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Real option game with a random regulator: the value of being preferred

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Abstract

We attempt to formalize a randomization procedure undertaken in pre-emptive real option games without simultaneous investment. This allows to propose a unified treatment of both real option games with and without simultaneous investment. This is done by introducing a random arbitrator with different parametrization. We then extend the study to an unfair arbitrator. This leads to competitive advantages in various asymmetrical situations. Relying on the results of [4], we apply the procedure to the risk-neutral and the risk-averse profiles in a stochastic pre-emptive real option game in complete market. The risk-averse case gives us the opportunity to study a new phenomenon we call aversion for confrontation, and its impact on the asymmetrical game.

1 Introduction

Real option games are commonly known as a collection of games with payoffs provided by the value function of some optimal control problems. This corresponds to an extensively studied situation, where two or more economical agents face with time a common project to invest into, and where there might be an advantage of being a *leader* (*pre-emptive* game) or a *follower* (*attrition* game). On these problems, one can enjoy a comprehensive broad scope of the literature provided by [1].
From the latter reference emerges a *standard real option game*: a symmetric duopoly pre-emption situation with a continuous Markov state variable, and risk-neutral agents. However, two approaches stand out on the final outcome of the game, or the mechanism of settlement, depending on the economical situation. It shall be pointed out if simultaneous action of the two agents results in a fair division of the project, as in the recent [4], or the election of one agent as the leader, as in the seminal [3]. If the project can be shared, there is a certain configuration when players do play a coordination game (when players want to be leader but prefer being follower rather than share the project incomes). Otherwise, there is a trigger state at which both players want to invest, but the role of leader is decided via the flipping of an even coin. This procedure turns out to be perfectly consistent for risk-neutral (and risk-averse) players, i.e., when independence axiom is valid. But this procedure also seems to come out of nowhere, simplifying the calculus and terminating the litigate between players in a rather direct and eluded way. Moreover, if players are risk-affine, the coin toss

procedure should incite players to act more eagerly. This behaviour, although not widespread in the rational choice literature, is more common in empirical studies of behavioural economics or theory of choice to describe very peculiar situations. We are clearly not investigating this situation in the present paper, but highlight the need for a formalization of the random procedure.
Motivated by this little problem, we concern ourselves in this paper with properly formalizing this ambiguity. We do that by simply introducing an arbitrator who decides what is happening in the case where both players act simultaneously, i.e., who is the leader and who is the follower. Starting from this minor idea, we actually reach several interesting key points of the real option game framework. Allowing the arbitrator to allow simultaneous investment also, we provide a unification of both endogenous attribution of roles with and without shared project. And then, allowing any weights on the arbitrator's decision, we make appearing new economical interactions.
Let us take a brief moment to describe one of those we have in mind. Assume that two economical agents are running for the investment in a project in time with possibility of simultaneous investment, as in [4]. In practice, even if they are accurately described as symmetrical, they would never act at the exact same time, providing that instantaneous action in a continuous time model is just an idealized situation. Instead, they show their intention to invest to a third party (a regulator, a public institution) at approximatively the same time. Assume now that this arbitrator has some flexibility in his judgement. For example, he can evaluate whose agent is the most suitable to be granted the project as a leader, regarding qualitative criteria. This situation might be illustrated in particular where environmental or health exigences are in line. When simultaneous investment is impossible, the real estate market example of [3] can also be cited again with in mind that safety constraints, but also aesthetic or confidence dimension, can intervene in the decision of a market regulator [1]. In those cases, if there is an asymmetry in the chance to be elected as a leader, perfectly informed players should take into account this strength or this weakness into their decision to invest or to differ. These are some of the economical situations the introduction of an unfair arbitrator can shed some light on. By setting the arbitrator to be totally unfair for one player, we enjoy the opportunity to price a kind of option of asymmetry, following the example of [4]. This option, weaker than the priority option in the latter reference, corresponds to the additional value of being elected leader in case of simultaneous move, i.e., the value of being preferred.
Let us present how the remaining of the paper proceeds. In the next section, we introduce the theoretical framework of real option game with an arbitrator. This abstract setting allows to introduce very easily the arbitrator as a specific asymmetry, and already provides the Nash equilibria in the different possible situations we encounter hereafter. Moreover, it gives a roadmap for alternative models, such as incomplete market and war of attrition games.
In Section 3, we apply the above result to the real option framework. Therefore, we will speak of *firms* to designate players, and the arbitrator of the game will be referred as the *regulator* on the project market. This is not an intention to create confusion, but to suit the economical situation it is supposed to describe. We rely on the ideal situation when the project cash flows are perfectly correlated to an asset price. We could then deduce that we fall in the setting of a complete market, but provided that firms cannot hedge or predict the decision of the regulator, this situation falls actually in the

