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ON THE CONTROL OF THE DIFFERENCE BETWEEN TWO BROWNIAN MOTIONS: A DYNAMIC COPULA APPROACH

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Abstract. We propose new copulae to model the dependence between two Brownian motions and to control the distribution of their difference. Our approach is based on the copula between the Brownian motion and its reflection. We show that the class of admissible copulae for the Brownian motions are not limited to the class of Gaussian copulae and that it also contains asymmetric copulae. These copulae allow for the survival function of the difference between two Brownian motions to have higher value in the right tail than in the Gaussian copula case. We derive two models based on the structure of the Reflection Brownian Copula which present two states of correlation ; one is directly based on the reflection of the Brownian motion and the other is a local correlation model. These models can be used for risk management and option pricing in commodity energy markets.

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1. Introduction

1.1. **Motivation.** Modeling dependence between risks has become an important problem in insurance and finance. An important application in risk management for commodity energy markets is the pricing of multi-asset options, and in particular the pricing of spread options. Spread options are used to model the returns of a plant, such as coal plant. A review on the spread options and on the pricing and hedging models is done by Carmona [4]. The simplest model used for derivative pricing and hedging on several underlying is the multivariate Black and Scholes model [5]. Each price is modeled by a geometric Brownian motion and the dependence between the different Brownian motions is modeled by a constant correlation matrix. The copula between the Brownian motions when they are linked by correlation is called a Gaussian copula. Copulae have many applications in finance and insurance, especially in credit derivative modeling. For instance, Li [20] used the Gaussian copula to model the dependence between time until default of different financial instruments. For more information on the use of copulae in finance, the reader can refer to [7].

Let X_t be the price of electricity at time t, Y_t the price of coal and H the heat rate (conversion factor) between the two. The income of the coal plant at time t can be modeled by $(X_t - HY_t - K)^+$ where K is a constant and corresponds to a fixed cost (we have neglected the price of carbon emissions). Coal is a combustible used to produce electricity and HY_t is the cost of one unit of coal used to produce one unit of electricity. Thus we expect to have $X_t > HY_t$, i.e. the price of electricity greater than the price of the unit of coal used to produce it, with a probability greater than $\frac{1}{2}$. Let us consider that the two commodities are modeled by an arithmetic Brownian motion with a zero drift under a risk neutral probability $\mathbb{P}$: $X_t = \sigma^X B^1_t$ and $HY_t = \sigma^Y B^2_t$ and we suppose that $\langle dB^1, dB^2\rangle_t = \rho dt$. The dependence between the two Brownian motions is modeled by a correlation, i.e a Gaussian copula. For $x \in \mathbb{R}$, we have

$$\mathbb{P}\left(X_t - HY_t \geq x\right) = \mathbb{P}\left(X_t - HY_t \leq -x\right).$$

and then, if $x \geq 0$,

$$\mathbb{P}\left(X_t - HY_t \geq x\right) \leq \frac{1}{2}$$

The distribution of the difference between the two prices is symmetric and moreover, the value of its survival function is limited to $\frac{1}{2}$ in the right tail. We would like to have higher values for this probability in order to enrich our modeling. The modeling of the dependence with a correlation does not allow to capture the asymmetry in the distribution of the difference of the prices and limits the values that can be achieved by its survival function. Today, it is common practice to use a factorial model [1] to model prices of commodities which is based on Brownian motions. Marginal models, i.e. when we consider only one commodity at the time, are enough performant for risk management. However, the dependence between them is modeled by a Gaussian copula, which is not enough to capture the asymmetry and the values taken by the survival function of their difference. Sklar's Theorem [27] states that the structure of dependence can be separated from the modeling of the marginals with the copula. Studying the impact of the structure of dependence on the modeling is equivalent to studying the impact of the copula.

Whereas copulae are very useful in a static framework where random variables are modeled, modeling with copulae is much more difficult in a dynamic framework, that is when processes are involved. In a discrete time framework, Patton [24] introduces the conditional copula which is a copula at time t defined conditionally on the information at time $t-1$. Fermanian and Wegkamp [13] generalize the concept of conditional copula. In a continuous time framework, Darsow et al. [9] consider the modeling of the time dependence by a copula. They give sufficient and necessary conditions for a copula to be the copula of a Markov process $X = (X_t)_{t\geq 0}$ between times t and s, i.e. the copula of (X_t, X_s), using the Chapman-Kolmogorov equation. We are more interested in the space dependence, that is the dependence between two different processes at a given time t. The question is studied by Jaworski and Krzywda [17]. They consider two Brownian motions and they are interested in copulae that make the bivariate process self-similar. They find necessary and sufficient conditions for the copula to be suitable for the Brownian motions deriving the Kolmogorov forward equation. The copula is linked by a local correlation function into a partial derivative equation. Further work have been done in the thesis of Bosc [3] where there are no constraints of self-similarity and it is not only limited to Brownian motions ; a more general partial derivative equation is found. More details about their work are given in Section 2.2. However, conditions for the copula to be suitable for the Brownian motions are very restrictive. An equivalent approach to the copula one is the coupling approach. A coupling of two stochastic processes is a bi-dimensional measure on the product space such that the marginal measures correspond to the ones of the stochastic processes. For more information on coupling, the reader can refer to [6]. One of the most important coupling is the coupling by reflection [21], based on the reflection of the Brownian motion which has some importance in this article.

1.2. **Objectives and results.** The objective of this article is to control the distribution of the difference between two Brownian motions at a given time t. The distribution of the difference between two Brownian motions B^1 and B^2 can be described by $x \mapsto \mathbb{P}\left(B^1_t - B^2_t \geq x\right)$, $x \in \mathbb{R}$, $t \geq 0$. If $B^1_t - B^2_t$ has a continuous cumulative distribution function, this function is the survival function of $B^1_t - B^2_t$ at point x. In particular, we want to find asymmetric distributions for $B^1 - B^2$ with more weight in the positive part than in the Gaussian copula case, i.e. $\mathbb{P}\left(B^1_t - B^2_t \geq \eta\right)$ greater than $\frac{1}{2}$ for a given $\eta > 0$. Since marginals of B^1_t and B^2_t are known, we control this distribution with the copula of $\left(B_t^1, B_t^2\right)$. One of the main issue is to work in a dynamical framework ; we then first need to extend the definition of copulae to Markovian diffusions. If we denote by $\mathcal{C}_B$ the set of admissible copulae for Brownian motions, which is properly defined in Section 2.2, our main goal is to study the range of the function

$$\begin{array}{rcccc} S_{\eta,t} & : & \mathcal{C}_B & \to & [0,1] \\ & & C & \mapsto & \mathbb{P}_C\left(B_t^1 - B_t^2 \geq \eta\right) \end{array}$$

denoted by $Ran\left(S_{\eta,t}\right)$ with $\mathbb{P}_C$ the probability measure associated to $\left(B^1, B^2\right)$ when $C \in \mathcal{C}_B$ and with $\eta > 0$ and $t \geq 0$ given.

Considering the set of Gaussian copulae, it is easy to prove that $\left[0, \Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right] \subset Ran\left(S_{\eta,t}\right)$ by controlling the correlation between the two Brownian motions with Φ the cumulative distribution function of a standard normal random variable. Furthermore, if we consider the restriction of $S_{\eta,t}$ to the set of Gaussian copulae $S_{\eta,t}|_{C_G^d}$, we have $Ran\left(S_{\eta,t}|_{C_G^d}\right) = \left[0, \Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right]$, see Proposition 14 (i) below.

Our major contribution is to construct a family of dynamic copulae in $\mathcal{C}_B$ that can achieve all the values between 0 and the supremum of $S_{\eta,t}$ on $\mathcal{C}_B$. We first prove that

$$\sup_{C \in \mathcal{C}_B} S_{\eta,t}\left(C\right) = 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)$$

in Proposition 14 (ii), implying that the Gaussian copulae can not describe all the values that can be achieved by $S_{\eta,t}$. This supremum is achieved with the copula of the Brownian motion and its reflection, which we call the Reflection Brownian Copula, and which a closed formula is given in Proposition 2. Deriving a new family of copulae that is described in Proposition 5 from the Reflection Brownian Copula, it is possible to achieve all the value between 0 and $2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)$, which means that

$$Ran\left(S_{\eta,t}\right) = \left[0, 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right];$$

this is the result of Proposition 14 (iii). Copulae used to achieve values in $Ran\left(S_{\eta,t}\right)$ present two states depending on the value of $B_t^1 - B_t^2$: one of positive correlation and one of negative one. These copulae are asymmetric and to our knowledge, these are the only asymmetric copulae suitable for Brownian motions available in the literature.

The structure of dependence of these copulae are too strong in the sense that the Brownian motions have a correlation of 1 in an infinite horizon. We derive one model based on the reflection of the Brownian motion where the dependence is relaxed: it is our multi-barrier correlation model. We define two barriers ν and η with $\nu < \eta$. We consider two independent Brownian motions X and B^Y, and we construct the Brownian motion Y^n that is correlated to $\tilde{X}^n$:

$$Y^n = \rho\tilde{X}^n + \sqrt{1-\rho^2}B^Y,$$

with $\tilde{X}^n$ the Brownian motion equal to $-X$ at the beginning and reflecting when $X - Y^n$ hits a two-state barrier equal to η before the first reflection and switching from η to ν or from ν to η at each reflection. The number of reflections is limited to n. We prove in Proposition 19 that $X - Y^n$ converges in law as $n \to \infty$ and in Corollary 18 that $\mathbb{P}\left(X_t - Y_t^n \geq x\right)$ increases for $x \in [\nu, \eta]$ when n increases if $\rho > 0$. We then consider the limit process which is of the form $X\left(\rho\right) - Y^N\left(\rho\right)$ with N a counting process, and $X\left(\rho\right)$ and $Y^N\left(\rho\right)$ two Brownian motions. When $X - Y^N$ is greater (resp. lower) than η (resp. ν), the correlation between X and Y^N is positive (resp. negative) and equal to ρ (resp. $-\rho$). The structure of dependence is then similar to the one of the Reflection Brownian Copula but relaxed. In Proposition 20, we prove that for $0 < z < \eta$,

$$\forall x \in \left[0, \Phi\left(\frac{-z}{2\sqrt{t}}\right) + \Phi\left(\frac{z-2\eta}{2\sqrt{t}}\right)\right], \exists\, \rho \in [-1,1] : \mathbb{P}\left(X_t\left(\rho\right) - Y_t^{N_t}\left(\rho\right) \geq z\right) = x.$$

Our model allows for $S_{z,t}$ to achieve all the values in $\left[0, \Phi\left(\frac{-z}{2\sqrt{t}}\right) + \Phi\left(\frac{z-2\eta}{2\sqrt{t}}\right)\right]$ which is strictly included in $Ran\left(S_{z,t}\right)$ but closed to it when z is closed to η. It allows to achieve higher values for $S_{z,t}$ and more asymmetry than in the Gaussian dependence case.

This model can be transposed to a local correlation model:

$$\begin{cases} dX_t = dB_t^X \\ dY_t = \rho\left(X_t - Y_t\right) dB_t^X + \sqrt{1 - \rho\left(X_t - Y_t\right)^2} dB_t^Y \end{cases}$$

with $\rho\left(x\right) = \rho_1$ if $x \leq \nu$ and $\rho\left(x\right) = \rho_2$ if $x \geq \eta$, which seems to be equivalent to the multi-barrier model when the two barriers have close values and $\rho_2 = -\rho_1 = \rho$.

1.3. **Structure of the paper.** In Section 2, we define the notion of dynamic copulae for Markovian diffusion processes and in particular for the case of two Brownian motions. We show that our definition includes several model of dependence present in the literature such as stochastic correlation models. In Section 3, we compute a copula called the Reflection Brownian Copula based on the dependence between a Brownian motion and its reflection and we derive new families of asymmetric copulae based on this copula. In Section 4, after showing the limitations of modeling the dependence between two random variables with symmetric copulae, we establish the results on the range of the function $S_{\eta,t}$, first in a static framework and then in a dynamical framework with Brownian motions. In Section 5 and Section 6, we construct models based on the structure of the Reflection Brownian Copula. The first one is the multi-barrier correlation model and is directly based on the reflection of the Brownian motion, the second one is a local correlation model. Section 4 and Section 5 are our major contributions. In Section 7, we apply our results to the modeling of the dependence between the price of two commodities which are electricity and coal. Proofs are given in Section 8.

2. Markov Diffusion Copulae

In finance and insurance, modeling of two dimensional processes is usually based on a 2 dimensional Brownian motion, that is when the structure of dependence between two 1 dimensional Brownian motions is modeled by a correlation. The copula of the two Brownian motions at a given time then belongs to the class of Gaussian copulae.

Let us recall that a function $C : [0,1]^2 \mapsto [0,1]$ is a copula if:

(i) C is 2-increasing, i.e. $C\left(u_2, v_2\right) - C\left(u_1, v_2\right) + C\left(u_1, v_1\right) - C\left(u_2, v_1\right) \geq 0$ for $u_2 \geq u_1, v_2 \geq v_1$ and $u_1, u_2, v_1, v_2 \in [0,1]$,
(ii) $C\left(u, 0\right) = C\left(0, v\right) = 0,\ u, v \in [0,1]$,
(iii) $C\left(u, 1\right) = u, C\left(1, u\right) = u,\ u \in [0,1]$.

We denote by $\mathcal{C}$ the set of copulae and by $\mathcal{C}_G$ the set of Gaussian copulae. $\mathcal{C}_G = \{C \in \mathcal{C} : \exists \rho \in [-1,1], C = C_{G,\rho}\}$ where $C_{G,\rho}$ denote the Gaussian copula with parameter ρ. We have

$$C_{G,\rho}(u,v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

with Φ the cumulative distribution function of a standard normal random variable and Φ_ρ the cumulative distribution function of a bivariate normal random variable with correlation ρ:

$$\Phi_\rho(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)} du dv.$$

In this section, we want to generalize the concept of copula which is adapted for random variables to a dynamical framework. We want to define the notion of copula for Markov diffusions in Section 2.1. In particular, we are interested in copulae suitable for Brownian motions in Section 2.2.

2.1. **Definition.**

In order to work in a dynamical framework, we need to extend the concept of copula to Markovian diffusions. Our definition is based on the work of Bielecki et al. [2] and gives a more general definition.

We recall that if $P = (P_t)_{t\geq 0}$ is a Markovian diffusion solution of the stochastic differential equation

$$dP_t = \mu(P_t)\,dt + \sigma(P_t)\,dW_t,$$

with $W = (W_t)_{t\geq 0}$ a standard Brownian motion, the infinitesimal generator $\mathcal{L}$ of P is the operator defined by

$$\mathcal{L}f(x) = \frac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x)$$

for f in a suitable space of functions including $\mathcal{C}^2$.

Definition 1 (Admissible copula for Markovian diffusions)**.** *We say that a collection of copula $C = (C_t)_{t\geq 0}$ is an admissible copula for the n real valued Markovian diffusions, $n \geq 2$, $\left(X^i\right)_{1\leq i\leq n}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if there exists a $\mathbb{R}^m$ Markovian diffusion $Z = \left(Z^i\right)_{1\leq i\leq m}$, $m \geq n$, defined on a probability extension of $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\begin{cases} \mathcal{L}\left(Z^i\right) = \mathcal{L}\left(X^i\right), 1 \leq i \leq n, \\ Z_0^i = X_0^i, 1 \leq i \leq n, \\ \text{for } t \geq 0, \text{ the copula of } \left(Z_t^i\right)_{1\leq i\leq n} \text{ is } C_t. \end{cases}$$

The strongest constraint to be admissible is that Z has to be a Markovian diffusion. Without this constraint, all the copulae are admissible. Sempi [26] studies the Brownian motions linked by a copula without this constraint. Definition 1 is consistent with the approach of [17] or [3] consisting of modeling dependence by a local correlation function. However, our approach is totally different.

2.2. **Brownian motion case.**

From now on, we work in a 2 dimensional framework and we denote by $\mathcal{C}_B$ the set of admissible copulae for Brownian motions, that is when X^1 and X^2 are Brownian motions. The only well known suitable copulae for Brownian motion are the Gaussian copulae.

We can extend the definition of $\mathcal{C}_G$ to a dynamical framework by defining

$$\mathcal{C}_G^d = \{(C_t)_{t\geq 0} : \exists (\rho_t)_{t\geq 0},\ \forall t \in \mathbb{R}^+\ C_t = C_{G,\rho_t} \text{ and } \rho_t \in [-1,1]\} \cap \mathcal{C}_B.$$

It is necessary to take the intersection with $\mathcal{C}_B$ because we do not know if conditions are needed on $(\rho_t)_{t\geq 0}$ for the copula to be admissible. We are not interested in this question in this paper. However, we know this intersection is not empty because $\{(C_t)_{t\geq 0} : \exists \rho \in [-1,1],\ \forall t \in \mathbb{R}^+\ C_t = C_{G,\rho}\} \subset \mathcal{C}_B$. One of our objective is to find copulae that are admissible for Brownian motion but that are not Gaussian copulae.

Jaworski and Marcin [17] prove that the set of admissible copulae for Brownian motions was not reduced to the Gaussian copulae. By linking local correlation and copula with the Kolmogorov backward equation, they find that a sufficient condition to be admissible is

$$\left|\frac{1}{2}e^{\frac{\Phi^{-1}(v)^2-\Phi^{-1}(u)^2}{2}}\frac{\partial^2_{u,u}C(u,v)}{\partial^2_{u,v}C(u,v)}+\frac{1}{2}e^{\frac{\Phi^{-1}(u)^2-\Phi^{-1}(v)^2}{2}}\frac{\partial^2_{v,v}C(u,v)}{\partial^2_{u,v}C(u,v)}\right|<1 \quad \forall(t,u,v)\in\mathbb{R}^+\times[0,1]^2 \tag{1}$$

when the copula does not depend on time. In particular, they prove that the extension of the FGM copula $C^{FGM}(u,v)=uv(1+a(1-u)(1-v)),a\in[-1,1]$ in a dynamical framework defined by $C_t(u,v)=C^{FGM}(u,v),\ t\geq 0$, is an admissible copula for Brownian motions. Bosc [3] has also found admissible copulae.

