electricity provider can alway make surveys in order to acquire this information.

2.1 Players' objectives and electricity particularity

Both players have their particular objectives:

- Agent's objective is to choose the level of electricity consumption c at any time t, which maximize his utility for electricity $u\left(t,x,c\right)$with respect to his type x minus the tariff $p(t,c)$ that he needs to pay for the electricity consumed

$$\max_{c}\left\{\int_{0}^{T} u\left(t,x,c\right)-p\left(t,c\right)dt\right\}.$$

- Principal's objective is to offer the tariffs which maximize his own profits: all payments she receives from consumers accepting the contract minus the costs for providing the total volume of electricity consumed by her clients. The provider can offer power either by buying on the electricity market or by producing it herself.

The tariffs designed by the Principal need to respect two conditions. The first one is the individual rationality of the Agents. Indeed, Agents are not forced to accept the contract offered by the Principal as they can pick alternative electricity providers, offering better conditions. This is taken into account in the model via a reservation utility H which represents the minimum level of satisfaction that an Agent needs to achieve in order to accept the contract. This reservation utility could interprets as an aggregation of competitors' offers or as a minimum level of utility imposed by public services, who offer a regulated tariff. The second condition that tariffs are required to respect is the so-called incentive compatibility condition:, which will be automatically satisfied, as there is no moral hazard in this model.

One particular feature of electricity product is the fact that it suffers from decreasing returns to scale: its marginal price increases with the total aggregate consumption. This comes from the fact that several technologies can be used in order to produce electricity. Some power plants have no or very low fuel costs such as renewables (hydro power plants, wind or solar production...) or nuclear production. These types of productions are chosen for satisfying base-load consumption. But when, the electricity consumption increases such as in peak hours, other power plants (coal, gas or fuel thermal plants for example) need to be turned on and their cost of production is much more expensive. The electricity spot price shows this strong base/peak patterns and can exhibit high spikes when capacity productions reach their limit with respect to consumption. For this reason, we impose in our model that the electricity production cost of the Principal depends on the consumption of all her clients. In fact, this cost should depend on the consumption of all consumers and not only on all clients of this provider but we assume that aggregate consumption of clients who select the provider is correlated to the consumption of the other consumers. This is justified since there are strong common preferences and behaviors among consumers, for example consumptions are higher during daytime than during the night.

2.2 Notations and model assumptions

We consider CRRA (Constant Relative Risk Aversion) utility function for the Agents.

$$u(t,x,c) = g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma},$$

where $\phi(t)$ represents the Agents' time preference for electricity. This factor is common to every Agent and typically represents for example the preference to have electricity during the daytime than during the middle of the night. $x \in [0,1]$ is the type of the Agent. We suppose that $\gamma < 1$ and we consider two different cases: $\gamma \in (0,1)$ and $\gamma < 0$. The function g_γ represents the willingness of the Agents to pay for their consumption depending on their type x and we take typically $g_\gamma(x) = x$ if $\gamma \in (0,1)$ and $g_\gamma(x) = 1-x$ if $\gamma < 0$. Graphic illustrations of the utility function are shown in figure 1. The case $\gamma \in (0,1)$ corresponds to the modeling of industrial Agents, whose utility grows to infinity if they can have infinite volume of electricity: they can always make their industrial capacities grow and generate more benefits whenever they have extra electricity. On the contrary, they can stop producing if they could not get any electricity or substitute it by another energy, which corresponds to a zero utility whenever $c = 0$. The case $\gamma < 0$ illustrates the residential Agents utility for whom electricity is a staple product: they can not avoid consuming electricity (they would get $-\infty$ utility). They also have a saturation for electricity : above a high volume of electricity, they do not gain much satisfaction with an extra volume because all their electrical needs are already fullfilled.

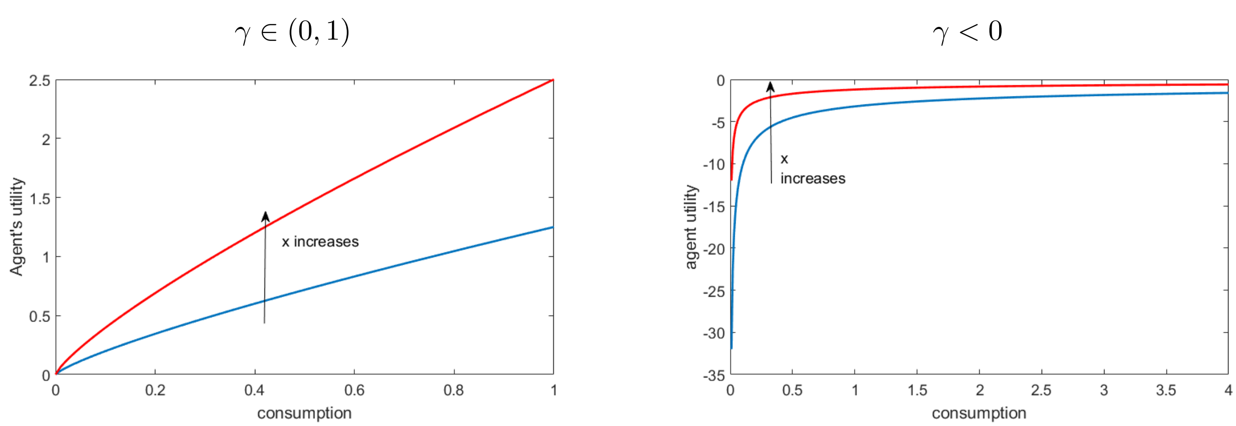


Figure 1: Agent's utility with respect to consumption for $\gamma \in (0,1)$ (left figure) and $\gamma < 0$ (right figure)

The indirect utility $p^\star(x)$ is the best level of utility that the Agent can obtain by signing the contract, which is entitled on the period $[0, T]$.

$$P^\star(x) := \int_0^T p^\star(t,x)dt = \sup_c \left\{ \int_0^T u(t,x,c) - p(t,c)dt \right\},$$

The Agent's decision to sign the contract depends on his reservation utility, denoted $H(x)$ for an agent of type x: he signs the contract with the Principal if and only if $P^\star(x) \geq H(x)$. We consider two cases for $H(x)$, either a constant function or a concave one verifying that the elasticity of reservation utility is smaller than the the elasticity of willingness to pay for consumption, i.e. $\frac{g_\gamma}{g'_\gamma} \leq \frac{H}{H'}$. The concavity of H indicates that competitors target principally the more efficient Agents. We denote by X^* the set of Agents who end up signing the contract:

$$X^\star(p^\star) := \{x \in [0,1],\ P^\star(x) \geq H(x)\}.$$

We suppose that Agents are uniformly distributed among the population and, as already mentioned, this feature is known by the Principal. We assume a convex cost of power production K:

$$K(t,c) = k(t)\frac{c^n}{n}.$$

The term $k(t)$ is positive and indicates the time dependence of electricity production costs: for example photovoltaic production occurs only at day, wind is blowing more during winter... The power $n > 1$ reflects the production fleet composition: for example when the fleet has expensive peak power plants n is high.

2.3 Optimal tariffs

In the setting we previously described, the Principal-Agent problem can be explicitly solved. We present in this part only a brief sketch of the argumentation and formal mathematical proofs are postponed to the remaining sections of the paper. Let's stress out that the problem is solved without imposing any *a priori* structure of the tariff function, except that it only depends on time and consumption. In order to be admissible, a tariff p should verified the individual rationality and incentive compatibility conditions, that is denoted $p \in \mathcal{P}$. Let us write formally the objective function of the Agents U_A and Principal U_P

$$U_A(p,x) := \sup_c \int_0^T \left(u(t,x,c(t)) - p(t,c(t))\right) dt = \int_0^T p^\star(t,x)dt.$$
$$U_P := \sup_{p \in \mathcal{P}} \int_0^T \left[\int_{X^\star(p^\star)} p(t, c^\star(t,x)) f(x)dx - K\left(t, \int_{X^\star(p^\star)} c^\star(t,x)f(x)dx\right)\right] dt.$$

The demonstration is performed into five main steps:

- For a given tariff p, the optimal responding consumption of the Agents can be determined as a function of $\frac{\partial p^\star}{\partial x}$. This is injected in the Principal problem which is now a problem expressed in term of $p^\star$ only.
- In order to solve the Principal problem, we first consider an alternative problem $\tilde{U}_P$, where we impose to the tariff p to be only continuous and non-decreasing, instead of being admissible.
- This alternative problem is simpler to solve. First, the structure of $X^\star$ is determined. Whenever H is constant, it is proven that $X^\star$ is of the form $[a_0, 1]$ meaning only the most efficient Agents select the contract. Whenever H is concave, $X^\star$ is of the form $[0, b_0] \cup [a_0, 1]$, meaning that the Principal not only selects the most efficient, but also the less efficient ones.

- Knowing the structure of $X^\star$, we rewrite $\frac{\partial p^\star}{\partial x}(t,x)$ as $f(t,a_0,b_0)$ using calculus of variations, we plug this expression into $\tilde{U}_P$ in order to determine a_0 and b_0.
- We finally verify whether the solution $p^\star$ for the alternative problem satisfies indeed the conditions of the initial problem, i.e. that it is admissible. By doing so, we are able to conclude that the solution $p^\star$ of the simpler problem, also solves the initial one of interest.

We compute that, whatever γ or H (constant or concave), the optimal tariff is a function of three components at most, namely a constant part p_3, a proportional part p_2 of the consumed power c, and a proportional part p_1 of c^γ

$$p(t,c) = p_1(t)c^\gamma + p_2(t)c + p_3(t).$$

This tariff is always a concave increasing function of the consumed power c. Next, an important observation is that this tariff is quite simple and close to current tariff structures proposed by electricity providers. Indeed, these tariffs are commonly split into a fixed charge in €, a volumetric charge in €/MWh and possibly a demand charge in €/MW. The fixed and the volumetric charges can depend on the maximum subscribed power which is another way to price the demand charge. The optimal tariffs the Principal offers in our settings are summarized in the following tables, where the explicit expressions for the functions $(p_i,\gamma)_i$, $(p_i^j)_{i,j}$ and $\hat{c}_i^\gamma(t)$ are respectively provided in Theorem 5.2 and Theorem 6.3 hereafter.

Table 1: Optimal tariff of residential consumers $\gamma < 0$

Selected Agents	H constant	H concave non decreasing
$[a_0,1]$ most effective Agents	$p_{2,\gamma}(t)c + p_{3,\gamma}(t)$	$p^1_{2,\gamma}(t)c + p^1_{3,\gamma}(t)$, for $c < \hat{c}_1^\gamma(t)$
$[b_0,a_0]$ intermediate Agents	not picked	not picked
$[0,b_0]$ least effective Agents	not picked	$p^3_{1,\gamma}(t)c^\gamma + p^3_{2,\gamma}(t)c + p^3_{3,\gamma}(t)$, for $\hat{c}_2^\gamma(t) < c$

Table 2: Optimal tariff of industrial consumers $\gamma \in (0,1)$

Selected Agents	H constant	H concave non decreasing
$[a_0,1]$ most effective Agents	$p_{1,\gamma}(t)c^\gamma + p_{2,\gamma}(t)c + p_{3,\gamma}(t)$	$p^1_{1,\gamma}(t)c^\gamma + p^1_{2,\gamma}(t)c + p^1_{3,\gamma}(t)$, for $c > \hat{c}_2^\gamma(t)$
$[b_0,a_0]$ intermediate Agents	not picked	not picked
$[0,b_0]$ least effective Agents	not picked	$p^3_{2,\gamma}(t)c + p^3_{3,\gamma}(t)$, for $c < \hat{c}_1^\gamma(t)$

Let us point out that the optimal tariffs are clearly interpretable, and the three components can be connected to electricity pricing standard issues. Clearly, in the obtained optimal tariff p_3 represents the fixed charge. The volumetric charge is the combination of a standard term $p_2(t)c$ plus $p_1(t)c^\gamma$. This last term is a way to charge more high demand consumers (indeed it only appears when c is high enough or when γ is positive, that is when the consumption of the Agent does not naturally saturate). Finally, no explicit demand charge appears but the coefficients p_i depend on the maximum subscribed power $\hat{c}_i^\gamma$, which limits the instantaneous power use and allows to charge more high power consumers. Let us point out that this method of electricity pricing is indeed implementable in practice, thanks to the recent spread ad development of smart meters, which enable a precise metering of electricity consumption and the dynamic management of maximum power (see for instance the USmartConsumer report [33], which states that at the end of 2016, 30% of overall European electricity meters were equipped with smart technology). In addition, the peak/off-peak issues are handled by the temporal structure of the tariff: each component depends on time t. Therefore, high power consumption within peak period will be overcharged compared to off-peak period. Let us also notice that the proportional part $p_1(t)$ to c^γ only depends on the Agent's utility parameters. Therefore this part should be common to any Principal whatever her cost of production, or the utility of reservation of the consumers.

The selected Agents are the most efficient when H is constant, which a classical result. But whenever H is concave, the Principal should also, in general, select Agents among the less efficient one. Indeed, when H is concave, reaching most efficient Agents is costly and it happens that getting less efficient Agents can in this case be profitable as they are easily satisfied. This type of feature seems to be uncommon in the Principal-Agent literature.

When H is constant, the tariff is a unique simple function of consumption, which even happens to be linear in c when $\gamma < 0$. When H is concave, the tariff is the combination of three functions depending on the level of consumption c. Nevertheless, observe that Agents who sign the contract never consume in the range $[\hat{c}_1^\gamma, \hat{c}_2^\gamma]$. Hence, the tariff portion there only acts as a repellent part.

3 Economic interpretations and numerical results

3.1 Examples when H is constant

For a constant utility reservation, more efficient Agents are selected, and we present numerical illustrations in Figure 2 for negative and positive γ. As previously explained, the tariff structure is linear with the consumption when $\gamma < 0$ and is concave otherwise, which is represented in the upper graphics of Figure 2. Middle graphics represent the utility Agents can obtain by signing the contract, depending on their type. If this utility level is smaller than their reservation utility (represented by the dashed line) they do not enter the contract and their consumption is null, as represented in the lower graphics. These utility representations also illustrate a classical result of informational rent: the most efficient Agents obtain a tariff inferior to what they are willing to pay, whereas the less efficient ones need to pay as much as they are able to, or are excluded.

3.2 Examples when H is concave

For a concave utility reservation, not only most efficient Agents are selected. We provide numerical illustrations where either the most or the less efficient Agents are selected on Figure 3. First, let us analyze the example when $H(x) = \sqrt{x}$ and $\gamma \in (0, 1)$, which corresponds to the left column. In this example, only the most efficient Agents sign the contract as they are the only ones obtaining a higher utility than their reservation one. As presented in the previous section, the tariff structure is the combination of three functions of consumption (upper graphics), but Agents who sign the contract only choose consumption such that $\hat{c}_2^\gamma < c$ which corresponds to the concave tariff part $p_1(t)^3 c^\gamma + p_2^3(t) c + p_3^3(t)$.

When $H(x) = \log(x)$ and $\gamma < 0$ (the right column of figure 3), only less efficient Agents take the contract. Indeed, the concavity of the reservation utility makes it profitable for the Principal to select these Agents, rather than the most efficient ones. The tariff structure is again the combination of three functions of

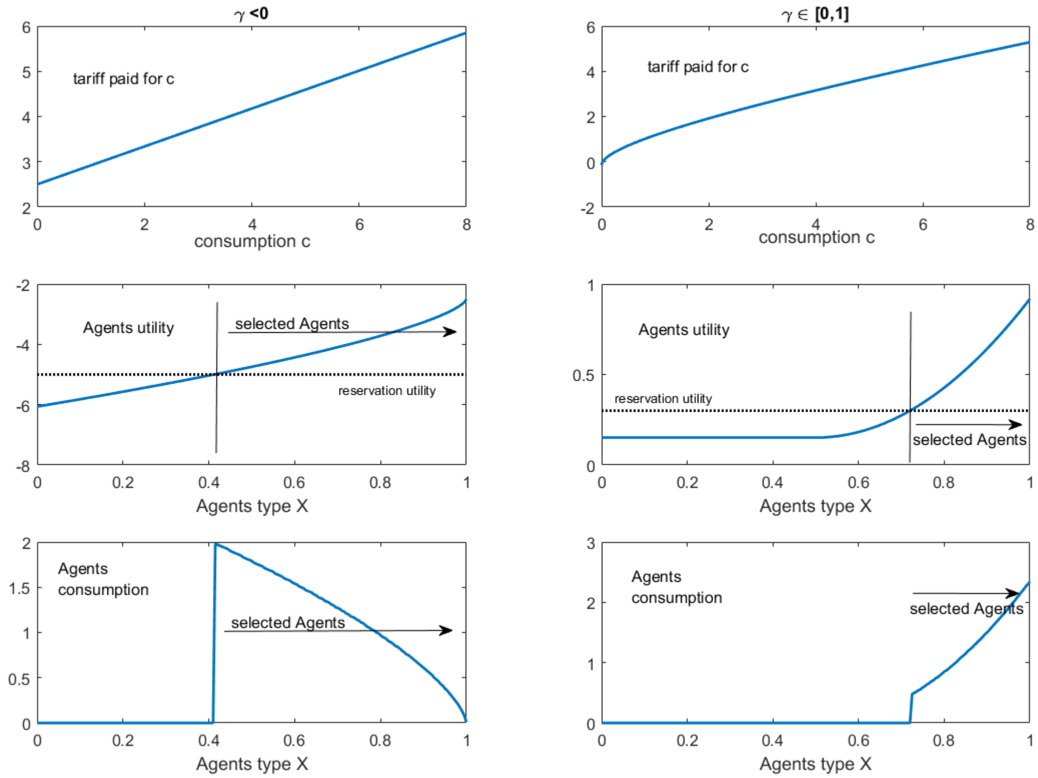


Figure 2: Tariffs paid against consumption (upper graphs), Agents' utility against type (middle graphs), and selected consumption against Agents' type (lower graphs); H constant.

consumption (upper graphics) but Agents who sign the contract only take consumption such that $\hat{c}_2^\gamma < c$ (of course $\hat{c}_2$ is different from the one in the previous example because we consider a different H). This again corresponds to the concave tariff part $p_1(t)^3 c^\gamma + p_2^3(t)c + p_3^3(t)$.

