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Abstract

We consider competitive capacity investment for a duopoly of two distinct producers. The producers are exposed to stochastically fluctuating costs and interact through aggregate supply. Capacity expansion is irreversible and modeled in terms of timing strategies characterized through threshold rules. Because the impact of changing costs on the producers is asymmetric, we are led to a nonzero-sum timing game describing the transitions among the discrete investment stages. Working in a continuous-time diffusion framework, we characterize and analyze the resulting Nash equilibrium and game values. Our analysis quantifies the dynamic competition effects and yields insight into dynamic preemption and over-investment in a general asymmetric setting. A case-study considering the impact of fluctuating emission costs on power producers investing in nuclear and coal-fired plants is also presented.

Keywords: Capacity Expansion, Continuous-time Games of Timing, Non-zero-sum Stopping Games, Power generation investments,

1. Introduction

The need to reduce carbon emission to achieve the 2 Celsius degree target puts under pressure power systems of many countries. Lowering the carbon content of electricity requires the development of competitive non-emissive energies for base-load generation. The most immediately viable alternative to provide dispatchable base-load power would be nuclear power plants. But, as shown in the 2005 and 2010 editions of the Projected Cost of Electricity Generation by the International Energy Agency, the relative competitiveness of nuclear power compared to coal-fired generation strongly depends on the existence of a material price for carbon emission. Indeed, a carbon price of 30 USD/tCO2 would definitively make nuclear power plants much more economical than coal-fired plants for electricity base-load generation. Unfortunately for the nuclear industry, as Figure 1 shows, the carbon price of the European Union Emission Trading System (EU-ETS) has fallen to a low of 5 €/tCO2 since mid-2012, and has not recovered since then to a value high enough to sustain emission reduction based on economic efficiency. Nevertheless, ongoing political developments, market design changes and technological advances might change this situation and benefit the nuclear producers. A crucial dilemma thus arises for the nuclear industry: either wait for a significant rise in the carbon price at the risk of base-load generation being preemptively taken by coal-fired plants, or intervene now at the cost of enduring short-term losses.

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Motivated by this context, we build a model to analyze a capacity expansion game in a competitive market. The game takes place between two players, representing sectors of electricity generators. Producer 1 invests in nuclear power plants with unit expansion cost K^1, while producer 2 invests in coal-fired plants with expansion cost K^2. We consider that those costs include the Operation & Maintenance costs since once the decision is made to invest, they become sunk costs. These investment costs are so massive that projects can be considered as a one-shot decision. To give an order of magnitude, the Hinkley Point Project of two nuclear power plants being built in the UK carries a cost of approximate 15 billion USD, and the cost of a 1 GW-capacity supercritical coal-fired plant is approximately 1 billion USD. Moreover, given the enormous sunk costs and plant lifetime of 40+ years, investments are viewed as irreversible. The aim of the article is then to analyze the resulting competitive investment to determine who and when will build new generating capacity.

In line with the above narrative, we focus on the carbon price X_t as the main state variable. Higher X_t benefits nuclear producers, while lower X_t benefits coal-fired plants. To reflect the significant uncertainties associated with the carbon price (see again Figure 1 which can be viewed as a historical trajectory of X_t), we work in a continuous-time stochastic setting. Thus, firms' investment strategies correspond to stopping times related to X_t. The game aspect of the model arises from the negative externality of capacity expansion. Namely, the competitive price is driven by the aggregate capacity of the producers, so that when one of the firms expands, electricity prices decline, hurting her competitor. This creates a preemptive motive for the investors and converts our framework into a non-zero-sum duopolistic game of timing. We assume that the firms make decisions to maximize their expected net present value of total future profits (NPV) in terms of the stochastic (X_t). Relying on the mechanism of a Nash equilibrium, we then characterize the competitive equilibrium by solving optimal stopping problems for one firm's best-response to her rival's actions. Importantly, depending on the competition strength, we find that both *threshold-type* and *preemptive* equilibria might arise.

Beyond the two profit-maximizing investors, we also aim to understand the role of the third-party regulator, or government in the game outcome. Carbon emission markets remain highly politicized, with a fluid market design. For instance, we can mention initiatives to prevent carbon price collapse, such as the Stability Reserve Mechanism in the ETS, and the United Kingdom carbon price floor of approximately 18 GBP/tCO2 institutionalized since 2016. France is following the same path. Thus, the establishment of a high and steady value for carbon strongly depends on the political will and ability of each state. Our purpose is thus to analyze the effect of such commitment on the market equilibrium. In particular, we are interested in the deviation of this equilibrium compared to the decision a benevolent planner would do.

As we discuss below, our setting yields a non-trivial extension of existing literature on stochastic timing games. Thus, our analysis is driven by methodological innovation and is relevant for other economic settings. In particular, it reflects a long-term research programme by the first and third authors on dynamic (i.e. with multi-stage strategies) non-zero-sum games.

1.1. Existing Literature

The general problem of capacity expansion under uncertainty has been extensively studied as a stochastic optimal control problem since the late 1950s (Luss, 1982). Due to massive costs of power generation, the theory of real options, which emerged in 1980s as a valuation technique for one-shot investments with high uncertainties is one of the classical and widely studied settings. Existing research has considered a variety of approaches to the choices faced by the firm, including singular control (Steg, 2012); impulse control (Aïd et al., 2016); timing control (Grenadier, 2000), and two-sided optimal switching control (Hamadène and Jeanblanc, 2007). Single-agent models for multi-stage capacity expansion were initiated in Dixit (1995) and Bar-Ilan et al. (2002).

To the extent that the profits of the firm are affected not only by her choices but also by decisions of others who produce substitutable or complementary goods, it is important to consider the strategic

Figure 1: Price (in euros per ton of CO_2) of the one year-ahead emission allowance on the EU-ETS. Source: TheIce.

interaction across firms. Assuming that firms take into account the other firms' reactions to their own actions and they know their rivals think the same way, expansion decisions in a competitive market can be treated as a *dynamic game*. This has been the motivation to combine real options frameworks with game theory. Early pioneering works to mix concepts from both theories were Smets (1993), who first introduced the effect of competition in the real option literature, and Williams (1993), who provided the first rigorous derivation of a Nash equilibrium in a real option framework. See also the books Grenadier and Rahl (2000); Chevalier-Roignant and Trigeorgis (2011).

The "standard" real option game features two symmetric firms competing to invest in a non-exclusive underlying project over the infinite (continuous) time horizon. The competition is of the leader/follower type: at the time of the first investment, one firm becomes the leader; the follower is then able to invest at a later date (Dixit and Pindyck, 1994; Huisman, 2013; Paxson and Pinto, 2005). We refer to the survey by Azevedo and Paxson (2014) who present a catalog of more than fifty articles dealing with variants of this setup. A simpler version is the preemption game introduced by Grenadier (2000), where two identical firms compete to be the first to initiate a new project.

To our knowledge the first paper to explicitly consider competitive capacity expansion was Bashyam (1996). Another notable contribution is Huisman and Kort (2015) who allow for joint optimization of the timing and project size, demonstrating that the first mover over-builds to delay entry of the other firm. Finally, in the context of competitive energy generation markets, we mention Takashima et al. (2008) who studied a preemption game to enter an electricity market, and Siddiqui and Takashima (2012) who considered multi-stage optimal expansion timing but with pre-determined investment order. See also the very recent work (Aïd et al., 2015) using a continuous-control framework.

In this article we keep some of the standard assumptions mentioned above, including the non-cooperative game aspect, fixed investment size, and two producers who produce perfectly-substitutable goods. However, in contra-distinction to most of the literature (though see Takashima et al. (2008) and De Angelis et al. (2015)), we consider *asymmetric* producers who have different preferences in terms of the underlying stochastic state X. This leads to a more general non-zero-sum game, and combines dynamic competition with (possibly) immediate preemption. Moreover, we consider the general multi-stage setup where investment order is fully endogenized. In particular, this brings forth an "apples-to-apples" quantification for (i) the value of being a guaranteed first-mover, which we dub the priority option as in Grasselli et al. (2013); and the (ii) cost of competition, i.e. relative to a cooperative solution. The latter case is viewed as a central planner who controls both firms and optimizes *aggregate* profits, cf. Aïd et al. (2014). As expected, we demonstrate that the threat of the rival investing causes firms to act preemptively and over-invest relative to either of the above alternatives, quantifying the preemption effect under a competitive framework.

As mentioned, we reduce the sequential game of timing to a series of (coupled) optimal stopping problems. A typical solution approach for optimal stopping problems driven by diffusion processes involves studies of free boundary problems or variational inequalities (see Oksendal (2013)). In this paper, we adapt the alternative technique of smallest concave majorants (see e.g. Dayanik and Karatzas (2003)), which has been used for zero-sum games in Peskir (2009), Ekström and Peskir (2008) and Lerche and Stich (2013), and for non-zero-sum games in recent preprints De Angelis et al. (2015); Attard (2015). A key advantage of the method is that it directly determines the value function, as well as the structure of the optimal stopping region, which allows us to apply explicit analytic recursion for the dynamic programming arguments.

The remainder of this paper is organized as follows. Section 2 formalizes the stages of the sequential timing game and connects game equilibrium to optimal stopping problems. In Section 3, we use the concave envelope method to give explicit solutions to these optimal stopping problems assuming linear payoff functions. Section 4 analyzes the Nash equilibria across stages of our game, establishing the threshold-type timing strategies. Numerical examples with the relative costs X_t driven by an Ornstein-Uhlenbeck (OU) process are discussed in Section 5. We discuss and suggest further studies in Section 6.

2. Problem Formulation

We consider a duopoly of two producers, dubbed firm 1 and firm 2. Each firm has options to irreversibly increase her current production capacity $Q^i(t)$ by paying a fixed lump-sum capital K^i, so as to generate more revenue. However, because the firms compete on the same market, expansion decisions of one firm carry negative externality (via lower market prices $P(t)$) for both of them, which leads to a non-zero-sum duopoly game.

2.1. Game Stages and Policies

To describe dynamic capacity expansion, we decompose the overall model into stages. Let $\vec{N}_t \in \{(N^1_t, N^2_t)_{t\geq 0} : N^i_t = 0, 1, \ldots, N^i_0\}$, where N^i_t counts how many expansion options remain for firm i, denote the game stage at date t. Irreversible investment implies that starting at $\vec{N}_0 = (N^1_0, N^2_0)$, each coordinate of $\vec{N}_t$ is *piecewise constant* and *non-increasing*. We postulate that firm capacities are fully determined by the investment game stage $Q^i(t) = Q^i(\vec{N}_t)$, and market price is solely a function of aggregate supply $Q(t) := Q^1(t)+Q^2(t)$ (here equated with aggregate production capacity). This is equivalent to assuming constant (or at least deterministic) demand, which is not far from the truth for base-load electricity generation where revenue is determined by fixed long-term contracts. It follows that price is a function of game stage, $P(t) = P(\vec{N}_t)$. Consequently $Q(\vec{N}_s) \geq Q(\vec{N}_t)$, $P(\vec{N}_s) \leq P(\vec{N}_t)$ for $s \geq t$ are *piecewise constant* as well. For typographical convenience, we henceforth use subscripts (n_1, n_2) to index above stage quantities, e.g. $Q^i_{n_1,n_2} \equiv Q^i(\vec{n})$.

As a simple example, one may take $Q^i_{n_1,n_2} = \underline{q}^i + (\Delta Q)(N^i_0 - n_i)$ at a stage $(n_1,\, n_2)$, where ΔQ is the size of each expansion, and $\underline{q}^i$'s are the initial capacities. For the clearing prices, a typical setting is a linear inverse-demand curve:

$$P_{n_1,n_2} = D(1 - \eta[Q^1_{n_1,n_2} + Q^2_{n_1,n_2}]), \tag{2.1}$$

where $\eta > 0$ is the demand elasticity and D is a price multiplier.

Remark 2.1. Since we will rely on dynamic programming-like arguments, the framework necessitates to specify the initial number of possible expansions (N_0^1, N_0^2). This can be justified by assuming a fixed demand curve, so that one can infer the maximum additional capacity that is economically feasible. For example, in the electricity generation market, it is standard to make forecasts on the new aggregate capacity that needs to be added, with the competition centered about who and when will expand (but not how far). If a total of $\Delta Q\bar{N}$ extra capacity is required, one can set $\vec{N}_0 = (\bar{N}, \bar{N})$.

To capture market uncertainty, we introduce the *relative cost* between the production expenses of the two firms as a one-dimensional diffusion process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the Îto stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \tag{2.2}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$. Denote by $\mathcal{D} := (\underline{d}, \bar{d})$, with $-\infty \leq \underline{d} < \bar{d} \leq +\infty$, the domain of X_t and $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by X_t. The coefficients $b : \mathcal{D} \to \mathbb{R}$ and $\sigma : \mathcal{D} \to \mathbb{R}_{++}$ are assumed to be Lipschitz so as to ensure a unique strong solution to (2.2).

When X_t is large, firm 1 has the comparative advantage in production, while when X_t is close to $\underline{d}$ firm 2 has the advantage. Since X_t lives on the real line, this monotonicity assumption is rather natural. Specifically we shall assume that production costs of firm 1 decrease (linearly) in X_t, while the production costs of firm 2 increase (linearly) in X_t, which leads to profit rates of the form

$$\begin{aligned} \pi^1_{n_1,n_2}(X_t) &= (P_{n_1,n_2} - C^1 + \rho^1 X_t)Q^1_{n_1,n_2}, \\ \pi^2_{n_1,n_2}(X_t) &= (P_{n_1,n_2} - C^2 - \rho^2 X_t)Q^2_{n_1,n_2}, \end{aligned} \tag{2.3}$$

at stage (n_1, n_2), where $C^i, \rho^i > 0$ are the firm-specific fixed production cost, and sensitivity of relative costs to X, respectively. The *monotonicity* of $\pi^i(\cdot)$ in x will be important for the analytic derivations in the sequel.

By expanding capacity, firms shift the game to a subsequent stage and henceforth change the profit rates they receive. We assume that the firms evaluate their decisions based on the total net present value of future profits (NPV), namely the expected future cashflow discounted at an exogenous, constant interest rate $r > 0$, minus the discounted lump-sum costs K^i paid at each expansion epoch. Given strategies $\boldsymbol{\alpha}^i$ of the firms and $\vec{N}_0 = (n_1, n_2)$, the NPV of firm i is

$$J^i_{n_1,n_2}(x; \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2) := \mathbb{E}\left[\int_0^\infty e^{-rs}\pi^i_{N_s^1,N_s^2}(X_s)\,ds - \sum_{j=1}^{n_1} K^i \cdot e^{-r\mathcal{I}^i_j} \,\middle|\, X_0 = x, \vec{N}_0 = (n_1, n_2)\right], \tag{2.4}$$

where the j-th investing time $\mathcal{I}^i_j$ of firm i is defined as:

$$\mathcal{I}^i_j = \inf\left\{s > \mathcal{I}^i_{j-1} : N^i_{s-} - N^i_s > 0\right\}, \tag{2.5}$$

for $j = 1, \cdots, n_1$ and $\mathcal{I}^i_0 := 0$. For brevity, we denote $J^i_{n_1,n_2}(x; \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ by $J^i_{n_1,n_2}(x)$ henceforth.

Definition 2.2. (*Game Policies*) We postulate actions of the firms to be of time-stationary Feedback Perfect State (FPS) or Markov type (see a detailed exposition in Ch. 3 of Carmona (2016)), namely the strategy set of firm i is

$$\mathcal{A}^i = \left\{\boldsymbol{\alpha}^i := \alpha^i(X_t, \vec{N}_t)\right\}, \qquad i = 1, 2. \tag{2.6}$$

The time-homogeneity of X_t and the feedback form of prices in terms of $\vec{N}$ is the natural motivation to restrict attention to time-homogenous investment strategies. In turn, this leads to time-stationary expected profits which only depend on the initial X_0 and $\vec{N}_0$, see (2.4).

Since the market price declines as aggregate capacity rises, investment by firm i will take place only once X_t moves sufficiently towards her preferred direction. Accordingly, we model capacity investment in terms of timing strategies. Due to the piecewise-constant feature of $\vec{N}_t$, it is sufficient to consider the $\mathbb{F}$-stopping times τ^i for the expansion epochs at each stage (n_1, n_2). As a result, the strategy sets can be represented as:

$$\mathcal{A}^1 = \left\{ \boldsymbol{\alpha}^1 := \left(\tau^1_{n_1,n_2}\right) \big| \, n_1 > 0, \forall n_2 \right\}, \quad \mathcal{A}^2 = \left\{ \boldsymbol{\alpha}^2 := \left(\tau^2_{n_1,n_2}\right) \big| \, n_2 > 0, \forall n_1 \right\}, \tag{2.7}$$

maintaining the structure of (2.6). This leads to a recursive formulation of equilibrium expected profits in analogue to dynamic programming.

Let us consider the *interior* stages where both firms have at least one expansion option left, namely $\vec{N}_0 = (n_1, n_2)$ where $n_1, n_2 > 0$. Thanks to (2.6) and the strong Markov property of X, the NPV of firm 1 can be decomposed as (letting $\underline{\tau} := \tau^1_{n_1,n_2} \wedge \tau^2_{n_1,n_2}$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$)

$$\begin{aligned} J^1_{n_1,n_2}(x, \boldsymbol{\alpha}) = \mathbb{E}_x \Bigg[\int_0^{\underline{\tau}} & e^{-rs} \pi^1_{n_1,n_2} (X_s) \, ds \\ & + \Big\{ e^{-r\underline{\tau}} \mathbb{1}_{\{\tau^1_{n_1,n_2} < \tau^2_{n_1,n_2}\}} \left(J^1_{n_1-1,n_2} \left(X_{\tau^1_{n_1,n_2}}, \boldsymbol{\alpha} \right) - K^1_{n_1} \right) \\ & + e^{-r\underline{\tau}} \mathbb{1}_{\{\tau^1_{n_1,n_2} > \tau^2_{n_1,n_2}\}} J^1_{n_1,n_2-1} \left(X_{\tau^2_{n_1,n_2}}, \boldsymbol{\alpha} \right) \\ & + e^{-r\underline{\tau}} \mathbb{1}_{\{\tau^1_{n_1,n_2} = \tau^2_{n_1,n_2}\}} \left(J^1_{n_1-1,n_2-1} \left(X_{\tau^1_{n_1,n_2}}, \boldsymbol{\alpha} \right) - K^1_{n_1} \right) \Big\} \Bigg]. \end{aligned} \tag{2.8}$$

We use the shorthand notation $\mathbb{E}_x \{\cdot\} := \mathbb{E} \{\cdot | X_0 = x\}$ and the subscript of $J^i_{n_1,n_2}$ to indicate the conditioning on $\vec{N}_0 = (n_1, n_2)$. In the *boundary* stages $(n_1, 0)$, or similarly $(0, n_2)$, one firm has no options left, which can be represented via e.g. $\tau^2_{n_1,0} = +\infty$ in (2.8), removing the last two cases/terms. For future use we introduce the static discounted future cashflows D^i's. The latter capture the situation where capacities are forever fixed, which is associated with the first term in (2.8):

$$\begin{aligned} D^i_{n_1,n_2}(x) &:= \mathbb{E}_x \left[\int_0^\infty e^{-rs} \pi^i_{n_1,n_2} (X_s) \, ds \right] \\ &= Q^i_{n_1,n_2} \times \left\{ \frac{P_{n_1,n_2} - C_i}{r} + (-1)^{i+1} \rho_i \int_0^\infty e^{-rt} \mathbb{E} \left[X_t \,|\, X_0 = x \right] \, dt \right\}. \end{aligned} \tag{2.9}$$

2.2. Game Equilibrium

Because decisions of one firm affect the other through the joint dependence of J^i's on $\vec{N}_t$, capacity expansion becomes a non-zero-sum stochastic game driven by the state variable X_t and endogenous game stage $\vec{N}_t$. To describe optimal behavior in this game we rely on the standard concept of Markov *Nash equilibrium.* Set $\mathbb{A} := \mathcal{A}^1 \otimes \mathcal{A}^2$. For $\boldsymbol{\alpha} = (\boldsymbol{\alpha^1}, \, \boldsymbol{\alpha^2}) \in \mathbb{A}$ and $\boldsymbol{\beta^i} \in \mathcal{A}^i$, we denote by $(\boldsymbol{\alpha^{-i}}, \, \boldsymbol{\beta^i})$ the strategy where firm i switches from $\boldsymbol{\alpha^i}$ to $\boldsymbol{\beta^i}$, while her rival retains $\boldsymbol{\alpha^{-i}}$.