Let us consider two independent Brownian motions B^1 and Z defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$. Definition 1 includes several models for Brownian motions used in the literature.

Deterministic correlation. Let us consider a function $t\mapsto\rho(t)$ defined on $\mathbb{R}^+$ with values in $[-1,1]$. Let $B^2_t=\int_0^t\rho(s)\,dB^1_s+\int_0^t\sqrt{1-\rho(s)^2}dZ_s$.

B^2 is a Brownian motion and the dynamic copula defined at each time t by the copula of $\left(B^1_t,B^2_t\right)$ is in $\mathcal{C}_B$.

Local correlation. Let us consider a function $(x,y)\mapsto\rho(x,y)$ defined on $\mathbb{R}^+$ with values in $[-1,1]$ and measurable. If the stochastic differential equation

$$dB^2_s=\rho\left(B^1_t,B^2_t\right)dB^1_s+\sqrt{1-\rho\left(B^1_t,B^2_t\right)^2}dZ_s$$

has a strong solution, the dynamic copula defined at each time t by the copula of $\left(B^1_t,B^2_t\right)$ is in $\mathcal{C}_B$ by the Lévy characterization of Brownian motion.

Stochastic correlation. Let us consider a Markovian diffusion $\rho=(\rho_s)_{s\geq0}$ independent of $\left(B^1,Z\right)$ locally square integrable and with values in $[-1,1]$.

We can extend the probability space and the filtration generated by $\left(B^1,Z\right)$. The stochastic process B^2 defined by $B^2_t=\int_0^t\rho(s)\,dB^1_s+\int_0^t\sqrt{1-\rho(s)^2}dZ_s$ is a Brownian motion and the dynamic copula defined at each time t by the copula of $\left(B^1_t,B^2_t\right)$ is in $\mathcal{C}_B$.

We can also consider a correlation diffusion driven by B^1, Z and an independent Brownian motion. If the system of stochastic differential equations has a strong solution, the copula is still in $\mathcal{C}_B$.

Contrary to the approaches of Jaworski and Macin [17], Bosc [3] or Bielecki et al. [2], Definition 1 includes stochastic correlation models. However, we need for the stochastic correlation to be a Markovian diffusion which is not needed in a general case ; the stochastic correlation has only to be progressively measurable.

3. Reflection Brownian Copula

In this section, our objective is to construct Markov Diffusion Copulae defined in Section 2. We construct a new copula based on the reflection of the Brownian motion. We show that the copula between the Brownian motion and its reflection is adapted to a dynamical framework and is a suitable copula for Brownian motions. Furthermore, we give a closed formula of this copula in Section 3.1. To our knowledge, this copula has not been studied in detail and it is the new copula suitable for Brownian motions. We also construct new families of copulae by extension of the Reflection Brownian Copula in Section 3.2.

3.1. **Closed formula for the copula.** In this section, we study the copula between the Brownian motion and its reflection. Since its reflection is also a Brownian motion, the copula is a good candidate for being in $\mathcal{C}_B$.

Let us consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis (right continuity and completion) and $B = (B_t)_{t\geq 0}$ a Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$. We denote by $\tilde{B}^h$ the Brownian motion reflection of B on $x = h$ with $h \in \mathbb{R}$, i.e. $\tilde{B}^h_t = -B_t + 2(B_t - B_{\tau^h})\mathbf{1}_{t\geq \tau^h}$ with $\tau^h = \inf\{t \geq 0 : B_t = h\}$. Thus, $\tilde{B}^k$ is a $\mathcal{F}$ Brownian motion according to the reflection principle (see [18, Theorem 3.1.1.2, p. 137]). Proposition 2 gives the copula of $\left(B, \tilde{B}^h\right)$.

We denote by $M(u,v) = \min(u,v)$ and $W(u,v) = \max(u+v-1,0)$, $u,v \in [0,1]$ the upper and lower Frechet copulae. We recall that Φ denotes the cumulative distribution function of a standard normal random variable.

Proposition 2. *Let $h > 0$. The copula of $(B, \tilde{B}^h)$, $\left(C_t^{ref,h}\right)_{t\geq 0}$, is defined by*

$$C_t^{ref,h}(u,v) = \begin{cases} v & \text{if} \quad \Phi^{-1}(u) - \Phi^{-1}(v) \geq \frac{2h}{\sqrt{t}} \\ W(u,v) + \Phi\left(\Phi^{-1}(M(u,1-v)) - \frac{2h}{\sqrt{t}}\right) & \text{if} \quad \Phi^{-1}(u) - \Phi^{-1}(v) < \frac{2h}{\sqrt{t}} \end{cases} \tag{2}$$

and $\left(C_t^{ref,h}\right)_{t\geq 0} \in \mathcal{C}_B$. We call this copula the Reflection Brownian Copula.

3.2. **Extensions.** In this section, we give methods to construct new admissible copulae for Brownian motions from the Reflection Brownian Copula.

Proposition 3 and its proof gives an approach to construct different admissible copulae for Brownian motions based on the Reflection Brownian Copula considering a correlated Brownian motion to the reflection of the Brownian motion.

Proposition 3. *Let $h > 0$ and $\rho \in (0,1)$. The copula*

$$C_t(u,v) = \begin{cases} \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v) + \frac{2\rho h}{\sqrt{t}}\right) + v - \Phi\left(\Phi^{-1}(v) + \frac{2\rho h}{\sqrt{t}}\right) & \text{if} \quad u \geq \Phi\left(\frac{h}{\sqrt{t}}\right) \\ \Phi_{-\rho}\left(\Phi^{-1}(u), \Phi^{-1}(v)\right) + \Phi_\rho\left(\Phi^{-1}(u) - \frac{2h}{\sqrt{t}}, \Phi^{-1}(1-v) - \frac{2\rho h}{\sqrt{t}}\right) + \\ \quad \Phi_\rho\left(\Phi^{-1}(u) - \frac{2h}{\sqrt{t}}, \Phi^{-1}(v)\right) - \Phi\left(\Phi^{-1}(u) - \frac{2h}{\sqrt{t}}\right) & \text{if} \quad u < \Phi\left(\frac{h}{\sqrt{t}}\right) \end{cases}$$

is in $\mathcal{C}^B$.

Contrary to the Reflection Brownian Copula, this copula is non degenerated in the sense that we have two distinct sources of randomness. Indeed, in the Reflection Brownian Copula case, if we know the trajectory of the Brownian motion, we also know the one of its reflection.

Remark 4. *In the case $\rho = 0$, we still have a copula which is the independent copula and then that is in $\mathcal{C}_B$.*

An other way to construct admissible copulae is to consider a random barrier. By enlarging the filtration, the copula of the two processes is an admissible copula and it can be computed by integrating the copula of the Reflection Brownian motion according to the law of the barrier. The result is given in Proposition 5.

Proposition 5. *Let ξ be a positive random variable with law having a density and $\overline{F}^{\xi}$ its survival function. The copula*

$$C_t^{\xi}(u,v) = v - \int_{-\infty}^{\Phi^{-1}(M(1-u,v))} \frac{e^{\frac{-w^2}{2}}}{\sqrt{2\pi}} \overline{F}^{\xi}\Big(\frac{\sqrt{t}}{2}\left(\Phi^{-1}\left(M\left(u,1-v\right)\right)-w\right)\Big)dw$$

is in $\mathcal{C}^B$.

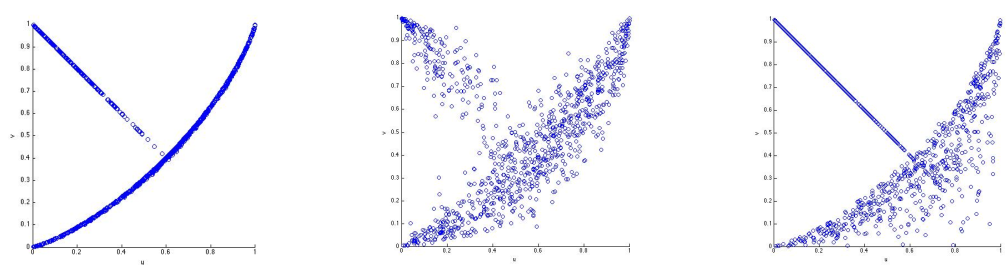
Example 6 below gives a copula with closed formula built with the method of Proposition 5.

Example 6. *Let $\xi \overset{d}{=} h + X$ with $h \in \mathbb{R}$ and X a random variable following an exponential law with parameter $\lambda > 0$. We have $\overline{F}^{\xi}(x) = \begin{cases} 1 & \text{if} \quad x \le h \\ e^{-\lambda(x-h)} & \text{if} \quad x > h \end{cases}$ and the copula*

$$(3) \quad C_t^{exp,h,\lambda}(u,v) = W(u,v) + \min\Big[\Phi\Big(\Phi^{-1}\left(M\left(1-u,v\right)\right) - \frac{2h}{\sqrt{t}}\Big), M(u,1-v)\Big]$$
$$- \Phi\Big(\min\Big[\Phi^{-1}\left(M\left(1-u,v\right)\right) - \frac{2h}{\sqrt{t}}, \Phi^{-1}\left(M\left(u,1-v\right)\right)\Big] - \frac{\lambda\sqrt{t}}{2}\Big)e^{\lambda h + \frac{\lambda^2 t}{4} + \frac{\lambda\sqrt{t}}{2}\Phi^{-1}(M(u,1-v))}$$

is in $\mathcal{C}_B$.

The methods of Proposition 3 and 5 could be used simultaneously to construct new classes of admissible copulae. Figure 1 represents the Reflection Brownian Copula and some of its extensions.



(A) (B) (C)

FIGURE 1. *The Reflection Brownian Copula $C^{ref,h}$ and some of its extensions at time $t = 1$ with $h = 2$. Figure 1a is the Reflection Brownian Copula. Figure 1b is the extension considering a Brownian motion correlated to the reflection of the first Brownian with a correlation $\rho = 0.95$, which is the copula of Proposition 3. Figure 1c is the extension in the case of a random barrier following an exponential law with parameter $\lambda = 2$, which is the copula of Example 3.*

4. Control of the distribution of the difference between two Brownian motions

Let B^1 and B^2 be two standard Brownian motions defined on a common filtered probability space $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{t\geq 0},\ \mathbb{P}_C)$ with $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis and where $\mathbb{P}_C$ is the probability measure associated to $\left(B^1, B^2\right)$ and $C = (C_t)_{t\geq 0} \in \mathcal{C}_B$ is the copula of $\left(B^1, B^2\right)$. In this section, we are interested in the distribution of the difference between B^1 and B^2, i.e. the function $x \mapsto \mathbb{P}_C\left(B^1_t - B^2_t \geq x\right)$ for $t > 0$ and in particular in the right tail of this distribution, i.e. when $x > 0$. Since the marginal of B^1 and B^2 are known, this function is entirely determined by the copula of $\left(B^1, B^2\right)$. Our goal is to find the range of values that can be achieved by this function at a given $x > 0$. Given $\eta > 0$ and $t \geq 0$, we define the function

$$
\begin{array}{rccc} S_{\eta,t} & : & \mathcal{C}_B & \to & [0,1] \\ & & C & \mapsto & \mathbb{P}_C\left(B^1_t - B^2_t \geq \eta\right). \end{array} \tag{4}
$$

Remark 7. *$\mathbb{P}_C$ is a probability measure that verifies $\mathbb{P}_C\left(B^1_t \leq x, B^2_t \leq y\right) = C_t\left(\Phi\left(\frac{x}{\sqrt{t}}\right), \Phi\left(\frac{y}{\sqrt{t}}\right)\right)$ for $x, y \in \mathbb{R}$. However, C does not describe entirely $\mathbb{P}_C$. Indeed, C describe the dependence between B^1_t and B^2_t at a given time t but not between B^1_s and B^2_t with $s \neq t$ for instance.*

Our objective is to control the value of this function at a given time t by controlling the dependence between the two Brownian motions. For this, we first study the range of this function $Ran\left(S_{\eta,t}\right)$. We show that the Reflection Brownian Copula defined in Section 3 and its extensions allow us to control $S_{\eta,t}$ and to achieve all the values in $Ran\left(S_{\eta,t}\right)$. After showing the limitations of symmetric copulae for the control of $S_{\eta,t}$ in Section 4.1, we give a result about $Ran\left(S_{\eta,t}\right)$ in a static case in Section 4.2, i.e. in the case of two Gaussian random variables. Most of results of Section 4.1 and Section 4.2 are classic for the sum of random variables ; we adapt them to the difference case. Finally, we give the main result concerning the range of $S_{\eta,t}$ in Section 4.3.

4.1. Impact of symmetry on $S_{\eta,t}$.

In this section, we show that modeling the dependence between two random variables with symmetric copulae limits the values that can be taken by the distribution of the difference between two random variables. It imposes some constraints on this distribution. Using asymmetric copulae is then necessary to control $S_{\eta,t}$. We also show that we can find asymmetric copulae suitable for Brownian motions.

Definition 8. *A copula C is symmetric if $C(u,v) = C(v,u)$, $u, v \in [0,1]$. We denote by $\mathcal{C}_s$ the set of symmetric copulae.*

Note that $\mathcal{C}_G \subset \mathcal{C}_s$ with $\mathcal{C}_G$ the set of Gaussian copulae.

If X and Y are two random variables with continuous cumulative distribution functions, we denote by $C^{X,Y}$ the copula of (X,Y). Sklar's Theorem [27] guarantees the existence and the unicity of $C^{X,Y}$. Proposition 9 gives properties on the distribution the difference of two random variables if their copula is symmetric.

Proposition 9. *Let X and Y be two real valued random variables defined on the same probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ with copula $C^{X,Y}$ and with continuous marginal distribution functions F^X and F^Y. If $F^X = F^Y$ and $C^{X,Y} \in \mathcal{C}_s$ then $\mathbb{P}(X - Y \leq -x) = \mathbb{P}(X - Y \geq x)$.*

We can extend the definition of symmetry and asymmetry to Markov Diffusion Copulae: we denote by $\mathcal{C}^d_a = \{(C_t)_{t\geq 0} : \forall t \geq 0, C_t \in \mathcal{C}_s\}$ the set of symmetric Markov Diffusion Copulae and by $\mathcal{C}^d_s = \{(C_t)_{t\geq 0} : \forall t \geq 0, C_t \in \mathcal{C} \setminus \mathcal{C}_s\}$ the set of asymmetric Markov Diffusion Copulae.

Corollary 10. *For $\eta > 0$ and $t > 0$, we have:*

$$Ran\left(S_{\eta,t}|_{\mathcal{C}_s^d}\right) \subset \left[0, \frac{1}{2}\right]$$

with $S_{\eta,t}|_{\mathcal{C}_s^d}$ the restriction of $S_{\eta,t}$ to $\mathcal{C}_s^d$.

Proof If we consider two Brownian motions B^1 and B^2 with dynamic copula $C \in \mathcal{C}_s^d$, we have according to Proposition 9: $\mathbb{P}\left(B_t^1 - B_t^2 \geq x\right) = \mathbb{P}\left(B_t^1 - B_t^2 \leq -x\right)$. However, $\mathbb{P}\left(B_t^1 - B_t^2 \geq x\right) + \mathbb{P}\left(B_t^1 - B_t^2 \leq -x\right) \leq 1$ if $x \geq 0$. Then we have the constraint $\mathbb{P}\left(B_t^1 - B_t^2 \geq x\right) \leq \frac{1}{2}$. □

In particular, since the Gaussian copula is symmetric, it is not possible to obtain asymmetry in the distribution of $B_t^1 - B_t^2$ at each time t when the dependence between two Brownian motions is given by a correlation structure. Limiting the modeling of the dependence to the Gaussian copula or to symmetric copulae makes the distribution of their difference symmetric and limits the value of $S_{\eta,t}$.

Modeling the dependence by an asymmetric copula is then necessary to have higher values than $\frac{1}{2}$ for $S_{\eta,t}$. We have

$$\mathcal{C}_B \cap \mathcal{C}_a^d \neq \emptyset.$$

Indeed, the Reflection Brownian Copula defined in Equation (2) is in $\mathcal{C}_B$ and is asymmetric. The set of admissible copulae for Brownian motion is not reduced to the set of Gaussian copulae and furthermore it contains an asymmetric copula which is the Reflection Brownian Copula. Jaworski and Marcin [17] and Bosc [3] have proven the existence of symmetric suitable copulae for Brownian motions. However, they did not find asymmetric copulae suitable for Brownian motions. We can also show that extensions of the Brownian Reflection Copula defined in Section 3.2 are asymmetric. To our knowledge, these copulae are the only asymmetric copulae suitable for Brownian motions in the literature.

Remark 11. *Copulae constructed in Section 3.2 can also be used as a method to construct asymmetric copulae, which is not always evident.*

4.2. **The Gaussian Random Variables Case.** Let us consider two standard normal random variables X and Y defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P}_C)$ where $\mathbb{P}_C$ is the probability measure associated to the copula C of (X, Y). Since the laws of the marginals of X and Y are fixed, the probability measure only depends on the copula of (X, Y), which justifies the notation $\mathbb{P}_C$. In this section, we study the control of the distribution of the difference $\mathbb{P}_C\left(X - Y \geq \eta\right)$ for a given η. We need to adapt the definition of $S_{\eta,t}$ for the static case, i.e. when the copula are not dynamic. We define the function

$$\begin{array}{cccc} \tilde{S}_\eta & : & \mathcal{C} & \rightarrow & [0,1] \\ & & C & \mapsto & \mathbb{P}_C\left(X - Y \geq \eta\right) \end{array}$$

for a given $\eta > 0$.