[1] A less ethical application is the price of bribe, although it seems difficult to imagine such a situation in reality with perfect knowledge of players, which is the restricted case we consider here. We don't doubt the reader will find better and more interesting applications than this one.

incomplete market framework. We then assume that firms are *risk-neutral*, as in many other papers on the subject. This section is the opportunity to make the case study of the different parametrizations of the regulator, in order to see the effect of the market advantage provided by the latter. Singular parametrizations allow to retrieve standard cases, or purely asymmetrical situations in the regulator preferences. One insightful corollary phenomenon of the latter case is that two possible equilibrium strategies can be equally relevant. We end the section discussing this situation.
We continue and end the study in Section 4 by assuming now that firms are *risk-averse*, and have a constant absolute risk aversion. This allows us to focus on the effect of the additional randomness induced by the regulator decision and the coordination game on firms decision to invest. Since payoffs are already computed in a complete market setting, the risk aversion is reduced to an aversion for confrontation. We prove that in the complete market setting, the risk-averse case is a continuous extension of the risk-neutral case. We also emphasize the effect of aversion for confrontation in the asymmetrical case.
We conclude the paper in Section 5. One last thing shall be said before getting to the heart of the matter. The real option framework implies the derivation of value functions corresponding to expected project values. For this matter, we follow and rely heavily on the recent [4], where the complete is fully explored. We thus avoid technical well-known mathematical developments and focus on the interactions between firms. We prefer to ease reading fluidity by developing as little as possible mathematics along the text, which are actually straightforward computations. We acknowledge [4] as a starting point for our study, and refer the reader to this article for many details, as also a very good first incursion in the real option game treated with formality. We apologize for such a subjective suggestion and the omission of the impressive amount of literature that has been provided on the topic since the seminal paper of [3]. Again, we refer to [1] for a fairer and more exhaustive homage to the many contributions to real option games.

2 The continuous Markov pre-emption game

2.1 Framework, arbitrator and strategies

Time is represented by the positive half-line $t \geq 0$. We consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} := (\mathcal{F}_t)$ is a filtration satisfying usual nice properties. We consider a state process Y, $\mathbb{F}$-adapted, continuous and Markov, and takes values in $\mathbb{R}$. The real option game with an arbitrator is the following. Two symmetrical players, endowed with a utility function U, start at time 0 with an option to exercise at an arbitrary time $t \geq 0$. The first one to exercise the option receives the payoff $L(t, Y_t)$ (as *leader*). The other player then receives the payoff $F(t, Y_t)$ (as *follower*). If both players act at the same time, an *arbitrator* intervenes instantaneously by choosing one alternative among three: she can give $L(t, Y_t)$ to first player and $F(t, Y_t)$ to second player, or the converse, or give an equal payoff $S(t, Y_t)$ to both players. More explicitly L, F and S are the *expected utility* of some future cash-flows depending on the evolution of the state process. The latter being Markov, the dependence is reduced to the actual value Y_t at time t, as we will see it in application. If nothing happens at some time t, the system evolves and is infinitely repeated as time goes by, see Fudenberg & Tirole [3] for the seminal formalization of the continuous-time game. We assume that agents cannot predict the decision of the arbitrator. Hence, the latter is given by random variables (A_1, A_2). We assume that this variables are defined on 3 states space orthogonal to Ω, meaning that (A_1, A_2) is independent of the filtration $\mathbb{F}$. These are defined by

$$(A_1, A_2)(t, Y_t) := \begin{cases} (L, F)(t, Y_t) & \text{with probability } q_1 \\ (F, L)(t, Y_t) & \text{with probability } q_2 \\ (S, S)(t, Y_t) & \text{with probability } q_S := 1 - q_1 - q_2 \end{cases} \tag{2.1}$$

Let's fix t and Y_t and denote $L := L(t, Y_t)$, $F := F(t, Y_t)$ and $S := (t, Y_t)$. One might think that payoffs L, F and S can take arbitrary values, but for the pre-emptive game, the taxonomy of situations provides three standard cases. We now consider two among them, which are the non ambiguous cases for the regular framework:

(d) When $F > L > S$, both players desire to acquire the payoff F. Neither player wants to act first, so that our situation is the same as in the standard case. Players *differ* action.