3.3 Impact of competition when H is constant

When H increases because competition is for example more intense, the Principal adapts his tariff in order to remain competitive. In that case, the Principal mainly decreases the constant part $p_{3,\gamma}$ of its tariff in order to attract consumers, see the left graphic of Figure 5 when $\gamma < 0$). The consumers selecting this new tariff obtain better conditions and as such consume more power, because it is cheaper, see the same example on the left graphic of Figure 4 when $\gamma < 0$. Therefore, when the Principal decreases his tariff, he does not decrease it enough in order to keep the same quantity of consumers: he accepts to retain less of them, but the selected ones do consume more, as represented on the right graphic of Figure 4. Nevertheless, the utility of the Principal decreases with competition, see the right part of Figure 5. At the extreme, the Principal even offers no tariff whenever H is too high. Observe that in this example, the fixed part of the tariff represents more than a half of the total cost of electricity for the consumers.

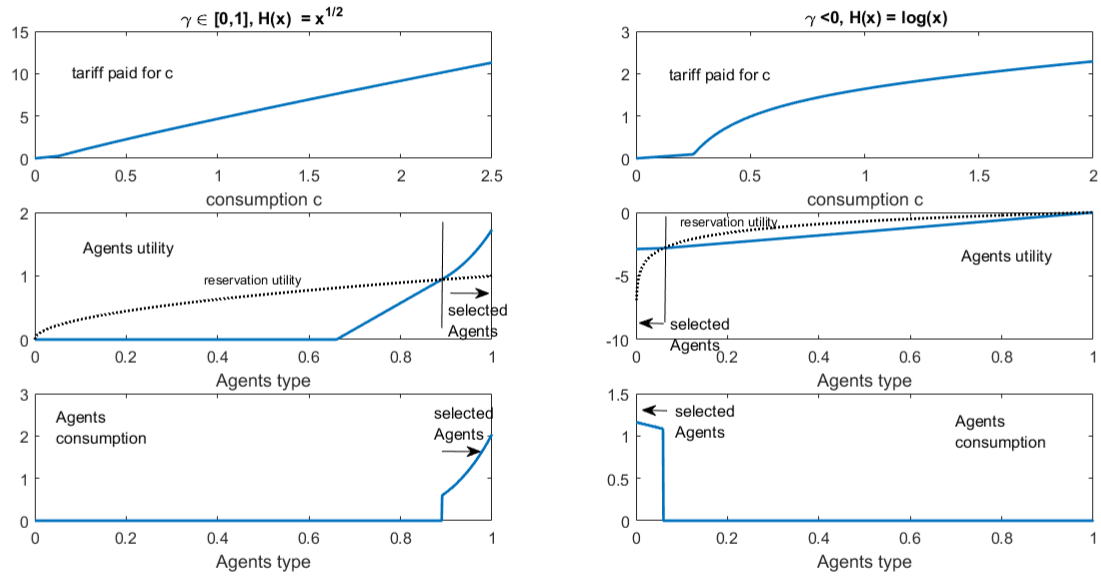


Figure 3: Tariffs against consumption (upper graphs), Agents's utility against type (middle graphs), and selected consumption against types (lower graphs); H concave.

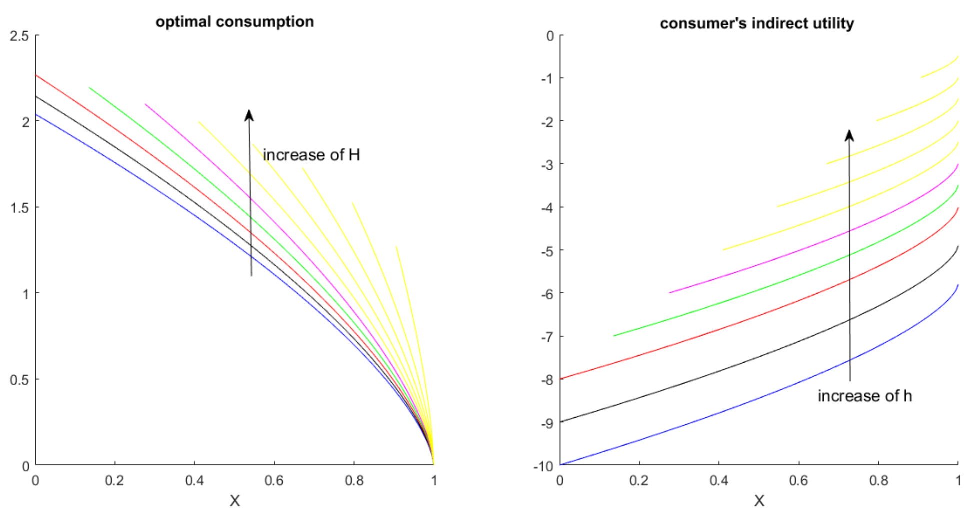


Figure 4: Evolution of Agents' utility (left) and consumption (right) against type; H increases and $\gamma < 0$.

3.4 Impact of cost of production when H is constant

For an increase of the cost of production K, the Principal also adapts his tariff in order to reflect this cost modification. In that case, the Principal mainly increases the proportional part $p_{2,\gamma}$ of its tariff so that he continues to attract consumers, see the example for constant H and $\gamma < 0$ on the left part of Figure 7. Consumers who select this new tariff are offered worse conditions, and as such consume less power, because it is more expensive, see the same example on the left graphic of Figure 6. In addition, less consumers select the contract. Therefore, the utility of the Principal decreases with the cost of production, as illustrated in the right part of Figure of 6.

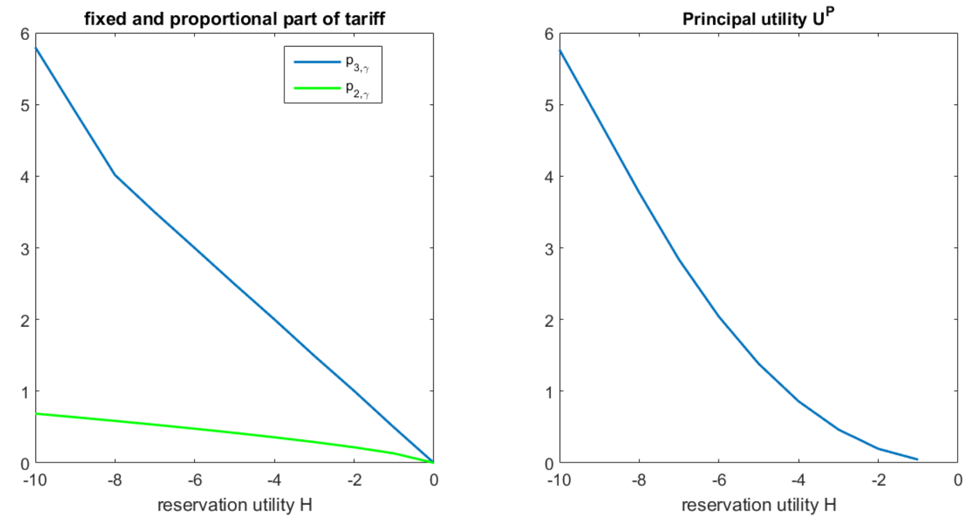


Figure 5: Evolution of tariff's components (left) and Principal utility U^P (right); H increases and $\gamma < 0$.

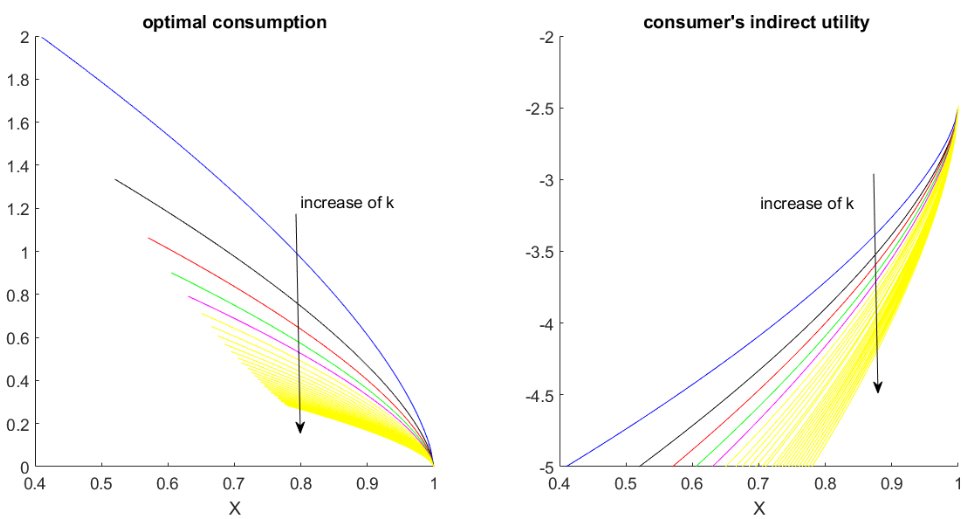


Figure 6: Agents' utility (left) and consumption (right) against type; K increases and $\gamma < 0$.

4 Model specification

We now turn to a more precise presentation of the model and try to present it in a rather general setting. In particular, the specific assumptions on the shape on the utility, type distribution or cost functions will only be introduced later, in order present our results in a more explicit fashion.

In this model, the Principal is a power company, whose purpose is to offer to its clients a collection of tariffs in order to maximise its profits. The time horizon $T > 0$ is fixed. The following notations will be used throughout the article

- $\mathcal{C}$ represents the admissible levels of consumption for the Agents, and will either be $\mathbb{R}_+$ or $\mathbb{R}_+^*$ depending on the utility function of the Agents.
- $p : [0, T] \times \mathcal{C} \longrightarrow \mathbb{R}_+$ is the tariff proposed by the Principal, such that $p(t, c)$ represents the price of electricity at time t corresponding to a level of consumption c.

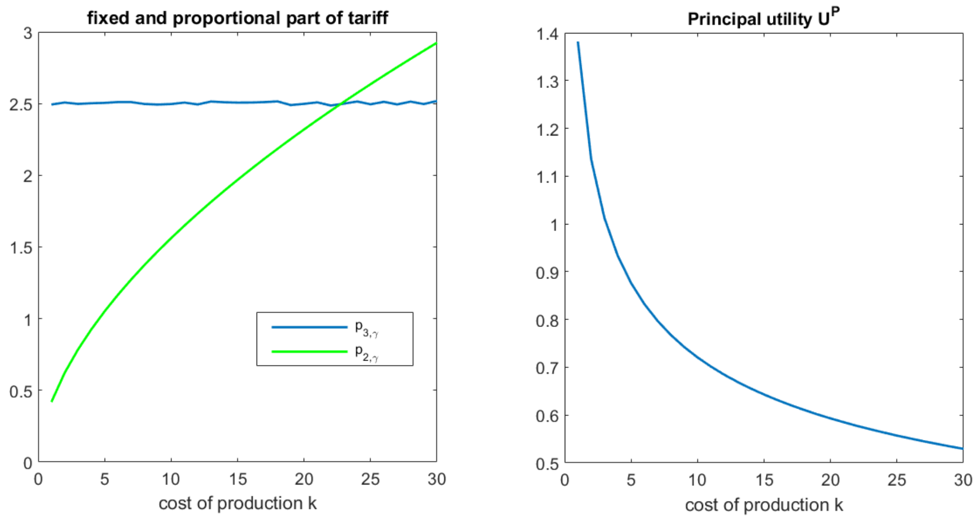


Figure 7: Tariff's components (left) and Principal utility U^P (right) against K; $\gamma < 0$.

- $K : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R}_+$ is the cost of production of electricity for the Principal, such that $K(t,c)$ represents the cost at time t for an aggregate level of production c. We assume that K is continuous in t, increasing, continuously differentiable and strictly convex in c.

- x is the Agent's type, assumed to take values in some subset X of $\mathbb{R}$.

- $c : [0,T] \times X \longrightarrow \mathcal{C}$ is the consumption function, such that $c(t,x)$ represents the consumption of electricity by an Agent of type x at time t.

- $u : [0,T] \times X \times \mathcal{C} \longrightarrow \mathbb{R}$ is the utility function of the Agents, such that $u(t,x,c)$ represents the utility obtained by an Agent of type x at time t when he consumes c. We will always assume that the map $c \longmapsto u(t,x,c)$ is non-decreasing and concave for every $(t,x) \in [0,T] \times X$. Moreover, the map u is assumed to be jointly continuous, such that $x \longmapsto u(t,x,c)$ is non-decreasing and differentiable Lebesgue almost everywhere for every $(t,c) \in [0,T] \times \mathcal{C}$, and such that $c \longmapsto \frac{\partial u}{\partial x}(t,x,c)$ is invertible. Finally, we assume that if $\mathcal{C} = \mathbb{R}_+$, the value $u(t,x,0) \in \mathbb{R}_+$ is independent of x, and if $\mathcal{C} = \mathbb{R}_+^*$ that $\lim_{c \to 0} u(t,x,c) = -\infty$, for every $(t,x) \in [0,T] \times X$. In other words, all the Agents have the same utility when they do not consume electricity.

- $f : X \longrightarrow \mathbb{R}_+$ is the distribution of the Agent's type over the population. As is customary in adverse selection problems, f is supposed to be known by the Principal.

4.1 Agent's problem

Let us start by defining the consumption strategies that the Agents are allowed to use. A consumption strategy c will be said to be admissible, which we denote by $c \in \mathfrak{C}$, if it is a Borel measurable map from $[0,T]$ to $\mathcal{C}$. Given a tariff p, that is a map from $[0,T] \times \mathbb{R}_+$ to $\mathbb{R}$, proposed by the Principal, an Agent of type $x \in X$ determines his consumption by solving the following problem

$$U_A(p,x) := \sup_{c \in \mathfrak{C}} \int_0^T \big(u(t,x,c(t)) - p(t,c(t))\big)dt. \tag{4.1}$$

The tariff that the Principal can offer to the Agents has to satisfy the incentive compatibility (IC) and the individual rationality (IR) conditions. In our setting, there is no moral hazard, so that the incentive compatibility condition is automatically satisfied. Furthermore, the (IR) condition can be expressed through the set $X(p)$ of the types of Agents which accept the contract p, which can be defined as

$$X(p) := \{x \in X,\ U_A(p,x) \geq H(x)\},$$

with a continuous and non-decreasing function H which represents the reservation utility of the Agents of different types, that is to say the utility that the Agents can hope to obtain by subscribing their power contract with a competitor. Agents for which the map $U_A(p,\cdot)$ is smaller than H will not accept the contract offered by the Principal.

We are now ready to give our definition of admissible tariffs, which uses vocabulary from $u-$convex analysis. We have regrouped all the pertinent results and definitions in Appendix A, for readers not necessary familiar with this theory.

Definition 4.1. *A tariff $p : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R}$ will be said to be admissible, denoted by $p \in \mathcal{P}$, if it satisfies*

(i) For any $(t,x) \in [0,T] \times X(p)$ the set $\partial^\star p^\star(t,x)$ is non–empty.

(ii) The map $x \longmapsto p^\star(t,x)$ is continuous on X, differentiable Lebesgue almost everywhere, for every $t \in [0,T]$, and satisfies

$$\int_0^T \int_X \left|\frac{\partial p^\star}{\partial x}\right|(t,x)dxdt < +\infty.$$

(iii) If one defines the map $c^\star : [0,T] \times [0,1] \longrightarrow \mathbb{R}_+$ by

$$c^\star(t,x) = \left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)} \left(\frac{\partial p^\star}{\partial x}(t,x)\right), \tag{4.2}$$

then the restriction of p to $\{(t,c) \in [0,T] \times \mathcal{C},\ \exists x \in X(p),\ c = c^\star(t,x)\}$ is $u-$convex.

Let us comment on the above definition. First of all, the regularity assumptions are mainly technical. The main point here is that since only the clients with type in $X(p)$ are going to accept the contract, the Principal will only have to face consumptions chosen by these clients. Besides, as we are going to prove in the the next proposition, this optimal consumption is exactly $c^\star(t,x)$. Therefore, any consumption $c \in \mathcal{C}$ which does not belong to the pre–image of $X(p)$ will never have to be considered by the Principal. In particular, there is a degree of freedom when defining the value of p there. Indeed, if clients of some type x reject the contract p, they will reject any contract with a higher price. This is the reason why we do not impose the admissible tariffs to be $u-$convex on $\mathcal{C}$ but only on the corresponding pre–image of the set $X(p)$.