Remark 12. *$\mathbb{P}_C$ is defined by $\mathbb{P}_C\left(X \leq x, Y \leq y\right) = C\left(\Phi\left(x\right), \Phi\left(y\right)\right)$ for $x, y \in \mathbb{R}$.*

In particular, we look for an upper bound of $\tilde{S}_\eta$. Lower bound is trivial and is achieved by the copula $M\left(u, v\right) = \min\left(u, v\right)$. Note that this copula is equivalent to having correlation 1 between the two random variables and corresponds to a case of comonotonicity. The problem is similar to the one consisting in finding bounds on the distribution of the sum. Makarov [22] finds bounds on the cumulative distribution function of the sum of two random variables at a given point given the marginals. Rüschendorf [25] proves this result using optimal transport theory. Frank et al. [15] prove the same result using copulae and find a copula that achieves the bound. Furthermore, the results are extended to dimensions greater than 2 and to the cumulative distribution function of $L(X,Y)$ where L is a non decreasing continuous function in X and Y with X and Y two random variables. Finding these bounds have several applications in finance and insurance such as finding bounds on value-at-risk [12].

In Proposition 13, we study the range of values taken by $\tilde{S}_\eta$. In particular, we look for an upper bound when the copula is taken among the set of Gaussian copulae then among all the copulae. We also find the range of $\tilde{S}_\eta$. In order to maximize $\tilde{S}_\eta(C)$ over all the copulae, we use the approach of Frank et al. [15] with copulae.

Proposition 13. *Let $\eta > 0$.*

Let

$$C^r(u,v) = \begin{cases} M(u-1+r,v) & \textit{if } (u,v) \in [1-r,1]\times[0,r], \\ W(u,v) & \textit{if } (u,v) \in [0,1]^2 \setminus ([1-r,1]\times[0,r]) \end{cases}$$

with $r = 2\Phi\left(\frac{-\eta}{2}\right)$.

We have:

(i) *Ran* $\left(\tilde{S}_\eta|_{\mathcal{C}_G}\right) = \left[0, \Phi\left(\frac{-\eta}{2}\right)\right]$ *with $\tilde{S}_\eta|_{\mathcal{C}_G}$ the restriction of $\tilde{S}_\eta$ to $\mathcal{C}_G$,*

(ii) $\sup_{C\in\mathcal{C}} \tilde{S}_\eta(C) = 2\Phi\left(\frac{-\eta}{2}\right)$ *and the supremum is achieved with C^r,*

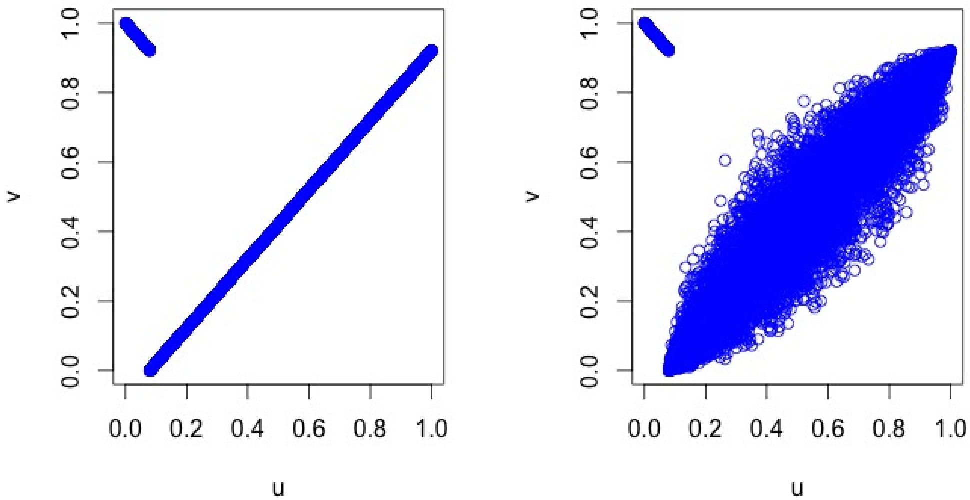
(iii) *Ran* $\left(\tilde{S}_\eta\right) = \left[0, 2\Phi\left(\frac{-\eta}{2}\right)\right]$.

If we only consider the set of Gaussian copulae, $\tilde{S}_\eta$ can only achieve the values in $\left[0, \Phi\left(\frac{-\eta}{2}\right)\right]$. If we consider all the copulae, values in $\left[\Phi\left(\frac{-\eta}{2}\right), 2\Phi\left(\frac{-\eta}{2}\right)\right]$ can also be achieved. Indeed, we can use the family of copulae constructed in Proposition 13 to achieve these values. It has a particular structure: it is divided in two parts according to the value of the first random variable. One state corresponds to a positive correlation and the upper bound is achieved in the comonotonic case. The other state corresponds to the countermonotonic case.

The family of copulae constructed in Proposition 13 are patchwork copulae [11]. Given a copula C, a patchwork copula is constructed by changing the value of C in a subrectangle of the unit square and replacing it with an other copula. In our case, we consider the countermonotonic copula and we change its values in the rectangle $[1-r,1]\times[0,r]$, replacing it by a Gaussian copula with parameter ρ. The copula achieving the bound corresponds to $\rho = 1$ and in this particular case, the copula is called a shuffle of M copula [23]. Figure 2 shows illustration of the copulae family constructed in Proposition 13 with a correlation of 1 and a correlation of -0.95.

If we consider two Brownian motions B^1 and B^2, B^1_t and B^2_t at a given time t are Gaussian random variables with variance t. Proposition 13 can be applied with $\eta' = \frac{\eta}{\sqrt{t}}$. Modeling the dependence of Brownian motions with a Gaussian copula then limits us in terms of values taken by $S_{\eta,t}$. In particular, it is not possible to have probabilities greater than $\frac{1}{2}$ which was already proven with the symmetry property of Gaussian copulae.

In this section, we showed the limits of the Gaussian copulae and that it was possible to achieve new values for $\tilde{S}_\eta$ or to put asymmetry in the distribution of the difference with the use of different



(A) $\rho = 1$ (B) $\rho = 0.95$

FIGURE 2. *Patchwork copula $C^r(u,v)$ presenting two states depending on the value of u: the first one corresponds to the Gaussian copula with correlation equal to -1, the second one to the Gaussian copula with correlation equal to ρ, with $\rho = 1$ or $\rho = 0.95$. r is equal to $2\Phi\left(\frac{-\eta}{2}\right)$ with $\eta = 0.2$.*

types of copulae. However, the copulae were used to model the dependence between the two Gaussian variables, i.e. two Brownian motions at given time t. We do not know if the copulae are suitable to model the dependence between $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$, that is in a dynamical framework.

4.3. **The Brownian Motion Case.** Proposition 14 gives a time dynamical version of Proposition 13.

Proposition 14. *Let $\eta > 0$ and $t > 0$. We have:*

(i) *$Ran\left(S_{\eta,t}|_{\mathcal{C}_G^d}\right) = \left[0, \Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right]$ with $S_{\eta,t}|_{\mathcal{C}_G^d}$ the restriction of $S_{\eta,t}$ to $\mathcal{C}_G^d$,*

(ii) *$\sup\limits_{C\in\mathcal{C}_B} S_{\eta,t}\left(C\right) = 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)$ and the supremum is achieved with $C^{ref,\frac{\eta}{2}}$ which is the Reflection Brownian Copula defined by Equation (2),*

(iii) *$Ran\left(S_{\eta,t}\right) = \left[0, 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right]$.*

We have found a copula which maximizes $S_{\eta,t}$ at each time t which is admissible for Brownian motions. This copula is also a solution to the problem $\sup\limits_{C\in\mathcal{C}} \tilde{S}_{\frac{\eta}{\sqrt{t}}}\left(C\right)$ and give an alternative solution of the supremum copula of Proposition 13. We also notice than $Ran\left(S_{\eta,t}\right) = Ran\left(\tilde{S}_{\frac{\eta}{\sqrt{t}}}\right)$. The constraint to be in $\mathcal{C}_B$ does not change the solution of our problem, values that can be achieved are the same but the copula are not the same. As in the copula of Proposition 13, the copula has two states: one of comonotonicity and one of countermonotonicity, depending here on the value of the $B_t^1 - B_t^2$. Figure 3 represents the Reflection Brownian Copula at time $t = 1$ with a reflection at $\frac{\eta}{2} = 0.1$. We can see that the structure is the same than the copula of Figure 2a. However, in Figure 2a the two lines are in separated parts of the square and in Figure 3, there is a part of the square where they are both present. This is due to the fact that $\tilde{B}_t$ is not a deterministic function of B_t but a deterministic function of B_t and $\sup_{s \le t} B_s$.

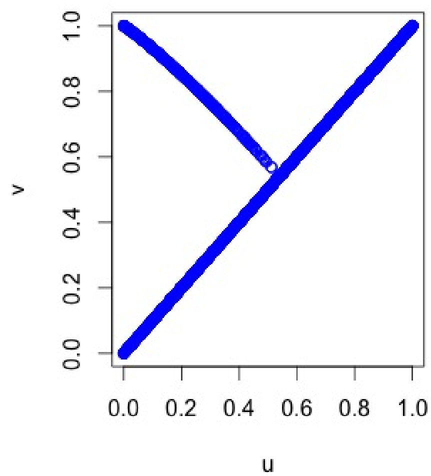


Figure 3. *Reflection Brownian Copula* $C_t^{ref,\frac{\eta}{2}}$ *at time* $t = 1$ *with* $\eta = 0.2$

Part (iii) of Proposition 14 gives us a way to control $S_{\eta,t}$. Furthermore, when the copula is the Reflection Brownian one, the probability for $B_t^1 - B_t^2$ to have strictly higher value than η is equal to 0 and there is a discontinuity at η ; copulae of part (iii) allow us to solve this issue. The copulae become suboptimal but still achieves higher values than in the Gaussian copula case.

Result of Proposition 14 (ii) can be interpreted with coupling. Let X be a stochastic process. Let X^a and X^b be processes with the dynamic of X such that $X_0^a = a$ and $X_0^b = b$. A coupling is said successful if $T = \inf\{t \ge 0 : X_t^a = X_t^b\} < \infty$ almost surely. T is called the coupling time. In our situation, the two Brownian motions start at 0 and are coupled when $B_t = \tilde{B}_t^{\frac{\eta}{2}} + \eta$ which is equivalent to consider one Brownian starting at 0 and the other starting at η. We have the coupling inequality:

$$\|\mathbb{Q}_a(t) - \mathbb{Q}_b(t)\| \le 2\mathbb{P}(T > t) \tag{5}$$

with $\|\|$ the total variation norm and $\mathbb{Q}_a(t)$ the distribution of X_t^a (same for $\mathbb{Q}_b(t)$ and X_b^t). In case of equality for (5), the coupling is said to be optimal [16]. The coupling by reflection [21], consisting of taking the reflection of the Brownian motion according to the hyperplane $x = \frac{a+b}{2}$, is optimal for Brownian motion. Hsu and Sturm [16] prove that in the case of Brownian motions, it is the only optimal Markovian coupling (definition 15).

Definition 15. [16] *Let $X = (X_1, X_2)$ be a coupling of Brownian motions. Let $\mathcal{F}^X$ be the filtration generated by X. We say that X is a Markovian coupling if for each $s \geq 0$, conditional on $\mathcal{F}_s^X$, the shifted process $\{(X_1(t+s), X_2(t+s)), t \geq 0\}$ is still a coupling of Brownian motions (now starting from $(X_1(s), X_2(s))$).*

In the optimal case, $\mathbb{P}(T > t)$ is minimal. The coupling by reflection can then be interpreted as the fastest way for the two processes to be equal. In our case, it is the fastest way for the $B^1 - B^2$ to be greater than η.

We found an admissible copula for Brownian motions which has the property to be asymmetric and to achieve upper bound for $S_{\eta,t}$. We have also constructed new families of asymmetric copulae allowing us to control the value of $S_{\eta,t}$. To our knowledge, no other asymmetric copulae admissible for Brownian motions has been found.

5. Multi-barrier correlation model

In Section 4, we have found dynamic copulae that allows us to control $S_{\eta,t}$ defined by Equation (4). However, the dependence between the two Brownian motions when it is modeled by these copulae is degenerated in the sense that the difference between the two Brownian motions becomes constant in an infinite horizon. In this section, we construct a model based on the reflection of the Brownian motion which does not present this degeneracy but which allows higher values for $S_{\eta,t}$ than in the Gaussian copula case. As seen in Section 3, the Reflection Brownian Copula contains two states depending on the value of the difference between the two Brownian motions: one of comonotonicity and one of countermonotonicity, that is correlation equal to 1 and -1.

We relax this strong dependence by diminishing the correlation in absolute value. Thus, we want to have two Brownian motions X and Y with the following correlation structure: if the value of $X - Y$ is under a certain level that we denote by ν, X and Y have a negative correlation $-\rho$ and if it is over an other level denoted by η, their correlation is positive and equal to ρ. One way to obtain this structure is to start with two Brownian motions having a negative correlation. When the difference between them reaches the barrier η, Y reflects and the correlation becomes positive. If the correlation is positive (resp. negative) and $X - Y$ reaches ν (resp. η), Y reflects and the correlation becomes negative (resp. positive). The number of reflection that can happen is a parameter of our model denoted by n. Y is then correlated to a reflection of X reflecting each time the difference between the two reaches one of the two barriers. Figure 4 gives an illustration of our model.

5.1. **Model.** Let B^X and B^Y be two independent Brownian motions defined on a common filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual properties. We will denote indifferently B^X by X.

Let $\eta > 0$, $\nu < \eta$ and $\rho \in [-1, 1]$.

Let $\alpha_k = \begin{cases} 0 \text{ if } k = 0 \\ \eta \text{ if } k \text{ odd} \\ \nu \text{ if } k \text{ even}, k \neq 0 \end{cases}$.

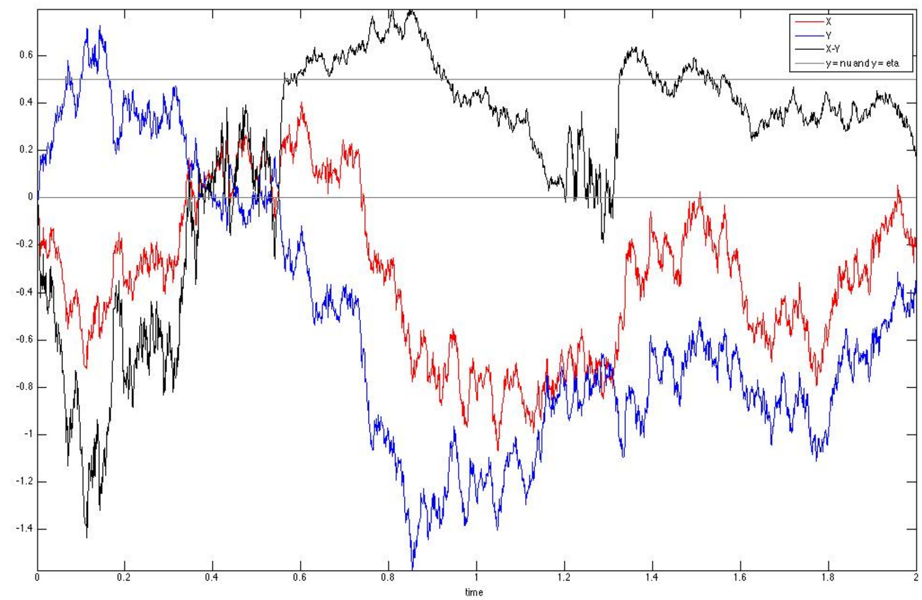


FIGURE 4. *One trajectory of X, Y^n, $X - Y^n$ in the multi-barrier correlation model with $\nu = 0$, $\eta = 0.5$, $\rho = 0.9$ and $n = \infty$.*

Let $\left(\tilde{B}^k\right)_{k\geq 0}$, $\left(Y^k\right)_{k\geq 0}$ and $(\tau_k)_{k\geq 0}$ be defined by

$$\begin{cases} \tau_0 = 0 \\ \tilde{B}^0 = -B^X \\ Y_t^0 = \rho\tilde{B}^0 + \sqrt{1-\rho^2}B^Y \end{cases},$$

$$\begin{cases} \tau_k = \inf\{t \geq \tau_{k-1} : B_t^X - Y_t^{k-1} = \alpha_k\} \quad k \geq 1 \\ \tilde{B}^k = \mathcal{R}(\tilde{B}^{k-1}, \tau_k) \quad k \geq 1 \\ Y^k = \rho\tilde{B}^k + \sqrt{1-\rho^2}B^Y \quad k \geq 1, \end{cases}$$

where $\mathcal{R}(B,\tau)$ is the reflection Brownian motion of B with the reflection happening at time τ and τ a stopping time, i.e. $\mathcal{R}(B,\tau)_t = -B_t + 2(B_t - B_\tau)\mathbf{1}_{t\geq\tau}$.

Proposition 16. *(i) $\left(Y^k\right)_{k\geq 0}$ is a sequence of $(\mathcal{F}_t)_{t\geq 0}$ Brownian motions and $(\tau_k)_{k\geq 0}$ is a sequence of $(\mathcal{F}_t)_{t\geq 0}$ stopping times.*

(ii) For $t > 0$,

(6)

$$X_t - Y_t^n = \begin{cases} \left(1+(-1)^k\rho\right)\left(B_t^X - B_{\tau_k}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_k}^Y\right) + \alpha_k, \; \tau_k \leq t \leq \tau_{k+1}, \; 0 \leq k \leq n \\ (1+(-1)^n\rho)\left(B_t^X - B_{\tau_{n+1}}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_{n+1}}^Y\right) + \alpha_{n+1}, \; \tau_{n+1} \leq t \end{cases}.$$

(iii) We have

$$\tau_k \overset{d}{=} \inf\{t \geq 0 : B_t = u_k\} \tag{7}$$

where

$$\begin{cases} u_0 = 0 \\ u_k = \frac{\eta}{\sqrt{2(1+\rho)}} + \frac{(\eta-\nu)}{\sqrt{2}}\Big(\frac{\lfloor \frac{k}{2} \rfloor}{\sqrt{1-\rho}} + \frac{\lfloor \frac{k-1}{2} \rfloor}{\sqrt{1+\rho}}\Big) & k \geq 1 \end{cases} \tag{8}$$

with B a standard Brownian motion and $\lfloor . \rfloor$ the floor function.