Definition 2.3. (*Nash Equilibrium*) Let $J^i(x, \cdot)$ denote the NPV received by firm i with $X_0 = x$. The pair $\boldsymbol{\alpha^*} = (\boldsymbol{\alpha^{1,*}}, \, \boldsymbol{\alpha^{2,*}}) \in \mathbb{A}$ is said to be a *Nash equilibrium* of the game, if for $i \in \{1, \, 2\}, \; \forall \boldsymbol{\beta^i} \in \mathcal{A}^i$:

$$J^i(x, \, \boldsymbol{\alpha^{*-i}}, \boldsymbol{\beta^i}) \leq J^i(x, \boldsymbol{\alpha^*}) \quad \forall x, \tag{2.10}$$

and the corresponding $J^i(x, \boldsymbol{\alpha}^*)$ is named the *equilibrium* game value of firm i.

Let $V^i_{n_1,n_2}(x) := J^i_{n_1,n_2}(x, \boldsymbol{\alpha}^*)$ denote the equilibrium game values at game stage $(n_1, \, n_2)$. Recursive construction of J^i's revealed by equation (2.8) and the time homogeneity of the state process X_t motivate dynamic programming methods that characterize $V^i_{n_1,\, n_2}$ by looking at the single-stage timing game defined by $\tau^i_{n_1,n_2}$'s. The resulting stopping time (if it exists) which yields the single-stage equilibrium is in turn part of the dynamic equilibrium strategy of firm i at stage $(n_1,\, n_2)$, and so denoted by $\tau^{i,*}_{n_1,n_2}$. Indeed, fixing $\tau^{-i,*}_{n_1,n_2}$, the stopping time $\tau^{i,*}_{n_1,n_2}$ is the maximizer of the RHS in (2.8). This can be seen most simply in the boundary stages, where one of the strategy sets is empty and (2.10) reduces to the traditional situation of a single-agent optimization. In our context, this optimization is an optimal stopping problem for X_t via $\tau^1_{n_1,0}$ (resp. τ^2_{0,n_2}):

$$V^1_{n_1,0}(x) = D^1_{n_1,0}(x) + \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} \left[V^1_{n_1-1,0}(X_\tau) - D^1_{n_1,0}(X_\tau) - K^1_{n_1} \right] \right\}, \tag{2.11}$$

$$V^2_{0,n_2}(x) = D^2_{0,n_2}(x) + \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} \left[V^2_{0,n_2-1}(X_\tau) - D^2_{0,n_2}(X_\tau) - K^2_{n_2} \right] \right\}, \tag{2.12}$$

where $\mathcal{T} := \mathcal{T}_{[0,+\infty)}$ denotes the collection of all $\mathbb{F}$-stopping times with values in $[0,+\infty)$, and D^i's are from (2.9). For the firm who has no remaining expansion options, her game value is obtained as

$$V^1_{0,n_2}(x) = D^1_{0,n_2}(x) + \mathbb{E}_x \left[e^{-r\tau^{2,*}_{0,n_2}} \left\{ V^1_{0,n_2-1}\left(X_{\tau^{2,*}_{0,n_2}}\right) - D^1_{0,n_2}\left(X_{\tau^{2,*}_{0,n_2}}\right) \right\} \right], \tag{2.13}$$

$$V^2_{n_1,0}(x) = D^2_{n_1,0}(x) + \mathbb{E}_x \left[e^{-r\tau^{1,*}_{n_1,0}} \left\{ V^2_{n_1-1,0}\left(X_{\tau^{1,*}_{n_1,0}}\right) - D^2_{n_1,0}\left(X_{\tau^{1,*}_{n_1,0}}\right) \right\} \right]. \tag{2.14}$$

Notice that these game values are determined by their rivals' game strategies $\tau^{2,*}_{0,n_2}$ or $\tau^{1,*}_{n_1,0}$ respectively, thus there is no more any optimization.

In the interior game stages (n_1, n_2), by the Nash equilibrium criterion (2.10) the equilibrium strategy of each firm is the **best-response** to her rival's action, and we denote the resulting NPVs by $\widetilde{V}^i(\,\cdot\,;\tau^{-i})$. Namely, based on (2.8), given the rival's game policy as $\tau^2_{n_1,n_2}$ (resp. $\tau^1_{n_1,n_2}$), firm 1 (resp. firm 2) solves the optimal stopping problem:

$$\begin{aligned}\widetilde{V}^1_{n_1,n_2}(x;\tau^2_{n_1,n_2}) - D^1_{n_1,n_2}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \Big[& \mathbb{1}_{\{\tau>\tau^2_{n_1,n_2}\}} e^{-r\tau^2_{n_1,n_2}} \left\{ V^1_{n_1,n_2-1}\left(X_{\tau^2_{n_1,n_2}}\right) - D^1_{n_1,n_2}\left(X_{\tau^2_{n_1,n_2}}\right)\right\} \\ & + \mathbb{1}_{\{\tau<\tau^2_{n_1,n_2}\}} e^{-r\tau} \left\{ V^1_{n_1-1,n_2}(X_\tau) - D^1_{n_1,n_2}(X_\tau) - K^1_{n_1} \right\} \\ & + \mathbb{1}_{\{\tau=\tau^2_{n_1,n_2}\}} e^{-r\tau} \left\{ V^1_{n_1-1,n_2-1}(X_\tau) - D^1_{n_1,n_2}(X_\tau) - K^1_{n_1} \right\} \Big], \end{aligned} \tag{2.15}$$

$$\begin{aligned}\widetilde{V}^2_{n_1,n_2}(x;\tau^1_{n_1,n_2}) - D^2_{n_1,n_2}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \Big[& \mathbb{1}_{\{\tau>\tau^1_{n_1,n_2}\}} e^{-r\tau^1_{n_1,n_2}} \left\{ V^2_{n_1-1,n_2}\left(X_{\tau^1_{n_1,n_2}}\right) - D^2_{n_1,n_2}(X_{\tau^2_{n_1,n_2}})\right\} \\ & + \mathbb{1}_{\{\tau<\tau^1_{n_1,n_2}\}} e^{-r\tau} \left\{ V^2_{n_1,n_2-1}(X_\tau) - D^2_{n_1,n_2}(X_\tau) - K^2_{n_2} \right\} \\ & + \mathbb{1}_{\{\tau=\tau^1_{n_1,n_2}\}} e^{-r\tau} \left\{ V^2_{n_1-1,n_2-1}(X_\tau) - D^2_{n_1,n_2}(X_\tau) - K^2_{n_2} \right\} \Big]. \end{aligned} \tag{2.16}$$

Observe that simultaneous investment can be ruled out since on the event $\{\tau = \tau^2_{n_1,n_2}\}$ it is strictly dominated by the strategy of first waiting $\tau > \tau^2_{n_1,n_2}$ and then optimally investing as a follower: $V^1_{n_1,n_2-1} \geq V^1_{n_1-1,n_2-1} - K^1_{n_1}$. Assuming that the suprema above are attained, we obtain the best-response policy $\tilde{\tau}^i_{n_1,n_2} = \tilde{\tau}^i_{n_1,n_2}(\tau^{-i}_{n_1,n_2})$ that maximizes (2.15)-(2.16), where we emphasize the dependence on the rival's strategy. The condition for a Nash equilibrium at (n_1, n_2) (as defined in Definition 2.3) is then characterized as a fixed point of the best-response strategies: $\tau^{1,*}_{n_1,n_2} = \tilde{\tau}^1_{n_1,n_2}(\tau^{2,*}_{n_1,n_2})$ and $\tau^{2,*}_{n_1,n_2} = \tilde{\tau}^2_{n_1,n_2}(\tau^{1,*}_{n_1,n_2})$. By induction on the discrete stages (n_1, n_2), we may patch these local equilibria to construct a global one:

$$\mathcal{A}^* = \left\{ \left(\boldsymbol{\alpha}^{1,*},\, \boldsymbol{\alpha}^{2,*}\right) : \begin{array}{l} \boldsymbol{\alpha}^{1,*} = \left(\tilde{\tau}^1_{n_1,n_2}\left(\tau^{2,*}_{n_1,n_2}\right),\, n_1 > 0, \forall n_2\right) \\ \boldsymbol{\alpha}^{2,*} = \left(\tilde{\tau}^2_{n_1,n_2}\left(\tau^{1,*}_{n_1,n_2}\right),\, n_2 > 0, \forall n_1\right) \end{array} \right\}. \tag{2.17}$$

3. Method of Solution

To tackle the problems introduced in Section 2.2, we impose further conditions on the state process X, in particular that the boundaries of the domain $\mathcal{D} = (\underline{d}, \bar{d})$ are *natural*, and that X is *regular* in $\mathcal{D}$. Informally this means that starting at any $x \in \mathcal{D}$, X will reach any $y \in \mathcal{D}$ with positive probability, while $\underline{d}, \bar{d}$ cannot be reached in finite time; see Ch. 2 of Borodin and Salminen (2012) for detailed expositions.

The set of optimal stopping problems in Section 2.2 is narrowed to considering

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h(X_\tau) \right\}, \quad \text{and} \tag{3.1}$$

$$V_R(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ \mathbb{1}_{\{\tau < \tau_R\}} e^{-r\tau} h(X_\tau) + \mathbb{1}_{\{\tau > \tau_R\}} e^{-r\tau_R} l(X_{\tau_R}) \right\}, \tag{3.2}$$

where τ_R is restricted to be an *exit time* associated to a given interval $R := (a, \bar{d})$ or $R := (\underline{d}, a)$, $a \in \text{int}(\mathcal{D})$, $h(\cdot)$ is the *first-mover* payoff, and $l(\cdot)$ corresponds to the resulting *second-mover/exit* payoff. We assume both h and l are *monotone* functions and h is twice differentiable a.e. on $\mathcal{D}$ (while l need not be smooth).

The method we use in this work characterizes the value functions via the smallest concave majorant associated to transformed first-mover payoff h, see e.g. Dayanik and Karatzas (2003), De Angelis et al. (2015). A key step of this smallest concave majorant method involves the transformations:

$$\psi(x) := \frac{F}{G}(x), \qquad \varphi(x) := \frac{G}{F}(x) \tag{3.3}$$

where F and G are defined in the following Lemma (see (Rogers and Williams, 2000, vol.II, p.292)).

Lemma 3.1. Recalling (2.2), the infinitesimal generator of (X_t) is

$$\mathcal{L} = b(x)\frac{d}{dx} + \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2}. \tag{3.4}$$

Define the first passage time of (X_t) associated to the level a by $\tau_a = \inf\{t \geq 0 : X_t = a\}$. For $r > 0$, there exist an *increasing* $F(x)$ and a *decreasing* $G(x)$ solutions to the ODE:

$$(\mathcal{L} - r)u(x) = 0, \qquad x \in \mathcal{D}. \tag{3.5}$$

These solutions are positive, continuous, strictly monotone and convex, linearly independent, and are related to the expectation of the Laplace transform of τ_a as follows:

$$\mathbb{E}_x \left\{ e^{-r\tau_a} \right\} = \begin{cases} \frac{F(x)}{F(a)}, & \text{if } x \leq a, \\ \frac{G(x)}{G(a)}, & \text{if } x \geq a. \end{cases} \tag{3.6}$$

Such functions F and G are called fundamental solutions of (3.5). Moreover for $\underline{d}$ and $\bar{d}$ natural boundary points one also has (see Borodin and Salminen (2012), Sec.2)

$$\lim_{x \downarrow \underline{d}} F(x) = 0, \qquad \lim_{x \downarrow \underline{d}} G(x) = +\infty, \qquad \lim_{x \uparrow \bar{d}} F(x) = +\infty, \qquad \lim_{x \uparrow \bar{d}} G(x) = 0. \tag{3.7}$$

Example 3.2. Suppose that (X_t) is a Geometric Brownian motion,

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \tag{3.8}$$

with $\mathcal{D} = (0, \infty)$, $\mu < r$ and $\sigma > 0$. The fundamental solutions F and G for (3.8) are:

$$F(x) := x^{\eta_+}, \qquad G(x) := x^{\eta_-}, \tag{3.9}$$

where η_+ and η_- are the positive and negative roots of the quadratic equation $\frac{\sigma^2}{2}\eta(\eta - 1) + \mu\eta - r = 0$.

Alternatively, suppose that (X_t) is an Ornstein-Uhlenbeck process

$$dX_t = \mu(\theta - X_t)dt + \sigma dW_t, \tag{3.10}$$

with $\mathcal{D} = \mathbb{R}$, μ, $\sigma > 0$ and $\theta \in \mathbb{R}$. The fundamental solutions for (3.10) are:

$$F(x) := \int_0^\infty u^{\frac{r}{\mu}-1} e^{\sqrt{\frac{2\mu}{\sigma^2}}(x-\theta)u - \frac{u^2}{2}} du, \quad \text{and} \quad G(x) := \int_0^\infty u^{\frac{r}{\mu}-1} e^{-\sqrt{\frac{2\mu}{\sigma^2}}(x-\theta)u - \frac{u^2}{2}} du. \tag{3.11}$$

Notice that direct differentiation yields that $F'(x) > 0$, $F''(x) > 0$, $G'(x) < 0$, $G''(x) > 0$. It follows that both $F(x)$ and $G(x)$ are strictly positive and convex, and $F(x)$ is strictly increasing while $G(x)$ is strictly decreasing. They are also symmetric around the mean-reverting level θ.

It follows from properties of F and G that ψ (resp. φ) : $\mathcal{D} \longmapsto \mathbb{R}^+$ is positive, strictly *increasing* (resp. *decreasing*), continuous, and twice differentiable on $\mathcal{D}$. Define the ψ-transform operator Ψ as:

$$\Psi h(y) := \begin{cases} \frac{h}{G} \circ \psi^{-1}(y), & \text{if } y > 0, \\ \lim_{x \downarrow \underline{d}} \dfrac{h(x)}{G(x)}, & \text{if } y = 0, \end{cases} \tag{3.12}$$

and similarly the φ-transform operator Φ by

$$\Phi h(z) := \begin{cases} \frac{h}{F} \circ \varphi^{-1}(z), & \text{if } z > 0, \\ \lim_{x \uparrow \bar{d}} \dfrac{h(x)}{F(x)}, & \text{if } z = 0. \end{cases} \tag{3.13}$$

Applying the operator Ψ (Φ resp.) transforms the optimal stopping problem from x coordinate to the $y = \psi(x)$ ($z = \varphi(x)$ resp.) coordinate.

Recalling the optimal stopping problem (3.2), if x were to be in the region R, since X is regular in $\mathcal{D}$, it reaches the R-boundary a with positive probability. Therefore, one can consider X as a truncated process on the domain $\bar{R} := [a, \bar{d})$ or $(\underline{d}, a]$, where the boundary at a is *absorbing*, i.e. the process X is stopped when it reaches the level a. Accordingly, we impose a boundary condition on the first-mover payoff h by defining

$$\hat{h}_R(x) := \begin{cases} h(x), & \text{if } x \in R, \\ l(x), & \text{if } x = a. \end{cases} \tag{3.14}$$

Thanks to the work of Dayanik and Karatzas (2003) and De Angelis et al. (2015), if x were to be in the region $R = (a, \bar{d})$ (resp. $R = (\underline{d}, a)$), the value function $V_R(x)$ is associated to the smallest concave majorant of transformed payoff $\Psi\hat{h}_R(y)$ (resp. $\Phi\hat{h}_R(z)$) over $\psi(\bar{R}) = [\psi(a), +\infty)$ (resp. $\varphi(\bar{R})$), denoted by $\mathcal{W}\Psi_{\bar{R}}\hat{h}(y)$ (resp. $\mathcal{W}\Phi_{\bar{R}}\hat{h}(z)$). If x were to be in the exit region $\mathcal{D} \setminus R$ (i.e $\tau_R = 0$), the value function $V_R(x)$ is equivalent to the instant second-mover payoff $l(x)$. To recap, we state the following proposition for the case $R = (a, \bar{d})$.

Proposition 3.3. The value function $V_R(x)$ defined in (3.2) can be admitted as follows:

$$V_R(x) = \begin{cases} G(x) \cdot \left[\mathcal{W}\Psi_{\bar{R}}\hat{h} \circ \psi(x)\right], & \text{if } x \in R, \\ l(x), & \text{if } x \in \mathcal{D} \setminus R. \end{cases} \tag{3.15}$$

Notice that the unconstrained value function $V(x)$ defined in (3.1) corresponds to the special case that $R = \mathcal{D}$ and $\hat{h} \equiv h$.

The behavior at a is crucial for the existence of an optimal stopping rule. If $l(a) \geq h(a)$, then $\lim\limits_{x \searrow a} V_R(x) = l(a)$ and an optimal stopping rule τ^* can be defined as

$$\Gamma := \{x \in \bar{R} : V_R(x) = h(x)\} \qquad \text{and} \qquad \tau^* := \inf\{t \geq 0 : X_t \in \Gamma\}. \tag{3.16}$$

However, if $l(a) < h(a)$, then the payoff $\hat{h}$ has a negative jump at the boundary a, and therefore the value function is also discontinuous there: $\lim_{x \searrow a} V_R(x) > l(a)$ (see a case study sketched in Figure 2(a)). This down-jump rules out (3.16) since in fact one ought to stop before reaching $a \in \Gamma$. In that case, there is no optimal stopping time; however for any $\varepsilon > 0$, an ε-optimal rule can be defined as the first hitting time of $\Gamma^\varepsilon = \{x \in \bar{R} : V_R(x) \leq h(x) + \varepsilon\}$.

Remark 3.4. An alternative method is to use the method of variational inequalities which reduces to looking for solutions of the ODE

$$(\mathcal{L} - r)V_R(x) = 0,$$

subject to certain free boundary and smooth pasting conditions. Assuming smoothness of the resulting solution (which is likely to fail based on the above discussion), one can apply verification to connect back to the optimal stopping problem. However, this method requires a priori assumptions about the shape of the continuation region (see Oksendal (2013)) and leads to multiple analytic challenges. See also the recent work by Aïd et al. (2016) regarding the challenges in applying smooth pasting conditions in impulse games.

In order to characterize corresponding smallest concave majorant, some regularity of the underlying payoff functions are required. Similar to De Angelis et al. (2015), we introduce the following classes of functions:

Definition 3.5. Let $\mathcal{L}$ be the infinitesimal generator of the state process X_t in (3.4). Let $\mathcal{H}$ be the class of real valued functions $h \in \mathcal{C}^2(\mathcal{D})$ such that

$$\limsup_{x \to \underline{d}} \left| \frac{h(x)}{G(x)} \right| = 0 = \limsup_{x \to \bar{d}} \left| \frac{h(x)}{F(x)} \right|, \tag{3.17}$$

$$\text{and} \qquad \mathbb{E}_x \left[\int_0^\infty e^{-rt} \left| (\mathcal{L} - r) h\left(X_t\right) \right| dt \right] < \infty, \tag{3.18}$$

for all $x \in \mathcal{D}$. We denote by $\mathcal{H}_{\text{inc}}$ (resp. $\mathcal{H}_{\text{dec}}$) the set of all $h \in \mathcal{H}$ such that $x \mapsto (\mathcal{L} - r)h(x)$ is strictly *positive* (resp. *negative*) on $(\underline{d}, b_h)$ and strictly *negative* (resp. *positive*) on $(b_h, \bar{d})$ for some $b_h \in \mathcal{D}$.

We now apply operators Ψ and Φ defined in Section 3 to the classes $\mathcal{H}_{\text{inc}}$ and $\mathcal{H}_{\text{dec}}$. The following lemma states key properties of the resulting transformed payoff functions. Proof of the first statement can be done by multiple approaches and we give one in Appendix A; for the rest of the statements we refer to (De Angelis et al., 2015, Lemma 3.1).

Lemma 3.6. Let $h \in \mathcal{H}_{\text{inc}}$ (resp. $\mathcal{H}_{\text{dec}}$) and set $\hat{y} := \psi(b_h)$ (resp. $\hat{z} := \varphi(b_h)$). Then the transformed function $\hat{H} := \Psi h$ (resp. Φh):

(i) is *convex* on $[0, \hat{y})$ (resp. $[0, \hat{z})$) and *concave* on $(\hat{y}, +\infty)$ (resp. $(\hat{z}, +\infty)$),

(ii) satisfies $\hat{H}(0+) = 0$ and $\hat{H}'(0+) = -\infty$,

(iii) has a unique global minimum at some $\bar{y} \in [0, \hat{y})$ (resp. $\bar{z} \in [0, \hat{z})$) and $\lim_{y \to \infty} \hat{H}(y) = +\infty$, hence it is monotonic increasing on $(\hat{y}, +\infty)$ (resp. $(\hat{z}, +\infty)$).

Example 3.2. (continued): Under an OU model (3.10) and given a linear function $h(x) = ax + c$, we have

$$(\mathcal{L} - r)h(x) = a \cdot b(x) - rh(x) = a\mu(\theta - x) - r(ax + c) = -a(\mu + r)x + a\mu\theta - rc.$$

Conditions (3.17) and (3.18) can be easily verified given the fundamental solutions F, G stated in (3.11). Therefore, when $a > 0$ such linear h's are increasing and are in $\mathcal{H}_{\text{inc}}$, while for $a < 0$ h's are decreasing and are in $\mathcal{H}_{\text{dec}}$. Under a GBM model (3.8), one can similarly verify that when $a > 0, c < 0$ such linear h's are in $\mathcal{H}_{\text{inc}}$, while for $a < 0, c > 0$ they are in $\mathcal{H}_{\text{dec}}$.