Our main result in this section is

Proposition 4.1. *For every $p \in \mathcal{P}$ and for almost every $x \in X(p)$, we have*

$$U_A(p,x) = \int_0^T p^\star(t,x)dt,$$

and the optimal consumption of Agents of type x at any time $t \in [0,T]$ is given by $c^\star(t,x)$ defined in (4.2). In particular, $X(p)$ can be defined through $p^\star$ only as follows

$$X(p) = X^\star(p^\star) := \left\{x \in X,\ P^\star(x) := \int_0^T p^\star(t,x)dt \geq H(x)\right\}.$$

Proof. Since the space of admissible strategies for the Agent is decomposable and the integrand is normal when p is admissible (see Definitions 14.59 and 14.27 in Rockafellar and Wets [29] and also the particular case 14.29 of a Carethéodory integrand), we have from Theorem 14.60 in [29] that the solution of problem (4.1) is given by pointwise optimization. Moreover, $\partial^\star p^\star(t,x)$ is non–empty for every $(t,x)\in[0,T]\times X(p)$, so we have that every optimal consumption strategy $c^\star:[0,T]\longrightarrow\mathbb{R}_+$ satisfies $c^\star(t)\in\partial^\star p^\star(t,x)$ for almost every $t\in[0,T]$ and

$$p^\star(t,x)=u(t,x,c^\star(t))-p(t,c^\star(t)).$$

Since $u(t,x,0)$ does not depend on x, the envelop Theorem ensures that the map $x\longmapsto p^\star(t,x)$ is differentiable Lebesgue almost everywhere and that we have for almost every $(t,x)\in[0,T]\times X(p)$

$$\frac{\partial u}{\partial x}(t,x,c^\star(t))=\frac{\partial p^\star}{\partial x}(t,x). \tag{4.3}$$

Indeed, if $c^\star(t)>0$, that is the classical envelop Theorem. Otherwise, when $\mathcal{C}=\mathbb{R}_+$, it is immediate to check, using the fact that $u(t,x,0)$ so that both terms in (4.3) are then actually equal to 0.

Then, since the map $c\longmapsto\frac{\partial u}{\partial x}(t,x,c)$ is invertible, we have for almost every $(t,x)\in[0,T]\times X(p)$ that $\partial^\star p^\star(t,x)$ is a singleton, and the optimal consumption is $c^\star:[0,T]\times X(p)\longrightarrow\mathbb{R}_+$ defined in (4.2). $\square$

4.2 The Principal's problem

The Principal sets a tariff $p\in\mathcal{P}$ as a solution to his maximization problem

$$U_P:=\sup_{p\in\mathcal{P}}\int_0^T\left[\int_{X(p)}p(t,c^\star(t,x))f(x)dx-K\left(t,\int_{X(p)}c^\star(t,x)f(x)dx\right)\right]dt. \tag{4.4}$$

Using the results of Section 4.1, we can rewrite this problem in terms of $p^\star$ only as

$$\begin{aligned}U_P=\sup_{p\in\mathcal{P}}\int_0^T\Bigg[\int_{X^\star(p^\star)}&\left(u\left(t,x,\left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)\right)-p^\star(t,x)\right)f(x)dx\\&-K\left(t,\int_{X^\star(p^\star)}\left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)f(x)dx\right)\Bigg]dt,\end{aligned} \tag{4.5}$$

Now notice that by (4.3), $p^\star$ is actually non-decreasing (since $x\longmapsto u(t,x,c)$ is non-decreasing for every $(t,c)\in[0,T]\times\mathcal{C}$). Let us then consider the space C^+ of maps g, such that for every $t\in[0,T]$, $x\longmapsto g(t,x)$ is continuous and non-decreasing with

$$\int_0^T\int_X\left|\frac{\partial g}{\partial x}(t,y)\right|dydt<+\infty.$$

We shall actually consider the problem $\widetilde{U}_P\geq U_P$, defined by

$$\begin{aligned}\widetilde{U}_P:=\sup_{p^\star\in C^+}\int_0^T\Bigg[\int_{X^\star(p^\star)}&\left(u\left(t,x,\left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)\right)-p^\star(t,x)\right)f(x)dx\\&-K\left(t,\int_{X^\star(p^\star)}\left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)f(x)dx\right)\Bigg]dt,\end{aligned} \tag{4.6}$$

where we have forgotten the implicit link existing between p and $p^\star$, which explains why we have in general $\widetilde{U}_P\geq U_P$. We will see in the frameworks described below that we can give conditions under which the two problems are indeed equal. The main advantage of $\widetilde{U}_P$ is that it no longer contains the condition that $p^\star$ has to be $u-$convex, a constraint that is not easy to consider in full generality.

Besides, we also emphasize that since the elements of C^+ are non-decreasing with respect to x, they are also differentiable Lebesgue almost everywhere, so that $\widetilde{U}_P$ is indeed well defined. Our aim now will be to compute $\widetilde{U}_P$. However, the present framework is far too general to hope obtaining explicit solutions, which are of the utmost interest in our electricity pricing model, so that we will concentrate our attention on the case of Agents with power-type utilities. In the next Section, we will focus on the particular case where the reservation utility H is constant, and shall consider the more general case where it may depend on the Agents' type in Section 6.

5 Constant reservation utility

In this section, we shall use the following standing assumptions

Assumption 5.1. (i) $X = [0,1]$.

(ii) *We have for every* $(t,x,c) \in [0,T] \times X \times \mathcal{C}$

$$u(t,x,c) = g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma},$$

for some $\gamma \in (-\infty,0) \cup (0,1)$*, some map* $g_\gamma : X \longrightarrow \mathbb{R}_+$ *which is continuous, increasing if* $\gamma \in (0,1)$*, decreasing if* $\gamma \in (-\infty,0)$*, and for some continuous map* $\phi : [0,T] \longrightarrow \mathbb{R}_+^\star$.

Let us comment on this modeling choice for the utility function. The term $g_\gamma(x)$ represents the willingness of the Agents to pay for their consumption, *i.e.* their need for energy depends on their type. The term ϕ is common to every type of Agents and represents the fact that (almost) everyone is eager to consume at the same time (for example during the day rather than at night). Furthermore, we consider both the cases $\gamma \in (0,1)$, which would be the classical power utility function, as well as the case $\gamma < 0$, which corresponds to a situation where Agents actually cannot avoid consuming electricity, as it would provide them a utility equal to $-\infty$, which may be seen as more realistic. As discussed previously, taking $\gamma \in (0,1)$ identifies to considering industrial Agents, as $\gamma < 0$ more typically refers to residential Agents.

Equation (4.2) now can be written as

$$c^\star(t,x) = \left(\frac{\gamma}{\phi\,(t)\,g_\gamma'(x)}\frac{\partial p^\star}{\partial x}\,(t,x)\right)^{\frac{1}{\gamma}}. \tag{5.1}$$

By inserting the previous expression in equation (4.4) and using that

$$p(t,c^\star(t,x)) = g_\gamma(x)\phi(t)\frac{c^\star(t,x)^\gamma}{\gamma} - p^\star(t,x),$$

the problem to solve can now be expressed as

$$\begin{aligned}\widetilde{U}_P = \sup_{p^\star\in C^+}\int_0^T \Bigg[\int_{X^\star(p^\star)}&\left(\frac{g_\gamma(x)}{g_\gamma'(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right)f(x)dx\\ &- K\Bigg(t,\int_{X^\star(p^\star)}\left(\frac{\gamma}{\phi(t)g_\gamma'(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}}f(x)dx\Bigg)\Bigg]dt.\end{aligned} \tag{5.2}$$

We consider in this section a further simplification, related to the reservation utility of the Agents, which we suppose to be independent of their type. This assumption will be relieved in Section 6 hereafter.

Assumption 5.2. *The reservation utility H is actually independent of x, that is*

$$H(x) =: H, \text{ for every } x \in [0,1].$$

Under Assumptions 5.1 and 5.2, the (IR) condition reduces to

$$X^\star(p^\star) = \left\{x \in [0,1], \ \int_0^T p^\star(t,x)dt \geq H\right\}.$$

Since $p^\star$ is non-decreasing in x we have for any $x_0 \in [0,1]$ that

$$\int_0^T p^\star(t,x_0)dx \geq H \implies \int_0^T p^\star(t,x)dx \geq H, \ \forall x \geq x_0.$$

Therefore, the set $X^\star(p^\star)$ has necessarily the form

$$X^\star(p^\star) = [x_0, 1],$$

where $x_0 \in [0,1]$ needs to be determined and verifies, by continuity, that $P^\star(x_0) = H$. This means that the Principal will only offer contracts to Agents of type greater than x_0. The problem (5.2) can therefore be written equivalently

$$\begin{aligned}\widetilde{U}_P = \sup_{x_0\in[0,1]} \sup_{p^\star\in C^+(x_0)} \int_0^T \Bigg[&\int_{x_0}^1 \left(\frac{g_\gamma(x)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)dx \\ &- K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\right)\Bigg]dt, \end{aligned} \tag{5.3}$$

with

$$C^+(x_0) := \left\{p^\star \in C^+, \ \int_0^T p^\star(t,x_0)dt = H\right\} = \left\{p^\star \in C^+, \ X^\star(p^\star) = [x_0,1]\right\}.$$

Let us end this section with the following sufficient condition of $u-$convexity when Assumption 5.1 holds. Its proof is deferred to Appendix B.

Lemma 5.1. *Let Assumption 5.1 hold and suppose in addition that g_γ is concave if $\gamma \in (0,1)$ and convex if $\gamma \in (-\infty, 0)$. Let $\psi : [0,T] \times X \longrightarrow \mathbb{R}$ be a map such that $x \longmapsto \psi(t,x)$ is non-decreasing and convex. Then ψ is $u-$convex. Furthermore, if we take*

$$g_\gamma(x) := \begin{cases} x, \text{ if } \gamma \in (0,1), \\ 1-x, \text{ if } \gamma < 0, \end{cases}$$

then any $u-$convex function $[0,T] \times X \longrightarrow \mathbb{R}$ is convex.

5.1 General distribution of costs and Agent types

Denote by F the cumulative distribution function of the types of Agents. By integration by parts we have for every $x_0 \in [0,1]$ and every $p^\star \in C^+(x_0)$

$$\begin{aligned}&\int_0^T \left[\int_{x_0}^1 \left(\frac{g_\gamma(x)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)dx - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\right)\right]dt \\ &= \int_0^T \left[\int_{x_0}^1 \left(\frac{g_\gamma(x)}{g'_\gamma(x)}f(x) + F(x) - 1\right)\frac{\partial p^\star}{\partial x}(t,x)dx - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\right)\right]dt \\ &\quad + (F(x_0) - 1)\int_0^T p^\star(t,x_0)dt.\end{aligned}$$

We therefore end up with the maximization problem

$$\widetilde{U}_P = \sup_{x_0\in[0,1]} \sup_{p^\star\in C^+(x_0)} \int_0^T \bigg[\int_{x_0}^1 \frac{\big(g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\big)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)dx \\ - K\bigg(t, \int_{x_0}^1 \bigg(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\bigg)^{\frac{1}{\gamma}} f(x)dx\bigg)\bigg]dt + (F(x_0)-1)H. \tag{5.4}$$

We can now state our main result of this section, wholes proof is postponed to Appendix B. It requires the introduction of the following function

$$\ell(x_0) := \int_{x_0}^1 \left(\frac{\big[g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\big]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}} dx,\ \ x_0\in[0,1].$$

Theorem 5.1. *Let Assumptions 5.1 and 5.2 hold. We have*

(i) The maximum in (5.3) *is attained for the maps*

$$p^\star(t,x) = p^\star(t,x_0^\star) + \int_{x_0^\star}^x \frac{g'_\gamma(y)}{\gamma}\left(\frac{\phi(t)^{\frac{1}{\gamma}}\big[g_\gamma(y)f(y)+g'_\gamma(y)F(y)-g'_\gamma(y)\big]^+}{f(y)\frac{\partial K}{\partial c}(t,A(t,x_0^\star))}\right)^{\frac{\gamma}{1-\gamma}} dy,\ \ x\in[0,1],$$

where $A(t,x_0)$ is defined

$$A(t,x_0) := \int_{x_0}^1 \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx.$$

and $x_0^\star$ is any maximizer of the map and

$$[0,1]\ni x_0 \longmapsto \int_0^T \left(\frac{\partial K}{\partial c}(t,A(t,x_0))m(t,x_0) - K(t,\gamma m(t,x_0))\right)dt + (F(x_0)-1)H,$$

with

$$m(t,x_0) := \frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{1}{1-\gamma}}},\ \ (t,x_0)\in[0,T]\times[0,1],$$

and $t\longmapsto p^\star(t,x_0^\star)$ is any map such that

$$\int_0^T p^\star(t,x_0^\star)dt = H.$$

For instance, one can choose $p^\star(t,x_0^\star) := H/T$, $t\in[0,T]$.

(ii) Define p for any $(t,c)\in[0,T]\times\mathbb{R}_+$ by

$$p(t,c) = \sup_{x\in[0,1]}\left\{g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma} - p^\star(t,x)\right\}.$$

If the map defined on $[0,1]$ by

$$x\longmapsto g'_\gamma(x)\left(\frac{\big[g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\big]^+}{f(x)}\right)^{\frac{\gamma}{1-\gamma}},$$

is non-decreasing, then $p^\star$ is $u-$convex, and p is the optimal tariff for the problem (4.4)*. Furthermore, the Principal only signs contracts with the Agents of type $x \in [x_0^\star, 1]$.*

(iii) Finally, in the case $\gamma \in (0,1)$, if f is non-increasing and the map

$$\beta : x \longmapsto \frac{(g_\gamma(x)f(x) + g'_\gamma(x)F(x) - g'_\gamma(x))}{f^\gamma(x)},$$

is increasing over the set $L := \{x \in [0,1],\ \beta(x) > 0\}$, then $x_0^\star$ is unique and is characterized by the equation

$$\left(\frac{1-\gamma}{\gamma}\right) \frac{\phi(t)^{\frac{1}{1-\gamma}} \beta(x_0^\star)}{\left(\frac{\partial K}{\partial c}(t, A(t, x_0^\star))\right)^{\frac{\gamma}{1-\gamma}}} = f(x_0^\star)H.$$

The same result holds in the case $\gamma \in (-\infty, 0)$ if f is non-decreasing and β is decreasing over L.

5.2 An explicit example

We insist on the fact that the tariff p defined in Theorem 5.1 is $u-$convex by definition, and it is finite since it is written as a supremum of a continuous function over a compact set. In order to verify that $p \in \mathcal{P}$, one therefore only needs to make sure that $p^\star$ is indeed the $u-$transform of p (which is the case if $p^\star$ is $u-$convex) and satisfies the other required properties. We will consider here a simplified framework where all the computations can be done almost explicitly.

Assumption 5.3. *The cost function K is given, for some $n > 1$, by*

$$K(t,c) := k(t)\frac{c^n}{n},\ (t,c) \in [0,T] \times \mathbb{R}_+,$$

for some map $k : [0,T] \longrightarrow \mathbb{R}_+^\star$. Moreover, the distribution of the type of Agents is uniform, that is $f(x) = 1$, and we impose $g_\gamma(x) := x\mathbf{1}_{\gamma \in (0,1)} + (1-x)\mathbf{1}_{\gamma < 0}$, for every $x \in [0,1]$.

Under Assumption 5.3, we then have

$$A(t, x_0) = \left(\frac{\phi(t)}{k(t)}\right)^{\frac{1}{n-\gamma}} \ell^{\frac{1-\gamma}{n-\gamma}}(x_0),$$

and the maximization problem becomes

$$\widetilde{U}_P = \sup_{x_0 \in [0,1]} \left\{ \left(\frac{1}{\gamma} - \frac{1}{n}\right) \int_0^T \left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}} dt\ \ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0 - 1)H \right\}.$$

Define

$$B_\gamma(T) := \left(\frac{1}{\gamma} - \frac{1}{n}\right) \int_0^T \left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}} dt,\ \ \Phi(x_0) := B_\gamma(T)\ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0 - 1)H,$$

where we emphasize that since $n > 1$, when $\gamma \in (0,1)$, we easily have that $B_\gamma(T) > 0$, while $B_\gamma(T) < 0$ when $\gamma < 0$. Furthermore, we remind the reader that when $\gamma > 0$, the reservation utility of the Agents is necessarily non-negative, while it has to be negative when $\gamma < 0$, since the utility function itself is negative.

Our result rewrites in this case

Theorem 5.2. *Let Assumptions 5.1, 5.2 and 5.3 hold.*

(i) If $\gamma \in (0,1)$, then, the optimal tariff $p \in \mathcal{P}$ is given for any $(t,c) \in [0,T] \times \mathbb{R}_+$ by

$$p(t,c) = \phi(t)\frac{c^\gamma}{2\gamma} + \left(\left(\frac{\phi(t)}{2}\right)^{\frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}} c - \frac{H}{T} + M(t)(2x_0^\star - 1)^{\frac{1}{1-\gamma}},$$

where

$$M(t) = \frac{1-\gamma}{2\gamma}\left(\frac{2(2-\gamma)}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}} \left(\frac{\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}} \left(1 - (2x_0^\star - 1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}},$$

and where $x_0^\star$ is the unique solution in $(1/2, 1)$ of the equation

$$H = 2nA_\gamma(T)\frac{2-\gamma}{n-\gamma}(2x_0^\star - 1)^{\frac{1}{1-\gamma}} \left(1 - (2x_0^\star - 1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

Furthermore, only the Agents of type $x \geq x_0^\star$ will accept the contract.

(ii) If $\gamma < 0$, then, the optimal tariff $p \in \mathcal{P}$ is given for any $(t,c) \in [0,T] \times \mathbb{R}_+$ by

$$p(t,c) = -\gamma c\left(-\frac{\phi(t)}{\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{1-\gamma}{\widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} - \frac{H}{T} - \widehat{M}(t)(1 - \widehat{x}_0^\star)^{\frac{1}{1-\gamma}},$$

where

$$\widehat{M}(t) = -\frac{1-\gamma}{\gamma}\left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}} \left(\frac{2^\gamma\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}} (1 - \widehat{x}_0^\star)^{-\frac{\gamma(2-\gamma)(n-1)}{(n-\gamma)(1-\gamma)}},$$

and where

$$\widehat{x}_0^\star := \left(1 - \left(\frac{n-\gamma}{n(1-\gamma)B_\gamma(T)}H\right)^{\frac{n-\gamma}{n(1-\gamma)+\gamma}} \left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{-\gamma(n-1)}{n(1-\gamma)+\gamma}} 2^{\frac{-n}{n(1-\gamma)+\gamma}}\right)^+.$$

Furthermore, only the Agents of type $x \geq \widehat{x}_0^\star$ will accept the contract.