5.2. Results on the distribution of the difference between the two Brownian motions.

Proposition 17. *Let $t > 0$ and $x \in \mathbb{R}$. The sequence $p_n(t,x) = \mathbb{P}\left(X_t - Y_t^n \geq x\right)$ verifies*

$$p_0(t,x) = \Phi\Big(\frac{-x}{\sqrt{2\,(1+\rho)\,t}}\Big), \tag{9}$$

(10)

$$p_{n+1}(t,x) = \begin{cases} p_n(t,x) + \Phi\Big(\frac{x-\alpha_{n+1}}{\sqrt{2(1+(-1)^n\rho)t}} - \frac{u_{n+1}}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\alpha_{n+1}}{\sqrt{2\left(1+(-1)^{n+1}\rho\right)t}} - \frac{u_{n+1}}{\sqrt{t}}\Big) & \text{if} \quad x < \alpha_{n+1} \\ p_n(t,x) + \Phi\Big(\frac{x-\alpha_{n+1}}{\sqrt{2(1+(-1)^n\rho)t}} + \frac{u_{n+1}}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\alpha_{n+1}}{\sqrt{2\left(1+(-1)^{n+1}\rho\right)t}} + \frac{u_{n+1}}{\sqrt{t}}\Big) & \text{if} \quad x \geq \alpha_{n+1} \end{cases}$$

with the sequence u_k defined by Equation (8).

Corollary 18. *Let $t > 0$. For $x \in [\nu, \eta]$, the sequence $p_n\left(t,x\right) = \mathbb{P}\left(X_t - Y_t^n \geq x\right)$ is increasing when $\rho > 0$ and decreasing when $\rho < 0$. When $\rho = 0$, $p_n(t,x)$ is constant for all $x \in \mathbb{R}$.*

Proof. Let $x \in [\nu, \eta]$ and let assume $\rho > 0$. If n is even,

$$p_{n+1}(t,x) - p_n(t,x) = \Phi\Big(\frac{x-\eta}{\sqrt{2\,(1+\rho)\,t}} - \frac{u_{n+1}}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\eta}{\sqrt{2\,(1-\rho)\,t}} - \frac{u_{n+1}}{\sqrt{t}}\Big) > 0.$$

If n is odd,

$$p_{n+1}(t,x) - p_n(t,x) = \Phi\Big(\frac{x-\nu}{\sqrt{2\,(1-\rho)\,t}} + \frac{u_{n+1}}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\nu}{\sqrt{2\,(1+\rho)\,t}} + \frac{u_{n+1}}{\sqrt{t}}\Big) > 0.$$

Then $p_n(t,x)$ is increasing. The proof is the same for $\rho < 0$. □

It is then possible to increase the value of $\mathbb{P}\left(X_t - Y_t^n \geq x\right)$ between ν and η by choosing a positive ρ and by increasing the number of reflections with this model. Let us study the convergence of the process $X - Y^n$ in Proposition 19.

Proposition 19. *Let $t > 0$ and $N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\tau_n \leq t}$. We have:*

- (i) *N_t is a $\left(\mathcal{F}_t\right)_{t\geq 0}$ counting process,*
- (ii) *$N_t < \infty$ almost surely,*
- (iii) *$\forall n \geq N_t$, $X_t - Y_t^n = X_t - Y_t^{N_t}$,*
- (iv) *$\mathbb{P}\left(X_t - Y_t^{N_t} \geq x\right) = \lim_{n\to\infty} p_n\left(t,x\right)$, $x \in \mathbb{R}$, $t \geq 0$,*

Proof (i) Since $(\tau_k)_{k\geq 0}$ is a sequence of $(\mathcal{F}_t)_{t\geq 0}$ stopping times, N_t is a $(\mathcal{F}_t)_{t\geq 0}$ counting process.
(ii) $\{N_t = n\} = \{\tau_n \leq t, \tau_{n+1} > t\}$ and then we have

$$\mathbb{E}(N_t) = \sum_{n=1}^{\infty} n\mathbb{P}(\tau_n \leq t, \tau_{n+1} > t) \leq \sum_{n=1}^{\infty} n\mathbb{P}(\tau_n \leq t)$$

According to Proposition 16 (iii), $\mathbb{P}(\tau_n \leq t) = 2\int_{\frac{u_n}{\sqrt{t}}}^{\infty} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy = 2\Phi\left(\frac{-u_n}{\sqrt{t}}\right)$. Since $\lim_{n\to\infty} u_n = \infty$ and $n = \underset{n\to\infty}{O}(u_n)$,

$$\mathbb{P}(\tau_n \leq t) = \underset{n\to\infty}{o}\left(e^{\frac{-u_n^2}{2t}}\right) = \underset{n\to\infty}{o}\left(\frac{1}{u_n^3}\right) = \underset{n\to\infty}{O}\left(\frac{1}{n^3}\right).$$

Then $n\mathbb{P}(\tau_n \leq t) = \underset{n\to\infty}{O}\left(\frac{1}{n^2}\right)$ and $\mathbb{E}(N_t) < \infty$ by comparison theorem of positive series, implying $N_t < \infty$ almost surely.
(iii) If $n \geq N_t$, the number of reflections of $X - Y^n$ between time 0 and time t is equal to N_t and $X_t - Y_t^n = X_t - Y_t^{N_t}$ almost surely.
(iv) Since for $n \geq N_t$ $X_t - Y_t^n = X_t - Y_t^{N_t}$, $X_t - Y_t^{N_t}$ is the limit in law of $X_t - Y_t^n$.

□

In proposition 20, we study the range of values that can be achieved by $\mathbb{P}\left(X_t - Y_t^{N_t} \geq z\right)$.

Proposition 20. *Let $t > 0$. We denote by $X(\rho) - Y^N(\rho)$ the process $X - Y^N$ when the correlation of the model is equal to ρ and the upper barrier to η. Let $\eta > z > 0$. We have*

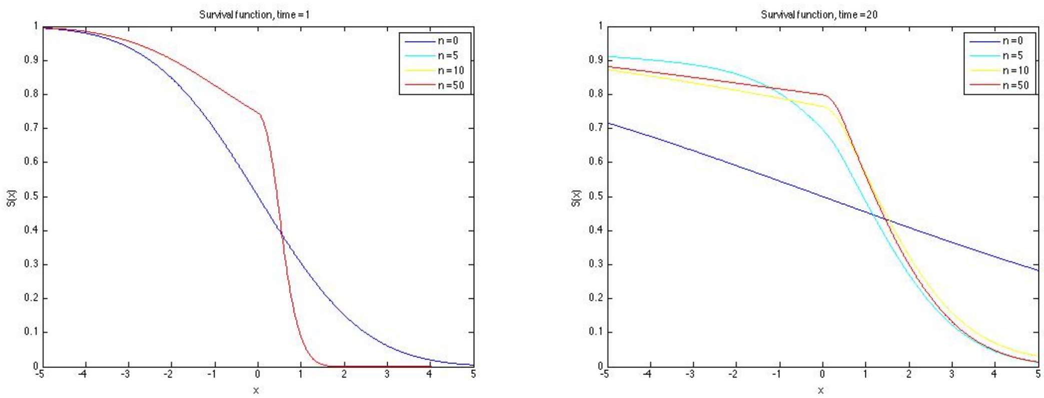
$$\forall x \in \left[0, \Phi\left(\frac{-z}{2\sqrt{t}}\right) + \Phi\left(\frac{z-2\eta}{2\sqrt{t}}\right)\right], \exists\, \rho \in [-1,1] : \mathbb{P}\left(X_t(\rho) - Y_t^{N_t}(\rho) \geq z\right) = x$$

Contrary to Proposition 14, it is not possible to control $\mathbb{P}\left(X_t - Y_t^{N_t} \geq z\right)$ between its bounds ; we can only achieve $\Phi\left(\frac{-z}{2\sqrt{t}}\right) + \Phi\left(\frac{z-2\eta}{2\sqrt{t}}\right)$ and not $2\Phi\left(\frac{-z}{2\sqrt{t}}\right)$. It would be true if z could be equal to η. The problem comes from a discontinuity when $\rho = 1$ and $z = \eta$. As z gets closer to η, the bound grows but we can not reach $z = \eta$. However, the values that we can achieve in this model are still better than in a Gaussian copula case where the supremum is equal to $\Phi\left(\frac{-z}{2\sqrt{t}}\right)$. In this model, the distribution of the difference between the two Brownian motions is less degenerated than the one of Proposition 14 in the sense that there exists two distinct sources of randomness at all time.

5.3. **Numerical illustrations.** Results of Proposition 18 are illustrated in Figure 5a.

The case $n = 0$ corresponds to the Gaussian case. We can see that in $[\nu, \eta]$, the survival function is increasing with n. In Figure 5a, the curves for $n = 5$, $n = 10$ and $n = 50$ are the same. At time $t = 1$, the probability to cross more than 5 barrier is very weak then the Brownian reflection reflects less than 5 times with a high probability. The convergence in n at small time is fast. In Figure 5b, we can observe the difference between the cases $n = 5$, $n = 10$ and $n = 50$ at time $t = 20$. The survival function continues to grow.

The results are confirmed with Figure 6. The higher the number of reflections is, more $X - Y^n$ is concentrated in the region $[\nu, \eta]$. However, in the positive part of the plan, $X - Y^n$ take lower values than in the Gaussian case. We also remark that the symmetry present in the case $n = 0$ disappears when n is higher.



(A) $t = 1$ (B) $t = 20$

FIGURE 5. *Survival function of $X - Y^n$ in the multi-barrier model at time t with parameters $\nu = 0$, $\eta = 0.5$ and $\rho = 0.9$ for different values of n*

This asymmetry can be also observed in the copula of (X_t, Y_t^n) illustrated in Figure 7.

As for the Reflection Brownian Copula, we observe two states. First one is under the diagonal and corresponds to a dependence close to the comonotonic case. The other one is over the diagonal and seems to be closer to an independent copula than to the countermonotonic case, except in the upper left corner where there is a strong dependence. We also observe a strong dependence in the upper right and lower left corners.

6. Local correlation model

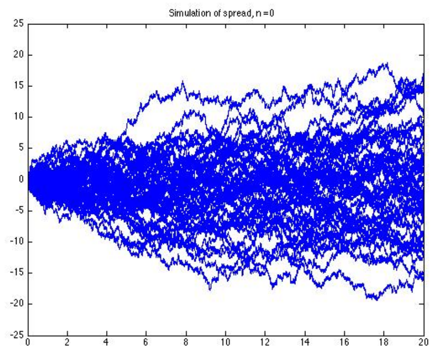
As in Section 5, we develop a model based on the two states structure of the Reflection Brownian Copula. However, we use a totally different approach where the reflection of the Brownian motion does not appear. Our model is a local correlation model, where the correlation depend on the value of the difference between the two Brownian motions and has two states, one if the difference is under a certain value and one if it is over an other value. Between the two, the correlation is taken such as the local correlation function is continuous and monotone.

The concept of local correlation is directly derived from the one of local volatility. In a Black and Scholes framework, the volatility is constant with the maturity and strikes which is not coherent with the implied volatilities from call and put option prices. Dupire introduce the local volatility in order to have a price model which is compatible with the volatility smiles and which is a complete market model [10]. Langnau introduces local correlation model which is the generalization of local volatility for a multi-dimensional framework [19].

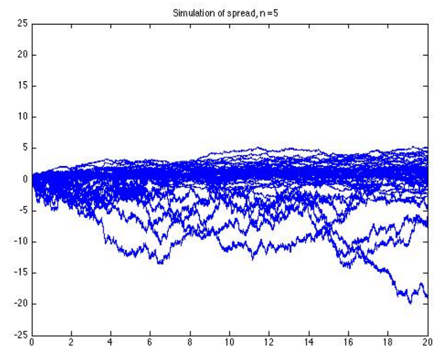
6.1. **Model.** Let B^X and B^Y be two independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Let η, ν, ρ_1 and ρ_2 be real numbers with $\eta > \nu$, $\mid \rho_1 \mid< 1$, $\mid \rho_2 \mid< 1$.

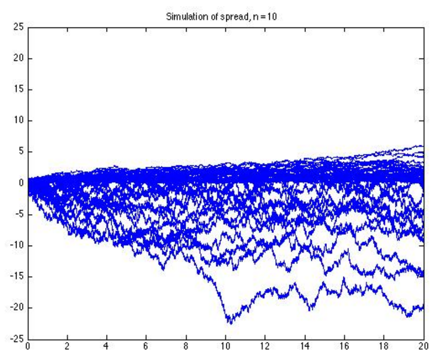
Let $\rho(x)$ be a continuous and monotone function such that $\rho(x) = \rho_1$ for $x \leq \nu$ et $\rho(x) = \rho_2$ for $x \geq \eta$.



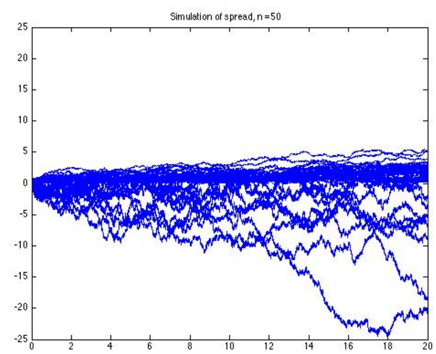
(A) $n = 0$



(B) $n = 5$



(C) $n = 10$



(D) $n = 50$

FIGURE 6. *50 simulations of $X - Y^n$ in the multi-barrier model between time 0 and 20 with parameters ν = 0, $\eta = 0.5$ and $\rho = 0.9$ and a time step of 0.001 for different values of n*

Let us consider the following system of stochastic differential equations:

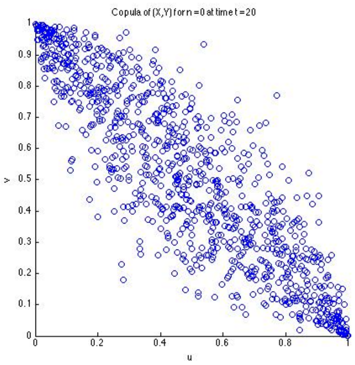
$$\left\{ \begin{array}{l} dX_t = dB_t^X \\ dY_t = \rho\left(X_t - Y_t\right) dB_t^X + \sqrt{1 - \rho\left(X_t - Y_t\right)^2} dB_t^Y \end{array} \right. \tag{11}$$

with $X_0 = 0$ and $Y_0 = 0$.

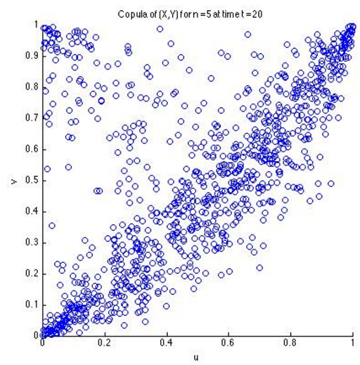
Proposition 21. *The system of stochastic differential equations (11) has an unique strong solution and the two components of the solution are Brownian motions.*

Proof As $\rho(x)$ and $\sqrt{1-\rho\left(x\right)^2}$ are Lipschitz on $\mathbb{R}$, $(x, y) \mapsto \begin{pmatrix} 1 & 0 \\ \rho\left(x-y\right) & \sqrt{1-\rho\left(x-y\right)^2} \end{pmatrix}$ is Lipschitz on $\mathbb{R}^2$, which is a sufficient condition for the system to have a strong solution.

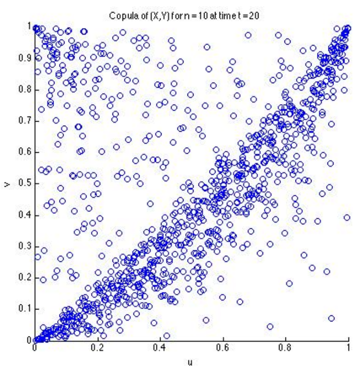
X is clearly a Brownian motion. By the Lévy characterization of the Brownian motion, Y is a Brownian motion. $\square$



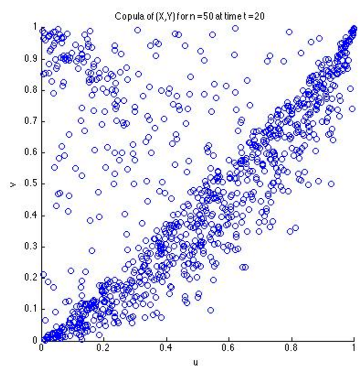
(A) $n = 0$



(B) $n = 5$



(C) $n = 10$



(D) $n = 50$

FIGURE 7. *Empirical copula of (X,Y) in the multi-barrier model at time* $t = 20$ *with parameters* $\nu = 0$*,* $\eta = 0.5$ *and* $\rho = 0.9$ *and a time step of 0.001 for different values of* n *done with 1000 simulations*

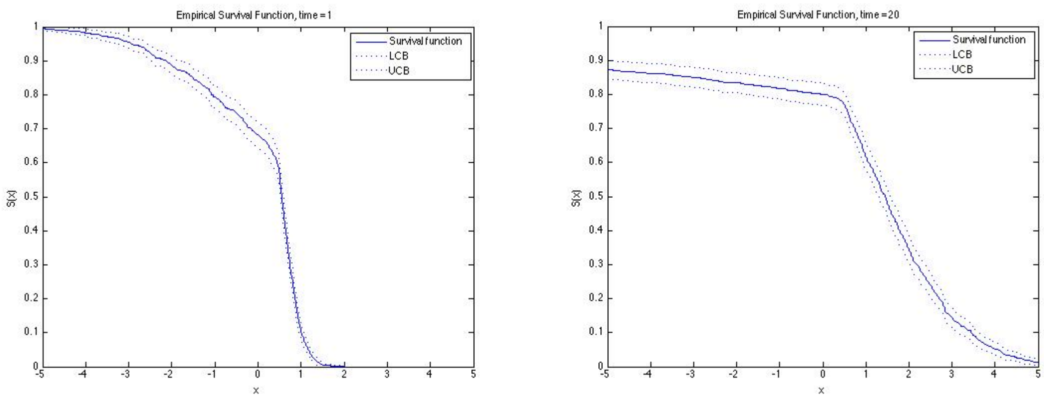
Remark 22. *Contrary to the model of Section 5, the process* (X, Y) *is Markovian. Indeed, in Section 5, we notice that* $X - Y$ *is not Markovian.*

6.2. **Numerical results.** In Figure 8, we compute the survival function of the $X_t - Y_t$ in the local correlation model at time $t = 1$ and $t = 20$. We use equivalent parameters to the numerical illustrations of the multi-barrier model. The local correlation function is chosen linear between ν and η.