Inspired by above examples, we henceforth assume that all $D^1_{n_1,n_2}$ are *increasing*, and their differences $D^1_{n_1-1,n_2} - D^1_{n_1,n_2} - K^1_{n_1}$ are contained in the class $\mathcal{H}_{\text{inc}}$, while all $D^2_{n_1,n_2}$ are *decreasing* and $D^2_{n_1,n_2-1} - D^2_{n_1,n_2} - K^2_{n_2}$ are contained in the class $\mathcal{H}_{\text{dec}}$. For the sake of concise exposition, we further concentrate on the case where the discounted cashflows $D^i_{n_1,n_2}(x)$ are *affine* in x. This essentially corresponds to the expectation $\mathbb{E}_x[X_t]$ being affine in x,

$$\mathbb{E}_x[X_t] := x \cdot A(t) + B(t). \tag{3.19}$$

Substituting into (2.9) leads to:

$$\begin{aligned} D^i_{n_1,n_2}(x) &= Q^i_{n_1,n_2} \times \left[\frac{P_{n_1,n_2} - C_i}{r} + (-1)^{i+1} \rho^i \int_0^\infty e^{-rs} \left\{ A(s)x + B(s) \right\} ds \right] \\ &= \zeta^i_{n_1,n_2} + (-1)^{i+1} \frac{\rho^i Q^i_{n_1,n_2}}{\delta} \cdot x, \end{aligned} \tag{3.20}$$

where ζ^i and δ are defined via

$$\zeta^i_{n_1,n_2} := Q^i_{n_1,n_2} \times \left[\frac{P_{n_1,n_2} - C_i}{r} + (-1)^{i+1} \rho^i \int_0^\infty e^{-rs} B(s) ds \right], \qquad \delta := \frac{1}{\int_0^\infty e^{-rs} A(s) ds}.$$

Example 3.2. (continued): Under a GBM model (3.8), we have $\mathbb{E}_x[X_t] = xe^{\mu t}$, and consequently D^i's are of the form (3.20) with $\int_0^\infty e^{-rs} B(s) ds = 0$, $\delta = r - \mu$.

Under an OU model (3.10), we have $\mathbb{E}_x[X_t] = xe^{-\mu t} + \theta\,(1 - e^{-\mu t})$, and consequently D^i's are of the form (3.20) with $\int_0^\infty e^{-rs} B(s) ds = \frac{\mu\theta}{r(r+\mu)}$, $\delta = r + \mu$.

4. Analytical Results

In this section, we will specify game strategies and game values of each firm at each game stage (n_1, n_2) by dynamic programming. The boundary game stages, at which only one firm has expansion option(s), can be solved directly as in (2.11)-(2.12), allowing us to determine (2.13)-(2.14) accordingly. For the interior game stages at which both firms have expansion options, we first derive their best-response to the rival's action and then obtain the equilibrium strategies via the *Nash equilibrium* fixed-point characterization.

4.1. Boundary Stages

In the scenarios where one firm has expansion options while her rival does not, deriving game values and policies boils down to solving a series of single-agent optimization problems (2.11)-(2.12). Inspired by the dynamic programming principle, we first consider corresponding problems at stage $(1, 0)$ for firm 1 and $(0, 1)$ for firm 2. Note that because capacities are forever constant after $\vec{N}_t = (0, 0)$, $V^i_{0,0}(x) \equiv D^i_{0,0}(x)$ for $i = 1, 2$.

In the game stage $(1, 0)$ firm 2 has already invested and firm 1 now optimizes her expected discounted profits. Substituting (3.20) into (2.11) for $n_1 = 1$, firm 1 solves the optimal stopping problem:

$$V^1_{1,0}(x) - D^1_{1,0}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h^1_{1,0}(X_\tau) \right\}, \tag{4.1}$$

where the first-mover payoff is:

$$h^1_{1,0}(x) = D^1_{0,0}(x) - D^1_{1,0}(x) - K^1_1 = \frac{\rho^1 \Delta Q^1_1}{\delta} \cdot x - \left(K^1_1 + \zeta^1_{1,0} - \zeta^1_{0,0} \right),$$

and we set ΔQ^1_1 as the expansion size of firm 1 when she has one option left. The payoff $h^1_{1,0}$ is linear and increasing in x, similar to a Call option payoff. Thus, this optimal stopping problem can be considered as an analogue to pricing a perpetual American Call. In order to solve this problem, we apply the operator Ψ (3.12) and then Proposition 3.3 with $R = \mathcal{D}$, see Appendix B.1 for the details.

Proposition 4.1. (*firm 1 at stage (1, 0)*) The value function associated to the optimal stopping problem (4.1) is admitted as:

$$V^1_{1,0}(x) = \begin{cases} D^1_{1,0}(x) + \frac{F(x)}{F(S^{1,*}_{1,0})} \cdot h^1_{1,0}\left(S^{1,*}_{1,0}\right), & \text{if } x \in (\underline{d},\, S^{1,*}_{1,0}), \\ D^1_{0,0}(x) - K^1_1, & \text{if } x \in [S^{1,*}_{1,0},\, \bar{d}). \end{cases} \tag{4.2}$$

The corresponding policy is characterized by a threshold-type stopping time

$$\tau^{1,*}_{1,0} = \inf\{t \geq 0 : X^x_t \geq S^{1,*}_{1,0}\}, \tag{4.3}$$

where the expansion threshold $S^{1,*}_{1,0}$ satisfies the equation

$$F(S^{1,*}_{1,0}) = \frac{h^1_{1,0}}{\left(h^1_{1,0}\right)'}(S^{1,*}_{1,0}) \times F'(S^{1,*}_{1,0}). \tag{4.4}$$

In the converse scenario, at stage $(0, 1)$ firm 2 possesses the only expansion option. Substituting (3.20) into (2.12) for $n_2 = 1$, she solves the following optimal stopping problem:

$$V^2_{0,1}(x) - D^2_{0,1}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h^2_{0,1}(X_\tau) \right\}. \tag{4.5}$$

The first-mover payoff is derived as:

$$h^2_{0,1}(x) = D^2_{0,0}(x) - D^2_{0,1}(x) - K^2_1 = -\frac{\rho^2 \Delta Q^2_1}{\delta} \cdot x - \left(K^2_1 + \zeta^2_{0,1} - \zeta^2_{0,0} \right),$$

where we set ΔQ^2_1 as the expansion size of firm 2 when she has one option left. Since $h^2_{0,1}$ is decreasing and linear in x, the single-agent optimizing problem can be considered as an analog to the perpetual American Put. The following Proposition readily follows, see Appendix B.2.

Proposition 4.2. (*firm 2 at stage (0, 1)*) The value function associated to the optimal stopping problem (4.5) is admitted as:

$$V^2_{0,1}(x) = \begin{cases} D^2_{0,0}(x) - K^2_1, & \text{if } x \in (\underline{d},\, S^{2,*}_{0,1}], \\ D^2_{0,1}(x) + \frac{G(x)}{G(S^{2,*}_{0,1})} \cdot h^2_{0,1}(S^{2,*}_{0,1}), & \text{if } x \in (S^{2,*}_{0,1},\, \bar{d}). \end{cases} \tag{4.6}$$

The corresponding policy is characterized by a threshold-type stopping time

$$\tau_{0,1}^{2,*} = \inf\{t \geq 0 : X_t^x \leq S_{0,1}^{2,*}\}, \tag{4.7}$$

where the expansion threshold $S_{0,1}^{2,*}$ satisfies the equation

$$G(S_{0,1}^{2,*}) = \frac{h_{0,1}^2}{\left(h_{0,1}^2\right)'}(S_{0,1}^{2,*}) \times G'(S_{0,1}^{2,*}). \tag{4.8}$$

Example 3.2. (continued) Under a GBM model (3.8), the game value of firm 1 at stage (1, 0) is derived as:

$$V_{1,0}^1(x) = \begin{cases} D_{1,0}^1(x) + \frac{K_1^1+\zeta_{1,0}^1-\zeta_{0,0}^1}{\eta_+-1}\left(\frac{x}{S_{1,0}^{1,*}}\right)^{\eta_+}, & \text{if } x \in (0,\, S_{1,0}^{1,*}), \\ D_{0,0}^1(x) - K_1^1, & \text{if } x \in [S_{1,0}^{1,*},\, +\infty), \end{cases}$$

where $S_{1,0}^{1,*} = \frac{\delta\left(K_1^1+\zeta_{1,0}^1-\zeta_{0,0}^1\right)\eta_+}{\rho^1 \Delta Q_1^1(\eta_+-1)}$ is the expansion threshold of firm 1 at stage (1, 0). The game value of firm 2 at stage (0, 1) is derived as:

$$V_{0,1}^2(x) = \begin{cases} D_{0,0}^2(x) - K_1^2, & \text{if } x \in (0,\, S_{0,1}^{2,*}], \\ D_{0,1}^2(x) + \frac{\left(K_1^2+\zeta_{0,1}^2-\zeta_{0,0}^2\right)}{1-\eta_-}\left(\frac{x}{S_{0,1}^{2,*}}\right)^{\eta_-}, & \text{if } x \in (S_{0,1}^{2,*},\, +\infty), \end{cases}$$

where $S_{0,1}^{2,*} = \frac{\delta\left(K_1^2+\zeta_{0,1}^2-\zeta_{0,0}^2\right)\eta_-}{\rho^2 \Delta Q_1^2(\eta_--1)}$ is the expansion threshold of firm 2.

Under an OU model (3.10), there is no explicit formula. The thresholds and game values can be obtained by plugging F (3.11) and G (3.11) into preceding propositions and solving the resulting equations numerically.

We now extend these propositions to the general boundary case, with a full proof in Appendix B.3.

Theorem 4.3. ***(Boundary Cases)*** *The game value of firm 1 at stage $(n_1,\, 0)$ and the game value for firm 2 at stage $(0,\, n_2)$ for $n_1,\, n_2 \geq 1$ are admitted as:*

$$V_{n_1,0}^1(x) = \begin{cases} D_{n_1,0}^1(x) + \frac{F(x)}{F(S_{n_1,0}^{1,*})} \cdot h_{n_1,0}^1\left(S_{n_1,0}^{1,*}\right), & \text{if } x \in (\underline{d},\, S_{n_1,0}^{1,*}), \\ V_{n_1-1,0}^1(x) - K_{n_1}^1, & \text{if } x \in [S_{n_1,0}^{1,*},\, \bar{d}), \end{cases} \tag{4.9}$$

$$V_{0,n_2}^2(x) = \begin{cases} V_{0,n_2-1}^2(x) - K_{n_2}^2, & \text{if } x \in (\underline{d},\, S_{0,n_2}^{2,*}], \\ D_{0,n_2}^2(x) + \frac{G(x)}{G(S_{0,n_2}^{2,*})} \cdot h_{0,n_2}^2(S_{0,n_2}^{2,*}), & \text{if } x \in (S_{0,n_2}^{2,*},\, \bar{d}), \end{cases} \tag{4.10}$$

with first-mover payoff functions

$$h_{n_1,0}^1(x) = V_{n_1-1,0}^1(x) - D_{n_1,0}^1(x) - K_{n_1}^1,$$
$$h_{0,n_2}^2(x) = V_{0,n_2-1}^2(x) - D_{0,n_2}^2(x) - K_{n_2}^2.$$

Their corresponding policies are characterized by threshold-type stopping times

$$\tau_{n_1,0}^{1,*} = \inf\{t \geq 0 : X_t^x \geq S_{n_1,0}^{1,*}\}$$
$$\tau_{0,n_2}^{2,*} = \inf\{t \geq 0 : X_t^x \leq S_{0,n_2}^{2,*}\},$$

where the series of optimal stopping levels $S^{1,*}_{n_1,0}$, $S^{2,*}_{0,n_2}$ satisfy the equations

$$F(S^{1,*}_{n_1,0}) = \frac{h^1_{n_1,0}}{\left(h^1_{n_1,0}\right)'}(S^{1,*}_{n_1,0}) \times F'(S^{1,*}_{n_1,0}), \tag{4.11}$$

$$G(S^{2,*}_{0,n_2}) = \frac{h^2_{0,n_2}}{\left(h^2_{0,n_2}\right)'}(S^{2,*}_{0,n_2}) \times G'(S^{2,*}_{0,n_2}). \tag{4.12}$$

Remark 4.4. We do not assume any order of the threshold sequences $\left(S^{1,*}_{n_1,0}\right)_{n_1\geq 1}$ or $\left(S^{2,*}_{0,n_2}\right)_{n_2\geq 1}$. If firm 1's thresholds are not increasing (resp. decreasing for firm 2), she would simultaneously exercise multiple expansion options if X_t moves in her preferred direction.

We have shown that the optimal game policies of the "follower" who is the only firm with expansion options left are of threshold-type. Her "leader" rival's game values are then accordingly determined from (2.13)-(2.14):

Corollary 4.5. *(Leader Game Value)* The game values of firm 1 at stage $(0, n_2)$ and game values of firm 2 at stage $(n_1, 0)$ for n_1, $n_2 \geq 1$ are admitted as:

$$V^1_{0,n_2}(x) = \begin{cases} V^1_{0,n_2-1}(x), & \text{if } x \in (\underline{d},\, S^{2,*}_{0,n_2}], \\ D^1_{0,n_2}(x) + G(x)\cdot\left[\frac{V^1_{0,n_2-1}-D^1_{0,n_2}}{G}\right]\left(S^{2,*}_{0,n_2}\right), & \text{if } x \in (S^{2,*}_{0,n_2},\, \bar{d}), \end{cases} \tag{4.13}$$

$$V^2_{n_1,0}(x) = \begin{cases} D^2_{n_1,0}(x) + F(x)\cdot\left[\frac{V^2_{n_1-1,0}-D^2_{n_1,0}}{F}\right]\left(S^{1,*}_{n_1,0}\right), & \text{if } x \in (\underline{d},\, S^{1,*}_{n_1,0}), \\ V^2_{n_1-1,0}(x), & \text{if } x \in [S^{1,*}_{n_1,0},\, \bar{d}). \end{cases} \tag{4.14}$$

4.2. Nash Equilibria at Interior Stages

In the scenarios that each firm has available expansion options, i.e. at stages (n_1, n_2) with n_1, $n_2 > 0$, the firms interact through the negative externality of expansion on the electricity price. In the given context of Nash equilibrium, we will obtain equilibrium policies as the fixed-point of the firms' best-response to each other's strategy. We first derive the solution at stage $(1, 1)$, then extend to an arbitrary interior stage.

4.2.1. Equilibria at Stage (1, 1)

Based on (2.15) with $n_1 = n_2 = 1$, firm 1's first-mover payoff is admitted as $h^1_{1,1}(x) = V^1_{0,1}(x) - D^1_{1,1}(x) - K^1_1$, and her second-mover payoff is $l^1_{1,1}(x) = V^1_{1,0}(x) - D^1_{1,1}(x)$. Following preceding results for boundary stages, one can easily verify that $l^1_{1,1}(x) > h^1_{1,1}(x)$ for x small enough, and $l^1_{1,1}(x) < h^1_{1,1}(x)$ for x large. Assuming that $h - l$ is strictly monotone, we accordingly define a "leadership" point $L^1_{1,1}$ where first-mover payoff of firm 1 equals her second-mover payoff:

$$L^1_{1,1} := \inf\{x \in \mathcal{D} : h^1_{1,1}(x) > l^1_{1,1}(x)\}. \tag{4.15}$$

The meaning of the leadership point arises from the competitive aspect: when $x \leq L^1_{1,1}$, firm 1 does not compete to be first, since she is in fact (instantaneously) better-off being a second mover. On the other hand, when $x > L^1_{1,1}$, firm 1 would prefer to be a leader than a follower. Similar considerations lead to the leadership threshold of firm 2:

$$L^2_{1,1} := \sup\{x \in \mathcal{D} : h^2_{1,1}(x) > l^2_{1,1}(x)\}. \tag{4.16}$$

Recall that the game strategy of firm 2 at stage $(1,1)$ is defined as a $\mathbb{F}$-stopping time $\tau^2_{1,1}$. Since the first-mover payoff of firm 2 is greater than her second-mover payoff if and only if the level of X_t is low, it is reasonable to assume that $\tau^2_{1,1}$ is of threshold-type:

$$\tau^2_{1,1} = \inf\{t \geq 0 : X_t \leq s_2\}, \tag{4.17}$$

i.e. expansion of firm 2 takes place once X_t drops below s_2.

Depending on relationship between $h^1_{1,1}(s_2)$ and $l^1_{1,1}(s_2)$, the payoff of firm 1 would experience a jump up/down at the exercise threshold of firm 2. In particular, in the case that $s_2 < L^1_{1,1}$, i.e. $h^1_{1,1} < l^1_{1,1}$ at $x = s_2$, firm 1 actually benefits from having firm 2 invest at s_2, accordingly is not incentivized to preempt when firm 2 intends to invest. She now solves the optimal stopping problem following (2.15):

$$\widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[\mathbb{1}_{\{\tau<\tau^2_{1,1}\}} e^{-r\tau}\left\{h^1_{1,1}(X_\tau)\right\} + \mathbb{1}_{\{\tau>\tau^2_{1,1}\}} e^{-r\tau^2_{1,1}}\left\{l^1_{1,1}(X_{\tau^2_{1,1}})\right\}\right]. \tag{4.18}$$

Proposition 4.6. (*threshold-type best-response of firm 1 at stage* $(1,1)$) If $s_2 < L^1_{1,1}$, the best-response of firm 1 associated to $\tau^2_{1,1} = \tau_{s_2}$ specified in (4.17) is the stopping time given by

$$\tau^1_{1,1}(s_2) = \inf\{t \geq 0 : X_t \geq S^1_{1,1}(s_2)\}, \tag{4.19}$$

where the optimal stopping level $S^1_{1,1}(s_2) := S_1 > s_2$ is a function of s_2, characterized by the following equation:

$$\begin{aligned}\left[\left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)G(S_1) - h^1_{1,1}(S_1)G(s_2)\right] F'(S_1) + \left[h^1_{1,1}(S_1)F(s_2) - \left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)F(S_1)\right] G'(S_1) \\ = \left(h^1_{1,1}\right)'(S_1)\left[G(S_1)F(s_2) - G(s_2)F(S_1)\right].\end{aligned} \tag{4.20}$$

Consequently, the optimal stopping problem (4.18) admits the value function

$$\widetilde{V}^1_{1,1}(x, s_2) = \begin{cases} V^1_{1,0}(x), & \text{if } x \in (\underline{d},\, s_2), \\ D^1_{1,1}(x) + \tilde{\omega}^1_{1,1}F(x) + \tilde{\nu}^1_{1,1}G(x), & \text{if } x \in (s_2,\, S_1), \\ V^1_{0,1}(x) - K^1_1, & \text{if } x \in (S_1,\, \bar{d}), \end{cases} \tag{4.21}$$

where $\tilde{\omega}^1_{1,1} := \tilde{\omega}^1_{1,1}(s_2)$ and $\tilde{\nu}^1_{1,1} := \tilde{\nu}^1_{1,1}(s_2)$ are defined as

$$\tilde{\omega}^1_{1,1} = \frac{h^1_{1,1}(S_1)G(s_2) - \left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)G(S_1)}{F(S_1)G(s_2) - F(s_2)G(S_1)}, \qquad \tilde{\nu}^1_{1,1} = \frac{\left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)F(S_1) - h^1_{1,1}(S_1)F(s_2)}{F(S_1)G(s_2) - F(s_2)G(S_1)}. \tag{4.22}$$

Conversely, in the case that $s_2 > L^1_{1,1}$, i.e. if $h^1_{1,1} > l^1_{1,1}$ at $x = s_2$, then firm 1 is better off preemptively exercising *right before* firm 2, since her first-mover payoff is higher than her second-mover one. Recalling the definition of the leadership points L^i, we see that firm 1 is incentivized to preempt immediately $\tau^1_{1,1} = 0$ when the state process is in $(L^1_{1,1}, s_2]$ (see also Kong and Kwok (2007)). On $(s_2, \bar{d})$, firm 1 again solves an optimal stopping problem following (2.15).

Proposition 4.7. (*preemptive best-response of firm 1*) If $s_2 > L^1_{1,1}$, the best-response of firm 1 is

$$\tau^{1,e}_{1,1}(s_2) = \inf\{t \geq 0 : L^1_{1,1} < X_t \leq (s_2+) \text{ or } X_t \geq S^{1,e}_{1,1}(s_2)\}, \tag{4.23}$$

where the optimal stopping level $S^{1,e}_{1,1}(s_2) := S^e_1 \geq s_2$ is a solution to (4.20).