6 Type-dependent reservation utilities

In this part we study the case where the reservation utility H is a general continuous and non-decreasing function of the Agent type $x \in [0,1]$. This case strongly differs from the previous section as the existence of a solution to the infinite-dimensional problem faced by the Principal is not guaranteed and we need to impose some additional structure to the set of admissible tariffs. Specifically, we will consider a new set of admissible tariffs which is contained in a reflexive Banach space and we will use the classical results from functional analysis to prove the existence of solutions to the Principal's problem. With that purpose in mind, we introduce the following Sobolev-like spaces.

Definition 6.1. *For any $\ell \geq 1$ and any open subset $\mathcal{O}$ of X, we denote by $W_x^{1,\ell}(\mathcal{O})$ the space of maps $q : [0,T] \times \mathcal{O} \longrightarrow \mathbb{R}$ for which there exists a null set $\mathcal{N}(q) \subset [0,T]$ (for the Lebesgue measure) satisfying that for every $t \in [0,T] \setminus \mathcal{N}(q)$ the map $x \longmapsto q(t,x)$ belongs to $W^{1,\ell}(\mathcal{O})$[1] and such that*

$$||q||_{\ell,\mathcal{O}} := \left(\int_0^T \int_{\mathcal{O}} |q(t,x)|^\ell dxdt\right)^{\frac{1}{\ell}} + \left(\int_0^T \int_{\mathcal{O}} \left|\frac{\partial q}{\partial x}(t,x)\right|^\ell dxdt\right)^{\frac{1}{\ell}} < \infty.$$

[1] That is to say the usual Sobolev space of maps admitting a weak first order derivative.

Remark 6.1. *For the rest of the paper, for every map q belonging to some space $W_x^{1,\ell}(\mathcal{O})$, the set $\mathcal{N}(q)$ will make reference to the one mentioned in Definition 6.1.*

For all the analysis of this section, we fix a number $m > 1$ such that $m\gamma < 1$. We are now ready to give our new definition of admissible tariffs.

Definition 6.2. *A tariff $p : [0,T]\times\mathbb{R}_+ \longrightarrow \mathbb{R}$ is said to be admissible (in the case when H is not constant), denoted by $p \in \widehat{\mathcal{P}}$, if in addition to Definition 4.1, it satisfies that $p^\star \in W_x^{1,m}(\overset{o}{X})$.*

In this new setting, the Principal offers a tariff $p \in \widehat{\mathcal{P}}$ which solves his maximization problem

$$\widehat{U}_P := \sup_{p\in\widehat{\mathcal{P}}} \int_0^T \left[\int_{X^\star(p^\star)} p(t, c^\star(t,x)) f(x) dx - K\left(t, \int_{X^\star(p^\star)} c^\star(t,x) f(x) dx\right)\right] dt. \tag{6.1}$$

Following the previous sections, we will consider the problem $\overline{U}_P \geq \widehat{U}_P$, defined by

$$\begin{aligned}\overline{U}_P = \sup_{p^\star\in\widehat{C}^+} \int_0^T & \left[\int_{X^\star(p^\star)} \left(\frac{g_\gamma(x)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)dx \right. \\ & \left. - K\left(t, \int_{X^\star(p^\star)} \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\right)\right] dt, \end{aligned} \tag{6.2}$$

where $\widehat{C}^+ = C^+ \cap W_x^{1,m}(\overset{o}{X})$. We aim at solving $\overline{U}_P$ and give conditions under which it coincides with $\widehat{U}_P$.

Now, moving to the reservation utility function H, recall that it determines the set $X^\star(p^\star)$. In order to avoid complex forms of this set we make the following assumption on g, H and f.

Assumption 6.1. *The functions g and H are such that for every $x \in [0,1]$*

$$\frac{g_\gamma(x)}{g'_\gamma(x)} \leq \frac{H(x)}{H'(x)}. \tag{6.3}$$

Moreover, the following maps

$$\begin{aligned} v_1(x) &:= g'_\gamma(x)\left(\frac{\left[g_\gamma(x)f(x) + g'_\gamma(x)F(x)\right]^+}{f(x)}\right)^{\frac{\gamma}{1-\gamma}}, \\ v_2(x) &:= g'_\gamma(x)\left(\frac{\left[g_\gamma(x)f(x) + g'_\gamma(x)F(x) - g'_\gamma(x)\right]^+}{f(x)}\right)^{\frac{\gamma}{1-\gamma}}, \end{aligned}$$

are non-decreasing on $[0,1]$.

Remark 6.2. *Condition (6.3) is equivalent to the elasticity of reservation utility being less than the elasticity of willingness to pay for consumption. For instance, in the case $\gamma \in (0,1)$, it is automatically satisfied when H is constant and if $g_\gamma(x) = x$ then (6.3) reduces to H being concave. Similarly, in the case $\gamma < 0$, (6.3) holds if $g_\gamma(x) = 1 - x$ and $H(x) = x^\alpha$ with $\alpha > 1$. On the other hand, if for instance, $f(x) = 1$ and $g(x) = x^\alpha$ with $\alpha \in (0,1]$, then v_1, v_2 are increasing when $\alpha \geq 1-\gamma$.*

The following proposition shows that when Condition (6.3) holds, it is actually never optimal for the Principal to propose a tariff for which the (IR) condition is binding on any measurable subset of $[0,1]$ with positive Lebesgue measure. Its proof is postponed to Appendix C.

Proposition 6.1. *Let Assumptions 5.1 and 6.1 hold, and let $p^\star \in \widehat{C}^+$ be any function such that the set*

$$Y^\star(p^\star) := \{x \in [0,1],\ P^\star(x) = H(x)\},$$

has positive Lebesgue measure. Then $p^\star$ is not optimal for problem (6.2).

In this section, we show that the problem splits into subintervals. Thanks to the previous proposition, we can consider without loss of generality functions $p^\star \in \widehat{C}^+$ such that the Lebesgue measure of $Y^\star(p^\star)$ is zero. For these functions, we define the set

$$\widehat{X}^\star(p^\star) := X^\star(p^\star) \setminus Y^\star(p^\star) = \{x \in [0,1],\ P^\star(x) > H(x)\},$$

which by continuity is an open subset of $[0,1]$. As $X^\star$ and $p^\star$ are continuous, we can replace all the integrals over $X^\star(p^\star)$ by integrals over $\widehat{X}^\star(p^\star)$ and we can write the latter set as a countable union of open disjoint intervals, that is

$$\widehat{X}^\star(p^\star) := [0,b_0) \cup \bigcup_{n\geq 1}(a_n,b_n) \cup (a_0,1],$$

for some $a_0 \in (0,1]$, $b_0 \in [0,1)$, and $0 < a_n < b_n < 1$, $\forall n \geq 0$. We denote $a := (a_n)_{n\geq 0}$, $b := (b_n)_{n\geq 0}$ and define $\mathcal{A}$ as the set of such that pairs (a,b). For any $(a,b) \in \mathcal{A}$, we also define the set

$$X^\star(a,b) = [0,b_0) \cup \bigcup_{n\geq 1}(a_n,b_n) \cup (a_0,1].$$

Remark 6.3. *The case $b_0 = 0$ stands for $P^\star(b_0) < H(b_0)$ and the case $a_0 = 1$ stands for $P^\star(a_0) < H(a_0)$. By continuity we have $P^\star(a_n) = H(a_n)$ and $P^\star(b_n) = H(b_n)$ for every $n \geq 1$.*

We can therefore write

$$\begin{aligned}\overline{U}_P = \sup_{(a,b)\in\mathcal{A}} \sup_{p^\star\in C^+(a,b)} \int_0^T \Bigg[\int_{X^\star(a,b)} &\left(\frac{g(x)}{g'(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)dx \\ &- K\Bigg(t, \int_{X^\star(a,b)}\left(\frac{\gamma}{\phi(t)g'(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\Bigg)\Bigg]dt, \end{aligned} \tag{6.4}$$

where $C^+(a,b)$ is given by all the maps $p^\star \in \widehat{C}^+$ such that $\widehat{X}^\star(p^\star) = X^\star(a,b)$. Define for every $(a,b) \in \mathcal{A}$ the operator $\Psi_{(a,b)} : C^+(a,b) \longrightarrow \mathbb{R}$ by

$$\begin{aligned}\Psi_{(a,b)}(p^\star) := \int_0^T \Bigg[\int_{X^\star(a,b)} &\left(\frac{g_\gamma(x)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)dx \\ &- K\Bigg(t, \int_{X^\star(a,b)}\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\Bigg)\Bigg]dt.\end{aligned}$$

We optimize first with respect to $p^\star$, and then study the problem

$$(P_{a,b}) \qquad \sup_{p^\star\in C^+(a,b)} \Psi_{(a,b)}(p^\star).$$

Our first result gives the existence of a solution to the above infinite-dimensional optimization problem, and requires the following assumption, which involves mainly the cost function. It is required in order to obtain nice coercivity properties

Assumption 6.2. *The cost function K satisfies the following growth condition*

$$K(t,c) \geq k(t)c^n, \ \forall c \in \mathcal{C},$$

where the map $k : [0,T] \longrightarrow \mathbb{R}_+$ is bounded from below by some constant $\underline{k} > 0$ and $n \geq 1$. Moreover, we have that

$$I := \inf \left\{ k(t) \left(\frac{\gamma f(x)}{\phi(t) g'_\gamma(x)} \right)^n, \ (t,x) \in [0,T] \times [0,1] \right\} > 0.$$

We now have

Proposition 6.2. *Let Assumptions 5.1 and 6.2 hold. For every $(a,b) \in \mathcal{A}$, the optimization problem $(P_{a,b})$ has at least one solution.*

Next, we obtain necessary optimality conditions for $(P_{a,b})$. Recalling from Remark 6.3 that the (IR) condition is binding at each a_n, b_n, by integration by parts we can rewrite $\Psi_{(a,b)}$ as

$$\begin{aligned}
\Psi_{(a,b)}(p^\star) = & \int_0^T \int_0^{b_0} \frac{\big(g_\gamma(x) f(x) + g'_\gamma(x) F(x)\big)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) dx dt \\
& + \int_0^T \int_{a_0}^1 \frac{\big(g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x)\big)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) dx dt \\
& + \sum_{n=1}^{\infty} \int_0^T \int_{a_n}^{b_n} \frac{\big(g_\gamma(x) f(x) + g'_\gamma(x) F(x)\big)}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x) dx dt \\
& - K\left(t, \int_{(0,b_0) \cup \bigcup_{n\geq 1}(a_n,b_n) \cup (a_0,1)} \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x) dx \right) dt \\
& + \sum_{n=1}^{\infty} F(a_n) H(a_n) - \sum_{n=1}^{\infty} F(b_n) H(b_n) - F(b_0) H(b_0) + (F(a_0) - 1) H(a_0).
\end{aligned} \tag{6.5}$$

To simplify notations, denote

$$A(t,a,b) := \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x) dx.$$

Theorem 6.1. *Let Assumptions 5.1 and 6.2 hold and let $p^\star$ be a solution of $(P_{a,b})$. Consider an interval $I = (x_\ell, x_r) \subseteq X^\star(a,b)$ such that $P^\star(x) > H(x)$ for every $x \in I$, $P^\star(x_\ell) = H(x_l)$ and $P^\star(x_r) = H(x_r)$. Then there exists a null set $\mathcal{N} \subset [0,T]$ and a constant μ_t for every $t \in [0,T] \setminus \mathcal{N}$ such that the following optimality condition is satisfied*

(i) In the case $I \subseteq (a_0, 1)$, for every $x \in I$ we have

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x) + g'_\gamma(x) \mu_t \right]^+}{f(x) \frac{\partial K}{\partial c}(t, A(t,a,b))} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}. \tag{6.6}$$

(ii) In the case $I \subseteq (0, b_0) \cup_{n\geq 1} (a_n, b_n)$, for every $x \in I$ we have

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) + g'_\gamma(x) \mu_t \right]^+}{f(x) \frac{\partial K}{\partial c}(t, A(t,a,b))} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}. \tag{6.7}$$

The proof of the Theorem 6.1 consists in several technical propositions which are given and proved in Section C.2 below.

The next and last proposition of this section proves the convexity of the solutions to problems $(P_{a,b})$ over intervals where the (IR) condition is not binding. This result will allow us to completely solve the Principal's problem in the next subsection, when the function H is strictly concave.

Proposition 6.3. *Let Assumptions 5.1, 6.1 and 6.2 hold. Let $p^\star$ be a solution to problem $(P_{a,b})$. Then $P^\star$ is convex on every interval over which $P^\star$ is strictly greater than H.*

Proof. From Assumption 6.1 and Theorem 6.1 we have that on every interval I over which $P^\star > H$, there exists a null set $\mathcal{N} \subset [0, T]$ such that for every $t \in [0, T] \setminus \mathcal{N}$, $x \longmapsto \frac{\partial p^\star}{\partial x}(t, x)$ is non-decreasing on I. Therefore $P^\star$ is convex on I since

$$\frac{\partial P^\star}{\partial x}(x) = \int_0^T \frac{\partial p^\star}{\partial x}(t, x)dt.$$

□

6.1 Strictly concave reservation utility

In this section we show that there are at most that most effective and less effective Agents who select the contract.

Assumption 6.3. *The map $x \longmapsto H(x)$ is strictly concave and non-decreasing.*

The main interest of Assumption 6.3 is the following simple result, which shows that we can always restrict our attention to sets $X^\star$ with a very simple form.

Proposition 6.4. *Let Assumptions 5.1, 6.1, 6.2 and 6.3 hold. Let $(a, b) \in \mathcal{A}$ be such that $0 < a_{n_0} < b_{n_0} < 1$ for some $n_0 \geq 1$. Then the solution to problem $(P_{a,b})$ is not optimal for problem* (6.2).

Proof. Let $p^\star$ be the solution of problem $(P_{a,b})$. We will prove that $P^\star \equiv H$ in the interval (a_{n_0}, b_{n_0}) and the result will follow from Proposition 6.1. Suppose not, then there exists $x_0 \in (a_{n_0}, b_{n_0})$ such that $P^\star(x_0) > H(x_0)$ and $p^\star$ is given by (6.6) in a neighbourhood around x_0, so $P^\star$ is increasing in that neighbourhood. By Proposition 6.3 we have that $P^\star(b_{n_0}) > H(b_{n_0})$, because on every interval which is contained in the set $\{x,\ P^\star(x) \geq H(x)\}$ the convex map $P^\star$ and the strictly concave map H can intersect at most at one point. This contradicts the fact that $p^\star \in C^+(a, b)$. □

In the rest of this section, we start by deriving a general solution under some implicit assumptions, and then show that the latter can be verified in the context of Assumption 5.3.

6.1.1 The general tariff

Proposition 6.4 implies that the solution of (6.4) is attained for some $p^\star$ satisfying $\widehat{X}^\star(p^\star) = [0, b_0) \cup (a_0, 1]$. We expect then the optimal tariff to look like the curve in Figure 8.

Let us then define the set

$$\mathcal{A}_2 := \left\{(a, b) \in [0, 1]^2,\ b \leq a\right\}.$$

Theorem 6.1 gives us only partial information about the solution of the problem $(P_{a,b})$, for $(a, b) \in \mathcal{A}_2$. Now that we assume in addition that H is strictly concave, we can actually complete that information with the following proposition, which tells us that the value of the constants μ_t is zero in the intervals of the form $[0, b)$ and $(a, 1]$. Its proof is postponed to the Appendix.

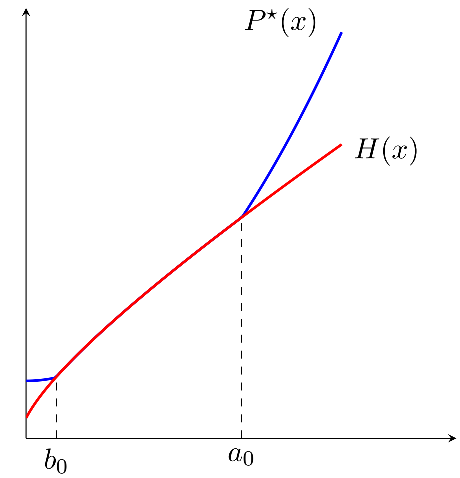


Figure 8: $\widehat{X}^{\star}(p^{\star})$ for strictly concave H.

Proposition 6.5. *Let Assumptions 5.1, 6.1, 6.2 and 6.3 hold. Let $p^{\star}$ be a solution of $(P_{a,b})$, for $(a,b) \in \mathcal{A}_2$, and I be as in Theorem 6.1. Then there exists a null set $\mathcal{N}' \subset [0,T]$ such that for every $t \in [0,T] \setminus \mathcal{N}'$ the optimality conditions from Theorem 6.1 hold with $\mu_t = 0$.*

Following the computations of Section 5, we define

$$A(t, a_0, b_0) := g_K^{(-1)}\Bigg(\phi(t)^{\frac{1}{1-\gamma}} \int_0^{b_0} \left(\frac{[g(x)f(x) + g'(x)F(x)]^+}{f^{\gamma}(x)}\right)^{\frac{1}{1-\gamma}} dx \\ + \phi(t)^{\frac{1}{1-\gamma}} \int_{a_0}^{1} \left(\frac{[g(x)f(x) + g'(x)F(x) - g'(x)]^+}{f^{\gamma}(x)}\right)^{\frac{1}{1-\gamma}} dx\Bigg).$$

The aim of the next proposition is similar in spirit to that of Proposition 6.4, in the sense that it allows to exclude many specifications of $(a,b) \in \mathcal{A}_2$ when one is interested in solving problem (6.2).