(e) When $L = F = S$, each firm is indifferent to receiving F or S, which is also equal to L. Both players act, whatever the decision of the arbitrator. We then retrieve the standard case: players *exercise.*

The third case, when $L > F > S$, is studied in the next subsection, as it presents the situation where both players want to act, but might prefer to receive F rather than S. It is the occasion to introduce a coordination game. The attentive reader might wonder what happens at end-points of the above intervals, when $F = L > S$ for example. This is postponed to the specific cases of Sections 3. One can easily extend the above procedure to the attrition game also.

2.2 Sub-game perfect Nash equilibria of the coordination game

The coordination game. In the particular case where $L > F > S$, players are in a coordination situation similar in many aspect to a duel or a prisoner's dilemma game. To study this situation specific to the continuous time setting, Fudenberg and Tirole [3] developped a method called *principle of rent equalization* based on economical arguments. A recent contribution of Thijssen & al. [5] extended the approach to the stochastic setting. It consists in extending the time domain to index pairs $(t, z) \in \mathbb{R}_+ \times \mathbb{Z}$, with the lexical order. That means that being at time t, with the payoffs defined above, we *extend* the time line by freezing time t and indefinitely repeating the following game:

	Play	Differ
Play	(S_1, S_2)	(L, F)
Differ	(F, L)	Repeat

Table 1: General coordination game

The situation when players act simultaneously, and the arbitrator is called upon, is actually a particular example of asymmetric game. Indeed in that situation, the outcome of the game is random for both players. Each player then has its own expected utility S_i for an outcome, given by

$$(S_1, S_2) := (q_1 L + q_2 S + q_S S, q_2 L + q_1 F + q_S S) . \tag{2.2}$$

Instantaneous strategies. Following [5], we introduce the formal framework of strategies for the coordination game. We raise the reader's awareness on the fact that due to the Markov structure t does not intervene in any value function. This is why we focus on Markov sub-game perfect equilibrium strategies. To obtain the latter, we proceed to the time extension $(t,z) \in \mathbb{R}_+ \times \mathbb{Z}$ of the model. The filtration is defined via $\mathcal{F}_{t,i} = \mathcal{F}_{t,j} \subseteq \mathcal{F}_{t',i}$ for any $t < t'$ and any $i \neq j$, and the state process is extended to $Y_{(t,z)} := Y_t$. A *simple strategy* for player $i \in \{1,2\}$ is defined as a pair of $\mathbb{F}$-adapted processes $(G^i_{(t,z)}, p^i_{(t,z)})$ taking values in $[0,1]^2$. The process $G^i_{(t,z)}$ must be a non-decreasing càdlàg process, and refers to the cumulative probability of firm i exercising before (t,z), whereas $p^i_{(t,z)}$ denotes the probability of exercising in the coordinate game of we introduce hereafter. As we noticed, Y is a continuous Markov process and that allows us to focus, without loss of generality, on Markov hitting-time strategies of the type $G^i_{(t,z)} \equiv G^i_t \equiv \mathbb{1}_{\{t \leq \tau(x^i)\}}$ with $\tau(x^i) := \inf\{t \geq 0 \ : \ Y_t \geq x^i\}$. Explicitly, $p^i_{(t,z)}$ is a strategy for the discretely repeated game at time t, so that it should be stationary: $p^i_{(t,z)} \equiv p^i_t \equiv p^i(Y_t)$, and not depend on the previous rounds of the game for $i = 1,2$. We openly address the conclusion of [4] on the use of strategies: the repeated discrete game is played only with $(p^i_{(t,z)})_{i=1,2}$ if $L(t,Y_t) > F(t,Y_t) > S(t,Y_t)$. The processes $(G^i_{(t,z)})_{i=1,2}$ will be kept for the endpoints case study, reported to further section. Let us now assume that

$$\max(p_1, p_2) > 0, \tag{2.3}$$

to remind us that $L > F$ and that we are in situation where players want to act. In this infinite 2-by-2 game, there are three possible outcomes:

- Player one receives payoff L with probability

$$a_1 := p_1(1-p_2) + (1-p_1)p_1(1-p_2)^2 + \ldots = p_1 \sum_{k=1}^{\infty} (1-p_1)^{k-1}(1-p_2)^k = \frac{p_1(1-p_2)}{p_1 + p_2 - p_1 p_2} \ .$$

- Player two receives payoff L with probability $a_2 := \dfrac{p_2(1-p_1)}{p_1 + p_2 - p_1 p_2}$.
- Players play simultaneously with probability $a_S := \dfrac{p_1 p_2}{p_1 + p_2 - p_1 p_2}$ and the arbitrator is invoked.