As for the multi-barrier model, we have an asymmetric distribution. Between ν and η, the survival function is over $\frac{1}{2}$ (Gaussian copula case). The survival function increases at the right of ν between time $t = 1$ and $t = 20$. Values are closed to the ones of the multi-barrier model.

As for the multi-barrier model, we observe that $X - Y$ is more concentrated in the region $[\nu, \eta]$ (Figure 9).

Figure 10 represents the copula of (X_t, Y_t) at time $t = 20$. It is similar to the one of the multi-barrier model.



(A) $t = 1$ (B) $t = 20$

FIGURE 8. *Empirical survival function of $X_t - Y_t$ in the local correlation model at time t with parameters $\nu = 0$, $\eta = 0.5$, $\rho_1 = -0.9$ and $\rho_2 = 0.9$ with interval confidence bounds at 99% and estimated with 1000 simulations and a step time of 0.001*

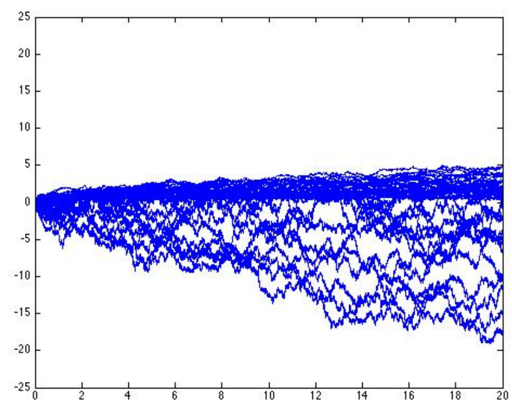


FIGURE 9. *50 simulations of $X - Y$ in the correlation local model between time $t = 0$ and $t = 20$ with parameters $\nu = 0$, $\eta = 0.5$, $\rho_1 = -0.9$ and $\rho_2 = -0.9$ and a time step of 0.001*

7. An application for joint modeling of commodity prices on energy market

In this section, we apply the multi-barrier model and the local correlation model to model jointly the price of two commodities, electricity and coal. Coal is a combustible used to produce electricity which implies an asymmetry in the distribution of the difference between the two prices ; it is more likely that price of coal is lower than price of electricity (in the same unit). Modeling the dependence with a Gaussian copula is then not adapted. An advantage of our models is that

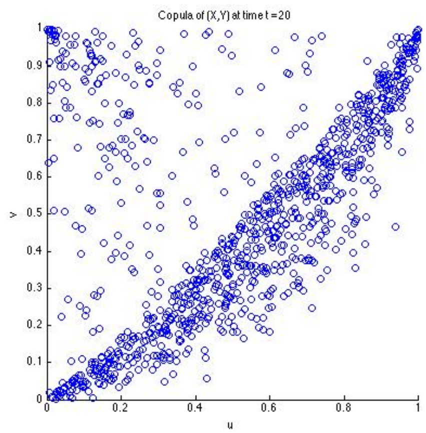


FIGURE 10. *Empirical copula of* (X_t, Y_t) *in the local correlation model at time* $t = 20$ *with parameters* $\nu = 0$*,* $\eta = 0.5$*,* $\rho_1 = -0.9$ *and* $\rho_2 = 0.9$ *and a time step of 0.001 with 1000 simulations*

it contains asymmetry in the distribution of the difference between the two prices. Furthermore, it allows not to change the marginal models.

7.1. **Model.** Let us consider a two-factor model for both electricity and coal. For more information on the two-factor model, we refer to the study of Benth and Koekebakker [1].

Let $f^E(t,T)$ (resp. $f^C(t,T)$) the forward price of the electricity (resp. coal) at time t with maturity T, that is the product that delivers electricity (resp. coal) at maturity T during one day. Stochastic differential equation (12) gives dynamic of these products.

$$\left\{\begin{array}{l} df^E(t,T)=f^E(t,T)\left(\sigma_s^E e^{-\alpha_s^E(T-t)}dB_t^{E,s}+\sigma_l^E dB_t^{E,l}\right) \\ df^C(t,T)=f^C(t,T)\left(\sigma_s^C e^{-\alpha_s^C(T-t)}dB_t^{C,s}+\sigma_l^C dB_t^{C,l}\right) \end{array}\right. \tag{12}$$

where $B^{E,s}$, $B^{E,l}$, $B^{C,s}$, $B^{C,l}$ are standard Brownian motions defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the dynamic of each commodity, there is one factor corresponding to the short term factor with a volatility $\sigma_s^i e^{-\alpha_s^i(T-t)}, i = E, C$ decreasing with time to maturity. This short term factor is used to model the Samuelson effect. The other factor is the long term factor with a constant volatility $\sigma_l^i, i = E, C$.

Products traded on the market have a delivery period, except for the spot. We denote by $f^i(t,T,\theta), i = E, C$ the price of the product at time t that delivers i at time T during a period θ. By absence of arbitrage opportunities, we have

$$f^i(t,T,\theta) = \frac{1}{\theta}\int_T^{T+\theta} f^i(t,u)\,du.$$

In the following, we will only consider n Month Ahead (nMAH), $n \geq 1$, which are products with a delivery period of one month and a delivery date which is the 1^{st} of the n^{th} following month from today.

Equation (13) gives the solutions of (12).

$$\left\{ \begin{array}{l} f^E(t,T) = f^E(0,T)\, e^{\int_0^t \sigma_s^E e^{-\alpha_s^E (T-s)} dB_s^{E,s} - \frac{1}{2}\int_0^t \left(\sigma_s^E\right)^2 e^{-2\alpha_s^E (T-s)} ds + \sigma_l^E B_t^{E,l} - \frac{1}{2}\left(\sigma_l^E\right)^2 t} \\ f^C(t,T) = f^C(0,T)\, e^{\int_0^t \sigma_s^C e^{-\alpha_s^C (T-s)} dB_s^{C,s} - \frac{1}{2}\int_0^t \left(\sigma_s^C\right)^2 e^{-2\alpha_s^C (T-s)} ds + \sigma_l^C B_t^{C,l} - \frac{1}{2}\left(\sigma_l^C\right)^2 t} \end{array} \right. \tag{13}$$

The spot price of electricity is given by $S_t^E = f^E(t,t)$ and the one of coal by $S_t^C = f^C(t,t)$. Then we have

$$\left\{ \begin{array}{l} S_t^E = f^E(0,t)\, e^{\int_0^t \sigma_s^E e^{-\alpha_s^E (t-s)} dB_s^{E,s} - \frac{1}{2}\int_0^t \left(\sigma_s^E\right)^2 e^{-2\alpha_s^E (t-s)} ds + \sigma_l^E B_t^{E,l} - \frac{1}{2}\left(\sigma_l^E\right)^2 t} \\ S_t^C = f^C(0,t)\, e^{\int_0^t \sigma_s^C e^{-\alpha_s^C (t-s)} dB_s^{C,s} - \frac{1}{2}\int_0^t \left(\sigma_s^C\right)^2 e^{-2\alpha_s^C (t-s)} ds + \sigma_l^C B_t^{C,l} - \frac{1}{2}\left(\sigma_l^C\right)^2 t} \end{array} \right. \tag{14}$$

We model the dependence as follow:

- $B^{E,s}$ and $B^{E,l}$ are independent,
- $B^{C,s}$ and $B^{C,l}$ are independent,
- $B^{E,s}$ and $B^{C,s}$ are independent,
- $B^{E,l}$ and $B^{C,l}$ are constructed following the multi-barrier correlation model defined in Section 5.

Usually, a constant correlation matrix is used to model the dependence between the 4 Brownian motions.

7.2. **Parameters.** We consider the parameters of the marginal laws given in Table 1. Units are taken according to the year. We use the forward prices on electricity and on coal during 2014 in France to estimate these parameters. The method used for estimation is the first one of [14].

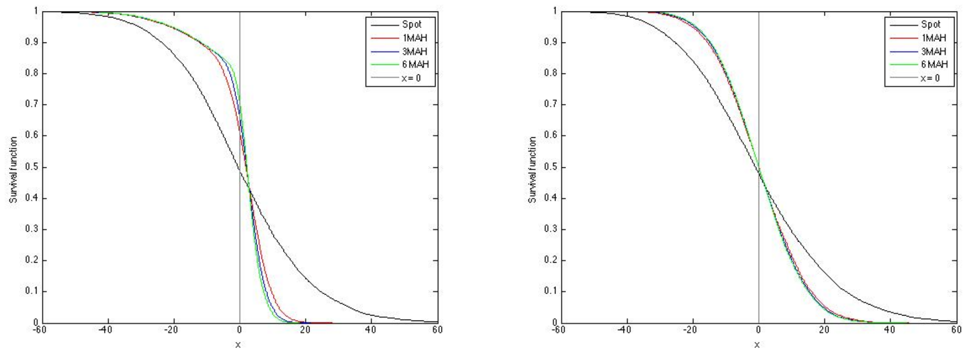
Parameters	Electricity	Coal
σ_l	10.2555%	9.2602%
σ_s	97.2925%	11.2134%
α_s	17.0363	2.07832

TABLE 1. *Parameters of the two-factor model for electricity and coal*

Parameters for the multi-barrier correlation model used to model the dependence between $B^{E,l}$ and $B^{C,l}$ are chosen arbitrarily ; we choose $\nu = 0$, $\eta = 0.5$, $\rho = 0.9$, $n = \infty$.

In the benchmark model where dependence between $B^{E,l}$ and $B^{C,l}$ is modeled by a constant correlation, the correlation is equal to 0.275546. The other correlation are equals to 0.

We assume that $f^E(0,T) - Hf^C(0,T) = 0$ and $f^E(0,T) = 100$ for all T (which does not represent the reality because we do not take into account the seasonality of the prices of electricity and coal)



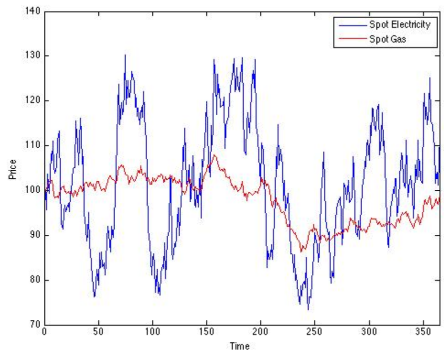
(A) *Multi-barrier correlation model* (B) *Benchmark model*

FIGURE 11. *Empirical survival function of the difference between the price of electricity and the price of coal at time $t = 335$ days estimated with 10000 simulations with a time step of $\frac{1}{24}$ days for different products (Spot, 1MAH, 3MAH, 6MAH) in the multi-barrier correlation model and in the benchmark model*

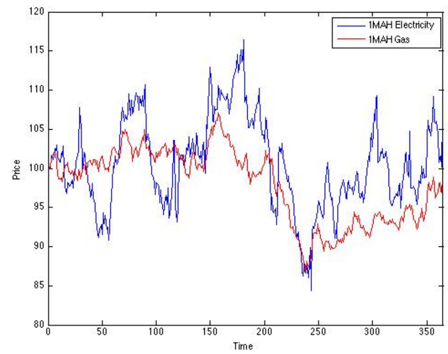
7.3. **Numerical results.** We are interested in the difference between $f^E(t,T)$ and $Hf^C(t,T)$. We only are interested in the multi-barrier correlation model ; results are the same for the local correlation model.

Figure 11 represents the survival function of the difference between spot, 1MAH, 3MAH, and 6MAH prices. In the multi-barrier correlation model, the probability for the difference between the two spot prices to be non negative is close to 50%, which is the same value than in the benchmark model. However, we have good results if we consider long term products as 1MAH, 3MAH and 6MAH: we have probabilities closed to 60% for the 1MAH, and 70% for the 3MAH and 6MAH in the multi-barrier correlation model whereas we have probabilities closed to 50% in the benchmark model. The probability increases with the time to maturity. In the case of spot prices, the volatilities of the prices of the commodities is dominated by the short term factor, which we do not control ; in the other cases, these volatilities are small and the long term factor which we control dominates. This explains that we do not increase a lot the probability for the difference between the spot prices to be non negative. We also observed that in the multi-barrier correlation model, the survival function decreases faster than in the benchmark model and probability of being superior to 20 is closed to 0, which is not the case in the benchmark model.

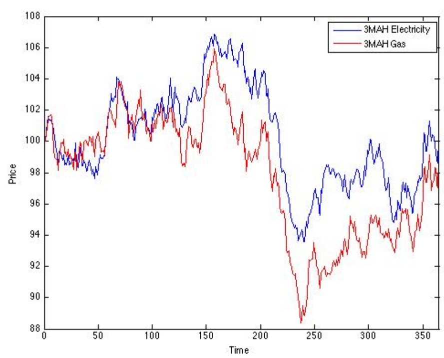
Figure 12 represents one trajectory of the different products. In the case of the spot prices, since electricity has a high volatility, it is difficult to control the difference between the two processes. For the other products, as the short term volatility decreases, we see that there is a control between the two processes.



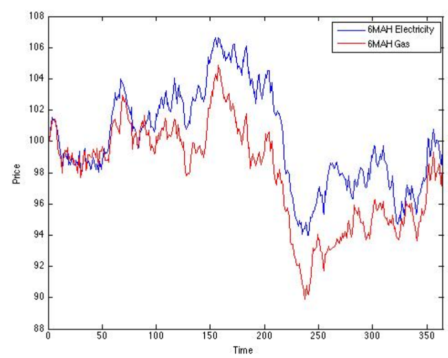
(A) *Spot prices of electricity and coal*



(B) *1MAH prices of electricity and coal*

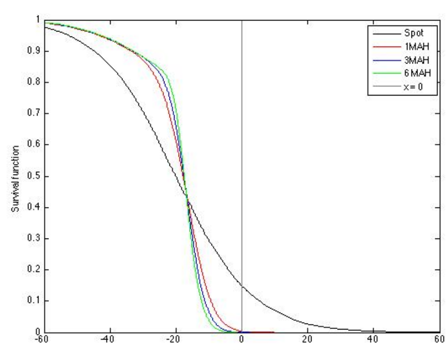


(C) *3MAH prices of electricity and coal*

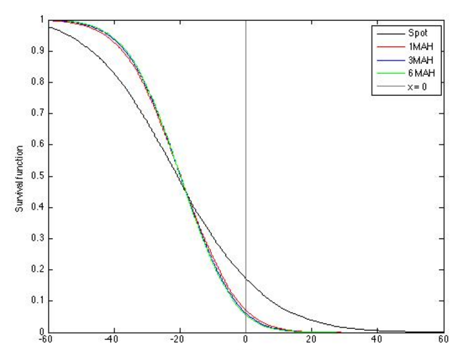


(D) *6MAH prices of electricity and coal*

FIGURE 12. *One year trajectory of electricity and coal products in the multi-barrier model with a time step of $\frac{1}{24}$ days*



(A) *Multi-barrier correlation model*



(B) *Benchmark model*

FIGURE 13. *Empirical survival function of the difference between the price of electricity and the price of coal at time $t = 335$ days estimated with 10000 simulations with a time step of $\frac{1}{24}$ days for different products (Spot, 1MAH, 3MAH, 6MAH) in the multi-barrier correlation model and in the benchmark model if the difference is equal to -20 at time $t = 0$*

Results are sensitive to initial conditions. If we choose $f^E(0,T) - Hf^C(0,T) = -20$ for instance, we will have a distribution that is concentrated around -20, because the difference between the price is a martingale. The probability to be greater than -20 is higher in the multi-barrier correlation model than in the benchmark model but the probability to be positive is lower than in the benchmark model: it is closed to 0 in the multi-barrier correlation model whereas it is closed to 10% in the benchmark model. Figure 13 represents the survival function of the difference between prices of electricity and coal for different products with $\nu = 0$ and $\eta = 0.5$. As we choose a barrier near 0, the survival function will be maximized around -20.

One way to improve the value of the survival function around 0 is to choose a higher η. The idea in our model is that we want $B^{C,l}$ to go over $B^{E,l} + \eta$, using correlation of -1 when the two prices are equals at time $t = 0$. We want for the price of the coal to go over the price of electricity, that happens when $f^E(t,T) = Hf^C(t,T)$, i.e. when $\sigma_l^E B_t^{E,l} - \sigma_l^C B_t^{C,l} = \log\left(\frac{f^E(0,T)}{Hf^C(0,T)}\right)$ if we neglect the short term factors. We have $\sigma_l^E \approx \sigma_l^C \approx \sigma = 0.1$ year$^{-}1$. Then, we want $B_t^{E,l} - B_t^{C,l} \approx \frac{1}{\sigma}\log\left(\frac{f^E(0,T)}{Hf^C(0,T)}\right)$. In the case with the same initial conditions, the right hand side term is equal to 0 and we choose a barrier of η. Heuristically, we then choose a barrier of $\eta' = \eta + \frac{1}{\sigma}\log\left(\frac{f^E(0,T)}{Hf^C(0,T)}\right) \approx 170.5$ and $\nu = 170$. Figure 14 gives the survival function of the different products in the multi-barrier correlation model with barriers $\nu = 170$ and $\eta = 170.5$.