Note that the infinitesimal preemption of firm 2 corresponds to "stopping at s_2+" which can be considered as a limit of ε-optimal strategies. This is because the value function on $(s_2, \bar{d})$ is admitted in terms of concave majorant which yields $\lim_{x \searrow s_2} \widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(s_2) = h^1_{1,1}(s_2) > l^1_{1,1}(s_2)$. Therefore, stopping at s_2 is too late and firm 1 prefers to preempt right before s_2. Proof of this proposition is in Appendix C.1, and very similar steps for the best-response of firm 2 are stated in Appendix C.2. As expected, the best-response of firm 2 depends on the relationship between threshold s_1 of firm 1 and $L^2_{1,1}$.

To determine Nash equilibria of these firms' strategies, we start by deriving the best-response of firm 2 corresponding to $\tau^{1,e}_{1,1}(s_2)$ defined in (4.23). Since the state process X_t is assumed to be regular in $\mathcal{D}$, from firm 2 perspective, $\tau^{1,e}_{1,1}(s_2)$ is indifferent from the situation that firm 1 invest at $\tau^1_{1,1} = \inf\{t \geq 0 : X_t > L^1_{1,1}\}$. From Appendix C.2, if $L^1_{1,1} \geq L^2_{1,1}$, the corresponding best-response of firm 2 is a threshold-type stopping time of threshold $S^2 \leq L^1_{1,1}$, which leads us to a threshold-type equilibrium (if it exists) following Proposition 4.6. Otherwise, if $L^1_{1,1} < L^2_{1,1}$, the corresponding best-response of firm 2 is admitted as

$$\tau^{2,e,*}_{n_1,n_2} = \inf\{t \geq 0 : L^1_{1,1} \leq X_t < L^2_{1,1} \text{ or } X_t < S^{2,e,*}_{1,1}\}, \tag{4.24}$$

where $S^{2,e,*}_{1,1} := S^{2,e}_{1,1}(L^1_{1,1})$. Note that firm 1 (resp. firm 2) is not incentivized to invest when $X_t = L^1_{1,1}$ (resp. $X_t = L^2_{1,1}$). Back to firm 1, since $L^2_{1,1} > L^1_{1,1}$, her best-response to $\tau^{2,e*}_{n_1,n_2}$ is then admitted by Proposition 4.7 as:

$$\tau^{1,e,*}_{1,1} = \inf\{t \geq 0 : L^1_{1,1} < X_t \leq L^2_{1,1} \text{ or } X_t \geq S^{1,e,*}_{1,1}\}, \tag{4.25}$$

where $S^{1,e,*}_{1,1} := S^{1,e}_{1,1}(L^2_{1,1})$. To summarize, when $L^1_{1,1} < L^2_{1,1}$ we always have the *preemptive equilibrium* defined by $\left(\tau^{1,e,*}_{1,1}, \tau^{2,e,*}_{1,1}\right)$. Under that equilibrium, one or more firms invest immediately when $L^1_{1,1} < x < L^2_{1,1}$, otherwise the investment happens either at the thresholds $S^{i,e,*}_{1,1}$ or at the leadership points $L^i_{1,1}$. It remains to specify the outcome of the first situation, $x \in (L^1_{1,1}, L^2_{1,1})$. This is similar to an infinitesimal coordination game which admits multiple solutions. One approach proposed by Grasselli et al. (2013) involves instantaneous mixed strategies and leads to the following proposition (see proof in Appendix C.3).

Proposition 4.8. (coordination game at stage $(1,1)$**)** Let $(p_1(x), p_2(x))$ be a mixed strategy profile, with $p_i(x)$ denoting the probability that firm i attempts to invest at $X_t = x$ over an infinitesimal round, played repeatedly. There are three equilibrium strategies:

(i) $(p^*_1(x), p^*_2(x)) = (0, 1)$;

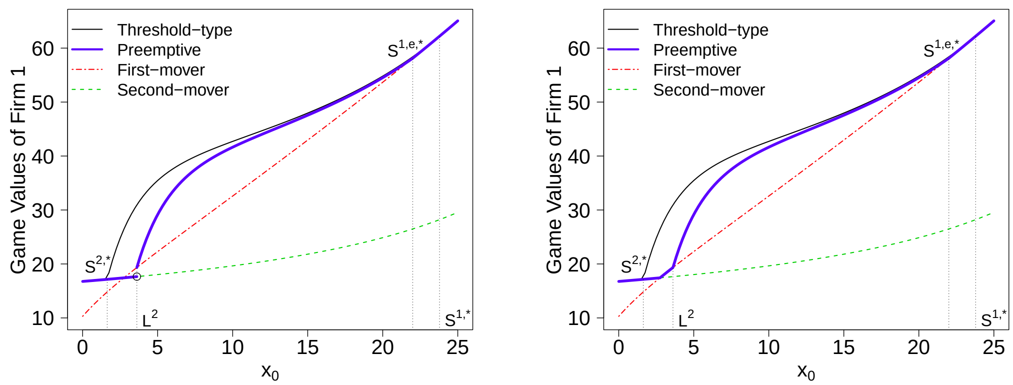
(ii) $(p^*_1(x), p^*_2(x)) = (1, 0)$;

(iii) $(p^*_1(x), p^*_2(x)) = \left(\frac{V^1_{0,1} - V^1_{1,0} - K^1_1}{V^1_{0,1} - D^1_{0,0}}(x), \frac{V^2_{1,0} - V^2_{0,1} - K^2_1}{V^2_{1,0} - D^2_{0,0}}(x)\right)$.

Note that there is a positive probability that the firms will invest *simultaneously* if they implement the third equilibrium, and firm 1 is more likely to invest when X_t is close to $L^2_{1,1}$ while firm 2 is more likely to invest when X_t is close to $L^1_{1,1}$. Moreover, the third equilibrium coincides with the first/second equilibrium when $x = L^1_{1,1}$ ($x = L^2_{1,1}$, resp.).

Choices (i) and (ii) above can be interpreted via a *preemptive priority* that pre-determines the winner of the instantaneous competition. For example, in our original economic example, a coal-fired plant is easier to build than a nuclear power plant, so one may assume that firm 2 has a preemptive priority, i.e. the coordination equilibrium selected is of type (i) above. Under that assumption firm

1 receives her second-mover value when $L_{1,1}^1 \le X_t < L_{1,1}^2$, which yields an upward jump in her resulting game value at $x = L_{1,1}^2$. In the converse scenario that firm 1 has a preemptive priority, she receives her first-mover value on $[L_{1,1}^1, L_{1,1}^2)$ and the resulting game value is continuous at $x = L_{1,1}^2$. Figure 2 illustrates these choices.



(a) (b)

Figure 2: Game values of the nuclear industry investor (cf. Section 5.4 for details) for both threshold-type equilibrium and preemptive equilibrium at stage $(2, 2)$ and $\mu = 0.23$. (Left: a) Firm 2 has a preemptive priority. (Right: b) Firm 1 has a preemptive priority.

Returning to Nash equilibria involving threshold-type strategies, the fixed-point characterization (2.17) boils down to solving the following system of equations:

$$\begin{cases} \left[l_{1,1}^1(S_2)G(S_1) - h_{1,1}^1(S_1)G(S_2)\right] F'(S_1) + \left[h_{1,1}^1(S_1)F(S_2) - l_{1,1}^1(S_2)F(S_1)\right] G'(S_1) \\ \qquad = \left(h_{1,1}^1\right)'(S_1)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right], \\ \left[h_{1,1}^2(S_2)G(S_1) - l_{1,1}^2(S_1)G(S_2)\right] F'(S_2) + \left[l_{1,1}^2(S_1)F(S_2) - h_{1,1}^2(S_2)F(S_1)\right] G'(S_2) \\ \qquad = \left(h_{1,1}^2\right)'(S_2)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right], \end{cases} \tag{4.26}$$

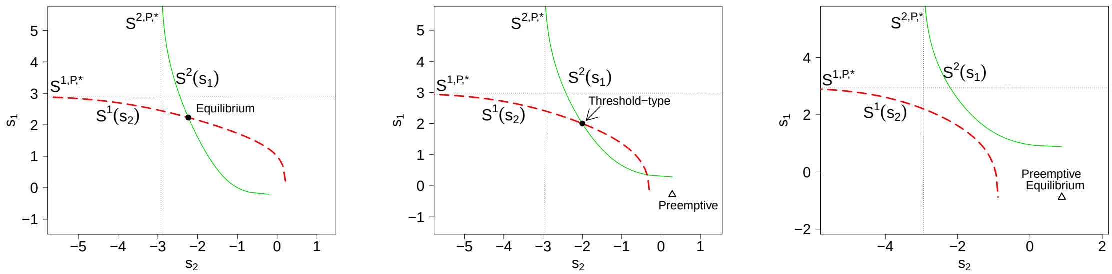
for $(S_1, S_2) \in [L_{1,1}^2, \bar{d}) \times (\underline{d}, L_{1,1}^1]$, and solutions to this system correspond to pairs of investment thresholds at stage $(1, 1)$. To discuss the existence of such equilibria, we state the following corollary characterizing the best-response curves.

Corollary 4.9. (best-response curves)

(i) For $s_2 \le L_{1,1}^1$ (resp. $s_1 \ge L_{1,1}^2$), the best-response function $s_2 \longmapsto S_{1,1}^1(s_2)$ (resp. $s_1 \longmapsto S_{1,1}^2(s_1)$) is continuous.

(ii) As $s_2 \downarrow \underline{d}$ (resp. $s_1 \uparrow \bar{d}$), the best-response of firm 1 (resp. firm 2) converges to a finite threshold $S_{1,1}^{1,P,*}$ (resp. $S_{1,1}^{2,P,*}$).

The first statement is a simple application of the implicit function theorem. An interpretation of $S^{i,P,*}$ is provided in Section 4.3. Existence of solutions to the system (4.26) then corresponds to existence of crossing points of these best-response curves. Depending on the relation between $L_{1,1}^1$ and $L_{1,1}^2$, there are three scenarios of the best-response curves, sketched in Figure 3.



(a) Scenario I (b) Scenario II (c) Scenario III

Figure 3: Equilibrium scenarios for the best-response curves. The red dashed lines represent best-response of firm 1, while the green solid lines represent best-response of firm 2. The dotted lines represent the limiting thresholds $S^{1,P,*}_{1,1}$, $S^{2,P,*}_{1,1}$ discussed in Section 4.3. Note that for $s_2 > L^1_{1,1}$ and $s_1 < L^2_{1,1}$ there is no threshold-type best-response and the "Δ" marks a preemptive equilibrium which corresponds to $(L^2_{1,1}, L^1_{1,1})$ in this numerical example.

Scenario I: $L^1_{1,1} > L^2_{1,1}$. In this case there is guaranteed at least one crossing point of the best-response curves, which corresponds to a threshold-type equilibrium at stage $(1, 1)$ (Figure 3(a)). Only threshold-type equilibria exist in this scenario, matching the setting studied by De Angelis et al. (2015, Section 3.1).

Scenario II: $L^1_{1,1} < L^2_{1,1}$ and the best-response curve cross (see Figure 3(b) which has 2 crossings). Consequently, both threshold-type equilibria and a preemptive equilibrium characterized by (4.24)-(4.25) exist.

Scenario III: $L^1_{1,1} < L^2_{1,1}$ and no crossing points between the best-response curves (Figure 3(c)), which implies that only a preemptive equilibrium exists.

If $L^1_{1,1} < L^2_{1,1}$ (i.e. beyond of Scenario I), existence of threshold-type equilibria is not guaranteed. From Corollary 4.9, one sufficient condition for existence is that $S^1_{1,1}(L^1_{1,1}) > L^2_{1,1}$ and $S^2_{1,1}(L^2_{1,1}) < L^1_{1,1}$. The latter condition does not actually hold in the numerical example sketched in Figure 3(b). Numerical examples in Section 5 suggest that Scenario III occurs under high volatility σ.

Remark 4.10. In Scenarios I & II there are multiple Nash MPEs, so equilibrium selection is an important issue. From monotonicity of payoff functions, typically a higher threshold of firm 1 and a lower threshold of firm 2 yield higher game values to both firms. This assumption combining with the sequential nature of investment decisions with a flavor of a Stackelberg competition preferences the *latest* equilibrium, i.e. selecting the highest threshold $S^1_{1,1}$ and the corresponding lowest threshold $S^2_{1,1}$. To understand the logic for this preference, consider two equilibria termed the *later* (higher threshold of firm 1 and lower threshold of firm 2) and the *earlier*. Now consider firm 1 currently at her early threshold $S^{1,erl}_{1,1}$ and contemplating whether to expand now, i.e. pick the early equilibrium, or wait. Conditional on firm 2 implementing $S^{2,lat}_{1,1}$, best-response optimality implies that

$$\widetilde{V}^1_{1,1}(S^{1,erl}_{1,1}; \tau^{2,lat}_{1,1}) = J^1_{1,1}(S^{1,erl}_{1,1}; \tau^{1,lat}_{1,1}, \tau^{2,lat}_{1,1}) \geq J^1_{1,1}(S^{1,erl}_{1,1}; \tau^{1,erl}_{1,1}, \tau^{2,lat}_{1,1}) = V^1_{0,1}(S^{1,erl}_{1,1}) - K^1_1.$$

So under that assumption, firm 1 can extract higher expected NPV by waiting and not investing immediately. Of course, there is a risk that the assumption is false, firm 2 will implement the early threshold $S^{2,erl}_{1,1}$, whereby firm 1 will lose by waiting. However, by symmetry, when *in the future*, X_t were to reach $S^{2,erl}_{1,1}$, firm 2 would face the same dilemma, and (by then knowing that firm 1 did not invest *in the past*) would also prefer to wait, in the hope of realizing the later equilibrium. It follows that the sequential nature of decisions encourages maximization of game values – each firm can rationally assume that *in the future* her rival will refrain from the earlier equilibrium, and hence rationally commit to waiting right now, and not expanding early. In effect, a firm can credibly signal to her rival that she is implementing the later equilibrium, yielding a higher game value to both.

Note that the above argument works when firms make decisions sequentially, but does not work for simultaneous actions where threat of preemption takes precedence. Namely, the Stackelberg logic cannot rule out preemptive equilibria. For example, in Scenario II, when X_t hits $L^2_{1,1}$, firm 1 has no time to signal that she prefers a threshold-type equilibrium, as she faces the immediate threat of firm 2 investing which would at once generate a loss in her (firm 1) NPV.

4.2.2. Equilibria at General Stages

To generalize to further interior stages (n_1, n_2) we assume that for all $n'_1 < n_1, n'_2 < n_2$, there is a threshold-type equilibrium (which has been selected, if necessary, among available choices) at stage (n'_1, n'_2). Under this assumption we can inductively apply the concave majorant method.

To fix ideas, consider stage $(2, 1)$; we use $(S^{1,*}_{1,1}, S^{2,*}_{1,1})$ to denote investment thresholds of the *threshold-type* equilibrium strategies adopted at stage $(1, 1)$. Then given firm 2's strategy with threshold $s_2 < L^1_{2,1}$, firm 1 solves:

$$\widetilde{V}^1_{2,1}(x, s_2) - D^1_{2,1}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\mathbb{1}_{\{\tau < \tau^2_{2,1}\}} e^{-r\tau} \left\{ h^1_{2,1}(X_\tau) \right\} + \mathbb{1}_{\{\tau > \tau^2_{2,1}\}} e^{-r\tau^2_{2,1}} \left\{ l^1_{2,1}(X_{\tau^2_{2,1}}) \right\} \right], \quad (4.27)$$

with $h^1_{2,1}(x) = V^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2$, where $V^1_{1,1}$ is the equilibrium game value received by firm 1 at stage $(1, 1)$. Since $S^{2,*}_{1,1} < S^{1,*}_{1,1}$ from Proposition 4.6, from (4.21) the corresponding first-mover payoff is derived as

$$h^1_{2,1}(x) = \begin{cases} D^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2 + V^1_{1,0}(x) - D^1_{1,1}(x), & \text{if } x \le S^{2,*}_{1,1}, \\ D^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2 + \omega^1_{1,1} F(x) + \nu^1_{1,1} G(x), & \text{if } S^{2,*}_{1,1} < x \le S^{1,*}_{1,1}, \\ D^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2 + h^1_{1,1}(x), & \text{if } x > S^{1,*}_{1,1}. \end{cases} \quad (4.28)$$

Since F and G terms do not contribute to $(\mathcal{L} - r)\, h^1_{2,1}$, and $(\mathcal{L} - r)\, h^1_{1,1}(x) < 0$ for $x > S^{1,*}_{1,1}$, we conclude that $h^1_{2,1}(x)$ is in the class $\mathcal{H}_{\text{inc}}$. Similarly, one can check that the first-mover payoff of firm 2, $h^2_{2,1}(x)$ is in the class $\mathcal{H}_{\text{dec}}$. Consequently, Proposition 3.3 and Lemma 3.6 allow us to apply similar arguments as Proposition 4.6 and Appendix C.2 to derive threshold-type best-response of these firms. Similar arguments yield

Theorem 4.11. *Let $h^i_{n_1,n_2}$ and $l^i_{n_1,n_2}$ be the first-mover payoffs and second-mover payoffs associated to optimal stopping problems* (2.15)*-*(2.16)*, for $i = 1, 2$. The threshold-type equilibrium policies implemented by the firms at stage (n_1, n_2) are the stopping times*

$$\tau^{1,*}_{n_1,n_2} = \inf\{t \ge 0 : X_t \ge S^{1,*}_{n_1,n_2}\},$$
$$\tau^{2,*}_{n_1,n_2} = \inf\{t \ge 0 : X_t \le S^{2,*}_{n_1,n_2}\},$$

where $\left(S^{1,*}_{n_1,n_2}, S^{2,*}_{n_1,n_2}\right)$ is a solution to the system of equations

$$\begin{cases} \left[l^1_{n_1,n_1}(S_2)G(S_1) - h^1_{n_1,n_1}(S_1)G(S_2)\right] F'(S_1) + \left[h^1_{n_1,n_1}(S_1)F(S_2) - l^1_{n_1,n_1}(S_2)F(S_1)\right] G'(S_1) \\ \qquad = \left(h^1_{n_1,n_1}\right)'(S_1) \left[G(S_1)F(S_2) - G(S_2)F(S_1)\right], \\ \left[h^2_{n_1,n_1}(S_2)G(S_1) - l^2_{n_1,n_1}(S_1)G(S_2)\right] F'(S_2) + \left[l^2_{n_1,n_1}(S_1)F(S_2) - h^2_{n_1,n_1}(S_2)F(S_1)\right] G'(S_2) \\ \qquad = \left(h^2_{n_1,n_1}\right)'(S_2) \left[G(S_1)F(S_2) - G(S_2)F(S_1)\right]. \end{cases} \quad (4.29)$$

Consequently, the equilibrium game values are

$$V^1_{n_1,n_2}(x) = \begin{cases} V^1_{n_1,n_2-1}(x), & \text{if } x \in (\underline{d},\ S^{2,*}_{n_1,n_2}], \\ D^1_{n_1,n_2}(x) + \omega^1_{n_1,n_2} F(x) + \nu^1_{n_1,n_2} G(x), & \text{if } x \in (S^{2,*}_{n_1,n_2},\ S^{1,*}_{n_1,n_2}), \\ V^1_{n_1-1,n_2}(x) - K^1_{n_1}, & \text{if } x \in [S^{1,*}_{n_1,n_2},\ \bar{d}), \end{cases} \tag{4.30}$$

$$V^2_{n_1,n_2}(x) = \begin{cases} V^2_{n_1,n_2-1}(x) - K^2_{n_2}, & \text{if } x \in (\underline{d},\ S^{2,*}_{n_1,n_2}], \\ D^2_{n_1,n_2}(x) + \omega^2_{n_1,n_2} F(x) + \nu^2_{n_1,n_2} G(x), & \text{if } x \in (S^{2,*}_{n_1,n_2},\ S^{1,*}_{n_1,n_2}), \\ V^2_{n_1-1,n_2}(x), & \text{if } x \in [S^{1,*}_{n_1,n_2},\ \bar{d}), \end{cases} \tag{4.31}$$

where

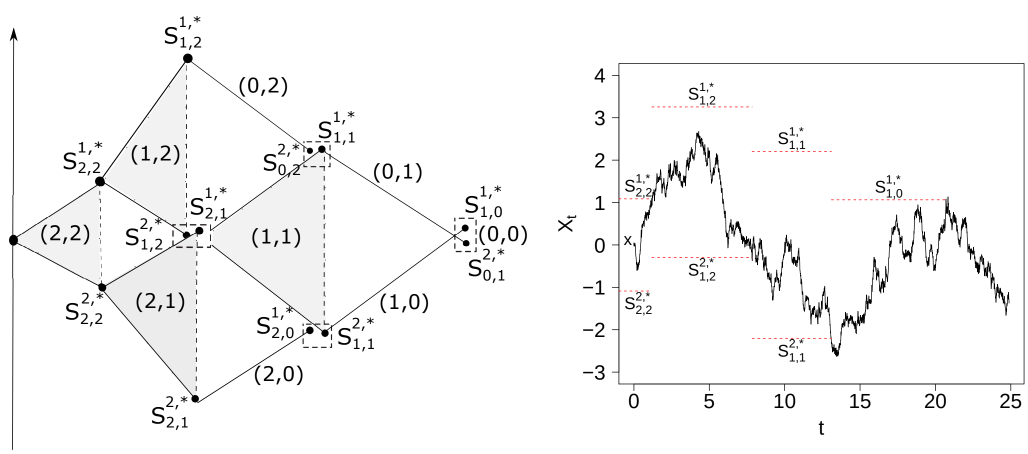
$$\omega^1_{n_1,n_2} = \frac{h^1_{n_1,n_2}(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - l^1_{n_1,n_2}(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}, \tag{4.32}$$

$$\nu^1_{n_1,n_2} = \frac{l^1_{n_1,n_2}(S^{2,*}_{n_1,n_2})F(S^{1,*}_{n_1,n_2}) - h^1_{n_1,n_2}(S^{1,*}_{n_1,n_2})F(S^{2,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}, \tag{4.33}$$

$$\omega^2_{n_1,n_2} = \frac{l^2_{n_1,n_2}(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - h^2_{n_1,n_2}(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}, \tag{4.34}$$

$$\nu^2_{n_1,n_2} = \frac{h^2_{n_1,n_2}(S^{2,*}_{n_1,n_2})F(S^{1,*}_{n_1,n_2}) - l^2_{n_1,n_2}(S^{1,*}_{n_1,n_2})F(S^{2,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}. \tag{4.35}$$

To recap, the overall dynamic expansion game proceeds in discrete stages. At each interior stage, there are two thresholds $S^{i,*}_{n_1,n_2}$, which determine the investment level of firm $i = 1, 2$. Figure 4(a) shows a schematic for all the different thresholds starting at $\vec{N}_0 = (2, 2)$. To better visualize the game evolution, a simulated state trajectory is presented in Figure 4(b) with the firms' thresholds for the case $\Delta Q^i = 0.25$ (in which interior stage equilibria correspond to Scenario II and we assume the firms implement the latest threshold-type equilibrium). The firms' equilibrium policies determine a two-sided exit region for each interior game stage and one-sided exit region for the boundary cases. As the state process X hits one of the firms' expansion threshold, the game jumps to the subsequent stage and yields a new exit region.