The game has three different outcomes and the exit of the repeated game without the action of at least one player is not permitted. For the continuous time game, this implies that in the present situation of confrontation there must be instantaneous emergence of at least one player. It is one of the delicate issues of the continuous time formulation. One can actually imagine that it is possible to have a differing behaviour for both players in time t, by stating a finite number of rounds in the above game. Then by density of the half real line $\{t \geq 0\}$, the probability that both players act at the same time is null if they use mixed strategies. Therefore, they are incited to follow a pure strategy in order to receive payoff L, but in that case symmetry leads to a sharing and a payoff S for both players, so that they shall use a mixed strategy instead. This paradox is settled by the infinite (discrete) game within the infinite (continuous) game.

Nash equilibria. Each player regards the expected utility provided by each possible decision, and then chooses a mixed strategy in consequence. The confrontation being instantaneous, players know that just after it they will receive a specific expected utility of the form L, F or S_i. Therefore, they are taking into account the uncertainty of their own strategy and the uncertainty of the arbitrator's decision. They will to maximize the quantity

$$E_1 := a_1 L + a_2 F + a_S S_1 \tag{2.4}$$

for player one, $E_2 := a_2 L + a_1 F + a_S S_2$ for player two. As explained in Appendix C of [4], we can look for Nash equilibria by first fixing p_2 and differentiate E_1^* with respect to p_1 to obtain the optimal value of p_1 for player one:

$$\frac{\partial E_1}{\partial p_1}(p_1, p_2) = \frac{p_2(L-F) + p_2^2(S_1 - L)}{(p_1(1-p_2) + p_2)^2} \tag{2.5}$$

Paying attention to this expression, one can then see that, by denoting

$$P_2 := \frac{L-F}{L-S_1} = \frac{L-F}{q_2(L-F) + q_S(L-S)} \ , \tag{2.6}$$

the sign of (2.5) is the sign of $p_2 - P_2$. We have a similar discrimination for second player by symmetry of expressions, necessitating to introduces the value $P_1 = (L-F)/(L-S_2)$. We then look for naive equilibria in the following manner:

(i) If $p_2 > P_2$, the optimal p_1 is 0. Then E_2 should not depend on p_2, and the situation is stable for any pair $(0, p_2)$ with p_2 in $(P_2, 1]$.

(ii) If $p_2 = P_2$, E_1 is constant and p_1 can take any value. If $p_1 < P_1$, then by symmetry p_2 should take value 1, leading to case (i). If $p_1 = P_1$, E_2 is constant and either $p_2 = P_2$, or we fall in case (i) or (iii). The only possible equilibrium is thus (P_1, P_2).

(iii) If $p_2 < P_2$, E_1 is increasing with p_1 and player one shall play with probability $p_1 = 1 > P_1$. Therefore p_2 optimizes E_2 when being 0, and E_1 becomes independent of p_1. Altogether, situation stays unchanged if $p_1 \in (P_1, 1]$ or if $p_1 = 0$. Otherwise, if $p_1 \leq P_1$, we fall back into cases (i) and (ii). The equilibria here are $(p_1, 0)$ with $p_1 \in (P_1, 1]$, and the trivial case $(0, 0)$.

Recalling constraint (2.3), we get rid of case $(0, 0)$. Coming back to the issue of the game when k goes to infinity in (t, k), three situations emerge from the above calculation. Two of them are pure coordinated equilibria, of the type $(a_1, a_2) = (1, 0)$ or $(0, 1)$, which can be produced by pure coordinated strategies $(p_1, p_2) = (a_1, a_2)$ for simplicity. The third one is a mixed equilibrium given by $(p_1, p_2) := (P_1, P_2)$, for which (a_1, a_2, a_S) follows according to its definition.