Proposition 6.6. *Let Assumptions 5.1, 6.1, 6.2 and 6.3 hold. Let $p^{\star}$ be a solution of $(P_{a,b})$. If either*

$$\Xi_{\gamma}(a_0, b_0) := \int_0^T \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_{\gamma}(a_0)f(a_0) + g'_{\gamma}(a_0)F(a_0) - g'_{\gamma}(a_0)\right]^+}{f(a_0)\frac{\partial K}{\partial c}(t, A(t, a_0, b_0))}\right)^{\frac{\gamma}{1-\gamma}} \frac{g'_{\gamma}(a_0)}{\gamma} dt < H'(a_0),$$

or

$$\Psi_{\gamma}(a_0, b_0) := \int_0^T \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_{\gamma}(b_0)f(b_0) + g'_{\gamma}(b_0)F(b_0)\right]^+}{f(b_0)\frac{\partial K}{\partial c}(t, A(t, a_0, b_0))}\right)^{\frac{\gamma}{1-\gamma}} \frac{g'_{\gamma}(b_0)}{\gamma} dt > H'(b_0),$$

then the solution to problem $(P_{a,b})$ is not optimal for problem (6.2).

Judging by the results of Proposition 6.6, it is natural to define $\mathcal{A}'_2$ as the set of all the pairs $(a,b) \in \mathcal{A}_2$ for which

$$\Xi_{\gamma}(a_0, b_0) \geq H'(a_0), \;\; \Psi_{\gamma}(a_0, b_0) \leq H'(b_0).$$

Thanks to Proposition 6.6, we have thus reduced the problem $(P_{a,b})$ to

$$\sup_{(a_0,b_0) \in \mathcal{A}'_2} \int_0^T \left[\frac{\phi(t)^{\frac{1}{1-\gamma}} \ell(a_0, b_0)}{\gamma \left(\frac{\partial K}{\partial c}(t, A(t, a_0, b_0))\right)^{\frac{\gamma}{1-\gamma}}} - K\left(t, \frac{\phi(t)^{\frac{1}{1-\gamma}} \ell(a_0, b_0)}{\left(\frac{\partial K}{\partial c}(t, A(t, a_0, b_0))\right)^{\frac{1}{1-\gamma}}}\right)\right] dt + \theta(a_0, b_0),$$

where we abused notations and defined the corresponding functions

$$\begin{aligned}\ell(a_0,b_0) :=& \int_0^{b_0}\left(\frac{[g(x)f(x)+g'(x)F(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx + \int_{a_0}^1\left(\frac{[g(x)f(x)+g'(x)F(x)-g'(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx,\\ \theta(a_0,b_0) :=& -F(b_0)H(b_0)+(F(a_0)-1)H(a_0),\end{aligned}$$

Since all these maps are continuous on $[0,1]^2$, the supremum over the compact set above is attained at some $(a_0^\star,b_0^\star)\in\mathcal{A}_2'$. We have therefore proved

Theorem 6.2. *Let Assumptions 5.1, 6.1, 6.2 and 6.3 hold. We have*

(i) The maximum in (6.2) *is attained for the map*

$$p^\star(t,x)=\begin{cases}\dfrac{H(b_0^\star)}{T}-\dfrac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,a_0^\star,b_0^\star))\right)^{\frac{\gamma}{1-\gamma}}}\displaystyle\int_x^{b_0^\star}v_1(y)dy, \text{ if } x\in[0,b_0^\star),\\ \tilde{p}^\star(t,x), \text{ if } x\in[b_0^\star,a_0^\star],\\ \dfrac{H(a_0^\star)}{T}+\dfrac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,a_0^\star,b_0^\star))\right)^{\frac{\gamma}{1-\gamma}}}\displaystyle\int_{a_0^\star}^{x}v_2(y)dy, \text{ if } x\in(a_0^\star,1],\end{cases}$$

where $\tilde{p}^\star(t,x)$ is any continuous and non-decreasing map (with respect to x) such that

$$\int_0^T\tilde{p}^\star(t,b_0^\star)dt=H(b_0^\star),\ \int_0^T\tilde{p}^\star(t,a_0^\star)dt=H(a_0^\star),\ \int_0^T\tilde{p}^\star(t,x)dt<H(x),\ \text{for all } x\in(b_0^\star,a_0^\star).$$

(ii) Define p, for any $(t,c)\in[0,T]\times\mathbb{R}_+$, by

$$p(t,c):=\sup_{x\in[0,1]}\left\{g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma}-p^\star(t,x)\right\}.$$

If $p^\star$ is $u-$convex on $X^\star(p^\star)$, then p is the optimal tariff for the problem (6.1)*. Furthermore, the Principal only signs contracts with the Agents of type $x\in[0,b_0^\star]\cup[a_0^\star,1]$.*

6.1.2 Power type cost function

Exactly as in the case where H was independent of x, the computations become much simpler as soon as Assumption 5.3 holds. Let's note $R_\gamma(a_0,b_0)=1+(2b_0)^{\frac{2-\gamma}{1-\gamma}}-((2a_0-1)^+)^{\frac{2-\gamma}{1-\gamma}}$ if $\gamma\in(0,1)$ and $R_\gamma(a_0,b_0)=1-((1-2b_0)^+)^{\frac{2-\gamma}{1-\gamma}}+(2-2a_0)^{\frac{2-\gamma}{1-\gamma}}$ if $\gamma<0$. Then, the functions ℓ and A are given, for any $(t,a_0,b_0)\in[0,T]\times\mathcal{A}_2'$, by

$$\ell_\gamma(a_0,b_0)=\frac{1-\gamma}{2(2-\gamma)}R_\gamma(a_0,b_0),\ A(t,a_0,b_0)=\left(\frac{\phi(t)}{k(t)}\right)^{\frac{1}{n-\gamma}}\ell(a_0,b_0)^{\frac{1-\gamma}{n-\gamma}}.$$

So, in order to obtain $(a_0^\star,b_0^\star)$ we have to solve

$$\sup_{(a_0,b_0)\in\mathcal{A}_2'}\left(\frac{1}{\gamma}-\frac{1}{n}\right)\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}dt\ \ell(a_0,b_0)^{\frac{n(1-\gamma)}{n-\gamma}}-b_0H(b_0)+(a_0-1)H(a_0). \tag{6.8}$$

Let us now compute the associated tariff p and check that p indeed belongs to $\mathcal{P}$ and that its $u-$transform is $p^\star$. Fix some $t \in [0, T]$ and define

$$N_\gamma := \frac{2^{\frac{\gamma}{1-\gamma}}(1-\gamma)}{\gamma}\left(\frac{2(2-\gamma)}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}}\left(\frac{\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}} R_\gamma(a_0, b_0)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

Recall that by Proposition 6.6, the following inequalities must be satisfied

(i) If $\gamma \in (0, 1)$

$$\frac{((2a_0^\star-1)^+)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star, b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}dt \geq H'(a_0^\star),\ \frac{(2b_0^\star)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star, b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}dt \leq H'(b_0^\star). \tag{6.9}$$

(ii) If $\gamma < 0$

$$-\frac{(2(1-a_0^\star)^+)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star, b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}dt \geq H'(a_0^\star),\ -\frac{((1-2b_0^\star)^+)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star, b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}dt \leq H'(b_0^\star). \tag{6.10}$$

Notice in particular that when $\gamma \in (0, 1)$, (6.9) implies that $a_0^\star > 1/2$, since H is increasing. With similar computations as in Section 5.2, we compute that

(i) If $\gamma \in (0, 1)$

$$p^\star(t, x) = \begin{cases} \dfrac{H(b_0^\star)}{T} - N_\gamma\left((b_0^\star)^{\frac{1}{1-\gamma}} - x^{\frac{1}{1-\gamma}}\right),\ \text{if } x \in [0, b_0^\star), \\ \tilde{p}^\star(t, x),\ \text{if } x \in [b_0^\star, a_0^\star], \\ \dfrac{H(a_0^\star)}{T} + N_\gamma\left(\left(x - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} - \left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}\right),\ \text{if } x \in (a_0^\star, 1], \end{cases}$$

(ii) If $\gamma < 0$

$$p^\star(t, x) = \begin{cases} \dfrac{H(b_0^\star)}{T} - N_\gamma\left(\left(\dfrac{1}{2} - b_0^\star \wedge \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} - \left(\dfrac{1}{2} - x \wedge \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}\right),\ \text{if } x \in [0, b_0^\star), \\ \tilde{p}^\star(t, x),\ \text{if } x \in [b_0^\star, a_0^\star], \\ \dfrac{H(a_0^\star)}{T} + N_\gamma\left((1-x)^{\frac{1}{1-\gamma}} - (1-a_0^\star)^{\frac{1}{1-\gamma}}\right),\ \text{if } x \in (a_0^\star, 1], \end{cases}$$

Actually, in this case, the map $p^\star$ will be $u-$convex if and only if the following implicit assumption holds.

Assumption 6.4. *The solutions $(a_0^\star, b_0^\star)$ of (6.8) are such that*

$$b_0^\star \leq a_0^\star - \frac{1}{2}.$$

Our main result in this case reads.

Theorem 6.3. *Let Assumptions 5.1, 5.3, 6.1, 6.2, 6.3 and 6.4 hold, then the optimal tariff $p \in \widehat{\mathcal{P}}$ is given for any $(t,c) \in [0,T] \times \mathbb{R}_+$, when $\gamma \in (0,1)$ by*

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} - \dfrac{H(a_0^\star)}{T},\ \text{if } L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} < c, \\ \tilde{x}^\star(c)\phi(t)\dfrac{c^\gamma}{\gamma} - \tilde{p}^\star(t,\tilde{x}(c)),\ \text{if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)L_\gamma(t)^{\gamma-1}c - \dfrac{H(b_0^\star)}{T} + N_\gamma(b_0^\star)^{\frac{1}{1-\gamma}},\ \text{if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}, \end{cases}$$

and when $\gamma < 0$ by

$$p(t,c) = \begin{cases} \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma(1-a_0^\star)^{\frac{1}{1-\gamma}} - \dfrac{H(a_0^\star)}{T},\ \text{if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\dfrac{c^\gamma}{\gamma} - \tilde{p}^\star(t,\tilde{x}(c)),\ \text{if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\dfrac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)\dfrac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(\dfrac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} - \dfrac{H(b_0^\star)}{T},\ \text{if } L_\gamma(t)\left(\dfrac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} < c. \end{cases}$$

where $L_\gamma(t) := \left(\frac{\gamma N_\gamma}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}$ and where $(a_0^\star, b_0^\star)$ are maximizers of

$$\sup_{(a_0,b_0)\in\mathcal{A}_2'} C(T)R_\gamma(a_0,b_0)^{\frac{n(1-\gamma)}{2-\gamma}} - b_0H(b_0) + (a_0-1)H(a_0).$$

Furthermore, the Principal will only choose clients with type $x \in [0, b_0^\star] \cup [a_0^\star, 1]$.

6.2 General reservation utility

In this section, we want to point out that the assumption of the reservation utility function H being strictly concave is not mandatory in order to solve problem (6.2), and we intend to explain in which other cases we can hope to solve it.

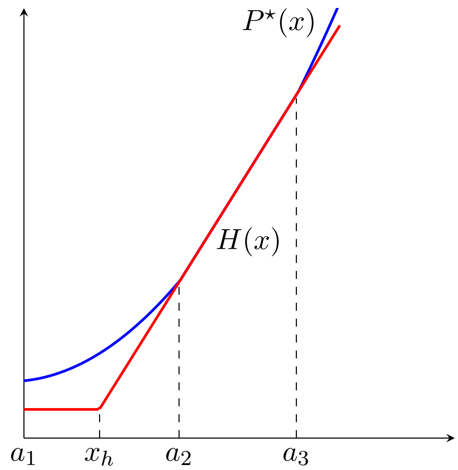
In order to reduce problem (6.2) to a finite dimensional problem, we just need H to have at most a finite number of intersecting points with a strictly convex function. If H were to satisfy this property, then we would be able to prove a result similar to Proposition 6.4, and we would know that the optimal set $\hat{X}^\star(p^\star)$ is a finite union of intervals contained in $[0,1]$.

The next step then is to prove that the Lagrange multipliers μ_t in Theorem 6.1 are equal to zero, using for instance local perturbations as we did to prove Proposition 6.5. This would allow to solve the optimality conditions (6.6) and (6.7) by using the corresponding auxiliary map $A(a,b)$. We illustrate this in the following example.

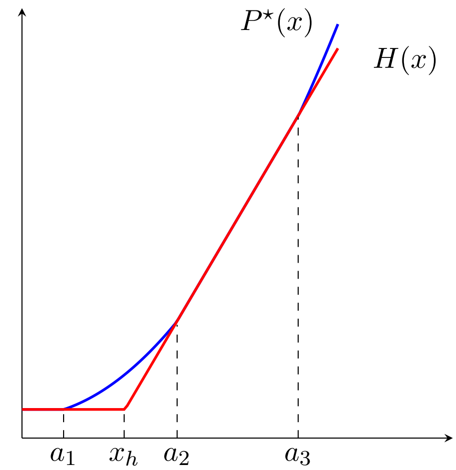
Example 6.1. *Suppose that for some $\alpha, \beta \geq 0$ and $x_h \in [0,1]$, H has the form*

$$H(x) = \begin{cases} \beta,\ \text{if } x \in [0, x_h], \\ \alpha(x - x_h) + \beta,\ \text{if } x \in [x_h, 1]. \end{cases}$$

Such a reservation utility accounts for the fact that all the Agents, whatever their appetence for power consumption is, should at least receive a minimal level of utility, in this case β. Though in general two



(a) Case $a_1 = 0$.



(b) Case $a_1 > 0$.

Figure 9: $X^\star(p^\star)$ for a "constant-linear" H.

convex functions can intersect at countably many points, given the specific form of H, it can intersect an increasing and convex function at at most three points, as shown in Figure 9.

Therefore, we deduce that $X^\star(p^\star)$ has the following form

$$X^\star(p^\star) = [a_1, a_2] \cup [a_3, 1], \text{ for some } 0 \leq a_1 \leq a_2 \leq a_3 \leq 1.$$

We define then the set

$$\mathcal{A}_3 := \big\{(a, b, c) \in [0, 1]^2,\ a \leq b \leq c\big\}.$$

After proving that the Lagrange multipliers μ_t in Theorem 6.1 are equal to zero, Problem $(P_{a,b})$ becomes, abusing notations slightly

$$\sup_{(a_1,a_2,a_3)\in\mathcal{A}_3} \int_0^T \left[\frac{\phi(t)^{\frac{1}{1-\gamma}}\ell(a_1,a_2,a_3)}{\gamma\left(\frac{\partial K}{\partial c}(t, A(t,a_1,a_2,a_3))\right)^{\frac{\gamma}{1-\gamma}}} - K\left(t, \frac{\phi(t)^{\frac{1}{1-\gamma}}\ell(a_1,a_2,a_3)}{\left(\frac{\partial K}{\partial c}(t, A(t,a_1,a_2,a_3)\right)^{\frac{1}{1-\gamma}}}\right)\right] dt + \theta(a_1,a_2,a_3),$$

where for any $(t, a_1, a_2, a_3) \in [0, T] \times \mathcal{A}_3$

$$\ell(a_1,a_2,a_3) := \int_{a_1}^{a_2}\left(\frac{[g_\gamma(x)f(x)+g'_\gamma(x)F(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx + \int_{a_3}^{1}\left(\frac{[g_\gamma(x)f(x)+g'_\gamma(x)(F(x)-1)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx,$$
$$\theta(a_1,a_2,a_3) := F(a_1)H(a_1) - F(a_2)H(a_2) + (F(a_3)-1)H(a_3),$$
$$A(t,a_1,a_2,a_3) := g_K^{(-1)}\bigg(\phi(t)^{\frac{1}{1-\gamma}}\int_{a_1}^{a_2}\left(\frac{[g_\gamma(x)f(x)+g'_\gamma(x)F(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx$$
$$+\ \phi(t)^{\frac{1}{1-\gamma}}\int_{a_3}^{1}\left(\frac{\big[g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\big]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx\bigg).$$

7 Conclusion

In this paper, we provide an explicit formulation of optimal electricity tariffs, from the point of view of a power company offering her service to consumers facing alternatives for satisfying energy requirements. It covers the adverse selection feature of Agents, which has been empirically highlighted in the literature [26], as electricity consumers tend to select time-dependent tariff whenever they consume little during peak hours. From this perspective, our results differ from the existing analysis in the literature on electricity pricing, which mainly focused on the point of view of a monopoly needing to recover its costs, such as in the Ramsey-Boiteux tarification described in [4] or in [27]. Our main results is the derivation of time-dependent and consumption-based optimal tariffs. They act in several ways in order to reflect the increasing marginal cost of electricity production with the aggregate consumption, by

- making electricity more expensive in peak hours;
- increasing the marginal price of high consumption power units;
- excluding effectively some of the consumers, by offering a tariff, which is too expensive in comparison to turning to alternative retailers or sources of energy, or simply to dropping out of business altogether.