(a) Stages

(b) Sample Trajectory of X

Figure 4: (Left) Sketch of the various stage thresholds as a function of (n_1, n_2). (Right) A sample trajectory of X with $X_0 = 0, \vec{N}_0 = (2, 2)$. The corresponding game evolution is (2, 2) → (1, 2) → (1, 1) → (0, 1) → (0, 0) with expansions at the first hitting times of the corresponding thresholds.

4.3. Predetermined Expansion Priority

In a competitive situation, the threat of the rival investing first causes the firms to act preemptively. As a result, competition leads to loss of value compared to a first-best strategy without any rivalry. To quantify this loss, we compare the derived equilibrium game values to the setting where the order of investment is pre-assigned. In the latter model, one firm is granted a *priority* option (Grasselli et al., 2013) meaning that she is allowed to single-handedly optimize her investment level without worrying about preemption. After the pre-assigned leader invests, the rival obtains a chance to invest as well. Thus, the priority option removes the preemption threat, but still maintains the multi-stage competition aspect.

With multiple investment options one may consider a combination of several priority options; to fix ideas we focus on the simplest situation where each firm starts with one expansion option $\vec{N}_0 = (1,1)$, and therefore priority grants *leadership* status, making the rival a follower. Assuming that the priority option is given to firm 1, her decision now reduces to solving the optimal stopping problem:

$$V_{1,1}^{1,P}(x) - D_{1,1}^1(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h_{1,1}^1 \left(X_\tau^x \right) \right\}, \tag{4.36}$$

where the payoff function is specified in (C.1)—after her investment the game will be in stage $(0,1)$ with the associated game value $V_{0,1}^1$.

Proposition 4.12. (*Policy and value function with priority option*) The value function associated to the optimal stopping problem (4.36) is:

$$V_{1,1}^{1,P}(x) = \begin{cases} D_{1,1}^1(x) + \frac{F(x)}{F(S_{1,1}^{1,P,*})} \cdot h_{1,1}^1 \left(S_{1,1}^{1,P,*} \right), & \text{if } x \in (\underline{d},\, S_{1,1}^{1,P,*}), \\ D_{0,1}^1(x) - K^1, & \text{if } x \in [S_{1,1}^{1,P,*},\, \bar{d}). \end{cases} \tag{4.37}$$

The corresponding investing policy is $\tau_P^{1,*} = \inf\{t \geq 0 :\, X_t^x \geq S_{1,1}^{1,P,*}\}$, where the optimal stopping level $S_{1,1}^{1,P,*}$ solves $F(S_{1,1}^{1,P,*}) \times \left(h_{1,1}^1\right)' (S_{1,1}^{1,P,*}) = h_{1,1}^1(S_{1,1}^{1,P,*}) \times F'(S_{1,1}^{1,P,*})$.

The proof matches that of Proposition 4.1, and hence is omitted. An important property is that the optimal priority threshold $S_{1,1}^{1,P,*}$ is no less than the leader's threshold $S_{1,1}^{1,*}$, which implies that competition causes preemption: if $X_0 \in (S_{1,1}^{1,*}, S_{1,1}^{1,P,*})$ then firm 1 chooses to invest now even though without competition she would be better off to wait until X rises up to $S_{1,1}^{1,P,*}$.

Note that pre-assigning firm 1 as the first-mover is mathematically equivalent to taking $s_2 \to \underline{d}$, i.e. best-response when firm 2 never invests. It follows that $S_{1,1}^{1,P,*} > S^1(s_2)$ for $\forall s_2$, i.e. $S^{1,P,*}$ is the limiting value of the best-response curve $\lim_{s_2 \searrow \underline{d}} S^1(s_2)$, see the earlier Figure 3.

4.4. Central Planner

A different perspective on competition is offered by considering the difference between the primary non-cooperative setting and its cooperative analogue. The latter can be thought of as a central planner (or state-controlled holding company) that jointly optimizes the aggregate expected profits. Since there is no more rivalry, this "monopoly" model reduces to a classical sequential real option problem; a related problem was treated in Aïd et al. (2014).

Treatment of the cooperative investment problem is analogous to the problems considered after we aggregate the profit rates via

$$\pi_{n_1,n_2}^M(x) := \pi_{n_1,n_2}^1(x) + \pi_{n_1,n_2}^2(x). \tag{4.38}$$

If π^i's are linear in x, then so is π^M and hence the solution structure remains the same. In particular, in states $(1,0),(0,1)$ we have investment thresholds $S_{1,0}^{M,*}, S_{0,1}^{M,*}$. To handle the investment decision in stage $(1,1)$ and beyond, we can view it as optimizing the *two-sided* stopping time $\tau_{1,1}^{1,M} \wedge \tau_{1,1}^{2,M}$, where $\tau_{1,1}^{1,M}$ is the time to invest in firm 1-expansion, while $\tau_{1,1}^{2,M}$ is the time to invest in firm 2:

$$V_{1,1}^{M}(x) = D_{1,1}^{1}(x) + D_{1,1}^{2}(x) + \sup_{\tau_{1,1}^{1,M},\, \tau_{1,1}^{2,M} \in \mathcal{T}} \mathbb{E}_x \left[\mathbb{1}_{\{\tau_{1,1}^{1,M} < \tau_{1,1}^{2,M}\}} e^{-r\tau_{1,1}^{1,M}} \left\{ h_{1,1}^{1,M}\left(X_{\tau_{1,1}^{1,M}} \right) \right\} + \mathbb{1}_{\{\tau_{1,1}^{1,M} > \tau_{1,1}^{2,M}\}} e^{-r\tau_{1,1}^{2,M}} \left\{ h_{1,1}^{2,M}\left(X_{\tau_{1,1}^{2,M}} \right) \right\} \right]. \quad (4.39)$$

Appendix D presents the resulting solution for $V_{1,1}^{M}$ and the optimal investment thresholds $S_{1,1}^{i,M,*}$ that define $\tau_{1,1}^{i,M}$. Since the cooperative solution is first-best, $V_{1,1}^{M} \geq V_{1,1}^{1} + V_{1,1}^{2}$, see Figure 5(b).

5. Numerical Examples

We assume in all following numerical examples that when available, the (latest) threshold-type equilibria are selected at each stage. Economically this means that the firms are not very aggressive and refrain from preemptive equilibrium strategies.

5.1. Dynamic Preemption and Over-investment for 1-Shot Expansions

In this section, we use a symmetric example to compare competitive investment strategies to their counterparts where competition is constrained (priority option) or firms cooperate. To focus on the preemption effect, we assume that each firm possesses only one option to expand her capacity. The firm parameters are identical, except that one prefers positive X_t and the other negative X_t.

Parameter	Meaning	Value
θ	mean-reversion level	0
μ	mean-reversion rate	0.06
σ	volatility	0.60
r	interest rate	0.03
ρ_i	cost sensitivity	± 1.60
$Q_{1,1}^{i}$	initial capacity of firm i	1.00
$Q_{0,0}^{i}$	expanded capacity of firm i	1.50
K^{i}	expansion cost	5
x_0	initial state of X_t	0

Table 1: Numerical setting for Section 5.1.

The relative cost X_t mean-reverts is an OU process as in Example 3.2 with zero mean-reversion level $\theta = 0$. As a consequence of these choices, all the equilibrium thresholds will be symmetric about $x = 0$. The price model is:

$$P_{n_1,n_2} = 30\big(1 - 0.17(Q_{n_1,n_2}^{1} + Q_{n_1,n_2}^{2})\big).$$

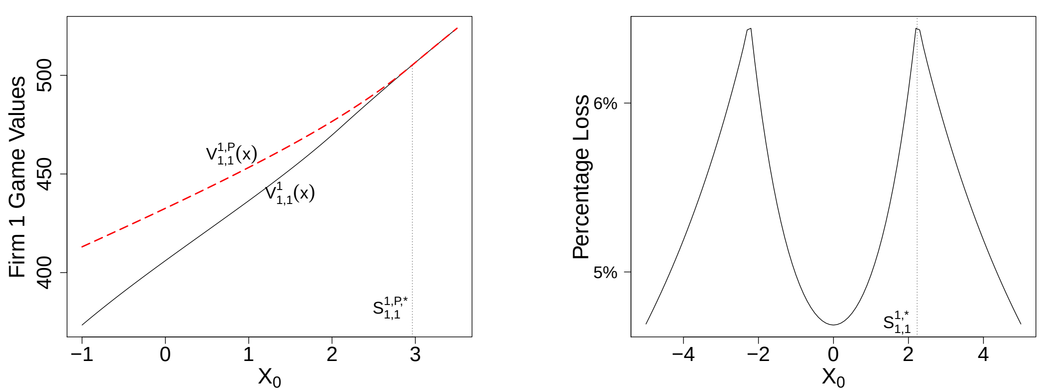
Starting with stage $(1,1)$ we compare three competition models: (i) non-cooperative game, where both firms compete to become the “leader” by investing first, the follower then has a chance to invest second; (ii) priority case where firm 1 is pre-determined to be the leader and hence can optimize her threshold $S_{1,1}^{1,P,*}$ without worrying about threat of preemption; (iii) cooperative game or the central planner model where the aggregate profit of the two firms is optimized. Thanks to the symmetry present in the example, in the competitive model the thresholds are symmetric about zero $S_{1,1}^{1,*} = -S_{1,1}^{2,*}$; also the second-investment thresholds are the same in case (i) and (ii) since the follower does not care if there was an initial priority option or not.

	Non-cooperative Game	Predetermined Leader	Central Planner
First-stage Policy	$S_{1,1}^{1,*} = 2.1822$	$S_{1,1}^{1,P,*} = 2.964$	$S_{1,1}^{1,M,*} = 2.886$
Second-stage Policy	$S_{1,0}^{1,*} = -0.0387$	—	$S_{1,0}^{M} = 10.5209$
Expected time of the first investment	$m_{S_{1,1}^{2,*},S_{1,1}^{1,*}}(0)$ $=16.125$	$m_{S_P^{1,*}}(0) = 107.448$	$m_{S_{1,1}^{1,M,*},S_{1,1}^{2,M,*}}(0)$ $=39.372$
Expected time of the second investment	$m_{S_{0,1}^{2,*}}(S_{1,1}^{1,*})$ $=18.943$	$m_{S_{0,1}^{2,*}}(S_{1,1}^{1,P,*}) = 23.269$	$m_{S_{0,1}^{2,M,*}}(S_{1,1}^{1,M,*})$ $= 7.280 \times 10^8$

Table 2: Equilibrium thresholds of firm 1 and expected times of sequential investments. By symmetry, equilibrium thresholds of firm 2 are the same values with opposite signs at each game stage. Stage $(1,1)$ equilibrium corresponds to Scenario I and is therefore unique.

With parameter values stated in Table 1, the equilibrium thresholds of firm 1 associated to each competition model are presented in Table 2. For example, her game strategy at stage $(1,1)$ is: $\tau_{1,1}^{1,*} = \inf\{t \geq 0 : X_t \geq S_{1,1}^{1,*} = 2.1822\}$, and so forth. We remark that under these parameters, stage $(1,1)$ yields Scenario I and the resulting threshold-type equilibrium is unique.

We observe that firm 1's threshold in a competitive market, $S_{1,1}^{1,*} = 2.1822$, is less than her equilibrium thresholds corresponding to other situations, which shows that competition leads to earlier expansion. With a priority option, firm 1 would not invest until $X_t \geq 2.964$, and under central planner, firm 1 would not invest until $X_t \geq 2.886$. In other words, when the state process X_t is in $(S_{1,1}^{1,*}, S_{1,1}^{1,P,*}) = (2.182, 2.964)$, the firm over-invests immediately, rather waiting for her first-best (i.e. non-competitive) threshold. Figure 5 quantifies the resulting impacts on expected profits. The left panel compares $V_{1,1}^1$ to $V_{1,1}^{1,P}$—note that the two are equal for $x \geq S_{1,1}^{1,P,*}$. The right panel shows $\left(1 - \frac{V_{1,1}^1(x)+V_{1,1}^2(x)}{V_{1,1}^M(x)}\right)$ which is the difference between the net profit of the central planner and the sum of two competitive firms' net profit. Cooperation increases profits, and the above ratio quantifies the aggregate loss caused by competitive preemption. This loss is maximized when the initial state $X_0 = x$ is equal to the expansion thresholds $S_{1,1}^{i,*}$, whereby one of the firms overinvests immediately.



(a) Game values of firm 1

(b) Percentage loss

Figure 5: Impact of competition on game values. (Left) Equilibrium game value of firm 1 predetermined as the leader $V_{1,1}^{1,P}$ (red dashed curve), versus firm 1 game value in a competitive market $V_{1,1}^1$ (solid black). (Right) Percentage loss $1 - (V_{1,1}^1(x_0) + V_{1,1}^2(x_0))/V_{1,1}^M(x_0)$ in the firms' aggregate profit due to competition.

To convert the above thresholds into a more economic context, we compute the average timing of an investment. For example, the first investment takes place at $\tau^1_{1,1} \wedge \tau^2_{1,1}$. The respective expected value can be obtained by viewing this quantity as the first exit time from an interval $(a, b) \supset x$, $\tau_{ab} = \inf\{t \geq 0 : X^x_t \leq a \text{ or } X^x_t \geq b\}$. Denote its expectation as $m(x; a, b) := \mathbb{E}_x[\tau_{ab}]$. Then the expected time of the first investment in a competitive market, the priority case, or the central planner are $m(0; S^{2,*}_{1,1}, S^{1,*}_{1,1})$, $m(0; -\infty, S^{1,P,*}_{1,1})$ and $m(0; S^{2,M,*}_{1,1}, S^{1,M,*}_{1,1})$, respectively. We also will consider the time between the first and the second investments (i.e. between the leader and follower times).

Remark 5.1. Analytic evaluation of $m(x; a, b)$ is possible. Consider the expected first passage time of X_t to level a, $m_a(x) := \mathbb{E}_x[\tau_a]$, where $\tau_a = \inf\{t \geq 0 : X^x_t = a\}$. Fixing a and applying Dynkin's formula, m_a solves the ordinary differential equation:

$$\mathcal{L}m_a(x) + 1 = 0, \qquad x < a, \tag{5.1}$$

with the boundary condition $m_a(a) = 0$. In particular, for an OU process, we solve (5.1):

$$m_a(x) = \mathbb{E}_x[\tau_a] = \frac{\sqrt{2\pi}}{\mu} \int_{(x-\theta)\sqrt{\frac{2\mu}{\sigma^2}}}^{(a-\theta)\sqrt{\frac{2\mu}{\sigma^2}}} \Phi(z)\, e^{\frac{1}{2}z^2} dz. \tag{5.2}$$

For $x > b$, by symmetry of an OU process:

$$m_b(x) = \mathbb{E}_x[\tau_b] = \frac{\sqrt{2\pi}}{\mu} \int_{(\theta-x)\sqrt{\frac{2\mu}{\sigma^2}}}^{(\theta-b)\sqrt{\frac{2\mu}{\sigma^2}}} \Phi(z)\, e^{\frac{1}{2}z^2} dz. \tag{5.3}$$

Then $m_{a,b}(x) = \mathbb{E}_x[\tau_{ab}]$ is given by (Darling and Siegert, 1953)

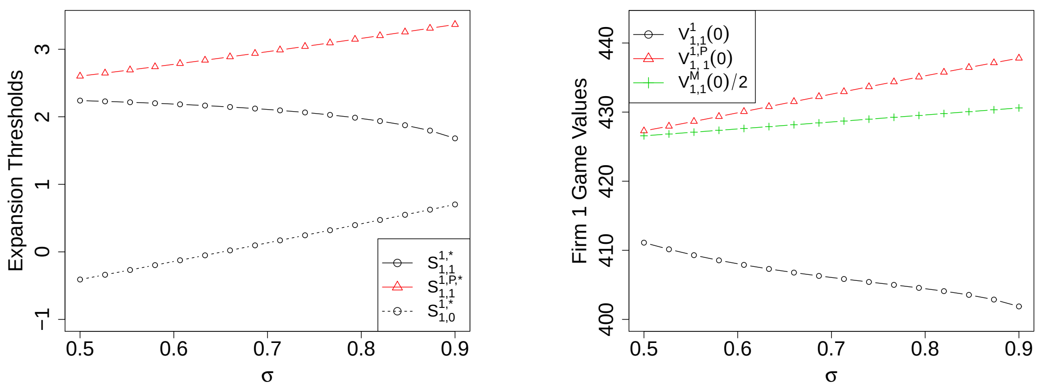
$$m_{a,b}(x) = \frac{m_a(x)m_b(a) + m_b(x)m_a(b) - m_a(b)m_b(a)}{m_b(a) + m_a(b)}. \tag{5.4}$$

Table 2 shows that there is in fact a very significant wedge between average investment under competition, and average investment by the central planner. An interesting observation is that the expected time of the second investment for a central planner (7.280×10^8) is so long that it is almost equivalent that the central planner will invest only once. With an initial state $X_0 = 0$, the expected time to *finish* expansion in a competitive market, $m_{S^{2,*}_{1,1}, S^{1,*}_{1,1}}(0) + m_{S^{2,*}_{0,1}}(S^{1,*}_{1,1}) = 35.068$ is also much shorter compared to the priority/cooperative analogues, so competition not only creates first-stage preemption, but also hastens the overall capacity build-up in the industry.

5.2. Effects of Market Fluctuation

We next discuss the effect of market fluctuations which can be parameterized by the volatility σ of the OU process (3.10). Higher volatility of the relative costs X_t implies more fluctuations in market conditions.

Figure 6 shows that as the volatility σ increases, the expansion threshold at stage $(1, 0)$ $S^{1,*}_{1,0}$ of firm 1 increases, while her stage $(1, 1)$-threshold $S^{1,*}_{1,1}$ decreases. With a priority option, the corresponding threshold of firm 1 is positively related to σ. In Figure 6(b), the equilibrium expected profit of firm 1 increases if she gets first-mover priority, which coincides with the intuition that more market fluctuations lead to higher average revenue. In particular, with higher σ, the predetermined leader can wait longer until the state process moves to her preferred direction and then reap higher rewards. On the contrary, in a competitive market, game values *decline* as σ increases. This discrepancy highlights the effects of competition. Namely, in the face of a more volatile market, firms become more aggressive and expand capacity much sooner, to the extent that their expected net profits drop. We observe that for σ large, the preemptive equilibrium (4.24)-(4.25)) becomes the only available game strategy the firms can adopt (i.e. we are in scenario III from Section 4.2).