Influence of arbitrator's preferences. As in [4], we could invoke the trembling-hand deviation as a discriminatory method to pick up a specific behaviour for players. But before that, one shall ask whether value P_i is greater or equal to 1 for $i = 1, 2$, so that two possibilities out of three may not be possible. If we recall that $q_1 \geq q_2$, we directly obtain from (2.2) that

$$S_1 \geq S_2 \ . \tag{2.7}$$

According to equation (2.7), we have $L > F > S_1 \geq S_2$ so that $P_2 \geq P_1 \geq 0$. Therefore, using the same procedure to find Nash equilibria as above, we find the following possibilities:

(a) $P_1 < P_2 < 1$: the three equilibria from above calculation are $(0, 1)$, (P_1, P_2) and $(1, 0)$.

(b) $P_1 < 1 \leq P_2$: we find only one Nash equilibrium which is $(1,0)$.

(c) $1 \leq P_1 < P_2$: we find only one Nash equilibrium which is $(1,1)$.

Again, according to [4], (P_1, P_2) is the only mixed strategy in the last case, and thus the only trembling-hand subgame perfect equilibrium. It will correspond to the more natural strategy in the sequel, not only because of the robustness of this additional criterion and its pertinence in real world situations, but also because it is also the natural extension of the particular case of symmetry, where both players have the same strategy. We will however fall in a special situation in the next section where one of the three others might seem at least more relevant.

2.3 The two standard frameworks

If we take $q_S = 1$, Note that from (2.6), and by assuming that the symmetry of firms imposes $a_1 = a_2$, we indeed retrieve the case of [4], being the case where simultaneous investment is possible and the only decision of the regulator.

Formalization of the coin toss procedure. Assume that simultaneous investment is forbidden, i.e., $q_S = 0$. Following definitions (2.2) and (2.6), and assuming $q_i \in (0,1)$, we have $P_i = 1/q_i > 1$ for $i = 1, 2$. Therefore, only case (c) from above is possible, which is the situation commonly found in the framework using a coin toss procedure. Here $(p_1, p_2) = (1,1)$, giving the instantaneous situation $(a_1, a_2, a_S) = (1/2, 1/2, 0)$. To our knowledge, papers invoking a coin toss, i.e., an arbitrator, assume that both players have the same chance to be elected as a leader. There is however no need for the regulator to be *fair*. This is a foreseeable phenomenon, since for $L > F$, it is always more interesting to obtain $q_i L + (1 - q_i)F$ rather than F. Note that if players behave symmetrically here, the expected payoff is not the same for each one and depends on q_1. This will also be true for any convex combination instead of a linear one, meaning that the behaviour is expected to be the same with risk-averse players. However, for risk-affine players, this does not hold any more. We do not investigate this peculiar setting reserved to particular economical situations, although it might suit the real option framework, and leave this for further research. The case where $q_1 = 1$ is left for the next sections.

3 Risk neutral agents in a semi-complete market

We now apply this framework to a specific economical model. To make things simple, in this section, we investigate the investment decision when the project value is perfectly correlated with a traded asset in a complete market. To progressively study the effect of a regulator on firms decision, we focus on the different parametrizations of (q_1, q_2, q_S). The introduction of the regulator however makes the payoffs not replicable, so that we need a pricing criterion for firms. We assume here they are risk-neutral, i.e., their utility function is given by $U(x) = x$.

3.1 The model and value functions

We consider two firms that can invest in a similar project with random cash flows and a fixed initial sunk cost K. The cash-flows are the product of a demand variable process Y and an inverse demand curve $(D_{Q(t)})_{t\geq 0}$, where $Q(t)$ is the number of firms which have invested in the project by time t. The process D_Q can only take the following three values

$$D_0 = 0 < D_2 < D_1$$

and the stochastic process Y is now denoted $Y^{t,y}$ if it verifies

$$Y_t^{t,y} = y \text{ and } dY_s = Y_s(\nu ds + \eta dW_s), \quad s \geq t,$$

with $(W_t)_{t\geq 0}$ a standard Brownian motion under the historical measure $\mathbb{P}$. We assume that Y is perfectly correlated with a liquid traded asset P which dynamics is

$$dP_t = P_t(\mu dt + \sigma dW_t) = P_t(rdt + \sigma dW_t^{\mathbb{Q}})$$

where r is the constant interest rate of a riskless bank account at the disposal of each firm, and $W_t^{\mathbb{Q}} = W_t + \lambda t$ is a Brownian motion under the unique risk-neutral measure $\mathbb{Q}$ of the arbitrage-free market. The variable λ in its expression is the Sharpe ratio equal to $(\mu - r)/\sigma$.