Our tariff structure can compare to common electricity tariffs which are often linear functions of the consumed energy (see for example the Australia Queensland tariffs[2]). Indeed, our tariff is either linear or concave, in which case it could be approximated for example by a set of non-crossing linear functions of consumed power, these linear functions having a maximum power and a fixed charge increasing relative to this power limit. Interestingly, this kind of tariff structure looks quite similar to the regulated electricity tariff menu offered to residential consumers in France[3]. The main distinction is the concavity of our optimal tariff with respect to the power limit, whereas the French regulated tariff is convex. This difference may be induced by our limitation to offering a single tariff instead of a menu of contracts. By doing so, our tariff intends to address all consumers in a single function. As a future research project, we are planing to tackle the extension of our modeling to a menu of tariff, allowing more flexibility for handling adverse selection.

This paper is but the first step in the direction we have outlined. In addition to extend the analysis to a menu of tariffs, at this stage, there is no uncertainty in our model, and Agents commit for a definite period $[0, T]$. At a later stage, we plan to incorporate uncertainties in the model: there are fluctuations in the prices of primary energies and changes in uses of electricity, and domestic consumption is subject to weather conditions. Expanding the characteristics of Agents in order to model their ability to modulate their consumption is a possible extension; see for example [22] for an analysis of rational consumers who react imperfectly to prices or [30] for a model with a portion of consumers who cannot modify their consumption. One can also consider more sophisticated but existing contracts, allowing for instance the producer to cut supply a certain number of days during the year. One should also take into account the existence of a spot market for electricity, which creates arbitrage opportunities for the Agents. Most tantalizing are the congestion problems: peak hours are not seasonal, they result from aggregate behavior, which is strategic in nature, and call for an analysis in terms of mean field games.

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[2] Available at `https://www.dews.qld.gov.au/electricity/prices/tariffs`

[3] For a description see `https://www.kelwatt.fr/prix/tarif-bleu-edf#resume`

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A $u-$convex analysis

We first recall the definition of $u-$convexity (adapted to our context, we refer the reader to the monograph by Villani [34] on optimal transport theory for more details).

Definition A.1. *Let ψ be a map from $[0,T]\times X$ to $\mathbb{R}$. The $u-$transform of ψ, denoted by $\psi^\star:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}\cup\{+\infty\}$ is defined by*

$$\psi^\star(t,c):=\sup_{x\in X}\left\{u(t,x,c)-\psi(t,x)\right\},\ \textit{for any } (t,c)\in[0,T]\times\mathcal{C}.$$

Similarly, if φ is a map from $[0,T]\times\mathcal{C}$ to $\mathbb{R}$, its $u-$transform, still denoted by $\varphi^\star:[0,T]\times X\longrightarrow\mathbb{R}\cup\{+\infty\}$, is defined by

$$\varphi^\star(t,x):=\sup_{c\in\mathcal{C}}\left\{u(t,x,c)-\varphi(t,c)\right\},\ \textit{for any } (t,x)\in[0,T]\times X.$$

A map $\phi:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}\cup\{+\infty\}$ is then said to be $u-$convex if it is proper[4] *and if there exists some $\psi:[0,T]\times X\longrightarrow\mathbb{R}$ such that*

$$\phi(t,c)=\psi^\star(t,c),\ \textit{for any } (t,c)\in[0,T]\times\mathcal{C}.$$

Similarly, a map $\Phi:[0,T]\times X\longrightarrow\mathbb{R}\cup\{+\infty\}$ is said to be $u-$convex if it is proper and there exists some $\Psi:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}$ such that

$$\Phi(t,x)=\Psi^\star(t,x),\ \textit{for any } (t,x)\in[0,T]\times X.$$

We recall the following easy characterization of $u-$convexity.

Lemma A.1. *A map $\phi:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}\cup\{+\infty\}$ is $u-$convex if and only if*

$$\phi(t,c)=(\phi^\star)^\star(t,c),\ \textit{for any } (t,c)\in[0,T]\times\mathcal{C}.$$

A similar statement holds for maps $\Phi:[0,T]\times X\longrightarrow\mathbb{R}\cup\{+\infty\}$.

Proof. We only prove the first statement, the other one being exactly similar. The result is an easy consequence of the fact that for any map $\phi:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}\cup\{+\infty\}$, we have the identity

$$\phi^\star(t,x)=((\phi^\star)^\star)^\star(t,x),\ \text{for any } (t,x)\in[0,T]\times X.$$

Indeed, we have by definition that for any $(t,x)\in[0,T]\times X$

$$\begin{aligned}((\phi^\star)^\star)^\star(t,x)&=\sup_{c\in\mathcal{C}}\left\{u(t,x,c)-\sup_{x'\in X}\left\{u(t,x',c)-\sup_{c'\in\mathcal{C}}\left\{u(t,x',c')-\phi(t,c'))\right\}\right\}\right\}\\&=\sup_{c\in\mathcal{C}}\inf_{x'\in X}\sup_{c'\in\mathcal{C}}\left\{u(t,x,c)-u(t,x',c)+u(t,x',c')-\phi(t,c')\right\}.\end{aligned}$$

Choosing $x'=x$, we immediately get that $((\phi^\star)^\star)^\star(t,x)\leq\phi^\star(t,x)$, while the converse inequality is obtained by choosing $c=c'$. □

Next, we can define the notion of the $u-$subdifferential of a $u-$convex function.

[4] That is to say not identically equal to $+\infty$.

Definition A.2. *Let $\phi : [0,T]\times\mathcal{C} \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a $u-$convex function. For any $(t,c)\in[0,T]\times\mathcal{C}$, the $u-$subdifferential of ϕ at the point (t,c) is the set $\partial^\star\phi(t,c)\subset X$ defined by*

$$\partial^\star\phi(t,c) := \{x\in X,\ \phi^\star(t,x) = u(t,x,c)-\phi(t,c)\}\,.$$

Similarly, let $\psi : [0,T]\times X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a $u-$convex function. For any $(t,x)\in[0,T]\times X$, the $u-$subdifferential of ψ at the point (t,x) is the set $\partial^\star\psi(t,x)\subset\mathbb{R}_+$ defined by

$$\partial^\star\psi(t,x) := \{c\in\mathcal{C},\ \psi^\star(t,c) = u(t,x,c)-\psi(t,x)\}\,.$$

Notice that since the map u is continuous, a $u-$convex function is automatically lower-semicontinuous and its $u-$subdifferential is a closed set.

B Proofs of Section 5

Proof of Lemma 5.1. By Lemma A.1, we know that the $u-$convexity of ψ is equivalent to

$$\psi(t,x) = \sup_{c>0}\min_{y\in[0,1]}\left\{(g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y)\right\}. \tag{B.1}$$

First notice that since ψ is convex in y and $\frac{g_\gamma}{\gamma}$ is concave, then for any $(t,x,c)\in[0,T]\times[0,1]\times(0,+\infty)$, the map

$$f_{(t,x)}(y,c) := (g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y),$$

is convex in y. Furthermore, for any $(t,x,y)\in[0,T]\times[0,1]^2$, the map $c\longmapsto f_{(t,x)}(y,c)$ is monotone on $(0,+\infty)$ and therefore quasiconcave. Since $[0,1]$ is convex and compact, we can apply Sion's minimax theorem [31] to obtain that

$$\begin{aligned}\sup_{c>0}\min_{y\in[0,1]}\left\{(g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y)\right\} &= \min_{y\in[0,1]}\sup_{c>0}\left\{(g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y)\right\}\\ &= \min_{y\in[0,1]}\left\{+\infty\mathbf{1}_{x>y}+\psi(t,y)\right\}\\ &= \psi(t,x),\end{aligned}$$

since ψ is non-decreasing.

Finally, it is easy to see that when g_γ is defined as in the statement of the lemma, we have

$$\sup_{c>0}\min_{y\in[0,1]}\left\{(g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y)\right\} = \sup_{c>0}\min_{y\in[0,1]}\left\{(x-y)c+\psi(t,y)\right\},$$

which corresponds to the classical convex conjugate, hence the desired result by Fenchel-Moreau's theorem. □

Proof of Theorem 5.1. We optimize first with respect to $p^\star$. For fixed x_0, start by defining $\Psi_{x_0} : L^1([0,T]\times[0,1])\longrightarrow\mathbb{R}$ by

$$\begin{aligned}\Psi_{x_0}(p^\star) := \int_0^T\Bigg[&\int_{x_0}^1\frac{\left(g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\right)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)dx\\ &-K\left(t,\int_{x_0}^1\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}}f(x)dx\right)\Bigg]dt+(F(x_0)-1)H.\end{aligned} \tag{B.2}$$

Ψ_{x_0} is clearly continuous and Fréchet differentiable and it is also concave because K is convex in c. Furthermore, for any $q \in L^1([0,T]\times[0,1])$, we have

$$\begin{aligned}\Psi'_{x_0}(p^\star;q) = & \int_0^T \int_{x_0}^1 \frac{\left(g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\right)}{g'_\gamma(x)}\frac{\partial q}{\partial x}(t,x)dxdt \\ & -\int_0^T \int_{x_0}^1 \frac{\partial q}{\partial x}(t,x)\frac{1}{\gamma}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1-\gamma}{\gamma}}\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}} f(x)\frac{\partial K}{\partial c}\left(t,A(t,x_0)\right)dxdt,\end{aligned}$$

where we defined

$$A(t,x_0) := \int_{x_0}^1 \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx.$$

Since Ψ_{x_0} is concave, the necessary and sufficient optimality condition for the problem with fixed x_0 is

$$\Psi'_{x_0}(p^\star;q) \le 0,\ \forall q \in T_{C^+(x_0)}(p^\star), \tag{B.3}$$

where $T_{C^+(x_0)}(p^\star)$ denotes the tangent cone to the closed set $C^+(x_0)$ at the point $p^\star$ defined by

$$T_{C^+(x_0)}(p^\star) := \left\{z,\ \exists\varepsilon>0, \forall h\in[0,\varepsilon]\ \exists w(h)\in C^+(x_0),\ ||p^\star+hz-w(h)|| = \circ(h)\right\}.$$

Using local functions we see that the inequality (B.3) must be satisfied almost everywhere on $[0,T]\times[x_0,1]$, that is, for every $q \in T_{C^+(x_0)}(p^\star)$ and almost every $(t,x)\in[0,T]\times[x_0,1]$ we have

$$\frac{\partial q}{\partial x}(t,x)\left[\frac{(g_\gamma(x)f(x)+g'_\gamma(x)(F(x)-1))}{g'_\gamma(x)} - \frac{1}{\gamma}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1-\gamma}{\gamma}}\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}} f(x)\frac{\partial K}{\partial c}(t,A(t,x_0))\right] \le 0.$$

Therefore, the optimal $p^\star \in C^+(x_0)$ should verify

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}}\left[g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\right]^+}{f(x)\dfrac{\partial K}{\partial c}\left(t,A(t,x_0)\right)}\right)^{\frac{\gamma}{1-\gamma}}\frac{g'_\gamma(x)}{\gamma}, \tag{B.4}$$

with $a^+ = \max\{a,0\}$, the positive part operator. By the above equation, we must have

$$A(t,x_0)\left(\frac{\partial K}{\partial c}\left(t,A(t,x_0)\right)\right)^{\frac{1}{1-\gamma}} = \phi^{\frac{1}{1-\gamma}}(t)\int_{x_0}^1\left(\frac{\left[g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\right]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx.$$

Now, let

$$g_K(c) := c\left(\frac{\partial K}{\partial c}(t,c)\right)^{\frac{1}{1-\gamma}},\ c\ge 0.$$

Since K is strictly convex and increasing with respect to c, it can be checked directly that g_K is increasing as well (on $\mathbb{R}_+$), so that we deduce

$$A(t,x_0) = g_K^{(-1)}\left(\phi^{\frac{1}{1-\gamma}}(t)\int_{x_0}^1\left(\frac{\left[g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\right]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}dx\right).$$

Therefore the solution to the problem with fixed x_0 is given by (B.4). We have thus reduced the problem (5.4) to

$$\widetilde{U}_P = \sup_{x_0\in[0,1]}\int_0^T\left(\frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{\gamma}{1-\gamma}}} - K\left(t,\frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{1}{1-\gamma}}}\right)\right)dt + (F(x_0)-1)H.$$

Seen as a function of x_0, the right-hand side above is clearly a continuous function. It therefore attains its maximum over the compact set $[0,1]$ at some (possibly non unique) $x_0^\star$. We will abuse notations and denote by $x_0^\star$ a generic maximizer.

If $p^\star$ is $u-$convex, since p is also $u-$convex (by definition), then $p^\star$ is necessarily the $u-$transform of p and therefore $p \in \mathcal{P}$, which means that we actually have $\widetilde{U}_P = U_P$. For the uniqueness result, define

$$\alpha(x_0) := \int_0^T \left(\frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\gamma\left(\frac{\partial K}{\partial c}(t, A(t,x_0))\right)^{\frac{\gamma}{1-\gamma}}} - K\left(t, \frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\left(\frac{\partial K}{\partial c}(t, A(t,x_0))\right)^{\frac{1}{1-\gamma}}}\right)\right) dt + (F(x_0)-1)H.$$

Note that α does not attain its maximum over any interval outside L, because there ℓ is constant (and therefore $A(t,\cdot)$ too) and F is increasing. Then, since over L we have

$$\begin{aligned}
\alpha'(x_0) &= \int_0^T \left(\frac{1-\gamma}{\gamma}\right) \phi(t)^{\frac{1}{1-\gamma}} \ell'(x_0) \left(\frac{\partial K}{\partial c}(t, A(t,x_0))\right)^{\frac{-\gamma}{1-\gamma}} dt + f(x_0)H \\
&= -\int_0^T \left(\frac{1-\gamma}{\gamma}\right) \phi(t)^{\frac{1}{1-\gamma}} \beta(x_0) \left(\frac{\partial K}{\partial c}(t, A(t,x_0))\right)^{\frac{-\gamma}{1-\gamma}} dt + f(x_0)H.
\end{aligned}$$

Under the hypotheses of the theorem, α' is decreasing over L in each one of the two cases so α is strictly concave. □

Proof of Theorem 5.2. We divide the proof in two cases.

- **Case 1:** $\gamma \in (0,1)$

In this case we have

$$\ell(x_0) = \int_{x_0 \vee \frac{1}{2}}^1 (2x-1)^{\frac{1}{1-\gamma}}\, dx = \frac{1-\gamma}{2(2-\gamma)} \left(1 - \left((2x_0-1)^+\right)^{\frac{2-\gamma}{1-\gamma}}\right).$$

Hence, it is clear that $x_0 \longmapsto \Phi(x_0)$ is increasing in $[0, \frac{1}{2}]$, so that it suffices to solve

$$\sup_{x_0 \in [1/2, 1]} \left\{ B_\gamma(T) \ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0 - 1)H \right\}.$$

Let

$$y_0 := (2x_0 - 1)^{\frac{1}{1-\gamma}}, \; A_\gamma(T) := B_\gamma(T) \left(\frac{1-\gamma}{2(2-\gamma)}\right)^{\frac{n(1-\gamma)}{n-\gamma}}.$$

Defining the map $\overline{\Phi} : [0,1] \longrightarrow \mathbb{R}$ by

$$\overline{\Phi}(y_0) := \Phi\left(\frac{y_0^{1-\gamma}+1}{2}\right),$$

we deduce

$$\overline{\Phi}(y_0) = A_\gamma(T)\left(1 - y_0^{2-\gamma}\right)^{\frac{n(1-\gamma)}{n-\gamma}} + \frac{1}{2}\left(y_0^{1-\gamma} - 1\right) H.$$

Next, we can check directly that $\overline{\Phi}$ is concave on $[0,1]$, and we have for any $y_0 \in [0,1]$

$$\overline{\Phi}'(y_0) = \frac{1-\gamma}{2} y_0^{-\gamma} \left(H - 2nA_\gamma(T)\frac{2-\gamma}{n-\gamma} y_0 \left(1 - y_0^{2-\gamma}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}\right).$$

Denote finally for any $y_0 \in [0,1]$

$$\chi(y_0) := H - 2nA_\gamma(T)\frac{2-\gamma}{n-\gamma}(2x_0-1)^{\frac{1}{1-\gamma}}\left(1-y_0^{2-\gamma}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

We have for any $y_0 \in [0,1]$

$$\chi'(y_0) = -\frac{2n(2-\gamma)}{(n-\gamma)^2}A_\gamma(T)\left(1-y_0^{2-\gamma}\right)^{-\frac{n+\gamma(n-2)}{n-\gamma}}\left(n-\gamma+\gamma(n-1)(2-\gamma)y_0^{2-\gamma}\right) < 0,$$

since $\gamma \in (0,1)$. Thus, since in addition we have

$$\chi(0) = H > 0, \text{ and } \lim_{y_0\uparrow 1}\chi(y_0) = -\infty,$$

there is a unique $y_0^\star \in (0,1)$ (and thus a unique $x_0^\star \in (1/2,1)$) such that $\overline{\Phi}'(y_0^\star) = 0$, at which the maximum of $\overline{\Phi}$ is attained. Finally, we can compute explicitly $p^\star(t,x)$ for any $(t,x) \in [0,T]\times[0,1]$ as

$$p^\star(t,x) = \frac{H}{T} + M(t)\left(((2x-1)^+)^{\frac{1}{1-\gamma}} - (2x_0^\star-1)^{\frac{1}{1-\gamma}}\right),$$

where we defined for simplicity

$$M(t) := \frac{1-\gamma}{2\gamma}\left(\frac{2(2-\gamma)}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}}\left(\frac{\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}}\left(1-(2x_0^\star-1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

It can then be checked directly that for any $c \geq 0$, the map $x \longmapsto x\phi(t)c^\gamma/\gamma - p^\star(t,x)$ is concave on $[0,1]$ and attains its maximum at the point

$$x^\star(c) := \mathbf{1}_{c>\left(\frac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}} + \frac{1}{2}\left(1+\left(\frac{(1-\gamma)\phi(t)}{2\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}}c^{1-\gamma}\right)\mathbf{1}_{c\leq\left(\frac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}}.$$

Therefore, we deduce immediately that for any $(t,c) \in [0,T]\times\mathbb{R}_+$

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{\gamma} + M(t)\left((2x_0^\star-1)^{\frac{1}{1-\gamma}} - 1\right) - \dfrac{H}{T}, \text{ if } c > \left(\dfrac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}, \\ \phi(t)\dfrac{c^\gamma}{2\gamma} + \left(\left(\dfrac{\phi(t)}{2}\right)^{\frac{1}{1-\gamma}}\dfrac{1-\gamma}{\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}}c - \dfrac{H}{T} + M(t)(2x_0^\star-1)^{\frac{1}{1-\gamma}}, \text{ otherwise.} \end{cases}$$

Next, we notice that for any $(t,x) \in [0,T]\times[0,1]$, the map $c \longmapsto x\phi(t)c^\gamma/\gamma - p(t,c)$ is decreasing on $\mathbb{R}_+$ if $x < 1/2$, and that it is concave on $\mathbb{R}_+$ if $x \geq 1/2$, so that it attains its maximum at the point

$$c^\star(t,x) := \left(\frac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}(2x-1)^{\frac{1}{1-\gamma}}\mathbf{1}_{x\in(1/2,1]}.$$

It is also immediate that $p^\star$ is always non-decreasing and is convex, and therefore $u-$convex by Lemma 5.1, so much so that we conclude that $p \in \mathcal{P}$.