(a) (b)

Figure 6: Effect of cost volatility σ. (Left) Equilibrium thresholds of firm 1 $S^{1,*}_{1,1}, S^{1,*}_{1,0}, S^{1,P,*}_{1,1}$ as the volatility σ of (X_t) varies. (Right) Respective equilibrium game values of firm 1 at $X_0 = 0$ versus σ.

	firm 1	firm 2
Stage $(2,1)$	$S^{1,*}_{2,1} = 1.133$	$S^{2,*}_{2,1} = -2.1312$
Stage $(1,1)$	$S^{1,*}_{1,1} = 3.323$	$S^{2,*}_{1,1} = -1.2083$
Stage $(2,0)$	$S^{1,*}_{2,0} = -1.043$	—
Stage $(1,0)$	$S^{1,*}_{2,0} = 1.064$	—
Stage $(0,1)$	—	$S^{2,*}_{2,0} = 0.0387$

Table 3: Investment thresholds for the case $\Delta Q^1 = 0.25, \Delta Q^2 = 0.5$. Interior stage equilibria correspond to Scenario II with multiple threshold-type equilibria; according to Remark 4.10 we always pick the latest one.

5.3. Case Study: Impact of Multi-Part Investments

As discussed, investments in generation capacity are done on a very large-scale with multi-billion dollar commitments. These massive single-shot decisions carry a lot of risk, so more flexible technologies might be preferable (see also Huisman and Kort (2015)). We interpret flexibility as the ability to split a large investment into smaller ones, for example by sequentially installing several small plants. In this section we present a numerical example to discuss the respective effect of expansion size and the number of expansion options. This analysis also links the sequential, discrete-stage model herein to a continuous control formulation where capacity is added incrementally in infinitesimal amounts.

We maintain the symmetric parameter setting with the OU process X_t from the previous section. Capacity expansion is modeled by $Q^i_{n_1,n_2} = \bar{q} - (\Delta Q)n_i$, where $\bar{q}$ is the *terminal* capacity to be reached, and ΔQ is the unit investment. We now compare the previous single-expansion situation that used $\Delta Q = 0.5$ and $n_i \in \{0,1\}$, $\bar{q} = 1.5$, with a two-stage expansion for firm i, modeled by $\Delta Q^i = 0.25$ and $n_i \in \{0,1,2\}$. The expansion lump-sum costs K^i are proportional to the expansion size ΔQ^i, allowing a direct ceteris paribus comparison. We remark that with added flexibility, the game in interior stages now features Scenario II with multiple threshold-type equilibria.

5.3.1. Single-Firm Increased Flexibility

We first consider the case that only firm 1 is allowed to split her project, namely $\left(\Delta Q^1, \Delta Q^2\right) = (0.25, 0.5)$. The resulting best-response curves are sketched in Figure 7 and the equilibrium expansion thresholds are summarized in Table 3. Compared to the case $(\Delta Q^1, \Delta Q^2) = (0.5, 0.5)$ in Table 2, thanks to increased flexibility firm 1 will begin adding capacity sooner, and is much more likely now to invest first: $\mathbb{P}_0\left(\tau^{1,*}_{2,1} < \tau^{2,*}_{2,1}\right) = 0.6853$; recall that in the base case that probability was 50/50.

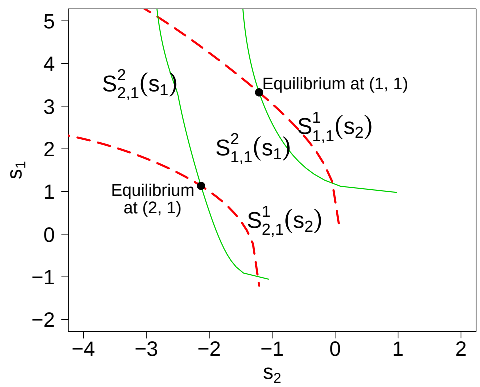
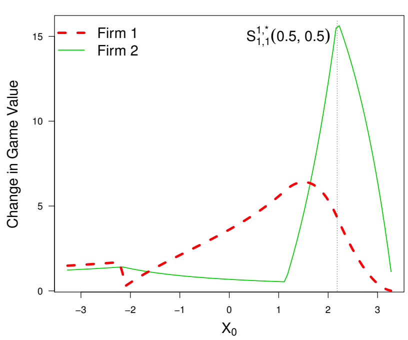
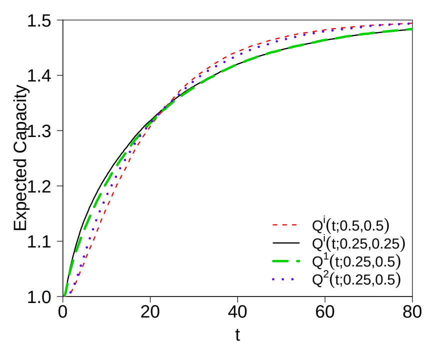


Figure 7: Best-response curves in the case $(\Delta Q^1, \Delta Q^2) = (0.25, 0.5)$. Interior stage equilibria correspond to Scenario II with multiple threshold-type equilibria; the latest ones are highlighted in the plot.

As expected, additional flexibility increases the game value of firm 1, see the red dashed line in Figure 8(a). The extra profit is maximized when the initial X_0 is between $S^{1,*}_{2,1}(0.25, 0.5)$ and $S^{1,*}_{1,1}(0.5, 0.5)$. Surprisingly, additional flexibility for firm 1 also increases game value of firm 2. This can be partly understood by supposing that $X_0 = S^{1,*}_{1,1}(0.5, 0.5)$, in which situation under $\Delta Q^1 = 0.5$ firm 1 will expand her capacity to 1.5 immediately, putting firm 2 into the undesirable "follower" state; with $\Delta Q^1 = 0.25$, the expansion is only to $Q^1_{1,2} = 1.25$, reducing the negative impact on firm 2. As a result, in this numerical example, *both* firms benefit from one of them gaining additional flexibility.



(a) $V^i_{2,1}(x; 0.25, 0.5) - V^i_{1,1}(x; 0.5, 0.5)$



(b) Expected Capacity

Figure 8: (Left: a) Impact of firm's 1 increased flexibility on game values received by each firm. The two "cusps" of the red dashed line are due to the game values not being smooth at the thresholds $S^{2,*}_{2,1}(0.25, 0.5)$ and $S^{2,*}_{1,1}(0.5, 0.5)$. (Right: b) The expected capacity $\mathbb{E}_0[Q^i(t)]$ of each firm starting with $X_0 = 0$.

5.3.2. Expected Capacity

Another question we are interested in is the expected capacity $\mathbb{E}_x[Q^i(t)]$ of each firm at time t, or equivalently the distribution of $N^i(t)$. The exact answer depends on $\mathbb{P}(\tau^i \le t)$ and requires computing the running maximum of an OU process which is not available in closed form. For our purposes we accordingly use Monte Carlo simulation to estimate the expected capacity of firm 1 in the cases $(\Delta Q^1, \Delta Q^2) = (0.25, 0.25)$, $(0.25, 0.5)$ and $(0.5, 0.5)$, assuming that $X_0 = 0$.

To compute $\mathbb{E}_x[Q^i(t)]$, we employ a Monte Carlo method based on the Euler scheme with $\Delta t = 1/120$ and 10000 simulated trajectories of the state process X. The estimated capacities at time t for each case are presented in Figure 8(b), from which we observe that added flexibility allows firms to smooth out their investment profiles over time, installing more capacity early on, and less (on average) later. Comparing the curves for $(\Delta Q^1, \Delta Q^2) = (0.25, 0.25)$ against those of $(\Delta Q^1, \Delta Q^2) = (0.5, 0.5)$ we see that smaller project size ΔQ makes aggregate capacity grow slower.

5.4. Case Study: Political Will for Meaningful Carbon Prices

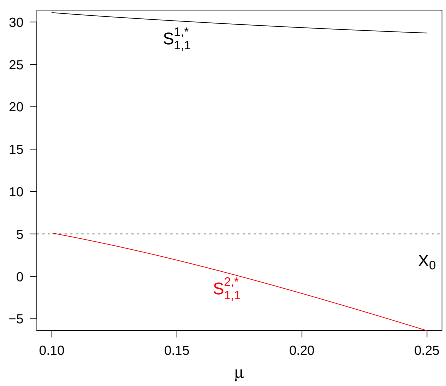
We return to the original economic example where firm 1 is the nuclear power generator and firm 2 is a coal-fired plant investor. With X_t representing the carbon emission price, higher X_t implies higher net profit made by the nuclear investors (who are carbon-neutral), while lower profit is made by the CO_2-emitting coal-fired plant.

Parameter	Value	Unit
Private discount rate r	10%	
Public discount rate r_{Public}	3%	
Nuclear expansion cost K^1	1400	USD/MWe
Coal expansion cost K^2	850	USD/MWe
Revenue rate $P_{1,1}$	24	USD/MWh
Revenue rate $P_{1,0}$	22	USD/MWh
Revenue rate $P_{0,1}$	22	USD/MWh
Revenue rate $P_{0,0}$	10	USD/MWh
Cost Sensitivity ρ	0.25	
Long-run carbon price θ	30	USD/tCO2
Political will μ	$[0.1, 0.25]$	
Initial carbon price X_0	5	USD/tCO2

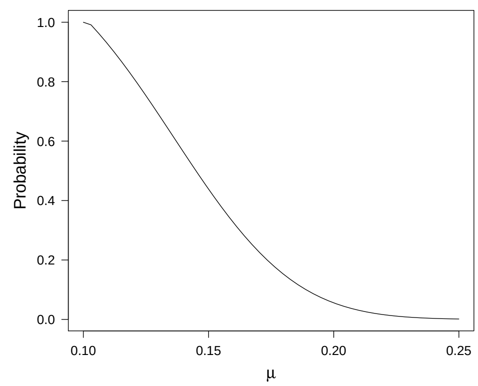
Table 4: Parameter values for Section 5.4.

As in the previous example, we model X_t as a mean-reverting OU diffusion. Such dynamics are interpreted in terms of the government policy to target a carbon price of $\$\theta$ per ton of CO_2. Market conditions generate fluctuations around this *long-run* average price, and the mean-reversion parameter μ represents the strength of the political will to keep prices around θ. Specifically, in light of recent experiences around the world, policy makers have tried to impose significant carbon prices ($\theta = 30$), while the actual prices have been rather low ($X_0 = 5$). The mean-reversion rate μ in (3.10) determines the expected time to reach the carbon price target, with the time-scale proportional to μ^{-1}.

In the short-run, market conditions are favorable for the coal-fired plants, reflected in the fact that their investment costs are lower, $K^2 < K^1$. In the long-run, the carbon price will rise and erode this favorable situation. For the social planner, the nuclear investment is therefore preferable (and can be justified through a lower social discount factor r_{Public}). However, private investors have much larger discounting $r = 10\%$. Therefore, depending on the political will, coal-fired investment might still be made in the near future. To sharpen this conflict, we assume that the leveraged



(a)



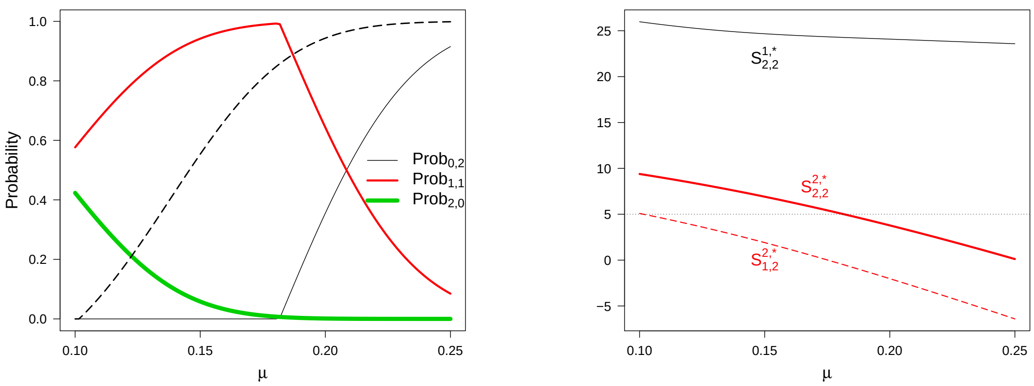
(b)

Figure 9: Game equilibrium at stage $(1, 1)$ for the nuclear-coal generation capacity market. Equilibrium is based on Scenario I with a unique threshold-type equilibrium. (Left) Investment thresholds $S_{1,1}^{1,*}, S_{1,1}^{2,*}$ as the mean-reversion rate μ varies. For convenience we also indicate the level of initial carbon price X_0. (Right) Probability that the coal-fired investor invests first $Prob_{1,0}$ as μ varies for the given $X_0 = 5$. For small μ the coal-fired producer is guaranteed to invest first; for large μ nuclear producer 1 is almost guaranteed to invest first, $Prob_{1,0} \simeq 0$.

costs of electricity generation (LCOE) and the nominal electricity prices are such that at most one investment is profitable. Thus, starting at stage $\vec{N}_0 = (1, 1)$, stages $(1, 0)$ and $(0, 1)$ are both absorbing. Consequently, the two firms are competing to make the first and only expansion (i.e. become the "leader" in this asymmetric single-shot setting). Namely, the coal-fired investor might want to preempt the base-load market before the carbon price makes her less competitive. Knowing this, one wonders whether the "green" nuclear power plant generator will hasten her own investment.

The nominal levels of prices P_{n_1,n_2} are designed in the following way. Noting r, the discount rate, the LCOE for player i is $p_i := \frac{r \cdot K_i}{N}$, where N is the number of hours per year to get a price in USD/MWh. In words, LCOE is the price level for which the net present value of building a new plant is zero. We have $p_2 \leq p_1$ and take $P_{1,1} > \max(p_1, p_2)$, but $P_{0,0} \ll \min(p_1, p_2)$. Thus, nominal prices after a first investment are set in such a way that once one player has invested, a second investment will lead to a nominal price much lower than the LCOE's of both players, making it unlikely that the price plus the carbon premium will rise above investment levels again. The intermediate nominal prices $P_{0,1}, P_{1,0}$ are right around p_i's, so that investment is possible, but is conditional on a favorable carbon price (low enough for the coal-fired investor, high-enough for the nuclear investor).

It turns out that with the above parameters, stage $(1, 1)$ leads to a unique non-preemptive equilibrium (scenario I). To explain the long-term structure of the market, we consider the *terminal stage* $\lim_{t\to\infty} \vec{N}_t$. With the parameter settings given, it is only profitable to make (exactly) one investment, so that $\lim_{t\to\infty} \vec{N}_t \in \{(1, 0), (0, 1)\}$. Figure 9 plots the probability $Prob_{0,1} = \mathbb{P}_{x_0}(\lim_{t\to\infty} \vec{N}_t = (0, 1))$ that the coal-fired producer is the one to build. We see that this quantity is highly sensitive to μ. If μ is too low, the competition will "choose" to preemptively build coal-fired plants ($S_{1,1}^{2,*} > X_0 = 5$), while the public decision-makers will still be struggling to establish a high and steady value of carbon price. As μ rises, the investment threshold of the coal-fired investor $S_{1,1}^{2,*}$ falls as she anticipates lower future profits, and hence demands larger short-term gains (possible only if carbon price is minimal) as compensation. Of course, with strong political will, carbon prices are unlikely to fall from $X_0 = 5$, so that the likelihood of coal-fired investor making an investment becomes negligible. This confirms the strong impact of policy-making on power plant investments. At the same time, the investment threshold of the nuclear investor is insensitive to μ, because nuclear capacity is not added until $X_t \simeq \theta$, whereby the mean-reversion rate is less relevant.



(a) (b)

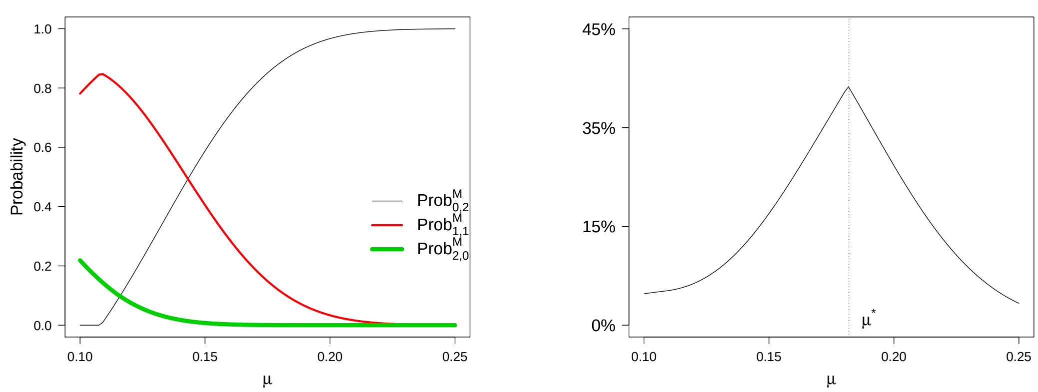
Figure 10: (Left: a) Distribution of the terminal stages $Prob_{n_1,n_2} := \mathbb{P}(\lim_{t\to\infty} \vec{N}_t = (n_1, n_2))$ under $\Delta Q^i = 0.25$. The solid lines represent probabilities of terminal stages in a competitive market. The dashed line represents the probability $Prob^e_{0,2} = \mathbb{P}(\lim_{t\to\infty} \vec{N}_t = (0,2)|\vec{N}_0 = (1,2))$ that two nuclear plants are built if a small nuclear plant is built at $X_0 = 5$ preemptively. (Right: b) Investment thresholds $S^{1,*}_{2,2}, S^{2,*}_{2,2}$ and $S^{2,*}_{1,2}$, as μ varies. For convenience we also indicate the level of initial carbon price $X_0 = 5$.

We next consider a multi-stage situation, whereby each of the two producers can build up to *two* equal-sized smaller-scale plants. We again assume that expansion costs are proportional to plant size, and also that the overall market demand economically supports aggregate capacity up to two plants. Specifically, we assume that with a single small plant, market price will decline to $P_{2,1} = P_{1,2} = 23$ and with any two small plants, $P_{2,0} = P_{0,2} = P_{1,1} = 22$, matching the large-plant setting in Table 4. Beyond that, investment becomes impractical, i.e. $P_{1,0}$ and $P_{0,1}$ are too low to ever be profitable. Therefore, starting at stage $(2,2)$, either (i) the nuclear producer builds 2 small nuclear plants; (ii) the coal-fired investor builds two small coal-fired plants; or (iii) each firm builds one plant apiece. The probabilities of the respective outcomes are labeled $Prob_{2,0}, Prob_{1,1}, Prob_{0,2}$, with $Prob_{n_1,n_2} := P(\lim_{t\to\infty} \vec{N}_t = (n_1, n_2))$. In contrast to the original large-scale investment competition, the initial competitive market at stage $(2,2)$ corresponds to Scenario II (unless μ is close to 0.1) supporting both a threshold- and preemptive-type equilibria. This occurs because smaller scale investments make nuclear investment profitable at lower carbon prices, sharpening the competition to install capacity first (algebraically it turns out that with given parameter values $L^1_{2,2} < L^2_{2,2}$). Stages $(2,1)$ and $(1,2)$ still correspond to Scenario I with a unique non-preemptive equilibrium.

We first assume the threshold-type equilibrium is selected. Figure 10 shows that the coal-fired investor will increase her investment threshold $S^{2,small,*}_{2,2}$ for a smaller project compared to the preceding single large plant $S^{2,large,*}_{1,1}$. As a result, for $\mu < \mu^* = 0.181$, the coal-fired investor is going to build one small plant at once and wait for a moment that the carbon price drops to expand her existing plant. Moreover, even for large μ, the coal-fired investor still has a good chance to build one small plant (leading to terminal stage $(1,1)$), which means that only a very strong policy can guide the market to exclusively "green" power plants.

From Figure 10(b) we also observe that the investment threshold of coal-fired investors at stage $(1,2)$ is significantly lower relative to the threshold at stage $(2,2)$. This is the opposite effect from what was observed in Section 5.3, due to the different relationship between stages and prices. Thus, one way for the public decision-makers to guide the industry could be via preempting the base-load market by a small "green" power plant at the initial time (e.g. built with government subsidies). In turn, the lowered electricity price reduces anticipated future profits of the coal-fired investor, and makes them less likely to ever invest (see also Bashyam (1996)). In Figure 10(a), as the dashed line shows, public decision-maker's preemption sharply increases the probability that two (small) nuclear power plants will be built. Another alternative for policy makers is to grant a priority option to nuclear investors at the first stage $(2,2)$. Distinguished from the preemption case, nuclear investors with a priority can simply wait until a high-enough carbon price to make their investment. Since the long-run carbon price is taken to be $\theta = 30$, once it is high, it will likely remain high. Consequently, a single priority option is enough to guide the market to the $(0,2)$ terminal stage, since the coal producer becomes very unlikely to invest in the $(1,2)$-stage.