The follower's problem. Assume that one of the two firms, say firm one, desires to invest at time t. If $Q(t) = 1$, then the available market for firm one is D_2. Having a complete financial market at her disposal to replicate the uncertain incomes of the project, the value of the latter at time t if $Y_t^{t,y} = y$ is given by the risk-neutral expectation of the project's discounted cash flows

$$V^F(t,y) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^{\infty} e^{-r(s-t)} D_2 Y_s^{t,y} ds\right] = \frac{D_2 y}{\eta\lambda - (\nu - r)} = \frac{D_2 y}{\delta} \tag{3.1}$$

with $\delta := \eta\lambda - (\nu - r)$. But firm one can wait to invest, and we suppose she can wait as long as she wants. She indeed have to pay the cost K at time τ she invests. In the financial literature, this is interpreted as a Russian call option of payoff $(D_2 Y_\tau/\delta - K)^+$. The value function of this option is given by

$$F(t,y) := \sup_{\tau\in\mathcal{T}_t} \mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau-t)}\left(\frac{D_2 Y_\tau^{t,y}}{\delta} - K\right)^+ \mathbb{1}_{\{\tau<+\infty\}} | \mathcal{F}_t\right] \tag{3.2}$$

where $\mathcal{T}_t$ denotes the collection of all $\mathbb{F}$-stopping times with values in $[t, \infty]$. We now recall Proposition 1 of [4] which gives the explicit solution to (3.2).

Proposition 3.1. *Assume $\delta > 0$. Then the solution to* (3.2) *is given by*

$$F(t,y) = \begin{cases} \frac{K}{\beta-1}\left(\frac{y}{Y_F}^{\beta}\right) & \text{if } y \leq Y_F, \\ \frac{D_2 y}{\delta} - K & \text{if } y > Y_F, \end{cases} \tag{3.3}$$

with a threshold Y_F given by

$$Y_F := \frac{\delta K \beta}{D_2(\beta - 1)} \tag{3.4}$$

and

$$\beta := \left(\frac{1}{2} - \frac{r-\delta}{\eta^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\eta^2}\right)^2 + \frac{2r}{\eta^2}} > 1. \tag{3.5}$$

The behaviour of the follower firm is thus quite explicit. She will differ investment until the demand reaches the level $Y_F = \beta/(\beta - 1)K > K$ which depends on the profitability of the project.

The leader's problem. Assume now that instead of having $Q(t) = 1$, $Q(t) = 0$. Firm one investing at time t will receive cash-flows associated to the level D_1 for some time, but she expects firm two to enter the market when the threshold Y_F is triggered. After that moment we call τ_F, both firms share the market and firm one receives cash flows determined by level D_2. The project value is thus

$$V^L(t,y) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau_F} e^{-r(s-t)}(D_1 \mathbb{1}_{\{s<\tau_s\}} + D_2 \mathbb{1}_{\{s\geq\tau_s\}})Y_s^{t,y} ds\right] = \frac{D_1 y}{\delta} - \frac{(D_1 - D_2)Y_F}{\delta}\left(\frac{y}{Y_F}\right)^{\beta} \quad (3.6)$$

where detailed computation can be found in [4]. This allows to characterize the leader's value function $L(t,y)$, i.e., the option to invest at time t for a demand y, as well as the value of the project $S(t,y)$ in the situation of simultaneous investment (Proposition 2 in [4]). Note that here no exercise time is involved as we consider the interest of exercising immediately.

Proposition 3.2. *Assume $\delta > 0$. Then*

$$L(t,y) = \begin{cases} \frac{D_1 y}{\delta} - \frac{(D_1-D_2)}{D_2}\frac{K\beta}{\beta-1}\left(\frac{y}{Y_F}\right)^{\beta} & \text{if } y < Y_F, \\ \frac{D_2 y}{\delta} - K & \text{if } y \geq Y_F, \end{cases} \quad (3.7)$$

If both firms invest simultaneously, we have

$$S(t,y) := \frac{D_2 y}{\delta} - K\ . \quad (3.8)$$

3.2 Equilibrium study with a biased regulator

It is now appropriate to recall the standard situations of the pre-emptive game we stated in Section 2. To this end, we need in this setting the following proposition, again formally proved in [4].