It can easily be shown that the following suboptimal but simpler tariff will give the same results in terms of selected Agents, optimal consumption and Principal's utility. Indeed as no consumer select

$c > \left(\frac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}$, to replace the relative tariff section by a higher function which means a more expensive contract does not modify the results. For any $(t,c) \in [0,T] \times \mathbb{R}_+$, the following tarif is also admissible

$$p(t,c) = \phi(t)\frac{c^\gamma}{2\gamma} + \left(\left(\frac{\phi(t)}{2}\right)^{\frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}} c - \frac{H}{T} + M(t)(2x_0^\star - 1)^{\frac{1}{1-\gamma}}$$

- **Case 2:** $\gamma \in (-\infty, 0)$

Now, we actually have

$$\ell(x_0) = 2^{\frac{1}{1-\gamma}} \int_{x_0}^1 (1-x)^{\frac{1}{1-\gamma}} dx = 2^{\frac{1}{1-\gamma}} \left(\frac{1-\gamma}{2-\gamma}\right)(1-x_0)^{\frac{2-\gamma}{1-\gamma}}.$$

The problem to solve is now

$$\sup_{x_0 \in [0,1]} \left\{ B_\gamma(T)\ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0 - 1)H \right\}.$$

It can be checked directly that the above map is actually strictly concave for $x_0 \in [0,1]$, and therefore that it attains its maximum at

$$\widehat{x}_0^\star := \left(1 - \left(\frac{H}{B_\gamma(T)} \frac{n-\gamma}{n(1-\gamma)}\right)^{\frac{n-\gamma}{n(1-\gamma)+\gamma}} \left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{-\gamma(n-1)}{n(1-\gamma)+\gamma}} 2^{\frac{-n}{n(1-\gamma)+\gamma}}\right)^+.$$

Finally, we can compute explicitly $p^\star(t,x)$ for any $(t,x) \in [0,T] \times [0,1]$ as

$$p^\star(t,x) = \frac{H}{T} + \widehat{M}(t)\left((1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}} - (1-x)^{\frac{1}{1-\gamma}}\right),$$

where we defined for simplicity

$$\widehat{M}(t) := -\frac{1-\gamma}{\gamma}\left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}} \left(\frac{2^\gamma \phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}} (1-\widehat{x}_0^\star)^{-\frac{\gamma(2-\gamma)(n-1)}{(n-\gamma)(1-\gamma)}}.$$

We deduce directly that in this case the map $x \longmapsto (1-x)\phi(t)c^\gamma/\gamma - p^\star(t,x)$ is concave, so that it attains its maximum on $[0,1]$ at

$$\widehat{x}^\star(c) := \left(1 - \left(-\frac{\phi(t)(1-\gamma)}{\gamma \widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} c^{1-\gamma}\right)^+,$$

so that

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{\gamma} - \dfrac{H}{T} - \widehat{M}(t)(1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}} + \widehat{M}(t), \text{ if } c > \left(-\dfrac{\gamma \widehat{M}(t)}{\phi(t)(1-\gamma)}\right)^{\frac{1}{\gamma}}, \\ -\gamma c\left(-\dfrac{\phi(t)}{\gamma}\right)^{\frac{1}{\gamma}} \left(\dfrac{1-\gamma}{\widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} - \dfrac{H}{T} - \widehat{M}(t)(1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}}, \text{ otherwise.} \end{cases}$$

It is also immediate in this case that $p^\star$ is always non-decreasing and is convex, and therefore $u-$convex by Lemma 5.1, so much so that we conclude that $p \in \mathcal{P}$.

It can easily be shown that the following suboptimal but simpler tariff will give the same results in terms of selected Agents, optimal consumption and Principal's utility because no consumer selects $c > \left(-\frac{\gamma\widehat{M}(t)}{\phi(t)(1-\gamma)}\right)^{\frac{1}{\gamma}}$. For any $(t,c) \in [0,T] \times \mathbb{R}_+$, the following tariff is also admissible

$$p(t,c) = -\gamma c\left(-\frac{\phi(t)}{\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{1-\gamma}{\widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} - \frac{H}{T} - \widehat{M}(t)(1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}}$$

$\square$

C Proofs of Section 6

C.1 Technical results

Proof of Proposition 6.1. Define the following functionals on $\widehat{C}^+$

$$\mathfrak{K}(p) := \int_0^T K\left(t, \int_{X^\star(p)}\left(\frac{\gamma}{\phi(t)g'(x)}\frac{\partial p}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\right)dt,$$
$$\mathfrak{J}(p) := \int_0^T \left[\int_{X^\star(p)}\left(\frac{g(x)}{g'(x)}\frac{\partial p}{\partial x}(t,x) - p(t,x)\right) f(x)dx\right] dt.$$

Proposition 6.1 states that if $Y^\star(p^\star)$ has positive measure, then $p^\star$ is not a maximizer of $p \longmapsto \mathfrak{J}(p) - \mathfrak{K}(p)$ over the set $\widehat{C}^+$. Indeed, in this case we can find an interval $[c,d] \subset Y^\star(p^\star)$ (remember that this is an open set with positive Lebesgue measure and that the latter is regular) and thus

$$\int_0^T p^\star(t,x)dt = H(x), \text{ for every } x \in [c,d].$$

Next, define

$$T^+ = \{t \in [0,T] : p^\star(t,c) < p^\star(t,d)\}.$$

Since H is strictly increasing we have that T^+ has positive Lebesgue measure. For every $t \in T^+$, define over $[c,d]$ a continuous and increasing function q satisfying $q(t,c) := p^\star(t,c)$, $q(t,d) := p^\star(t,d)$ and $q(t,x) < p^\star(t,x)$ over (c,d). Consider the following modification of $p^\star$.

$$\hat{p}(t,x) := \begin{cases} q(t,x), \text{ if } (t,x) \in T^+ \times [c,d], \\ p^\star(t,x), \text{ if } (t,x) \notin T^+ \times [c,d]. \end{cases}$$

We have that $X^\star(\hat{p}) = X^\star(p^\star) \setminus (c,d)$ and therefore $\mathfrak{K}(\hat{p}) < \mathfrak{K}(p^\star)$. Moreover,

$$\begin{aligned}
\mathfrak{J}(\hat{p}) &= \mathfrak{J}(p^\star) - \int_0^T \left[\int_c^d \left(\frac{g(x)}{g'(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)dx\right] dt \\
&= \mathfrak{J}(p^\star) - \int_c^d \left(\frac{g(x)}{g'(x)}H'(x) - H(x)\right) f(x)dx \\
&> \mathfrak{J}(p^\star),
\end{aligned}$$

where we used Assumption 6.1. Since $\hat{p}$ is also non-decreasing in x, $\hat{p} \in \widehat{C}^+$, and we conclude that $p^\star$ is not optimal. $\square$

Proof of Proposition 6.2. Note that in the optimization problem $(P_{a,b})$ we can without loss of generality restrict our attention to the feasible maps on $[0,T] \times X^\star(a,b)$. In other words, for fixed $(a,b) \in \mathcal{A}$, we define the closed and convex set $F_{a,b}$ as the set of maps $q \in W_x^{1.m}(X^\star(a,b))$ such that for every $t \in [0,T]$, $x \longmapsto q(t,x)$ is continuous and non-decreasing, $Q(x) := \int_0^T q(t,x)dt \geq H(x)$ for every $x \in X^\star(a,b)$ and $Q(a_n) = H(a_n)$, $Q(b_n) = H(b_n)$ for every $n \geq 1$.

We show that $\Psi_{(a,b)}$, seen on the Banach space $\big(W_x^{1,m}(X^\star(a,b)), ||\cdot||_{m,X^\star(a,b)}\big)$, is coercive on $F_{a,b}$.

- **Case 1:** $\gamma \in (0,1)$

Observe first that if $(q_n)_{n\in\mathbb{N}} \subset F_{a,b}$ is such that $||q_n||_{L^m([0,T]\times X^\star(a,b))} \underset{n\to\infty}{\longrightarrow} \infty$, then since for every $n \in \mathbb{N}$ the map $x \longmapsto \int_0^T q_n(t,x)dt$ is bounded from below by H on $X^\star(a,b)$, we have that

$$-\int_0^T \int_{X^\star(a,b)} q_n(t,x)f(x)dxdt \underset{n\to\infty}{\longrightarrow} -\infty. \tag{C.1}$$

Next, define

$$A := \int_{X^\star(a,b)} f(x)\left(\frac{\gamma}{\phi(t)g'(x)}\right)^{\frac{1}{\gamma}} dx,\ B := \int_0^T k(t)dt.$$

From Jensen's inequality for the maps $\psi_A(x) = x^{\frac{1}{\gamma m}}$, $\psi_B(x) = x^{\frac{n}{\gamma m}}$, we have that (recall that m is such that $\gamma m < 1$ and that $n > 1$)

$$\begin{aligned}
&\int_0^T K\left(t, \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial q_n}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\right)dt\\
\geq &\int_0^T k(t)\left(\int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial q_n}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx\right)^n dt\\
\geq &A^{n(1-\frac{1}{\gamma m})}\int_0^T k(t)\left(\int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}}\left|\frac{\partial q_n}{\partial x}(t,x)\right|^m f(x)dx\right)^{\frac{n}{\gamma m}} dt\\
\geq &A^{n(1-\frac{1}{\gamma m})}B^{(1-\frac{n}{\gamma m})}\left(\int_0^T \int_{X^\star(a,b)} k(t)\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}}\left|\frac{\partial q_n}{\partial x}(t,x)\right|^m f(x)dxdt\right)^{\frac{n}{\gamma m}}\\
\geq &A^{n(1-\frac{1}{\gamma m})}B^{(1-\frac{n}{\gamma m})}I\left|\left|\frac{\partial q_n}{\partial x}\right|\right|^{\frac{n}{\gamma}}_{L^m([0,T]\times X^\star(a,b))}.
\end{aligned}$$

We have therefore proved that, denoting by m' the conjugate of m

$$\Psi_{(a,b)}(q_n) \leq \left|\left|\frac{g_\gamma}{g'_\gamma}\right|\right|_{L^{m'}([0,T]\times X^\star(a,b))}\left|\left|\frac{\partial q_n}{\partial x}\right|\right|_{L^m([0,T]\times X^\star(a,b))} - A^{n(1-\frac{1}{\gamma m})}B^{1-\frac{n}{\gamma m}}I\left|\left|\frac{\partial q_n}{\partial x}\right|\right|^{\frac{n}{\gamma}}_{L^m([0,T]\times X^\star(a,b))}.$$

This with (C.1) implies clearly that $\Psi_{(a,b)}$ is indeed coercive in $F_{a,b}$.

- **Case 2:** $\gamma < 0$

In this case $\frac{g_\gamma}{g'_\gamma} < 0$. Since $\frac{\partial q_n}{\partial x}$ is non-negative, if $||\frac{\partial q_n}{\partial x}||_{L^m([0,T]\times X^\star(a,b))} \longrightarrow \infty$ we have that

$$\int_0^T \int_{X^\star(a,b)} \frac{g_\gamma(x)}{g'_\gamma(x)}\frac{\partial q_n}{\partial x}(t,x)f(x)dx\ dt \longrightarrow -\infty.$$

Then, from (C.1) and the positiveness of K we conclude that $\Psi_{(a,b)}$ is coercive in $F_{a,b}$.

To conclude the proof, note that $W_x^{1,m}(X^\star(a,b))$ is a reflexive Banach space, so the coercivity of $\Psi_{(a,b)}$ implies that it possesses at least a maximizer $p^\star$ in $F_{a,b}$. Therefore any $q \in W_x^{1.m}(0,1)$, continuous and non-decreasing in x, which coincides with $p^\star$ in $X^\star(a,b)$ and such that $Q(x) = \int_0^T q(t,x)dt$ satisfies $Q(x) < H(x)$ for $x \in [0,1] \setminus X^\star(a,b)$ is a solution to $(P_{a,b})$. □

We state the following Lemma before proving Proposition 6.5.

Lemma C.1. *Let $p^\star$ be a solution of $(P_{a,b})$. Define*

$$T^{b_0} = \left\{ t \in [0,T] : \frac{\partial p^\star}{\partial x}(t,x) = 0, \forall x \in (0,b_0) \right\}, \ T^{a_0} = \left\{ t \in [0,T] : \frac{\partial p^\star}{\partial x}(t,x) = 0, \forall x \in (a_0,1) \right\}.$$

If T^{a_0} has positive Lebesgue measure, then for every $x \in (0,b_0)$, $g(x)f(x) + g'(x)F(x) \le 0$. If T^{b_0} has positive Lebesgue measure, then for every $x \in (a_0,1)$, $g(x)f(x) + g'(x)F(x) - g'(x) \le 0$.

Proof. We consider the case in which T^{a_0} has positive Lebesgue measure. Suppose there exist $[x_1,x_2] \subset [a_0,1]$ such that for every $x \in [x_1,x_2]$

$$g(x)f(x) + g'(x)F(x) - g'(x) > 0.$$

Then, for any $q \in W_x^{1,m}(0,1)$ satisfying $q(t,x) = 0, \forall (t,x) \notin T^{a_0} \times [x_1,1]$, $x \longmapsto q(t,x)$ is increasing in $[x_1,x_2]$, $\forall t \in T^{a_0}$, and $q(t,x) = q(t,x_2)$, $\forall (t,x) \in T^{a_0} \times [x_2,1]$, the map $p^\star + \varepsilon q$ belongs to $C^+(a,b)$ for $\varepsilon \ge 0$. Therefore $\Psi'_{(a,b)}(p^\star;q) \le 0$,, which means

$$\int_{T^{a_0}} \int_{x_1}^{x_2} \frac{\partial q}{\partial x}(t,x) \frac{f(x)g(x) + g'(x)F(x)}{g'(x)} \le 0,$$

hence a contradiction. □

Proof of Proposition 6.5. Let us prove the case $I \subset (a_0,1)$, the case $I \subset (0,b_0)$ being similar. From the convexity of $P^\star$ on every interval over which $P^\star$ is strictly greater than H we deduce the existence of $c_0 \in [a_0,1)$ such that $P^\star(x) > H(x)$ for every $x \in (c_0,1]$ and $P^\star(x) = H(x)$ for every $x \in [a_0,c_0]$. It follows from Lemma C.1 that either T^{a_0} is a null set or for every $t \in T^{a_0}$ the optimality conditions from Theorem 6.1 hold with $\mu_t = 0$. Call $T_1 = [0,T] \setminus (\mathcal{N} \cup T^{a_0})$ and define for every $t \in T_1$

$$x_1(t) = \inf \left\{ x \in (c_0,1) : \frac{\partial p^\star}{\partial x}(t,x) > 0 \right\}.$$

We have that $p^\star(t,\cdot)$ is strictly increasing in $[x_1(t),1]$ and it is given by (6.6). Define next

$$T_1^+ := \{t \in T_1, \ \mu_t > 0\}, \ T_1^- := \{t \in T_1, \ \mu_t < 0\}.$$

We will prove that T_1^- and T_1^+ have Lebesgue measure equal to zero. Consider any map $q \in W_x^{1,m}(0,1)$ satisfying

$$\begin{cases} q(t,x) = 0, \forall (t,x) \notin T_1^- \times (x_1(t),1], \\ x \longmapsto q(t,x) \text{ is increasing in } [x_1(t),1], \forall t \in T_1^-. \end{cases}$$

Then $p^\star + \varepsilon q \in C^+(a,b)$ for every $\varepsilon \ge 0$, so $\Psi'(p^\star;q) \le 0$. Since

$$\Psi'(p^\star;q) = \int_{T_1^-} \int_{x_1(t)}^1 -\frac{\partial q}{\partial x}(t,x)\mu_t dxdt,$$

we conclude that T_1^- is a null set. Next, take any $\bar{x} \in (c_0, 1)$ such that $P^\star(\bar{x}) > H(\bar{x})$ and for every $t \in T_1^+$ redefine if necessary the point $x_1(t)$ in order to satisfy $x_1(t) \geq \bar{x}$. Define then $q : [0, T] \times [0, 1] \longrightarrow \mathbb{R}$ by

$$q(t, x) := 1_{\{t \in T_1^+,\ x \geq x_1(t)\}} \frac{\frac{\partial p^\star}{\partial x}(t, x_1(t))(x - x_1(t)) + p^\star(t, x_1(t)) - p^\star(t, x)}{p^\star(t, 1) - p^\star(t, x_1(t)) - \frac{\partial p^\star}{\partial x}(t, x_1(t))(1 - x_1(t))} \Delta.$$