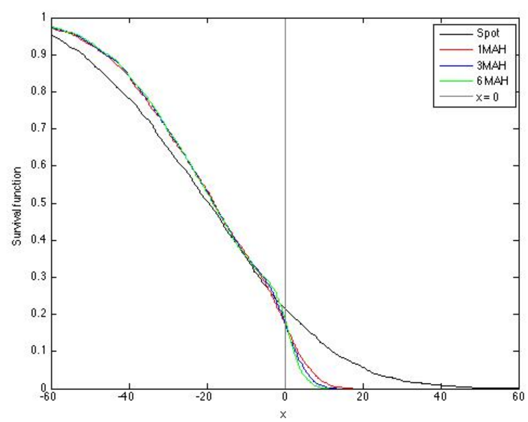


FIGURE 14. *Empirical survival function of the difference between the price of electricity and the price of coal at time* $t = 335$ *days estimated with 10000 simulations with a time step of* $\frac{1}{24}$ *days for different products (Spot, 1MAH, 3MAH, 6MAH) in the multi-barrier correlation model if the difference is equal to -20 at time* $t = 0$ *and with barriers* $\nu = 170$ *and* $\eta = 170.5$

We can see that around 0, the values of the survival function are much better than in the benchmark model: around 20% in the multi-barrier model and around 10% in the benchmark model. However, the values are still low. Indeed, even in the maximal case where the second Brownian motion is the reflection of the first one and the volatilities are equals, the probability for the difference between the Brownian motions to be positive knowing that one starts at $-x$, $x > 0$ and the other at 0 is equal to $2\Phi\left(\frac{-x}{2\sqrt{t}}\right)$ which decreases with x.

8. Proofs

8.1. **Preliminary results.** We start with well known results that will be useful for the proofs of propositions.

Lemma 23. *Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion on a filtered probability space $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big)$. We have:*

(i) *for $y \geq 0$,*

$$\mathbb{P}\Big(B_t \leq x, \sup_{s\leq t} B_s \leq y\Big) = \begin{cases} \Phi\Big(\frac{x}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-2y}{\sqrt{t}}\Big) & \text{if} \quad x < y \\ 2\Phi\Big(\frac{y}{\sqrt{t}}\Big) - 1 & \text{if} \quad x \geq y \end{cases},$$

(ii) *for $y \leq 0$,*

$$\mathbb{P}\Big(B_t \leq x, \inf_{s\leq t} B_s \leq y\Big) = \begin{cases} \Phi\Big(\frac{x}{\sqrt{t}}\Big) & \text{if} \quad x \leq y \\ 2\Phi\Big(\frac{y}{\sqrt{t}}\Big) - \Phi\Big(\frac{-x+2y}{\sqrt{t}}\Big) & \text{if} \quad x > y \end{cases}.$$

Proof The reader can refer to [18, Theorem 3.1.1.2, p. 137] for the proof of (i) and to Section 3.1.5 page 142 of [18, Section 3.1.5, p. 142] for the proof of (ii). □

Lemma 24. *Let $B^1 = \big(B^1_t\big)_{t\geq 0}$ and $B^2 = \big(B^2_t\big)_{t\geq 0}$ be two independent standard Brownian motion defined on a common filtered probability space $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big)$ with $(\mathcal{F}_t)_{t\geq 0}$ having all the good properties. Let $h \geq 0$ and $\tau^h = \inf\{t \geq 0 : B^2_t = h\}$. We have:*

$$\mathbb{P}\Big(B^1_t - B^1_{\tau^h} \leq x, \tau^h \leq t\Big) = \Phi\Big(\frac{x-h}{\sqrt{t}}\Big)\mathbf{1}_{x<0} + \Big(\Phi\Big(\frac{x+h}{\sqrt{t}}\Big) - 2\Phi\Big(\frac{h}{\sqrt{t}}\Big) + 1\Big)\mathbf{1}_{x\geq 0}.$$

Proof Conditional on $\{t \geq \tau^h\}$, $B^1_t - B^1_{\tau^h}$ is a Brownian motion independent to $\mathcal{F}_{\tau^h}$. Then

$$\mathbb{P}\Big(B^1_t - B^1_{\tau^h} \leq x, \tau^h \leq t\Big) = \mathbb{E}\Big(\Phi\Big(\frac{x}{\sqrt{t-\tau^h}}\Big)\mathbf{1}_{t\geq \tau^h}\Big).$$

The same argument can be used to prove that

$$\mathbb{P}\Big(B^2_t - B^2_{\tau^h} \leq x, \tau^h \leq t\Big) = \mathbb{E}\Big(\Phi\left(\frac{x}{\sqrt{t-\tau^h}}\right)\mathbf{1}_{t\geq \tau^h}\Big).$$

Then we have

$$\begin{aligned} \mathbb{P}\Big(B^1_t - B^1_{\tau^h} \leq x, \tau^h \leq t\Big) &= \mathbb{P}\Big(B^2_t - B^2_{\tau^h} \leq x, \tau \leq t\Big) \\ &= \mathbb{P}\Big(B^2_t \leq x + h, \sup_{s\leq t} B^2_s \geq h\Big). \end{aligned}$$

We can conclude using Lemma 23. □

8.2. **Proof of Proposition 2.** We have:

$$\mathbb{P}\Big(B_t \le x, \tilde{B}_t^h \le y\Big) = \mathbb{P}\Big(B_t \le x, \tilde{B}_t^h \le y, \sup_{s\le t} B_s \le h\Big) + \mathbb{P}\Big(B_t \le x, \tilde{B}_t^h \le y, \sup_{s\le t} B_s \ge h\Big). \tag{15}$$

We compute the first term of Equation (15):

$$\begin{aligned}
\mathbb{P}\Big(B_t \le x, \tilde{B}_t^h \le y, \sup_{s\le t} B_s \le h\Big) &= \mathbb{P}\Big(B_t \le x, -B_t \le y, \sup_{s\le t} B_s \le h\Big)\\
&= \mathbb{P}\Big(-y \le B_t \le x, \sup_{s\le t} B_s \le h\Big)\\
&= \Big(\mathbb{P}\Big(B_t \le x, \sup_{s\le t} B_s \le h\Big) - \mathbb{P}\Big(B_t \le -y, \sup_{s\le t} B_s \le h\Big)\Big)\mathbf{1}_{x+y>0}\\
&= \begin{cases} \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x-2h}{\sqrt{t}}\right) + \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(\frac{y+2h}{\sqrt{t}}\right) & \text{if} \quad x \le h, \\ & \qquad y \ge -h, \\ & \qquad x+y>0 \\ 2\Phi\left(\frac{h}{\sqrt{t}}\right) - 1 + \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(\frac{y+2h}{\sqrt{t}}\right) & \text{if} \quad x > h, \\ & \qquad y \ge -h \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

by application of Lemma 23. In the same way, we compute the second term of Equation (15):

$$\begin{aligned}
\mathbb{P}\Big(B_t \le x, \tilde{B}_t^h \le y, \sup_{s\le t} B_s \ge h\Big) &= \mathbb{P}\Big(B_t \le x, B_t \le y+2h, \sup_{s\le t} B_s \ge h\Big)\\
&= \mathbb{P}\Big(B_t \le \min\left(x, y+2h\right), \sup_{s\le t} B_s \ge h\Big)\\
&= \begin{cases} \Phi\left(\frac{\min(x,y+2h)-2h}{\sqrt{t}}\right) & \text{if} \quad \min\left(x,y+2h\right) < h \\ -2\Phi\left(\frac{h}{\sqrt{t}}\right) + 1 + \Phi\left(\frac{\min(x,y+2h)}{\sqrt{t}}\right) & \text{if} \quad \min\left(x,y+2h\right) \ge h \end{cases}.
\end{aligned}$$

Combining the last two equations, we obtain

$$\begin{aligned}
\mathbb{P}\left(B_t \le x, \tilde{B}_t^h \le y\right) &= \begin{cases} \Phi\left(\frac{\min(x,y+2h)-2h}{\sqrt{t}}\right) & \text{if} \quad x+y \le 0 \text{ or } \left(y \le -h, x+y>0\right) \\ \Phi\left(\frac{\min(x,y+2h)}{\sqrt{t}}\right) - \Phi\left(\frac{y+2h}{\sqrt{t}}\right) + \Phi\left(\frac{y}{\sqrt{t}}\right) & \text{if} \quad y > -h,\ x+y>0 \end{cases}\\
&= \begin{cases} \Phi\left(\frac{y}{\sqrt{t}}\right) & \text{if} \quad x-y \ge 2h \\ \Phi\left(\frac{x-2h}{\sqrt{t}}\right) & \text{if} \quad x-y<2h, x+y \le 0 \\ \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{y+2h}{\sqrt{t}}\right) + \Phi\left(\frac{y}{\sqrt{t}}\right) & \text{if} \quad x-y<2h, x+y>0 \end{cases} \qquad (16)\\
&= \begin{cases} \Phi\left(\frac{y}{\sqrt{t}}\right) & \text{if} \quad x-y \ge 2h \\ \max\Big(\Phi\left(\frac{y}{\sqrt{t}}\right) + \Phi\left(\frac{x}{\sqrt{t}}\right) - 1, 0\Big) + \min\Big(\Phi\left(\frac{x-2h}{\sqrt{t}}\right), \Phi\left(\frac{-y-2h}{\sqrt{t}}\right)\Big) & \text{if} \quad x-y<2h \end{cases}.
\end{aligned}$$

We conclude using $C_t^{ref,h}\left(u,v\right) = \mathbb{P}\Big(B_t \le \sqrt{t}\Phi^{-1}\left(u\right), \tilde{B}_t^h \le \sqrt{t}\Phi^{-1}\left(v\right)\Big)$.

8.3. **Proof of Proposition 3.** Recall that Φ_ρ denotes the bivariate cumulative distribution function of two standard normal variables correlated with $\rho \in [-1,1]$. We start with a technical lemma.

Lemma 25. *Let a, b and $x \in \mathbb{R}$. We have:*

(i)

$$\int_{-\infty}^{x} \Phi\left(au+b\right) \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}} du = \Phi_{\frac{-a}{\sqrt{a^2+1}}}\Big(\frac{b}{\sqrt{a^2+1}}, x\Big).$$

(ii)

$$\Phi_{\sqrt{1-\rho^2}}\left(x,y\right) = \Phi\left(y\right)\Phi\Big(\frac{x-\sqrt{1-\rho^2}y}{\rho}\Big) + \Phi\left(x\right) - \Phi_{\rho}\Big(x, \frac{x-\sqrt{1-\rho^2}y}{\rho}\Big),\ x,y\in\mathbb{R},\ \rho>0$$

(iii)

$$\Phi_{\rho}\left(x,y\right) = \Phi\left(y\right) - \Phi_{-\rho}\left(-x,y\right),\ x,y\in\mathbb{R}$$

Proof (i) Let X and Y two independent Gaussian random variables defined on the same probability space. We have:

$$\begin{aligned}\mathbb{P}\left(X\le aY+b, Y\le x\right) &= \mathbb{E}\left(\mathbf{1}_{Y\le x}\mathbb{P}\left(X\le aY+b \mid Y\right)\right)\\ &= \mathbb{E}\left(\mathbf{1}_{Y\le x}\Phi\left(aY+b\right)\right)\\ &= \int_{-\infty}^{x}\Phi\left(au+b\right)\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}du\end{aligned}$$

and

$$\mathbb{P}\left(X\le aY+b, Y\le x\right) = \mathbb{P}\Big(\frac{X-aY}{\sqrt{1+a^2}}\le\frac{b}{\sqrt{1+a^2}}, Y\le x\Big) = \Phi_{\frac{-a}{\sqrt{a^2+1}}}\Big(\frac{b}{\sqrt{a^2+1}}, x\Big).$$

(ii) Let $a<0,\ b,z\in\mathbb{R}$. We have:

$$\begin{aligned}\Phi_{\frac{-a}{\sqrt{a^2+1}}}\Big(\frac{b}{\sqrt{a^2+1}}, z\Big) &= \int_{-\infty}^{z}\Phi\left(au+b\right)\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}du\\ &= \Phi\left(az+b\right)\Phi\left(z\right) - a\int_{-\infty}^{z}\frac{e^{\frac{-(au+b)^2}{2}}}{\sqrt{2\pi}}\Phi\left(u\right)du\\ &= \Phi\left(az+b\right)\Phi\left(z\right) + \int_{az+b}^{+\infty}\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\Phi\Big(\frac{u-b}{a}\Big)du\\ &= \Phi\left(az+b\right)\Phi\left(z\right) + \Phi\Big(\frac{b}{\sqrt{1+a^2}}\Big) - \Phi_{\frac{1}{\sqrt{a^2+1}}}\Big(\frac{b}{\sqrt{a^2+1}}, ax+b\Big).\end{aligned}$$

We conclude by taking $a=\frac{-\rho}{\sqrt{1-\rho^2}}$, $b=x\sqrt{1+a^2}$, $z=\frac{y-b}{a}$.

(iii) Let X and Y two Gaussian random variables correlated with a correlation of ρ defined on the same probability space. We have:

$$\mathbb{P}\left(X\le x, Y\le y\right) = \mathbb{P}\left(-X\ge -x, Y\le y\right) = \mathbb{P}\left(Y\le y\right) - \mathbb{P}\left(-X\le -x, Y\le y\right).$$

□

We can now prove Proposition 3. Let $X=B$ and $Y=\rho\tilde{B}^h+\sqrt{1-\rho^2}Z$ where B and Z are two independent Brownian motions. X and Y are Brownian motions and we have

$$\begin{aligned}\mathbb{P}\left(X_t \le x, Y_t \le y\right) &= \mathbb{P}\Big(B_t \le x, \tilde{B}_t^h \le \frac{y-\sqrt{1-\rho^2}Z_t}{\rho}\Big)\\ &= \mathbb{E}\Big[\mathbb{P}\Big(B_t \le x, \tilde{B}_t^h \le \frac{y-\sqrt{1-\rho^2}Z_t}{\rho} \mid Z_t\Big)\Big].\end{aligned}$$

Since B is independent from Z, using Equation (16), we find that $\mathbb{P}\left(X_t \le x, Y_t \le y\right)$ is the sum of the three following terms:

(i)

$$\mathbb{E}\Big[\Phi\Big(\frac{y-\sqrt{1-\rho^2}Z_t}{\rho\sqrt{t}}\Big)\mathbf{1}_{Z_t \ge \frac{\rho(2h-x)+y}{\sqrt{1-\rho^2}}}\Big],$$

(ii)

$$\mathbb{E}\Big[\Phi\Big(\frac{x-2h}{\sqrt{t}}\Big)\mathbf{1}_{Z_t \le \frac{\rho(2h-x)+y}{\sqrt{1-\rho^2}}}\mathbf{1}_{Z_t \ge \frac{\rho x+y}{\sqrt{1-\rho^2}}}\Big],$$

(iii)

$$\mathbb{E}\Big[\Big(\Phi\Big(\frac{x}{\sqrt{t}}\Big) - \Phi\Big(\frac{y+2h\rho-\sqrt{1-\rho^2}Z_t}{\rho\sqrt{t}}\Big) + \Phi\Big(\frac{y-\sqrt{1-\rho^2}Z_t}{\rho\sqrt{t}}\Big)\Big)\mathbf{1}_{Z_t \le \frac{\rho(2h-x)+y}{\sqrt{1-\rho^2}}}\mathbf{1}_{Z_t \le \frac{\rho x+y}{\sqrt{1-\rho^2}}}\Big].$$

The first term (i) is equal to:

$$\mathbb{E}\Big[\Phi\Big(\frac{y-\sqrt{1-\rho^2}Z_t}{\rho\sqrt{t}}\Big)\Big] - \mathbb{E}\Big[\Phi\Big(\frac{y-\sqrt{1-\rho^2}Z_t}{\rho\sqrt{t}}\Big)\mathbf{1}_{Z_t \le \frac{\rho(2h-x)+y}{\sqrt{1-\rho^2}}}\Big].$$

Furthermore, using Lemma 25, we have

$$\begin{aligned}\mathbb{E}\Big[\Phi\Big(\frac{y-\sqrt{1-\rho^2}Z_t}{\rho\sqrt{t}}\Big)\mathbf{1}_{Z_t \le \frac{\rho(2h-x)+y}{\sqrt{1-\rho^2}}}\Big] &= \int_{-\infty}^{\frac{\rho(2h-x)+y}{\sqrt{(1-\rho^2)t}}} \Phi\Big(\frac{y-\sqrt{1-\rho^2}\sqrt{t}u}{\rho\sqrt{t}}\Big)\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}du\\ &= \Phi_{\sqrt{1-\rho^2}}\Big(\frac{y}{\sqrt{t}}, \frac{\rho(2h-x)+y}{\sqrt{(1-\rho^2)\,t}}\Big)\end{aligned}$$

and

$$\mathbb{E}\Big[\Phi\Big(\frac{y-\sqrt{1-\rho^2}Z_t}{\rho\sqrt{t}}\Big)\Big] = \Phi\Big(\frac{y}{\sqrt{t}}\Big).$$

We compute terms (ii) and (iii) using the same method and we find:

$$\begin{aligned}\mathbb{P}\Big(X_t \le x, Y_t \le y\Big) = \Phi\Big(\frac{y}{\sqrt{t}}\Big) &- \Phi_{\sqrt{1-\rho^2}}\Big(\frac{y}{\sqrt{t}}, \frac{\rho\,(2h-x)+y}{\sqrt{(1-\rho^2)\,t}}\Big) + \Phi_{\sqrt{1-\rho^2}}\Big(\frac{y}{\sqrt{t}}, \frac{\rho \min\,(2h-x,x)+y}{\sqrt{(1-\rho^2)\,t}}\Big)\\ &- \Phi_{\sqrt{1-\rho^2}}\Big(\frac{y+2h\rho}{\sqrt{t}}, \frac{\rho \min\,(2h-x,x)+y}{\sqrt{(1-\rho^2)\,t}}\Big) + \Phi\Big(\frac{x}{\sqrt{t}}\Big)\Phi\Big(\frac{\rho \min\,(2h-x,x)+y}{\sqrt{(1-\rho^2)\,t}}\Big)\\ &+ \Phi\Big(\frac{x-2h}{\sqrt{t}}\Big)\Big(\Phi\Big(\frac{\rho\,(\max\,(2h-x,x))+y}{\sqrt{(1-\rho^2)\,t}}\Big) - \Phi\Big(\frac{\rho x+y}{\sqrt{(1-\rho^2)\,t}}\Big)\Big).\end{aligned}$$

After some algebra, we find using Lemma 25:

$$\mathbb{P}\Big(X_t \le x, Y_t \le y\Big) = \begin{cases} \Phi_\rho\Big(\frac{y+2\rho h}{\sqrt{t}}, \frac{x}{\sqrt{t}}\Big) + \Phi\Big(\frac{y}{\sqrt{t}}\Big) - \Phi\Big(\frac{y+2\rho h}{\sqrt{t}}\Big) & \text{if} \quad x \ge h\\ \Phi_{-\rho}\Big(\frac{y}{\sqrt{t}}, \frac{x}{\sqrt{t}}\Big) + \Phi_\rho\Big(\frac{-y-2\rho h}{\sqrt{t}}, \frac{x-2h}{\sqrt{t}}\Big) + \Phi_\rho\Big(\frac{y}{\sqrt{t}}, \frac{x-2h}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-2h}{\sqrt{t}}\Big) & \text{if} \quad x < h\end{cases}$$

and the copula is equal to $\mathbb{P}\left(X_t \le \sqrt{t}\Phi^{-1}(u), Y_t \le \sqrt{t}\Phi^{-1}(v)\right)$.