Proposition 3.3. *Assume $\delta > 0$. There exists a unique point $Y_L \in (0, Y_F)$ such that*

$$\begin{cases} S(t,y) < L(t,y) < F(t,y) & \text{for } y < Y_L, \\ S(t,y) < L(t,y) = F(t,y) & \text{for } y = Y_L, \\ S(t,y) < F(t,y) < L(t,y) & \text{for } Y_L < y < Y_F, \\ S(t,y) = F(t,y) = L(t,y) & \text{for } y \geq Y_F\ . \end{cases} \quad (3.9)$$

We then retrieve two explicit cases, when $y < Y_L$, where the two firms wait to obtain a better return of the project, and when $y \geq Y_F$, leading to immediate commitment of the firms, whatever the decision of the regulator. These correspond respectively to cases (d) and (e) of previous Section. We now look more deeply into the case $Y_L \leq y \leq Y_F$.

Risk-neutral agents and semi-complete market. According to the Markovian framework we will without loss of generality fix $t = 0$ and consider the process $Y^{0,y}$. Henceforth, we omit the t dependency in value functions and strategies. Coming back to the situation where $y \in (Y_L, Y_F)$, the two firms are facing a coordination game similar to the one of subsection 2.2. Notice that the regulator intervenes in the coordination game only once and if both firms want to invest at the same time. Since we assumed that the regulator decision is independent of $\mathbb{F}$, the two firms face a market incompleteness. We thus introduce now the fact that firms are *risk-neutral*, i.e., they will estimate the value of the outcome A_i defined in formula (2.1) by taking the expectations given the law of (A_1, A_2):

$$\begin{cases} S_1(y) := q_1 L(y) + q_2 F(Y) + q_S S(y) & \text{for firm 1,} \\ S_2(y) := q_2 L(y) + q_1 F(Y) + q_S S(y) & \text{for firm 2.} \end{cases} \quad (3.10)$$

Values $L(y)$, $F(y)$ and $S(y)$ are still computed under the risk-neutral measure $\mathbb{Q}$ on Ω. That means that expectations of (3.10) are made under the *minimal entropy martingale measure* for this problem, and that uncertainty of the model follows a *semi-complete* market hypothesis, This means that if we reduce uncertainty to the market information, i.e., the filtration $\mathbb{F}$, then the market is considered as complete. It is a very convenient assumption when central limit theorem can be invoked for the orthogonal random variable, as in diversification for insurance purpose. See [2] for the formal definition and corresponding situations. Here it is not the case, and we include the risk-neutral assumption to obtain simple solutions. Firms value the arbitrator's intervention by just weighting each arbitrage-free price by its probability. The risk-averse case is postponed to the next section.

The case study. Since the cases $q_S = 1$ and $q_S = 0$ were studied in section 2, we now study the general case $0 < q_2 \leq q_1 < 1 - q_2$ of subsection 2.2 with respect to the value of y. In reference to [4], we provide a proposition allowing to define the different regions characterized by different Nash Equilibria. This can actually make explicit the evolution of the mixed strategy of players in the case $q_S = 1$ of [4], see equation (4.4) and the associated comments in the next section.

Proposition 3.4. *The functions P_2 and P_1 are increasing on $[Y_L, Y_F]$.*

Proof By taking $d_1(y) := L(y) - F(y)$ and $d_2(y) := S(y) - F(y)$, we get

$$P_i(y) = \frac{1}{q_i}\left[\frac{d_1(y)}{d_1(y) + \alpha_i(d_1(y) - d_2(y))}\right] \quad \text{and} \quad P_i'(y) = \frac{1}{q_i}\left[\frac{\alpha_i(d_1(y)d_2'(y) - d_2(y)d_1'(y))}{(d_1(y) + \alpha_i(d_1(y) - d_2(y)))^2}\right]$$

where

$$\alpha_i := q_S/q_i \leq 1 \text{ with } i \in \{1, 2\}. \tag{3.11}$$

We are thus interested in the sign of the quantity $g(y) := d_1(y)d_2'(y) - d_2(y)d_1'(y)$ which quickly leads to

$$\frac{g(y)\delta}{yD_2} = \left(\frac{y}{Y_F}\right)^{\beta-1}\left[(D_1 - D_2)(\beta + \frac{1}{\beta} - 2 - \frac{\delta K}{D_2})\right] + \frac{\delta K}{D_2}(D_1 - D_2) \ .$$

Since $\beta > 1$, the function $x(y)$ is an increasing function. Since $y \leq Y_F$, it suffices to verify that $(\delta K)/D_2 \geq (\beta + 1/\beta - 2 - (\delta K)/D_2)$, which is naturally the case for any β, to obtain that g is non-negative on the interval. $\square$