Since $p^\star(t, \cdot)$ is convex, we have that q is non-increasing, $p^\star(t, \cdot) + \varepsilon q(t, \cdot)$ is non-decreasing for $\varepsilon \sim 0$ and $p(t, 1) + q(t, 1) = p(t, 1) - \Delta$. Therefore $p^\star + \varepsilon q \in C^+(a, b)$ for $\varepsilon \sim 0$ so $\Psi'(p^\star; q) \leq 0$. Since

$$\Psi'(p^\star; q) = \int_{T_1^+} \int_{x_1(t)}^1 -\frac{\partial q}{\partial x}(t, x) \mu_t \, dx \, dt,$$

we conclude that the set T_1^+ has Lebesgue measure equal to zero. $\square$

Proof of Proposition 6.6. We show that under the conditions of the proposition, $P^\star \equiv H$ over some subset of $(0, a_0] \cup [b_0, 1)$ with positive Lebesgue measure and the result follows from Proposition 6.1. Suppose not, then for almost every $t \in [0, T]$ we have

$$\frac{\partial p^\star}{\partial x}(t, x) = \begin{cases} \left(\dfrac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) \right]^+}{f(x) \frac{\partial K}{\partial c}(t, A(t, a_0, b_0))} \right)^{\frac{\gamma}{1-\gamma}} \dfrac{g'_\gamma(x)}{\gamma}, \ x \in (0, b_0), \\ \left(\dfrac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x) \right]^+}{f(x) \frac{\partial K}{\partial c}(t, A(t, a_0, b_0))} \right)^{\frac{\gamma}{1-\gamma}} \dfrac{g'_\gamma(x)}{\gamma}, \ x \in (a_0, 1). \end{cases}$$

Thus either $\frac{\partial P^\star}{\partial x}(a_0) > H'(a_0)$ or $\frac{\partial P^\star}{\partial x}(b_0) < H'(b_0)$, which contradicts that $\hat{X}^\star(p^\star) = [0, b_0) \cup (a_0, 1]$. $\square$

Proof of Theorem 6.3. First of all, we recall that we have a degree of freedom in choosing the map $\tilde{p}$ to which p is equal on $[b_0^\star, a_0^\star]$, since it does not play any role in criterion that $p^\star$ maximises. Of course, if we want to be able to conclude, this map has to be $u-$convex in the end. Therefore, if we can choose it so that $p^\star$ is C^1 and convex in x, we can apply Lemma 5.1 and conclude that $p^\star$ is indeed $u-$convex. This can be made if and only if the derivative of $p^\star$ at $a_0^\star$ is greater or equal to the derivative of $p^\star$ at $b_0^\star$, which can be shown immediately to be equivalent to, regardless of the value of γ,

$$a_0^\star - \frac{1}{2} \geq b_0^\star.$$

Furthermore, if this is not satisfied, then $p^\star$ is not convex, and we can apply the second part of Lemma 5.1 to conclude that $p^\star$ is not $u-$convex.

We now divide the proof in two steps.

- **Case** (i): $\gamma \in (0, 1)$.

Given the discussion above, in this case the only thing we have to do is to compute p. Denote for simplicity

$$L_\gamma(t) := \left(\frac{\gamma N_\gamma}{(1 - \gamma)\phi(t)} \right)^{\frac{1}{\gamma}}.$$

We know that the map $x \longmapsto x\phi(t)c^\gamma/\gamma - p^\star(t, x)$ is concave on $[0, 1]$. Notice as well that since $a_0^\star > 1/2$, we have $1/2 \geq a_0^\star - 1/2 \geq b_0^\star$. We can then compute its maximum and obtain directly that it is attained at

$$x^\star(c) := \begin{cases} 1, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \dfrac{1}{2} + L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c), \text{ if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}, \end{cases}$$

where $\tilde{x}^\star(c)$ is any point in $[b_0^\star, a_0^\star]$ such that

$$\frac{\partial \tilde{p}^\star}{\partial x}(t, \tilde{x}^\star(c)) = \phi(t)\frac{c^\gamma}{\gamma}.$$

We deduce that

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{\gamma} - N_\gamma\left(2^{-\frac{1}{1-\gamma}} - \left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}\right) - \dfrac{H(a_0^\star)}{T}, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \phi(t)\dfrac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} - \dfrac{H(a_0^\star)}{T}, \text{ if } L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} < c \leq \dfrac{L_\gamma(t)}{2^{\frac{1}{1-\gamma}}}, \\ \tilde{x}^\star(c)\phi(t)\dfrac{c^\gamma}{\gamma} - \tilde{p}^\star(t, \tilde{x}(c)), \text{ if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)L_\gamma(t)^{\gamma-1}c - \frac{H(b_0^\star)}{T} + N_\gamma(b_0^\star)^{\frac{1}{1-\gamma}}, \text{ if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}. \end{cases}$$

As in the case H constant, it can easily be shown that the following simpler tariff is also admissible and produce the same results as no consumer selects $c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}$:

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} - \dfrac{H(a_0^\star)}{T}, \text{ if } L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} < c, \\ \tilde{x}^\star(c)\phi(t)\dfrac{c^\gamma}{\gamma} - \tilde{p}^\star(t, \tilde{x}(c)), \text{ if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)L_\gamma(t)^{\gamma-1}c - \frac{H(b_0^\star)}{T} + N_\gamma(b_0^\star)^{\frac{1}{1-\gamma}}, \text{ if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}. \end{cases}$$

• **Case** (ii): $\gamma < 0$. As in the previous case, our assumptions imply that $a_0^\star \geq 1/2$ and $1/2 \geq a_0^\star - 1/2 \geq b_0^\star$. We can then prove that the maximum of the map $x \longmapsto x\phi(t)c^\gamma/\gamma - p^\star(t,x)$ is attained at

$$x^\star(c) := \begin{cases} 1 - L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c), \text{ if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\dfrac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \dfrac{1}{2} - L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } L_\gamma(t)\left(\dfrac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ 0, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \end{cases}$$

where $\tilde{x}^\star(c)$ is any point in $[b_0^\star, a_0^\star]$ such that

$$\frac{\partial \tilde{p}^\star}{\partial x}(t, \tilde{x}^\star(c)) = \phi(t)\frac{c^\gamma}{\gamma}.$$

We deduce that

$$p(t,c) = \begin{cases} \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma(1-a_0^\star)^{\frac{1}{1-\gamma}} - \dfrac{H(a_0^\star)}{T}, \text{ if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\dfrac{c^\gamma}{\gamma} - \tilde{p}^\star(t,\tilde{x}(c)), \text{ if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\dfrac{1}{2}-b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)\dfrac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(\dfrac{1}{2}-b_0^\star\right)^{\frac{1}{1-\gamma}} - \dfrac{H(b_0^\star)}{T}, \text{ if } L_\gamma(t)\left(\dfrac{1}{2}-b_0^\star\right)^{\frac{1}{1-\gamma}} < c \leq \dfrac{L_\gamma(t)}{2^{\frac{1}{1-\gamma}}}, \\ \phi(t)\dfrac{c^\gamma}{\gamma} + N_\gamma\left(\left(\dfrac{1}{2}-b_0^\star\right)^{\frac{1}{1-\gamma}} - 2^{-\frac{1}{1-\gamma}}\right) - \dfrac{H(b_0^\star)}{T}, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}. \end{cases}$$

As previously, it can easily be shown that the following simpler tariff is also admissible and produce the same results as no consumer selects $c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}$:

$$p(t,c) = \begin{cases} \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma(1-a_0^\star)^{\frac{1}{1-\gamma}} - \dfrac{H(a_0^\star)}{T}, \text{ if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\dfrac{c^\gamma}{\gamma} - \tilde{p}^\star(t,\tilde{x}(c)), \text{ if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\dfrac{1}{2}-b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)\dfrac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(\dfrac{1}{2}-b_0^\star\right)^{\frac{1}{1-\gamma}} - \dfrac{H(b_0^\star)}{T}, \text{ if } L_\gamma(t)\left(\dfrac{1}{2}-b_0^\star\right)^{\frac{1}{1-\gamma}} < c. \end{cases}$$

$\square$

C.2 Proof of Theorem 6.1

We give here a series of result which once combined prove Theorem 6.1. To simplify the statements, we give them in a generic set (a_n, b_n), the generalization being straightforward. The first proposition shows that the existence of the interval I in the theorem allows us to localize Problem $(P_{a,b})$, and replace it by a simpler one, in which the constraint $P^\star(x) \geq H(x)$ for every $x \in X^\star(a,b)$ can be ignored.

Proposition C.1. *Let $p^\star$ be a solution of $(P_{a,b})$ and suppose there exists $x_1 \in (a_n, b_n)$ such that $P^\star(x_1) > H(x_1)$. Then, there exists $x_0 \in (a_n, b_n)$, $x_0 < x_1$, such that $p^\star$ is solution to the following problem*

$$(P_{x_0,x_1}) \qquad \sup_{q \in C(x_0,x_1)} \Psi_{(a,b)}^{x_0,x_1,p^\star}(q), \tag{C.2}$$

where

$$\begin{aligned} \Psi_{(a,b)}^{x_0,x_1,p^\star}(q) :=& \int_0^T \int_{x_0}^{x_1} \frac{g_\gamma(x)f(x) + g'_\gamma(x)F(x)}{g'_\gamma(x)} \frac{\partial q}{\partial x}(t,x)dxdt \\ &- \int_0^T K\left(t, \int_{x_0}^{x_1} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial q}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx + I_{(a,b)}^{x_0,x_1}(p^\star)\right)dt, \\ I_{(a,b)}^{x_0,x_1}(p^\star) :=& \int_{X^\star(a,b)\setminus(x_0,x_1)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)dx, \end{aligned}$$

and $C(x_0,x_1)$ denotes the set of maps $q \in W_x^{1,m}(x_0,x_1)$ such that

- *$x \longmapsto q(t,x)$ is continuous and increasing for every $t \in [0,T] \setminus \mathcal{N}(q)$.*

- $p^\star(t,x_0) + \int_{x_0}^{x_1} \frac{\partial q}{\partial x}(t,x)\ dx = p^\star(t,x_1)$ *for every* $t \in [0,T] \setminus \mathcal{N}(q)$.

Proof. Define

$$x_0 := \inf\left\{z \in X^\star(a,b),\ P^\star(x) \geq H(x_1) \text{ for every } x \in [z,x_1]\right\}.$$

By continuity we have that $x_0 < x_1$ and $P^\star(x_0) = H(x_1)$. Notice that the restriction of $p^\star$ to the set $[x_0,x_1]$ belongs to $C(x_0,x_1)$. Suppose the restriction is not a solution of (P_{x_0,x_1}), then there exists $q^\star \in C(x_0,x_1)$ such that $\Psi_{(a,b)}^{x_0,x_1,p^\star}(q^\star) > \Psi_{(a,b)}^{x_0,x_1,p^\star}(p^\star)$. Define then $\bar{p} : [0,T] \times [0,1] \longrightarrow \mathbb{R}$ by

$$\bar{p}(t,x) := \begin{cases} p^\star(t,x),\ x \notin [x_0,x_1], \\ p^\star(t,x_0) + \displaystyle\int_{x_0}^{x} \frac{\partial q^\star}{\partial x}(t,x)dx,\ x \in (x_0,x_1). \end{cases}$$

Then, for every $x \in [x_0,x_1]$

$$\int_0^T \bar{p}(t,x)dt \geq \int_0^T \bar{p}(t,x_0)dt \geq H(x_1) \geq H(x),$$

and it is straightforward that $\bar{p} \in C^+(a,b)$. This is a contradiction with the optimality of $p^\star$ in problem $(P_{a,b})$ because

$$\Psi_{(a,b)}(\bar{p}) = \Psi_{(a,b)}(p^\star) - \Psi_{(a,b)}^{x_0,x_1,p^\star}(p^\star) + \Psi_{(a,b)}^{x_0,x_1,p^\star}(q^\star).$$

$\square$

Now we state the optimality conditions for the problem (P_{x_0,x_1}).

Proposition C.2. *Let $p^\star$ be a solution of (P_{x_0,x_1}) with x_0, x_1 as in Proposition C.1. Then there exists a null set $\mathcal{N} \subset [0,T]$ and a constant μ_t for every $t \in [0,T] \setminus \mathcal{N}$ such that for every $x \in (x_0,x_1)$*

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) + g'_\gamma(x)\mu_t\right]^+}{f(x)\frac{\partial K}{\partial c}\left(t, A(t,a,b)\right)} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}. \tag{C.3}$$

Proof. Notice that the set $C(x_0,x_1)$ can be written as

$$C(x_0,x_1) = \left\{q \in W_x^{1,m}(x_0,x_1),\ g(q) \in C,\ h(q) = 0\right\},$$

where $g : W_x^{1,m}(x_0,x_1) \longrightarrow L^m([0,T] \times [x_0,x_1])$ is defined by $g(q) = \frac{\partial q}{\partial x}$, where C is the following convex cone $C := \{q \in L^m([0,T] \times [x_0,x_1]),\ q(t,x) \geq 0, \text{ a.e.}\}$ and $h : W_x^{1,m}(x_0,x_1) \longrightarrow L^m([0,T])$ is defined by

$$h(q) := \int_{x_0}^{x_1} \frac{\partial q}{\partial x}(\cdot,x)dx + p^\star(\cdot,x_0) - p^\star(\cdot,x_1).$$

It can be checked in the same way as in Remark 5 from [7], that their Assumption S is satisfied in this context. Furthermore, it is a classical result that the dual of $W_x^{1,m}(x_0,x_1)$ is $W_x^{1,m/(m-1)}(x_0,x_1)$.

Define now the Lagrangian $L : W_x^{1,m}(x_0,x_1) \times W_x^{1,m/(m-1)}(x_0,x_1) \times L^{\frac{m}{m-1}}(0,T) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} L(q,\lambda,\mu) := \Psi_{(a,b)}^{x_0,x_1,p^\star}(q) &+ \int_0^T \int_{x_0}^{x_1} \lambda(t,x) \frac{\partial q}{\partial x}(t,x)dxdt \\ &+ \int_0^T \mu(t) \left(\int_{x_0}^{x_1} \frac{\partial q}{\partial x}(t,x)dx + p^\star(t,x_0) - p^\star(t,x_1) \right) dt. \end{aligned}$$

Then, from Corollary 2 in [9] it follows that there exists $\lambda \in W_x^{1,m/(m-1)}(x_0, x_1)$, $\mu \in L^m(0,T)$ such that

$$\begin{cases} 0 = \dfrac{g_\gamma(x)f(x) + g'_\gamma(x)F(x)}{g'_\gamma(x)} - \dfrac{1}{\gamma}\left(\dfrac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1-\gamma}{\gamma}} \left(\dfrac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}} f(x)\dfrac{\partial K}{\partial c}(t, A(t,a,b)) \\ \qquad +\mu(t) + \lambda(t,x), \text{ a.e. in } [0,T]\times[x_0,x_1], \\ \lambda(t,x)\dfrac{\partial p^\star}{\partial x}(t,x) = 0, \ \lambda(t,x) \geq 0, \text{ a.e. in } [0,T]\times[x_0,x_1]. \end{cases}$$

Then, when $\frac{\partial p^\star}{\partial x}(t,x) > 0$ we have that $\lambda(t,x) = 0$ and

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}}\left[g_\gamma(x)f(x) + g'_\gamma(x)F(x) + g'_\gamma(x)\mu(t)\right]}{f(x)\frac{\partial K}{\partial c}(t, A(t,a,b))}\right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}.$$

In case $\frac{\partial p^\star}{\partial x}(t,x) = 0$ we have that

$$\frac{g_\gamma(x)f(x) + g'_\gamma(x)F(x)}{g'_\gamma(x)} + \mu(t) = -\lambda(t,x) \leq 0,$$

which ends the proof. □

We prove finally that the map μ does not depend on x_0, x_1 and is the same in the interval $I = (x_\ell, x_r)$.

Proposition C.3. *Let $I = (x_\ell, x_r) \subset (a_n, b_n)$ be as in Theorem 6.1. Then for any x_0, $x_1 \in I$, there exist a null set $\mathcal{N} \subset [0,T]$ and a constant μ_t for every $t \in [0,T] \setminus \mathcal{N}$ such that for every $x \in (x_0, x_1)$ (6.7) is satisfied.*

Proof. Let $y_0 := x_1$ and define by induction for $k \geq 0$

$$z_k := \inf\{z \in (a_n, b_n),\ P^\star(x) \geq H(y_k),\ \forall x \in [z, y_k]\},\ y_{k+1} := \frac{z_k + y_k}{2}.$$

By continuity we have that $P^\star(z_k) = H(y_k)$, so $y_{k+1} < y_k$ and the sequence $(y_k)_k$ converges necessarily to a_n. We conclude by applying Proposition C.2 to every interval (z_k, y_k) and noting that these intervals overlap themselves. □

Finance for Energy Market Research Centre

Institut de Finance de Dauphine, Université Paris-Dauphine
1 place du Maréchal de Lattre de Tassigny
75775 PARIS Cedex 16
www.fime-lab.org