Figure 11(a) shows that the social planner (equivalent to a cooperative game, or a generator monopoly) is likely to build two nuclear plants, consistent with the idea that "green" generation is more profitable in the long-run. Significantly, Figure 11(b) illustrates that the percentage loss caused by competition can be as high as 40% at moderate levels of μ (when a small coal-fired plant is built instantly). This confirms the anecdotal evidence of very significant losses incurred by producers in newly deregulated markets, and the accompanying capacity over-investment (due to the preemptive race to build first). It also illustrates the dramatic impact that the short-term driven competition can have on the long-term market organization; here over-investment drastically alters the mix of power plants likely to be built, hurting long-run profits of *both* producers. We also observe that such losses are reduced to almost zero for μ large enough, which again corroborates the strong impact of public policy.



(a)

(b)

Figure 11: (Left: a) Distribution of the terminal stage $\lim_{t\to\infty} \vec{N}_t$ under cooperative game model. (Right: b) Percentage loss caused by competition: $\left[1 - (V^1_{2,2} + V^2_{2,2})/V^M_{2,2}\right](X_0)$.

Coming back to the equilibrium type at stage $(2,2)$, suppose instead that the investors are aggressive and implement the preemptive equilibrium strategy. Since a coal-fired power plant is easier to build, it is natural to assume that the coal-fired investor possesses preemptive priority. For $\mu < \mu^* = 0.181$, this makes no difference: an aggressive coal investor will behave exactly the same as before because she will build a small plant at once and there is no preemptive equilibrium at stage $(2,1)$. For larger μ, it turns out that $L^2_{2,2} < X_0 = 5$, which prevents firm 2 from immediate investment, as the NPV of an expansion is negative. Meanwhile, the nuclear investor will choose to preempt right before the carbon emission price drops down to $L^2_{2,2}$ (see resulting game value of firm

1 in Figure 2(a)). Consequently, exclusively "green" power plants are more likely to be established.

Finally, we end this section by illustrating which equilibrium scenario takes place during stage $(2, 2)$ as the political will μ and carbon market fluctuation σ vary. As Figure 12 shows, we observe that only a preemptive equilibrium exists (Scenario III) under low carbon volatility σ. Also, the impact of μ is non-monotone: when μ is very small or very large, the competition between industries is less preemptive (as one industry is clearly ahead in the short-term) and hence threshold-type equilibria exist. However, for intermediate values of μ, the equal-strength competition raises benefits of aggressive investment and generates preemptive equilibria. The impact of σ is harder to explain and is ultimately linked to its recursive effect on the leadership thresholds $L^i_{2,2}$ of the two industries.

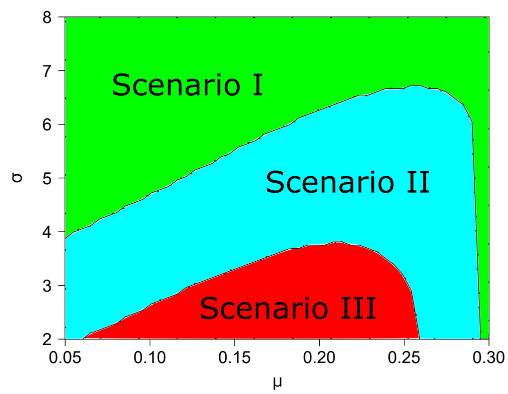


Figure 12: Resulting equilibrium types at stage $(2, 2)$ in the small-scale power plants case, as the political will μ and carbon volatility σ vary.

6. Conclusion & Future Works

In this article we have developed a framework for competitive capacity expansions among two distinct producers. The sequential nature of expansion options allows us to maintain a lot of tractability, while still offering several innovative insights. In particular, our analysis reveals multiple scenarios regarding the number and types of Nash equilibria. Our method of solution also makes a new connection between a class of non-zero-sum continuous-time stopping games and optimal stopping problems with exit values.

Applying the model to generation capacity expansion of energy producers, we are able to quantify several features of competitive markets including: loss due to competition (compared to central planner set-up); impact of price dynamics on competitive behavior; value of regulatory preferences via investment priority rights. We also show that within a competitive setting there could be complicated trickle-down effects from early decisions; for example an initial (sub-optimal) preemption by one firm could drastically alter the likely final outcome. These effects suggest policy mechanisms to influence long-term market organization through early actions.

The presented model is crucially limited to having two producers. This limitation arises both due to the assumed 1-dimensional system state X_t, whereby there can only be two "directions" for producer preferences, and the fact that Nash equilibria among more than two players are very challenging to analyze. It is therefore an open problem on how to extend the analysis to either multi-dimensional X_t or to oligopolies. In the former case, threshold-type strategies would have to be replaced by investment-curves, significantly complicating mathematical treatment.

One generalization that would be more tractable is a mean-field type model with infinitely many distinct producers; in that case the different thresholds of each producer will lead to a statedependent aggregate capacity $Q(x)$, effectively endogenizing the payoff function. Properly setting up the equilibrium conditions would be the main challenge for such extension. Another interesting modification is to allow firms to both expand and shrink their capacity (which would create additional linkages among the game values and introduce further dis-investment thresholds), or to consider continuous investment controls (either one-sided or two-sided).

Appendix A. Proof of Lemma 3.6

Proof. We take up the ψ-transform $\Psi h(y)$ as an example; the proof for φ-transform function can be done following the same scheme. The continuity and differentiability of $\Psi h(y)$ follow directly from those of h, G and ψ, and it is equivalent to show that

$$(\Psi h)''(y) = \frac{2}{\sigma^2(x)G(x)\left(\psi'(x)\right)^2}\left(\mathcal{L}-r\right)h(x), \qquad x = \psi^{-1}(y). \tag{A.1}$$

By definition of the operator Ψ (3.12),

$$(\Psi h)''(y) = \left(\frac{1}{\psi'(x)}(\frac{h}{G})'(x)\right)' = \frac{1}{(\psi'(x))^2}\left[(\frac{h}{G})''(x) - \frac{\psi''(x)}{\psi'(x)}(\frac{h}{G})'(x)\right]. \tag{A.2}$$

On the other hand, by direct differentiation (and dropping the x-argument for typographical convenience)

$$\begin{aligned}
(\mathcal{L}-r)\,h &= (\mathcal{L}-r)\left(\frac{h}{G}\cdot G\right) \\
&= b(x)(\frac{h}{G})'G + \frac{\sigma^2(x)}{2}\left[(\frac{h}{G})''G + 2(\frac{h}{G})'G'\right] + \frac{h}{G}\left(\mathcal{L}-r\right)G \\
&= \frac{\sigma^2(x)G}{2}\left[(\frac{h}{G})'' + \left(\frac{2b(x)}{\sigma^2(x)} + 2\frac{G'}{G}\right)\cdot(\frac{h}{G})'\right].
\end{aligned} \tag{A.3}$$

Meanwhile,

$$\begin{aligned}
(\mathcal{L}-r)\,F &= (\mathcal{L}-r)\left(G\frac{F}{G}\right) = (\mathcal{L}-r)\,(G\psi) \\
&= b(x)\psi'G + \frac{\sigma^2(x)G}{2}\psi'' + \sigma^2(x)\psi'G' + \psi\left(\mathcal{L}-r\right)G \\
&= b(x)\psi'G + \frac{\sigma^2(x)G}{2}\psi'' + \sigma^2(x)\psi'G' = 0.
\end{aligned} \tag{A.4}$$

Equations (A.3) and (A.4) follow from the fact that F and G are solutions to the ODE (3.5), and equation (A.4) yields that

$$-\frac{\psi''(x)}{\psi'(x)} = \frac{2b(x)}{\sigma^2(x)} + 2\frac{G'(x)}{G(x)}. \tag{A.5}$$

Substituting (A.5) into (A.3) and comparing with (A.2), we obtain (A.1) and complete the proof. In the case $h \in \mathcal{H}_{\text{dec}}$ and using φ-transformed $(\Phi h)(y)$ it follows by similar arguments that

$$(\Phi h)''(z) = \frac{2}{\sigma^2(x)G(x)\left(\varphi'(x)\right)^2}\left(\mathcal{L}-r\right)h(x), \qquad x = \varphi^{-1}(z). \tag{A.6}$$

Note that since $\varphi' < 0$, the interval $(0,\, b_h)$ on x coordinate corresponds to $(\varphi(b_h),\, +\infty)$ on $z = \varphi(x)$ coordinate, which completes the proof accordingly. $\square$

Appendix B. Boundary Stages

Appendix B.1. Proof of Proposition 4.1

Proof. This is a canonical single agent optimal stopping problem. In this work, we prove it following the work of Leung and Li (2015). Recall that the optimal stopping problem (4.1) corresponds to the case $R = \mathcal{D}$ discussed in Proposition 3.3. Applying operator Ψ to $h^1_{1,0}$, we obtain $H^1_{1,0}(y) := \Psi h^1_{1,0}(y)$, which is continuous and twice differentiable on $\psi(\mathcal{D}) = (0, +\infty)$. Meanwhile, denoting the smallest concave majorant of $H^1_{1,0}$ over $\mathbb{R}^+$ by $\mathcal{W}H^1_{1,0}(y)$, and referring to Proposition 3.3, we obtain

$$V^1_{1,0}(x) - D^1_{1,0}(x) = G(x) \cdot \left[\mathcal{W}H^1_{1,0} \circ \psi(x)\right], \qquad x \in \mathcal{D}. \tag{B.1}$$

Since $h^1_{1,0}$ is a linear increasing function which is in the class $\mathcal{H}_{\text{inc}}$, the transformed payoff $y \mapsto H^1_{1,0}(y)$ possesses properties stated in Lemma (3.6), namely it is convex on $[0, \psi\left(b^1_{1,0}\right))$ and concave on $(\psi\left(b^1_{1,0}\right), +\infty)$. Therefore, we conclude that there exists a unique number $y^* > \psi(b^1_{1,0})$, such that the smallest concave majorant $\mathcal{W}H^1_{1,0}$ is a straight line from the origin tangent to $H^1_{1,0}$ at $\left(y^*, H^1_{1,0}(y^*)\right)$ on $[0, y^*)$, and then coincides with $H^1_{1,0}$ on $[y^*, +\infty)$ (see Figure 13(a)):

$$\left(\mathcal{W}H^1_{1,0}\right)(y) = \begin{cases} y\frac{H^1_{1,0}(y^*)}{y^*}, & \text{if } y < y^*, \\ H^1_{1,0}(y), & \text{if } y \geq y^*. \end{cases} \tag{B.2}$$

Define $S^{1,*}_{1,0} := \psi^{-1}(y^*)$. By direct differentiation, we obtain

$$\frac{dH^1_{1,0}(y)}{dy}\bigg|_{y=\psi(S^{1,*}_{1,0})} = \frac{\left(h^1_{1,0}\right)'(S^{1,*}_{1,0})G(S^{1,*}_{1,0}) - h^1_{1,0}(S^{1,*}_{1,0})G'(S^{1,*}_{1,0})}{F'(S^{1,*}_{1,0})G(S^{1,*}_{1,0}) - F(S^{1,*}_{1,0})G'(S^{1,*}_{1,0})}.$$

To match the first derivative at the tangent point, it must hold that

$$\frac{H^1_{1,0}(y^*)}{y^*} = (H^1_{1,0})'(y^*), \tag{B.3}$$

where by (3.3) the LHS is admitted as

$$\frac{H^1_{1,0}(y^*)}{y*} = \frac{H^1_{1,0}(\psi(S^{1,*}_{1,0}))}{\psi(S^{1,*}_{1,0})} = \frac{h^1_{1,0}(S^{1,*}_{1,0})}{F(S^{1,*}_{1,0})}. \tag{B.4}$$

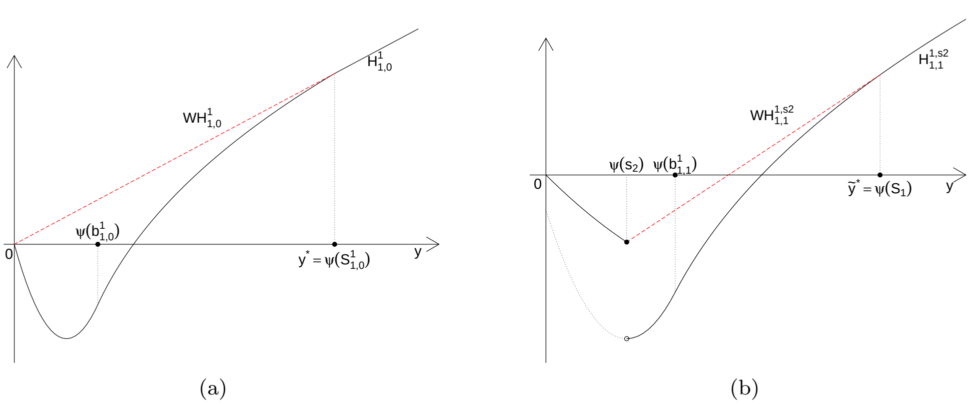


Figure B.13: (Left) $H^1_{1,0}$ and its smallest concave majorant $\left(\mathcal{W}H^1_{1,0}\right)$, sketched according to Lemma 3.6. (Right) The transformed payoff $H^{1;s_2}_{1,1}$ and its smallest concave majorant over $(\psi(s_2), +\infty)$.

Consequently, we can rewrite condition (B.3) in terms of $S_{1,0}^{1,*}$, and simplify it to (4.4). Substituting (B.4) into (B.2), we get

$$\mathcal{W}H_{1,0}^{1} \circ \psi(x) = \begin{cases} \psi(x)\frac{H_{1,0}^{1}(y^*)}{y^*} = \frac{F(x)}{G(x)}\frac{h_{1,0}^{1}(S_{1,0}^{1,*})}{F(S_{1,0}^{1,*})}, & \text{if } x \in (\underline{d},\, S_{1,0}^{1,*}), \\ H_{1,0}^{1}(\psi(x)) = \frac{h_{1,0}^{1}(x)}{G(x)}, & \text{if } x \in [S_{1,0}^{1,*},\, \bar{d}). \end{cases}$$

Combining above with (B.1) we obtain the expression for the value function $V_{1,0}^{1}(x)$ in (4.2). This also yields the structure of the optimal stopping region as (4.3) and the smooth pasting condition at the threshold $S_{1,0}^{1,*}$ via (B.3). □

Appendix B.2. Proof of Proposition 4.2

Proof. The optimal stopping problem (4.5) corresponds to the special case $R = \mathcal{D}$. Applying Φ operator and referring to Proposition 3.3 yields that

$$V_{0,1}^{2}(x) - D_{0,1}^{2}(x) = F(x) \cdot \left[\mathcal{W}H_{0,1}^{2} \circ \varphi(x)\right], \qquad x \in \mathcal{D}, \tag{B.5}$$

where $H_{0,1}^{2}(z) := \Phi h_{0,1}^{2}(z) = \frac{h_{0,1}^{2}}{F} \circ \varphi^{-1}(z)$ is continuous and twice differentiable on $\varphi(\mathcal{D}) = (0, +\infty)$, and $\mathcal{W}H_{0,1}^{2}$ is its smallest concave majorant in the z-coordinate. Since $h_{0,1}^{2}$ is in $\mathcal{H}_{\text{dec}}$, Lemma 3.6 implies that $z \mapsto H_{0,1}^{2}(z)$ possesses the same shape as $y \mapsto H_{1,0}^{1}(y)$ sketched in Figure 13(a), and consequently its smallest concave majorant $\mathcal{W}H_{0,1}^{2}$ has the same shape as $\mathcal{W}H_{1,0}^{1}$. Similar arguments as in the proof of Proposition 4.1 yield that

$$\mathcal{W}H_{0,1}^{2} \circ \varphi(x) = \begin{cases} \varphi(x)\frac{H_{0,1}^{2}(z^*)}{z^*} = \frac{G(x)}{F(x)}\frac{h_{0,1}^{2}(S_{0,1}^{2,*})}{G(S_{0,1}^{2,*})}, & \text{if } x \in (S_{0,1}^{2,*},\, \bar{d}), \\ H_{0,1}^{2}(\varphi(x)) = \frac{h_{0,1}^{2}(x)}{F(x)}, & \text{if } x \in (\underline{d},\, S_{0,1}^{2,*}], \end{cases}$$

where $S_{0,1}^{2,*}$ is obtained by matching the first derivative at $z^* := \varphi\left(S_{0,1}^{2,*}\right)$

$$\frac{H_{0,1}^{2}\left(z^*\right)}{z^*} = \left(H_{0,1}^{2}\right)'\left(z^*\right).$$

Finally, the value function $V_{0,1}^{2}(x)$, as well as the stopping region (4.7) stated in (4.6) is obtained by (B.5). □

Appendix B.3. Proof of Theorem 4.3

We use induction to prove the result for the case where only firm 1 has expansion options. Suppose that at stage $(n_1 - 1,\, 0)$ firm 1 implements game strategy characterized by threshold $S_{n_1-1,0}^{1,*}$ and receives game value $V_{n_1-1,0}^{1}$. Following (2.11), at stage $(n_1,\, 0)$ firm 1 solves an optimal stopping problem with first-mover payoff:

$$\begin{aligned} h_{n_1,0}^{1}(x) &= V_{n_1-1,0}^{1}(x) - D_{n_1,0}^{1}(x) - K_{n_1}^{1} \\ &= \begin{cases} D_{n_1-1,0}^{1}(x) - D_{n_1,0}^{1}(x) - K_{n_1}^{1} + \frac{h_{n_1-1,0}^{1}\left(S_{n_1-1,0}^{1}\right)}{F\left(S_{n_1-1,0}^{1}\right)}F(x), & \text{if } x < S_{n_1-1,0}^{1,*}, \\ D_{n_1-1,0}^{1}(x) - D_{n_1,0}^{1}(x) - K_{n_1}^{1} + h_{n_1-1,0}^{1}(x), & \text{if } x \geq S_{n_1-1,0}^{1,*}, \end{cases} \end{aligned}$$

where $\Delta Q_{n_1}^{1}$ is the expansion size of firm 1 when she has n_1 options and $h_{n_1-1,0}^{1}(x) = V_{n_1-2,0}^{1}(x) - D_{n_1-1,0}^{1}(x) - K_{n_1-1}^{1}$ is her first-mover payoff at stage $(n_1 - 1, 0)$ and is contained in the class $\mathcal{H}_{\text{inc}}$.

Also note that $h^1_{n_1,0}$ is smooth at the point $S^{1,*}_{n_1-,0}$ following the smooth pasting condition. This problem corresponds again the case $R = \mathcal{D}$ in Proposition 3.3 and therefore yields a value function

$$V^1_{n_1,0}(x) - D^1_{n_1,0}(x) = G(x) \cdot \left[\mathcal{W}H^1_{n_1,0} \circ \psi(x)\right], \tag{B.6}$$

where $H^1_{n_1,0}(y) := \Psi h^1_{n_1,0}(y)$ and $\mathcal{W}H^1_{n_1,0}(y)$ is its smallest concave majorant over $\mathbb{R}^+$. Since F is a solution to the ODE (3.5), the F term in $h^1_{n_1,0}$ does not contribute to $(\mathcal{L}-r)\, h^1_{n_1,0}$. From the assumption that all increasing linear functions are in $\mathcal{H}_{\text{inc}}$ and $(\mathcal{L}-r)\, h^1_{n_1-1,0}(x) < 0$ for $x \geq S^{1,*}_{n_1-1,0}$, we then conclude that $h^1_{n_1,0}(x)$ is in the class $\mathcal{H}_{\text{inc}}$ and there exists $b^1_{n_1,0}$, such that $H^1_{n_1,0}(y)$ is convex over $(0, \psi\left(b^1_{n_1,0}\right))$ and concave over $(\psi\left(b^1_{n_1,0}\right), +\infty)$, cf. Lemma 3.6. Repeating the proof of Proposition 4.1 then gives the game value and strategy of firm 1 stated in Theorem 4.3. Identical arguments work for firm 2, using the Φ-transform.