8.4. **Proof of Proposition 5.** Let f^{ξ} be the density of ξ. Let B be a Brownian motion independent from ξ. We enlarge the filtration of B to take into account ξ. We consider the reflection of the Brownian motion $\tilde{B}^{\xi}$. We have:

$$\mathbb{P}\left(B_t \leq \sqrt{t}\Phi^{-1}(u), \tilde{B}_t^{\xi} \leq \sqrt{t}\Phi^{-1}(v)\right) = \mathbb{E}\left[\mathbb{P}\left(B_t \leq \sqrt{t}\Phi^{-1}(u), \tilde{B}_t^{\xi} \leq \sqrt{t}\Phi^{-1}(v) \mid \xi\right)\right]. \tag{17}$$

Since B is independent from ξ, we have according to Proposition 2:

$$\begin{aligned}\mathbb{P}\left(B_t \leq \sqrt{t}\Phi^{-1}(u), \tilde{B}_t^{\xi} \leq \sqrt{t}\Phi^{-1}(v) \mid \xi\right) =& v\mathbf{1}_{\Phi^{-1}(u)-\Phi^{-1}(v)\geq \frac{2\xi}{\sqrt{t}}} + W(u,v)\mathbf{1}_{\Phi^{-1}(u)-\Phi^{-1}(v)< \frac{2\xi}{\sqrt{t}}} \\ &+ \Phi\left(\Phi^{-1}(M(u,1-v)) - \frac{2\xi}{\sqrt{t}}\right)\mathbf{1}_{\Phi^{-1}(u)-\Phi^{-1}(v)< \frac{2\xi}{\sqrt{t}}}\end{aligned}$$

Thus, the right hand side of Equation (17) is the sum of the three following terms:

(i)
$$\mathbb{E}\left[v\mathbf{1}_{\Phi^{-1}(u)-\Phi^{-1}(v)\geq \frac{2\xi}{\sqrt{t}}}\right] = vF^{\xi}\left(\sqrt{t}\frac{\Phi^{-1}(u)-\Phi^{-1}(v)}{2}\right),$$

(ii)
$$\mathbb{E}\left[W(u,v)\mathbf{1}_{\Phi^{-1}(u)-\Phi^{-1}(v)< \frac{2\xi}{\sqrt{t}}}\right] = W(u,v)\overline{F}^{\xi}\left(\sqrt{t}\frac{\Phi^{-1}(u)-\Phi^{-1}(v)}{2}\right),$$

(iii)
$$\mathbb{E}\left[\Phi\left(\Phi^{-1}(M(u,1-v)) - \frac{2\xi}{\sqrt{t}}\right)\mathbf{1}_{\Phi^{-1}(u)-\Phi^{-1}(v)< \frac{2\xi}{\sqrt{t}}}\right],$$

that we denote by I.

We have:

$$\begin{aligned} I &= \int_{\sqrt{t}\frac{\Phi^{-1}(u)-\Phi^{-1}(v)}{2}}^{+\infty} \Phi\left(\Phi^{-1}(M(u,1-v)) - \frac{2h}{\sqrt{t}}\right) f^{\xi}(h)\,dh \\ &= M(1-u,v)\overline{F}^{\xi}\left(\sqrt{t}\frac{\Phi^{-1}(u)-\Phi^{-1}(v)}{2}\right) \\ &- \frac{2}{\sqrt{t}}\int_{\sqrt{t}\frac{\Phi^{-1}(u)-\Phi^{-1}(v)}{2}}^{+\infty} \Phi^{'}\left(\Phi^{-1}(M(u,1-v)) - \frac{2h}{\sqrt{t}}\right)\overline{F}^{\xi}(h)\,dh. \end{aligned}$$

Adding the three terms of Equation (17), since $M(1-u,v) + W(u,v) = v$, we obtain:

$$\begin{aligned} C_t^{\xi}(u,v) &= v - \frac{2}{\sqrt{t}}\int_{\sqrt{t}\frac{\Phi^{-1}(u)-\Phi^{-1}(v)}{2}}^{+\infty} \Phi^{'}\left(\Phi^{-1}(M(u,1-v)) - \frac{2h}{\sqrt{t}}\right)\overline{F}^{\xi}(h)\,dh \\ &= v - \int_{-\infty}^{\Phi^{-1}(M(1-u,v))} \Phi^{'}(h)\,\overline{F}^{\xi}\left(\frac{\sqrt{t}}{2}\left(\Phi^{-1}(M(u,1-v)) - h\right)\right)dh \end{aligned}$$

with $\Phi^{'}(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$, which achieves the proof.

8.5. **Proof of Proposition 9.** [8, Proposition 2.1] states that

$$\mathbb{P}(X+Y \leq x) = \int_0^1 \partial_u C\left(u, F^Y\left(x - \left(F^X\right)^{-1}(u)\right)\right)du, x \in \mathbb{R}. \tag{18}$$

The existence of $\partial_u C\left(u, F^Y\left(x - \left(F^X\right)^{-1}(u)\right)\right)$ for $u \in [0,1]$ is assured by [8, Lemma 2.1].

We also have

$$F^{-Y}(y) = 1 - F^Y(-y),\ y \in \mathbb{R} \text{ and,} \tag{19}$$

$$C^{X,-Y}(u,v) = u - C(u, 1-v), \ (u,v) \in [0,1], \tag{20}$$

with $C^{X,-Y}$ the copula of $(X,-Y)$.

Equation (18) is also valid for $(X,-Y)$. Using Equation (19) and Equation (20), we have

$$\mathbb{P}(X - Y \le x) = \int_0^1 \left[1 - \partial_u C\left(u, F^Y\left(\left(F^X\right)^{-1}(u) - x\right)\right)\right] du, x \in \mathbb{R} \tag{21}$$

and

$$\mathbb{P}(X - Y > x) = \int_0^1 \partial_u C\left(u, F^Y\left(\left(F^X\right)^{-1}(u) - x\right)\right) du, x \in \mathbb{R}. \tag{22}$$

Let us suppose that $C^{Y,X} \in \mathcal{C}_s$ and that X and Y have the same continous marginal distribution function F. Let $C^{Y,X}$ be the copula of (Y,X). We have $C^{Y,X}(u,v) = C^{X,Y}(v,u)$. However, $C^{X,Y}(v,u) = C^{X,Y}(u,v)$ then $C^{Y,X}(u,v) = C^{X,Y}(u,v)$ and

$$\begin{aligned}
\mathbb{P}(X - Y \ge x) &= \mathbb{P}(Y - X \le -x) \\
&= \int_0^1 \left[1 - \partial_u C^{Y,X}\left(u, F\left((F)^{-1}(u) + x\right)\right)\right] du \\
&= \int_0^1 \left[1 - \partial_u C^{X,Y}\left(u, F\left((F)^{-1}(u) + x\right)\right)\right] du \\
&= \mathbb{P}(X - Y \le -x)
\end{aligned}$$

using Equation (21).

8.6. **Proof of Proposition 13.** (i) Let $\rho \in [-1,1]$. We have $\tilde{S}_\eta(C_{G,\rho}) = \Phi\left(\frac{-\eta}{\sqrt{2(1-\rho)}}\right)$. This function is decreasing in ρ and then the extremum are achieved for $\rho = 1$ and $\rho = -1$ and are equal to 0 and $\Phi\left(\frac{-\eta}{2}\right)$.

(ii) This is a direct application of the results of [15] where superior and inferior bounds on $\mathbb{P}(X + Y < \eta)$ are found and where X and Y are two random variables with known marginals. As

1) $\sup_{C \in \mathcal{C}} \mathbb{P}_C(X - Y \ge \eta) = 1 - \inf_{C \in \mathcal{C}} \mathbb{P}_C(X - Y < \eta)$,
2) $-Y$ and Y have the same law,

the bound is equal to

$$1 - \inf_{C \in \mathcal{C}} \mathbb{P}_C(X + Y < \eta).$$

The copula achieving the bound is defined by the transformation

$$C^{X,Y}(u,v) = u - C^{X,-Y}(u, 1-v).$$

(iii) We want to prove that for all x in $\left[0, 2\Phi\left(\frac{-\eta}{2}\right)\right]$, there exists C in $\mathcal{C}$ such that $\tilde{S}_\eta(C) = x$.

If $x \in \left[0, \Phi\left(\frac{-\eta}{2}\right)\right]$, we use a Gaussian copula with $\rho = 1 - \frac{1}{2^2}\left(\frac{\eta}{\Phi^{-1}(x)}\right)^2$.

Let us suppose that $x \in \left[\Phi\left(\frac{-\eta}{2}\right), 2\Phi\left(\frac{-\eta}{2}\right)\right]$. We use the copula C^r to construct a new class of copulae. As for C^r, we separate the square $[0,1]^2$ in two parts and to define a copula in each part of the square. We use the concept of patchwork copula defined by Durante et al. [11]. Let

$H = [1-r,1] \times [0,r]$, $H^c = [0,1]^2 \setminus H$ and $\rho \in (-1,1)$. Let $C^p_\rho(u,v)$ the patchwork copula defined by C_ρ in H and W in H^c:

$$\begin{aligned} C^p_\rho(u,v) &= \mu_W\left(([0,u]\times[0,v])\cap H^c\right) + rC_{G,\rho}\Big(\frac{1}{r}\max(u+r-1,0), \frac{1}{r}\min(v,r)\Big) \\ &= \left(W(u,v) - W(u,r)\right)\mathbf{1}_{v\geq r} + rC_{G,\rho}\Big(\frac{1}{r}\max(u+r-1,0), \min\big(\frac{v}{r},1\big)\Big) \end{aligned}$$

where μ_W is the measure induced by the copula W.

If we consider two standard normal random variables with copula C^p_ρ, the survival function of their difference at point x is equal, according to Equation (22), to

$$\begin{aligned} \int_0^1 \partial_u C^p_\rho\left(u, \Phi\left(\Phi^{-1}(u) - x\right)\right) du &= \int_0^1 \Big(\mathbf{1}_{u\geq\Phi(\frac{x}{2})} - \mathbf{1}_{u\geq 1-r}\Big)\mathbf{1}_{u\geq\Phi(\Phi^{-1}(r)+x)} du \\ &\quad + \int_{1-r}^1 \Phi\Big(\frac{\Phi^{-1}\big(\min\big(\frac{\Phi(\Phi^{-1}(u)-x)}{r},1\big)\big) - \rho\Phi^{-1}\left(\frac{u+r-1}{r}\right)}{\sqrt{1-\rho^2}}\Big) du \\ &= \Big(1 - r - \Phi\left(\frac{x}{2}\right)\Big)\mathbf{1}_{x\leq 2\Phi^{-1}(1-r)} \\ &\quad + \int_{1-r}^1 \Phi\Big(\frac{\Phi^{-1}\big(\min\big(\frac{\Phi(\Phi^{-1}(u)-x)}{r},1\big)\big) - \rho\Phi^{-1}\left(\frac{u+r-1}{r}\right)}{\sqrt{1-\rho^2}}\Big) du \end{aligned}$$

which is continuous at $x = \eta$. Thus, $\tilde{S}_\eta\left(C^p_\rho\right)$ is equal to the survival function of their difference at point η, which is:

$$\tilde{S}_\eta\left(C^p_\rho\right) = \int_{1-r}^1 \Phi\Big(\frac{\Phi^{-1}\big(\min\big(\frac{\Phi(\Phi^{-1}(u)-\eta)}{r},1\big)\big) - \rho\Phi^{-1}\left(\frac{u+r-1}{r}\right)}{\sqrt{1-\rho^2}}\Big) du.$$

Using the previous equation and dominated convergence theorem, we can prove that $\rho \mapsto \tilde{S}_\eta\left(C^p_\rho\right)$ is continuous on $(-1,1)$.

We have $C^p_1 = C^r$ and $C^p_{-1} = W$. Furthermore, we can show after some algebra that

$$\tilde{S}_\eta\left(C^p_\rho\right) \underset{\rho\to 1}{\longrightarrow} 2\Phi\Big(\frac{-\eta}{2}\Big) = \tilde{S}_\eta\left(C^p_1\right)$$

and

$$\tilde{S}_\eta\left(C^p_\rho\right) \underset{\rho\to -1}{\longrightarrow} \Phi\Big(\frac{-\eta}{2}\Big) = \tilde{S}_\eta\left(C^p_{-1}\right).$$

Then $\rho \mapsto \tilde{S}_\eta\left(C^p_\rho\right)$ is continuous on $[-1,1]$, which achieves the proof.

8.7. **Proof of Proposition 14.** (i) As the copulae of $\mathcal{C}^d_G$ are of the form $\left(C_{G,\rho_t}\right)_{t\geq 0}$, the demonstration of this part of the proposition is similar to the one of the static framework.

(ii) Let (B^1, B^2) be two Brownian motion with copula $C^{ref,\frac{\eta}{2}}$. B^2 is then the reflection of B^1 according to the stopping time $\tau = \inf\{t \geq 0 : B^1_t = \frac{\eta}{2}\} = \inf\{t \geq 0 : B^1_t - B^2_t = \eta\}$. For $t < \tau$, $B^1_t - B^2_t < \eta$ and for $t \geq \tau$, $B^1_t - B^2_t = \eta$. Thus, we have:

$$S_{\eta,t}\left(C^{ref,\frac{\eta}{2}}\right) = \mathbb{P}_{C^{ref,\frac{\eta}{2}}}(t \geq \tau) = \mathbb{P}_{C^{ref,\frac{\eta}{2}}}\Big(\sup_{s\leq t} B^1_s \geq \frac{\eta}{2}\Big) = 2\Phi\Big(\frac{-\eta}{2\sqrt{t}}\Big)$$

according to Lemma 23.

If $C \in \mathcal{C}_B$, the copula C_t is in $\mathcal{C}$ and then according to Proposition 13

$$\sup_{C \in \mathcal{C}_B} \mathbb{P}_C \left(B_t^1 - B_t^2 \geq \eta\right) \leq \sup_{C \in \mathcal{C}} \mathbb{P}_C \left(B_t^1 - B_t^2 \geq \eta\right) = 2\Phi\Big(\frac{-\eta}{2\sqrt{t}}\Big),$$

which concludes this part of the proof.

(iii) We want to prove that for all x in $\left[0, 2\Phi\left(\frac{-\eta}{2}\right)\right]$, there exists C in $\mathcal{C}$ such that $\tilde{S}_\eta(C) = x$. Let $x \in \left[0, 2\Phi\left(\frac{-\eta}{2\sqrt{T}}\right)\right]$.

If $x \in \left[0, \Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right]$, we consider the Gaussian dynamic copula with $(\rho_s)_{s\geq 0} = 1 - \frac{1}{2t}\left(\frac{\eta}{\Phi^{-1}(x)}\right)^2$ which is in $[-1,1]$ and we have $S_{\eta,t}(C_{G,\rho}) = x$.

If $x \in \left[\Phi\left(\frac{-\eta}{2\sqrt{T}}\right), 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)\right]$, we consider the copula $C^{exp,h,\lambda}$ defined by Equation (3). After some algebra, we find that

$$\mathbb{P}_{C^{exp,h,\lambda}}\left(B_t^1 - B_t^2 \geq x\right) = \begin{cases} 2\Phi\left(\frac{-x}{2\sqrt{t}}\right)e^{-\frac{\lambda x}{2}+\lambda h} - \Phi\left(\frac{-x}{2\sqrt{t}} - \frac{\lambda\sqrt{t}}{2}\right)e^{\frac{\lambda^2 t}{8} - \frac{\lambda x}{4} + \lambda h} & \text{if} \quad x \geq 2h \\ \Phi\left(\frac{-x}{2\sqrt{t}}\right) + \Phi\left(\frac{x-4h}{2\sqrt{t}}\right) - \Phi\left(\frac{x-4h}{2\sqrt{t}} - \frac{\lambda\sqrt{t}}{2}\right)e^{\frac{\lambda^2 t}{8} - \frac{\lambda x}{4} + \lambda h} & \text{if} \quad x < 2h \end{cases}$$

Then,

$$S_{\eta,t}\left(C^{exp,\frac{\eta}{2},\lambda}\right) = 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right) - \Phi\left(\frac{-\eta}{2\sqrt{t}} - \frac{\lambda\sqrt{t}}{2}\right)e^{\frac{\lambda^2 t}{8} + \frac{\lambda\eta}{4}}$$

As we have:

1) $\lambda \mapsto S_{\eta,t}\left(C^{exp,\frac{\eta}{2},\lambda}\right)$ is continuous on $[0,\infty)$,
2) $S_{\eta,t}\left(C^{exp,\frac{\eta}{2},\lambda}\right) \underset{\lambda \to 0}{\longrightarrow} \Phi\left(\frac{-\eta}{2\sqrt{t}}\right)$,
3) $S_{\eta,t}\left(C^{exp,\frac{\eta}{2},\lambda}\right) \underset{\lambda \to \infty}{\longrightarrow} 2\Phi\left(\frac{-\eta}{2\sqrt{t}}\right)$,

we can conclude.

8.8. **Proof of Proposition 16.** (i) This part of the proof can be done by induction.