Appendix C. Interior Stages

Appendix C.1. Proof of Proposition 4.6 and 4.7

From the definition of $\tau^2_{1,1}$ in (4.17), the optimization problem (4.18) corresponds to the case $R = \left(s_2,\, \overline{d}\right)$ in Proposition 3.3. The first-mover payoff is:

$$\begin{aligned} h^1_{1,1}(x) &= V^1_{0,1}(x) - D^1_{1,1}(x) - K^1_1 \\ &= \begin{cases} D^1_{0,0}(x) - D^1_{1,1}(x) - K^1_1, & \text{if } x \in (\underline{d},\, S^{2,*}_{0,1}), \\ D^1_{0,1}(x) - D^1_{1,1}(x) - K^1_1 + G(x)\cdot\left[\frac{D^1_{0,0}-D^1_{0,1}}{G}\right]\left(S^{2,*}_{0,1}\right), & \text{if } x \in (S^{2,*}_{0,1},\, \bar{d}). \end{cases} \end{aligned} \tag{C.1}$$

Given the strategy of firm 2 stated in (4.17), we define

$$\hat{h}^{1,s_2}_{1,1}(x) := 1_{(s_2,\bar{d})}(x) h^1_{1,1}(x) + 1_{(x=s_2)} l^1_{1,1}(x). \tag{C.2}$$

Applying the operator Ψ defined in (3.12), we denote the transformed function by $H^{1,s_2}_{1,1}(y) := \Psi\hat{h}^{1,s_2}_{1,1}(y)$, and its smallest concave majorant over $[\psi(s_2),\, +\infty)$ by $\mathcal{W}H^{1,s_2}_{1,1}$. Following Proposition 3.3, the corresponding value function is admitted as

$$\widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(x) = G(x)\cdot\left[\mathcal{W}H^{1,s_2}_{1,1} \circ \psi(x)\right], \qquad s_2 < x < \bar{d}. \tag{C.3}$$

Let us first consider the case $s_2 < L^1_{1,1}$. Since G is a solution to the ODE (3.5), we conclude that $h^1_{1,1}$ is in the class $\mathcal{H}_{\text{inc}}$. Following Lemma 3.6, there exists a fixed point $b^1_{1,1}$ such that $y \mapsto \Psi h^1_{1,1}(y)$ is convex on $\left(0,\, \psi(b^1_{1,1})\right)$ and concave on $\left(\psi(b^1_{1,1}),\, +\infty\right)$. Consequently (see Figure 13(b)), there exists a unique $\tilde{y}^* > \psi(b^1_{1,1})$, such that the smallest concave majorant $\mathcal{W}H^{1,s_2}_{1,1}(y)$ is a straight line from $\left(\psi(s_2),\, \Psi l^1_{1,1}(\psi(s_2))\right)$, tangent to $H^{1,s_2}_{1,1}(y)$ at $\left(\tilde{y}^*,\, \Psi h^1_{1,1}(\tilde{y}^*)\right)$ and then coincides with $H^{1,s_2}_{1,1}(y)$:

$$\mathcal{W}H^{1,s_2}_{1,1}(y) = \begin{cases} \Psi l^1_{1,1}(\psi(s_2)) + (y - \psi(s_2))\left(\Psi l^1_{1,1}\right)'(\tilde{y}^*), & \text{if } y \in [\psi(s_2),\, \tilde{y}^*), \\ \Psi h^1_{1,1}(y), & \text{if } y \geq \psi(\tilde{y}^*), \end{cases} \tag{C.4}$$

which yields the optimal stopping region characterized in (4.19). To match the first derivative at the tangent point, it must hold that

$$\frac{\Psi h^1_{1,1}(\tilde{y}^*) - \Psi l^1_{1,1}(\psi(s_2))}{\tilde{y}^* - \psi(s_2)} = \left(\Psi h^1_{1,1}\right)'(\tilde{y}^*). \tag{C.5}$$

Define $S_1 := \psi^{-1}(\tilde{y}^*)$. Substituting (C.2) and $\psi = \frac{F}{G}$ into the LHS of (C.5), we obtain:

$$\frac{\Psi h^1_{1,1}(\tilde{y}^*) - \Psi l^1_{1,1}(\psi(s_2))}{\tilde{y}^* - \psi(s_2)} = \frac{\frac{h^1_{1,1}(S_1)}{G(S_1)} - \frac{l^1_{1,1}(s_2)}{G(s_2)}}{\frac{F(S_1)}{G(S_1)} - \frac{F(s_2)}{G(s_2)}} = \frac{h^1_{1,1}(S_1)G(s_2) - l^1_{1,1}(s_2)G(S_1)}{F(S_1)G(s_2) - F(s_2)G(S_1)} := \tilde{\omega}^1_{1,1}. \qquad \text{(C.6)}$$

Differentiating the RHS directly, it follows that

$$\frac{dH^1_{1,1}(y)}{dy}\Big|_{\psi^{-1}(\tilde{y}^*)=S_1} = \frac{\left(h^1_{1,1}\right)'(S_1)G(S_1) - h^1_{1,1}(S_1)G'(S_1)}{F'(S_1)G(S_1) - F(S_1)G'(S_1)},$$

hence we can rewrite condition (C.5) in terms of S_1 and simplify it to equation (4.20). Finally, for $x \in (s_2,\, S_1)$, $\left(H^{1,s_2}_{1,1}\right)'(\tilde{y}^*) = \tilde{\omega}^1_{1,1}$ implies that

$$W^{1,s_2}_{1,1}(\psi(x)) = \Psi l^1_{1,1}(\psi(s_2)) + (\psi(x) - \psi(s_2))\tilde{\omega}^1_{1,1} \triangleq \tilde{\omega}^1_{1,1}\psi(x) + \tilde{\nu}^1_{1,1}, \qquad s_2 \le x < S_1,$$

where $\tilde{\omega}^1_{1,1}$ and $\tilde{\nu}^1_{1,1}$ can be verified to match (4.22). From Proposition 3.3, the value function is then admitted as

$$\widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(x) = \begin{cases} G(x)W^{1,s_2}_{1,1}(\psi(x)) = \tilde{\omega}^1_{1,1}F(x) + \tilde{\nu}^1_{1,1}G(x), & \text{if } s_2 < x < S_1, \\ h^1_{1,1}(x), & \text{if } x \ge S_1, \end{cases}$$

which coincides with (4.21), and completes the proof of Proposition 4.6.

Next, suppose that $s_2 > L^1_{1,1}$. For $L^1_{1,1} < x \le s_2$, firm 1 will try to preempt her rival since her corresponding first-mover payoff is higher than her second-mover payoff. For $x > s_2$, the value function of firm 1 is again admitted as (C.3) according to Proposition 3.3. However, since there is a *negative* jump at $y = \psi(s_2)$ in $H^{1,s_2}_{1,1}(y)$, the smallest concave majorant $\mathcal{W}H^{1,s_2}_{1,1}(y)$ is now a straight line from $\left(\psi(s_2),\, \Psi h^1_{1,1}(\psi(s_2))\right)$, tangent to $H^{1,s_2}_{1,1}(y)$ at $\left(\tilde{y}^*,\, \Psi h^1_{1,1}(\tilde{y}^*)\right)$ and then coincides with $H^{1,s_2}_{1,1}(y)$:

$$W^{1,s_2}_{1,1}(y) = \begin{cases} \Psi h^1_{1,1}(\psi(s_2)) + (y - \psi(s_2))\left(\Psi h^1_{1,1}\right)'(\tilde{y}^*), & \text{if } y \in (\psi(s_2),\, \tilde{y}^*), \\ \Psi h^1_{1,1}(y), & \text{if } y \ge \psi(\tilde{y}^*). \end{cases} \qquad \text{(C.7)}$$

And the first derivative is matched at the tangent point

$$\frac{\Psi h^1_{1,1}(\tilde{y}^*) - \Psi h^1_{1,1}(\psi(s_2))}{\tilde{y}^* - \psi(s_2)} = \left(\Psi h^1_{1,1}\right)'(\tilde{y}^*). \qquad \text{(C.8)}$$

Repeating the preceding steps then yields the threshold S^e_1. Note that if $s_2 \ge b^1_{1,1}$, equation (C.8) yields $S^e_1 = s_2$. Meanwhile, following from (C.3), $\lim\limits_{x \searrow s_2} \widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(s_2) = h^1_{1,1}(s_2) > l^1_{1,1}(s_2)$, which implies stopping at s_2 is too late, and firm 1 would prefer to preempt right before s_2. Therefore, with an ε-optimal stopping rule defined as

$$\Gamma^\varepsilon := \{x \in (s_2, \bar{d}) :\, \widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(x) \le \hat{h}(x) + \varepsilon\} \quad \text{and} \quad \tau^\varepsilon := \inf\{t \ge 0 :\, X_t \in \Gamma^\varepsilon\}, \qquad \text{(C.9)}$$

the best-response of firm 1 in this situation is $\lim\limits_{\varepsilon \searrow 0} \tau^\varepsilon$ which corresponds to (4.23).

Appendix C.2. Best-response of Firm 2 at stage (1, 1)

Similar to previous discussion, we start with the assumption that firm 1's policy is of threshold type: $\tau^1_{1,1} = \inf\{t \geq 0 : X_t \geq s_1\}$. For $s_1 > L^2_{1,1}$, firm 2 solves the optimal stopping problem:

$$\widetilde{V}^2_{1,1}(x, s_1) - D^2_{1,1}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\mathbb{1}_{\{\tau < \tau^1_{1,1}\}} e^{-r\tau} \left\{ h^2_{1,1}(X_\tau) \right\} + \mathbb{1}_{\{\tau > \tau^1_{1,1}\}} e^{-r\tau^1_{1,1}} \left\{ l^2_{1,1}(X_{\tau^1_{1,1}}) \right\} \right]. \quad \text{(C.10)}$$

The resulting *threshold-type* best-response of firm 2 is

$$\tau^2_{1,1}(s_1) = \inf\{t \geq 0 : X_t \leq S^2_{1,1}(s_1)\},$$

where the optimal stopping level is characterized as the solution to:

$$\begin{aligned} &\left[h^2_{1,1}(S_2)G(s_1) - \left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)G(S_2)\right] F'(S_2) + \left[\left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)F(S_2) - h^2_{1,1}(S_2)F(s_1)\right] G'(S_2) \\ &\quad = \left(h^2_{1,1}\right)'(S_2)\left[G(s_1)F(S_2) - G(S_2)F(s_1)\right]. \end{aligned} \quad \text{(C.11)}$$

Consequently, the optimal stopping problem (C.10) admits the value function

$$\widetilde{V}^2_{1,1}(x, s_1) = \begin{cases} V^2_{1,0}(x) - K^2_1, & \text{if } x < S_2(s_1), \\ D^2_{1,1}(x) + \tilde{\omega}^2_{1,1}F(x) + \tilde{\nu}^2_{1,1}G(x), & \text{if } x \in [S_2(s_1),\, s_]), \\ V^2_{0,1}(x), & \text{if } x > s_1, \end{cases} \quad \text{(C.12)}$$

where $\tilde{\omega}^2_{1,1} := \tilde{\omega}^2_{1,1}(s_1)$ and $\tilde{\nu}^2_{1,1} := \tilde{\nu}^2_{1,1}(s_1)$ are defined as

$$\tilde{\omega}^2_{1,1} = \frac{\left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)G(S_2) - h^2_{1,1}(S_2)G(s_1)}{F(s_1)G(S_2) - F(S_2)G(s_1)}, \qquad \tilde{\nu}^2_{1,1} = \frac{h^2_{1,1}(S_2)F(s_1) - \left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)F(S_2)}{F(s_1)G(S_2) - F(S_2)G(s_1)}. \quad \text{(C.13)}$$

For $s_1 < L^2_{1,1}$, firm 2 is incentivized to preempt when $s_1 \leq X_t < L^2_{1,1}$ or right before X_t hits s_1. To wit, the *preemptive* best-response of firm 2 is a "stopping time" admitted as

$$\tau^{2,e}_{1,1}(s_1) = \inf\{t \geq 0 : (s_1-) \leq X_t < L^2_{1,1} \text{ or } X_t \leq S^{2,e}_{1,1}(s_1)\}, \quad \text{(C.14)}$$

where the optimal stopping level $S^{2,e}_{1,1} := S^e_2 \leq s_1$ is a solution to (C.11).

The proof is a symmetric repetition of the proof of Proposition 4.6 and 4.7, except the fact that the first-mover payoff of firm 2, $h^2_{1,1}(x)$, is in the class $\mathcal{H}_{\text{dec}}$, and $z = \varphi(x)$ coordinate has opposite direction from y coordinate.

Appendix C.3. Proof of Proposition 4.8

Proof. The coordination game is used to model instantaneous competition without imposing simultaneous action. It is played over infinitely many rounds, each of which lasts an infinitesimal amount of time. If at any round at least one firm invests, the game stops. Otherwise, we move on to the next round. Firm strategies are assumed to be fixed, i.e. stationary, across rounds; namely firm i attempts to invest with probability $p_i(x) \in [0, 1]$. Given the strategy profile $(p_1(x), p_2(x))$, the outcome of a given round is that firm 1 invests first with probability $p_1(x)(1 - p_2(x))$, firm 2 invests first with probability $(1 - p_1(x))p_2(x)$, and both firms invest simultaneously with probability $p_1(x)p_2(x)$. The fourth outcome is that nobody invests and we continue to the next round. Over infinitely many rounds, the game will eventually terminate and the final outcome will be

$$P_{0,1}(x) = \frac{p_1(x)(1-p_2(x))}{p_1(x)+p_2(x)-p_1(x)p_2(x)} \qquad \text{(firm 1 invests first)},$$
$$P_{1,0}(x) = \frac{p_2(x)(1-p_1(x))}{p_1(x)+p_2(x)-p_1(x)p_2(x)} \qquad \text{(firm 2 invests first)},$$
$$P_{0,0}(x) = \frac{p_1(x)p_2(x)}{p_1(x)+p_2(x)-p_1(x)p_2(x)} \qquad \text{(simultaneous investment)}.$$

Consequently, the NPV of firm 1 is

$$\begin{aligned} V^1_{1,1}(x) &= P_{0,1}(x)(V^1_{0,1}(x)-K^1) + P_{1,0}(x)V^1_{1,0}(x) + P_{0,0}(x)(D^1_{0,0}(x)-K^1) \\ &= \frac{p_1(x)(V^1_{0,1}(x)-K^1) + p_2(x)V^1_{1,0}(x) - (V^1_{1,0}(x)+V^1_{0,1}(x)-D^1_{0,0}(x))p_1(x)p_2(x)}{p_1(x)+p_2(x)-p_1(x)p_2(x)}. \end{aligned} \tag{C.15}$$

Differentiating *w.r.t.* $p_1(x)$, we get

$$\frac{\partial V^1_{1,1}}{\partial p_1}(x) = \frac{p_2(x)(V^1_{0,1}(x)-V^1_{1,0}(x)) - (V^1_{0,1}(x)-D^1_{0,0}(x))p_2^2(x)}{(p_1(x)+p_2(x)-p_1(x)p_2(x))^2},$$

which is free of $p_1(x)$. Finally, we obtain the best-response strategy for firm 1 as

$$\begin{cases} \text{if } p_2(x) > \frac{V^1_{0,1}-V^1_{1,0}-K^1}{V^1_{0,1}-D^1_{0,0}}(x), & \text{then } p_1^*(x) = 0, \\ \text{if } p_2(x) < \frac{V^1_{0,1}-V^1_{1,0}-K^1}{V^1_{0,1}-D^1_{0,0}}(x), & p_1^*(x) = 1, \\ \text{if } p_2(x) = \frac{V^1_{0,1}-V^1_{1,0}-K^1}{V^1_{0,1}-D^1_{0,0}}(x), & p_1^*(x) \in (0,1) \quad \text{is free.} \end{cases}$$

Since in this scenario we have $V^1_{0,1}(x) - K^1 > V^1_{1,0} > D^1_{0,0} - K^1$, combining similar results obtained for firm 2, we obtain the three stated equilibrium strategies. □

Appendix D. Central Planner Cooperative Monopoly

Denote by $D^M_{n_1,n_2}(x) \triangleq \left(\zeta^1_{n_1,n_2} + \zeta^2_{n_1,n_2}\right) + \frac{\rho^1 Q^1_{n_1,n_2} - \rho^2 Q^2_{n_1,n_2}}{\delta} \cdot x$ the aggregate expected profits for the central planner in stage (n_1, n_2). In stage $(1,0)$ the payoff is

$$h^M_{1,0}(x) = D^M_{0,0}(X^x_\tau) - D^M_{1,0}(X^x_\tau) - K_1 = \frac{\rho^1 \Delta Q^1_1}{\delta} \cdot x - K^M_{1,0}$$

and using Proposition 4.1 the value function is

$$V^M_{1,0}(x) = \begin{cases} D^M_{1,0}(x) + \frac{F(x)}{F(S^M_{1,0})} \cdot h^M_{1,0}\left(S^M_{1,0}\right), & \text{if } x \le S^M_{1,0}; \\ D^M_{0,0}(x) - K^1, & \text{if } x > S^M_{1,0}, \end{cases} \tag{D.1}$$

where the optimal stopping level $S^M_{1,0}$ satisfies (4.4) after substituting $h^1_{1,0}$ by $h^M_{1,0}$. In stage $(0,1)$ the payoff is $h^M_{0,1}(x) = -\frac{\rho^2 \Delta Q^2_1}{\delta} \cdot x - K^M_{0,1}$. Using Proposition 4.2 the corresponding value function is:

$$V^M_{0,1}(x) = \begin{cases} D^M_{0,0} - K^2, & \text{if } x \le S^M_{0,1}; \\ D^M_{0,1}(x) + \frac{G(x)}{G(S^M_{0,1})} \cdot h^M_{0,1}(S^M_{0,1}), & \text{if } x > S^M_{0,1}, \end{cases} \tag{D.2}$$

where the optimal stopping level $S_{0,1}^M$ satisfies equation (4.8), substituting $h_{0,1}^2$ by $h_{0,1}^M$.

In stage $(1,1)$ the payoffs become $h_{1,1}^{1,M}(x) = V_{0,1}^M(x) - D_{1,1}^M(x) - K^1$ and $h_{1,1}^{2,M}(x) = V_{1,0}^M(x) - D_{1,1}^M(x) - K^2$, which can be easily verified to belong to $\mathcal{H}_{\text{inc}}$ and $\mathcal{H}_{\text{dec}}$, respectively. With $\tau_{1,1}^{2,M,*}$ fixed (resp. $\tau_{1,1}^{1,M,*}$), the optimal problem (4.39) converts to an optimal stopping problem with an exit level (4.18), where the function $h_{1,1}^{2,M}$ (resp. $h_{1,1}^{1,M}$) acts as the second-mover payoff. Consequently, the *first-stage policy* of the monopoly is the paired stopping time given by

$$\tau_{1,1}^{1,M,*} = \inf\{t \geq 0 : \; X_t \geq S_{1,1}^{1,M,*}\}, \qquad \tau_{1,1}^{2,M,*} = \inf\{t \geq 0 : \; X_t \leq S_{1,1}^{2,M,*}\}. \tag{D.3}$$

The overall value function of the central planner in stage (1,1) is:

$$V_{1,1}^M(x) = \begin{cases} V_{1,0}^M(x), & \text{if } x \in (\underline{d},\, S_{1,1}^{2,M,*}) \\ D_{1,1}^M(x) + \omega_M F(x) + \nu_M G(x), & \text{if } x \in (S_{1,1}^{2,M,*},\, S_{1,1}^{1,M,*}) \\ V_{0,1}^M(x), & \text{if } x \in (S_{1,1}^{1,M,*},\, \bar{d}) \end{cases} \tag{D.4}$$

where

$$\omega_M = \frac{h_{1,1}^{1,M,*}(S_{1,1}^{1,M,*})G(S_{1,1}^{2,M,*}) - h_{1,1}^{2,M,*}(S_{1,1}^{2,M,*})G(S_{1,1}^{1,M,*})}{F(S_{1,1}^{1,M,*})G(S_{1,1}^{2,M,*}) - F(S_{1,1}^{2,M,*})G(S_{1,1}^{1,M,*})},$$
$$\nu_M = \frac{h_{1,1}^{2,M,*}(S_{1,1}^{2,M,*})F(S_{1,1}^{1,M,*}) - h_{1,1}^{1,M,*}(S_{1,1}^{1,M,*})F(S_{1,1}^{2,M,*})}{F(S_{1,1}^{1,M,*})G(S_{1,1}^{2,M,*}) - F(S_{1,1}^{2,M,*})G(S_{1,1}^{1,M,*})}.$$

The thresholds $(S_{1,1}^{1,M,*},\, S_{1,1}^{2,M,*})$ solve the system of equations (compare to (4.29))

$$\begin{cases} \left[h_{1,1}^{2,M}(S_2)G(S_1) - h_{1,1}^{1,M}(S_1)G(S_2)\right] F'(S_1) + \left[h_{1,1}^{1,M}(S_1)F(S_2) - h_{1,1}^{2,M}(S_2)F(S_1)\right] G'(S_1) \\ \qquad = \left(h_{1,1}^{1,M}\right)'(S_1)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right]; \\ \left[h_{1,1}^{2,M}(S_2)G(S_1) - h_{1,1}^{1,M}(S_1)G(S_2)\right] F'(S_2) + \left[h_{1,1}^{1,M}(S_1)F(S_2) - h_{1,1}^{2,M}(S_2)F(S_1)\right] G'(S_2) \\ \qquad = \left(h_{1,1}^{2,M}\right)'(S_2)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right]. \end{cases} \tag{D.5}$$

The extension to general (n_1, n_2) stage is analogous.

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