(ii) For $\tau_0 = 0 \leq t \leq \tau_1$, $X_t - Y_t^n = (1+\rho)B_t^X - \sqrt{1-\rho^2}B_t^Y$. The equality holds for k = 0.

Let us suppose that the property true at rank $k < n+1$, that is

$$X_t - Y_t^n = \left(1 + (-1)^k \rho\right)\left(B_t^X - B_{\tau_k}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_k}^Y\right) + \alpha_k, \quad \tau_k \leq t \leq \tau_{k+1}.$$

If $\tau_k \leq t \leq \tau_{k+1}$, $Y_t^n = \rho\tilde{B}_t^k + \sqrt{1-\rho^2}B_t^Y$ then

$$X_t - \rho\tilde{B}_t^k - \sqrt{1-\rho^2}B_t^Y = \left(1 + (-1)^k \rho\right)\left(B_t^X - B_{\tau_k}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_k}^Y\right) + \alpha_k. \tag{23}$$

As $\tilde{B}_t^k$ does not change after time τ_{k+1}, this relationship remains true for all time greater than τ_k. At time τ_{k+1}, we have the equation

$$\alpha_{k+1} = \left(1 + (-1)^k \rho\right)\left(B_{\tau_{k+1}}^X - B_{\tau_k}^X\right) - \sqrt{1-\rho^2}\left(B_{\tau_{k+1}}^Y - B_{\tau_k}^Y\right) + \alpha_k. \tag{24}$$

Taking the difference between Equation (23) and Equation (24), we have

$$X_t - \rho\tilde{B}_t^k - \sqrt{1-\rho^2}B_t^Y = \left(1 + (-1)^k \rho\right)\left(B_t^X - B_{\tau_{k+1}}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_{k+1}}^Y\right) + \alpha_{k+1}.$$

Let $\tau_{k+1} \le t \le \tau_{k+2}$. If $k = n$, the proof is over because $Y_t^n = \rho \tilde{B}_t^n + \sqrt{1-\rho^2} B_t^Y$ for τ_{n+1}. Otherwise, $Y_t^n = \rho \tilde{B}_t^{k+1} + \sqrt{1-\rho^2} B_t^Y$ with $\tilde{B}_t^{k+1} = \mathcal{R}\left(\tilde{B}_t^k, \tau_{k+1}\right) = 2\tilde{B}_{\tau_{k+1}}^k - \tilde{B}_t^k$ and

$$\begin{aligned}
X_t - Y_t^n &= X_t - \rho \tilde{B}_t^{k+1} - \sqrt{1-\rho^2} B_t^Y \\
&= X_t - \rho \tilde{B}_t^k - \sqrt{1-\rho^2} B_t^Y + \rho(\tilde{B}_t^k - \tilde{B}_t^{k+1}) \\
&= X_t - \rho \tilde{B}_t^k - \sqrt{1-\rho^2} B_t^Y + 2\rho(\tilde{B}_t^k - \tilde{B}_{\tau_{k+1}}^k) \\
&= \left(1 + (-1)^k \rho\right)\left(B_t^X - B_{\tau_{k+1}}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_{k+1}}^Y\right) + \alpha_{k+1} + 2\rho\left(\tilde{B}_t^k - \tilde{B}_{\tau_{k+1}}^k\right).
\end{aligned}$$

Let $s, t > \tau_k$, we have

$$\begin{aligned}
\tilde{B}_t^k - \tilde{B}_s^k &= -\tilde{B}_t^{k-1} + 2\tilde{B}_{\tau_k}^{k-1} + \tilde{B}_s^{k-1} - 2\tilde{B}_{\tau_k}^{k-1} \\
&= -\left(\tilde{B}_t^{k-1} - \tilde{B}_s^{k-1}\right) = (-1)^k \left(\tilde{B}_t^0 - \tilde{B}_s^0\right) \\
&= (-1)^{k+1}\left(B_t^X - B_s^X\right).
\end{aligned}$$

Then $2\rho\left(\tilde{B}_t^k - \tilde{B}_{\tau_{k+1}}^k\right) = 2\rho\,(-1)^{k+1}\left(B_t^X - B_{\tau_{k+1}}^X\right)$ and we find that the property holds at rank $k+1$, which achieves the proof.

(iii) The property holds for $k = 1$.

Let us suppose that the property holds for $k = 2p+1$. $X_{\tau_k} - Y_{\tau_k}^n = \eta$ and τ_{k+1} is the first time greater than τ_k when $X_t - Y_t^n$ goes to ν. According to Equation (6),

$$\mathbb{P}\left(\tau_{k+1} \le t\right) = \mathbb{P}\Big(\inf_{\tau_k \le s \le t} (1-\rho)\left(B_s^X - B_{\tau_k}^X\right) - \sqrt{1-\rho^2}\left(B_s^Y - B_{\tau_k}^Y\right) + \eta \le \nu, t \ge \tau_k\Big).$$

If $t \ge \tau_k$, $(B_t^X - B_{\tau_k}^X)$ and $(B_t^Y - B_{\tau_k}^Y)$ are Brownian motions independent of $\mathcal{F}_{\tau_k}$. Then using Lemma 23 and Lemma 24, we have

$$\begin{aligned}
\mathbb{P}\left(\tau_{k+1} \le t\right) &= \mathbb{E}\Big(2\Phi\Big(\frac{\nu-\eta}{\sqrt{2(1-\rho)(t-\tau_k)}}\Big)\mathbf{1}_{t \ge \tau_k}\Big) \\
&= 2\mathbb{P}\left((1-\rho)\left(B_t^X - B_{\tau_k}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_k}^Y\right) \le \nu - \eta, t \ge \tau_k\right) \\
&= 2\Phi\Big(\frac{\nu-\eta}{\sqrt{2(1-\rho)t}} - u_k\Big).
\end{aligned}$$

This is the law of the stopping time $\tau = \inf\{t \ge 0 : B_t = u_k + \frac{\eta-\nu}{\sqrt{2(1-\rho)}}\}$ and the property holds for $k+1$. The proof is similar for $k = 2p$.

8.9. **Proof of Proposition 17.**

Lemma 26. *For $t > 0$, $x \in \mathbb{R}$,*

$$\mathbb{P}\left(X_t - Y_t^n \le x, t \ge \tau_n\right) = \begin{cases} \Phi\Big(\frac{x-\alpha_n}{\sqrt{2(1+(-1)^n\rho)t}} - \frac{u_n}{\sqrt{t}}\Big) & \text{if} \quad x < \alpha_n \\ \Phi\Big(\frac{x-\alpha_n}{\sqrt{2(1+(-1)^n\rho)t}} + \frac{u_n}{\sqrt{t}}\Big) - 2\Phi\left(\frac{u_n}{\sqrt{t}}\right) + 1 & \text{if} \quad x \ge \alpha_n \end{cases}$$

and

$$\mathbb{P}\left(X_t - Y_t^n \le x, t \ge \tau_{n+1}\right) = \begin{cases} \Phi\Big(\frac{x-\alpha_{n+1}}{\sqrt{2(1+(-1)^n\rho)t}} - \frac{u_{n+1}}{\sqrt{t}}\Big) & \text{if} \quad x < \alpha_{n+1} \\ \Phi\Big(\frac{x-\alpha_{n+1}}{\sqrt{2(1+(-1)^n\rho)t}} + \frac{u_{n+1}}{\sqrt{t}}\Big) - 2\Phi\Big(\frac{u_{n+1}}{\sqrt{t}}\Big) + 1 & \text{if} \quad x \ge \alpha_{n+1} \end{cases}.$$

Proof We have:

$$\mathbb{P}\left(X_t - Y_t^n \le x, t \ge \tau_n\right) = \mathbb{P}\Big(\left(1+(-1)^n\rho\right)\left(B_t^X - B_{\tau_n}^X\right) - \sqrt{1-\rho^2}\left(B_t^Y - B_{\tau_n}^Y\right) + \alpha_n \le x, t \ge \tau_n\Big)$$
$$= \mathbb{E}\Big(\Phi\Big(\frac{x-\alpha_n}{\sqrt{2\left(1+(-1)^n\rho\right)(t-\tau_n)}}\Big)\mathbf{1}_{t\ge\tau_n}\Big).$$

However, according to Equation (7), $\tau_n^n \sim \tau' = \inf\{t \ge 0 : B_t = u_n)\}$ where B_t is a standard Brownian motion. Then we have, using Lemma 24,

$$\begin{aligned}\mathbb{P}\left(X_t - Y_t^n \le x, t \ge \tau_n\right) &= \mathbb{E}\Big(\Phi\Big(\frac{x-\alpha_n}{\sqrt{2\left(1+(-1)^n\rho\right)(t-\tau')}}\Big)\mathbf{1}_{t\ge\tau'}\Big)\\ &= \Phi\Big(\frac{x-\alpha_n}{\sqrt{2\left(1+(-1)^n\rho\right)t}} - \frac{u_n}{\sqrt{t}}\Big)\mathbf{1}_{x<\alpha_n}\\ &+ \Big(\Phi\Big(\frac{x-\alpha_n}{\sqrt{2\left(1+(-1)^n\rho\right)t}} - \frac{u_n}{\sqrt{t}}\Big) - 2\Phi\left(\frac{u_n}{\sqrt{t}}\right) + 1\Big)\mathbf{1}_{x\ge\alpha_n}.\end{aligned}$$

The proof is the same for $\mathbb{P}\left(X_t - Y_t^n \le x, t \ge \tau_{n+1}\right)$.

□

We can now prove Proposition 17. We have:

$$\begin{aligned}p_{n+1}(t,x) - p_n(t,x) &= \mathbb{P}\left(X_t - Y_t^n \le x\right) - \mathbb{P}\left(X_t - Y_t^{n+1} \le x\right)\\ &= \mathbb{P}\left(X_t - Y_t^n \le x, \tau_{n+1} \le t\right) - \mathbb{P}\left(X_t - Y_t^{n+1} \le x, \tau_{n+1} \le t\right)\\ &+ \mathbb{P}\left(X_t - Y_t^n \le x, \tau_{n+1} \ge t\right) - \mathbb{P}\left(X_t - Y_t^{n+1} \le x, \tau_{n+1} \le t\right).\end{aligned}$$

For $\tau_{n+1} \ge t$, $X_t - Y_t^n$ and $X_t - Y_t^{n+1}$ are equals then

$$\mathbb{P}\left(X_t - Y_t^n \le x, \tau_{n+1} \ge t\right) = \mathbb{P}\left(X_t - Y_t^{n+1} \le x, \tau_{n+1} \ge t\right).$$

then we have

$$p_{n+1}(t,x) - p_n(t,x) = \mathbb{P}\left(X_t - Y_t^n \le x, \tau_{n+1} \le t\right) - \mathbb{P}\left(X_t - Y_t^{n+1} \le x, \tau_{n+1} \le t\right)$$

and we can conclude using Lemma 26.

8.10. **Proof of Proposition 20.** Let $\rho \in (-1,1)$ and $p_n(\rho,t,x)$ be the value of $p_n(t,x)$ when the correlation of the model is equal to ρ. We also denote $u_n(\rho)$ the value of u_n.

According to Equation (9) and Equation (10),

$$p_n(\rho,t,x) = \sum_{k=1}^{n} a_k(\rho,t,x)$$

with

$$a_1(\rho,t,x) = \begin{cases} -\Phi\Big(\frac{x-\eta}{\sqrt{2(1-\rho)t}} - \frac{\eta}{\sqrt{2(1+\rho)t}}\Big) + \Phi\Big(\frac{-x}{\sqrt{2(1+\rho)t}}\Big) + \Phi\Big(\frac{x-2\eta}{\sqrt{2(1+\rho)t}}\Big) & \text{if} \quad x < \eta\\ \Phi\Big(\frac{\eta-x}{\sqrt{2(1-\rho)t}} - \frac{\eta}{\sqrt{2(1+\rho)t}}\Big) & \text{if} \quad x \ge \eta\end{cases}$$

and

$$a_k(\rho,t,x) = \begin{cases} \Phi\Big(\frac{x-\alpha_k}{\sqrt{2\left(1+(-1)^{k-1}\rho\right)t}} - \frac{u_k(\rho)}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\alpha_k}{\sqrt{2\left(1+(-1)^{k}\rho\right)t}} - \frac{u_k(\rho)}{\sqrt{t}}\Big) & \text{if} \quad x < \alpha_k\\ \Phi\Big(\frac{x-\alpha_k}{\sqrt{2\left(1+(-1)^{k-1}\rho\right)t}} + \frac{u_k(\rho)}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\alpha_k}{\sqrt{2\left(1+(-1)^{k}\rho\right)t}} + \frac{u_k(\rho)}{\sqrt{t}}\Big) & \text{if} \quad x \ge \alpha_k\end{cases}$$

for $k \geq 2$.

First, we want to show that $\mathbb{P}\Big(X_t(\rho) - Y_t^{N_t}(\rho) \geq x\Big) = \sum_{n\geq 1} a_n(\rho,t,x)$ is continuous on $(-1,1)$.

Let $x \in \mathbb{R}$ and $\rho \in (-1,1)$. Since $\Phi(x) = 1 - \Phi(-x)$, we have $a_k(\rho,t,x) = \Phi\Big(\frac{\alpha_k - x}{\sqrt{2\left(1+(-1)^k\rho\right)t}} - \frac{u_k(\rho)}{\sqrt{t}}\Big) - \Phi\Big(\frac{\alpha_k - x}{\sqrt{2\left(1+(-1)^{k-1}\rho\right)t}} - \frac{u_k(\rho)}{\sqrt{t}}\Big)$ for $x \geq \alpha_k$ and $k \geq 2$. Then, for $k \geq 2$, we have

$$\begin{aligned}|a_k(\rho,t,x)| &\leq 2\Phi\Big(-\frac{|x-\alpha_k|}{\sqrt{2(1+|\rho|)t}} - \frac{u_k(\rho)}{\sqrt{t}}\Big)\\ &\leq 2\Phi^c\Big(\frac{|x-\alpha_k|}{\sqrt{2(1+|\rho|)t}} + \frac{u_k(\rho)}{\sqrt{t}}\Big)\\ &\leq 2\Phi^c\Big(\frac{u_k(\rho)}{\sqrt{t}}\Big)\end{aligned}$$

with $\Phi^c(x) = 1 - \Phi(x) = \Phi(-x)$. Furthermore, for $k \geq 2$, we find after some algebra that

$$u_k(\rho) \geq \frac{(\eta-\nu)(k-1)}{2\sqrt{2(1-\rho^2)}}.$$

Then, $|a_k(\rho,x,t)| \leq 2\Phi^c\Big(\frac{(\eta-\nu)(k-1)}{2\sqrt{2(1-\rho^2)t}}\Big)$. Furthermore, for $h > 0$, $\Phi^c(h) \leq \frac{1}{\sqrt{2\pi}h}e^{\frac{-h^2}{2}} \leq \frac{2}{\sqrt{2\pi h^3}}$. Then we obtain

$$|a_k(\rho,t,x)| \leq \frac{32\left(\sqrt{1-\rho^2}\right)^3}{\sqrt{\pi}(\eta-\nu)^3(k-1)^3} \tag{25}$$

and in particular, $|a_k(\rho,t,x)| \leq \frac{32}{\sqrt{\pi}(\eta-\nu)^3(k-1)^3}$. Then $\sum_{n\geq 2} a_n(\rho,t,x)$ normally converges on each compact of $(-1,1)$ and since $\rho \mapsto a_k(\rho,t,x)$ is continuous on $(-1,1)$, $\rho \mapsto \mathbb{P}\Big(X_t(\rho) - Y_t^{N_t}(\rho) \geq x\Big)$ is continous on $(-1,1)$ for all $x \in \mathbb{R}$.

According to Equation (25),

$$\sum_{n\geq 2}|a_n(\rho,t,x)| \leq \frac{32\left(\sqrt{1-\rho^2}\right)^3}{\sqrt{\pi}(\eta-\nu)^3}\sum_{n\geq 2}\frac{1}{(n-1)^3}.$$

Thus, $\sum_{n\geq 2} a_n(\rho,t,x) \underset{\rho\to\pm 1}{\longrightarrow} 0$ and we have:

$$\mathbb{P}\Big(X_t(\rho) - Y_t^{N_t}(\rho) \geq x\Big) \underset{\rho\to -1}{\longrightarrow} \mathbf{1}_{x<0} + \frac{1}{2}\mathbf{1}_{x=0}$$

$$\text{and } \mathbb{P}\Big(X_t(\rho) - Y_t^{N_t}(\rho) \geq x\Big) \underset{\rho\to 1}{\longrightarrow} \Big(\Phi\Big(\frac{-x}{2\sqrt{t}}\Big) + \Phi\Big(\frac{x-2\eta}{2\sqrt{t}}\Big)\Big)\mathbf{1}_{x<\eta} + \Phi\Big(\frac{-\eta}{2\sqrt{t}}\Big)\mathbf{1}_{x=\eta}.$$

Furthermore, $Y^N(-1) = X$ and $Y^N(1)$ is the reflection of B^X, then:

$$\mathbb{P}\Big(X_t(-1) - Y_t^{N_t}(-1) \geq x\Big) = \mathbf{1}_{x\leq 0}$$

$$\text{and } \mathbb{P}\Big(X_t(1) - Y_t^{N_t}(1) \geq x\Big) = \Big(\Phi\Big(\frac{-x}{2\sqrt{t}}\Big) + \Phi\Big(\frac{x-2\eta}{2\sqrt{t}}\Big)\Big)\mathbf{1}_{x\leq\eta}.$$

Thus, $\rho \mapsto \mathbb{P}\left(X_t(\rho) - Y_t^{N_t}(\rho) \geq x\right)$ is continuous at $\rho = -1$ for all $x \neq 0$ and at $\rho = 1$ for all $x \neq \eta$. $\rho \mapsto \mathbb{P}\left(X_t(\rho) - Y_t^{N_t}(\rho) \geq x\right)$ is continuous on $[-1, 1]$ for all $x \neq 0$ and $x \neq \eta$ which allows us to conclude.

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