Subregions. Note that the quantities q_1 and q_2 do not affect the monotonicity of P_2 and P_1. We already know that $P_2(y) \geq P_1(y)$. Looking at (2.6), we also have that $P_2(Y_L) = P_1(Y_L) = 0$ and $P_i(Y_F) = 1/(q_i + q_S) > 1$ for $i = 1, 2$. This implies that we retrieve the three subregions (a), (b) and (c) of previous section:

(a) the interval $[Y_L, Y_1)$ for some $Y_1 \in (Y_L, Y_F)$ such that $P_2(Y_1) = 1$, and thus on which $P_1(y) \leq P_2(y) < 1$. The point Y_1 verifies

$$F(Y_1) = q_1 L(Y_1) + q_2 F(Y_1) + q_S S(Y_1) = S_1(Y_1) \ . \tag{3.12}$$

For this interval, we are in the situation described in subsection 2.2 where three Nash equilibria are possible, and only $(p_1(y), p_2(y)) = (P_1(y), P_2(y))$ is kept, as it is the only trembling-hand perfect equilibrium. We also keep that one because it is the one involved in the fair situation of

[4], so that we can invoke an argument of continuity between situations. For this strategy, the probabilities of outcomes of the game are the following:

$$(a_1, a_2, a_S)(y) = \left(\frac{F - S}{L + F - 2S} - \frac{a_0}{\alpha_1}, \frac{F - S}{L + F - 2S} - \frac{a_0}{\alpha_2}, \frac{a_0}{q_S} \right)(y) \ . \tag{3.13}$$

with $a_0(y) = \frac{L-F}{L+F-2S}(y)$ being the probability of simultaneous investment in the case $q_S = 1$ of [4]. Applying (2.4), we notice that the rent equalization procedure of [3] holds in our general case and that the two players have for expected payoff

$$(E_1(y), E_2(y)) = (F(y), F(y)) \ . \tag{3.14}$$

The juncture with $[0, Y_L)$ is a delicate point. At the left of point Y_L, no firm wants to invest. At the right side of point Y_L, P_i converges to 0 when y converges to Y_L by above, for $i = 1, 2$. Therefore, so do (p_1, p_2) toward $(0, 0)$. A short calculation provides

$$\lim_{y \downarrow Y_L} \frac{a_1(y)}{a_2(y)} = 1 \quad \text{and} \quad \lim_{y \downarrow Y_L} a_s(y) = 0 \ . \tag{3.15}$$

Therefore, at the right of point Y_L, $(a_1(Y_L^+), a_2(Y_L^+), a_S(Y_L^+)) = (1/2, 1/2, 0)$. Calling the argument of [4] in our own way, there is a certain continuity of behaviour from the fact that simultaneous investment is improbable at point Y_L, and its probability of occurrence $a_S(y)$ is continuous at this point.

At end point Y_1, we find that $P_2(y)$ goes to 1, and (a_1, a_2, a_S) tends toward $(0, 1{-}P_1(Y_1), P_1(Y_1))$. Following (3.12), $E_1(y)$ goes to $F(Y_1)$. Consequences are dealt just below for the next interval, but as long as $y < Y_1$ the equilibrium is given by $(p_1(y), p_2(y)) = (P_1(y), P_2(y))$.

(b) The next case is given by interval $[Y_1, Y_2)$ for some $Y_2 \in [Y_1, Y_F)$, on which $P_1(y) < 1 \leq P_2(y)$. The reader will guess that Y_2 is chosen such that $P_1(Y_2) = 1$, and verifies

$$F(Y_2) = q_2 L(Y_2) + q_1 F(Y_2) + q_S S(Y_2) = S_2(Y_2) \ . \tag{3.16}$$

We just said that $E_1(Y_1) = F$. Being risk-neutral, the first firm prefers being a leader at time $\tau(Y_1)$, and is indifferent between being follower or letting the regulator chose. For the other firm, differ means receiving $F(Y_1)$ and exercising means letting the regulator give an expected payoff $q_1 F(Y_1) + q_2 L(Y_1) + q_S S(Y_1) < F$. Differing is her best option. That means that at point Y_1, the equilibrium is $(p_1, p_2) = (1, 0)$. It is clear that for $y < Y_2$, $F(y) > S_2(y)$, and since first firm has a definite advantage not to hesitate exercising her option, the second firm will differ exercise until $\tau(Y_2)$.

(c) On the interval $[Y_2, Y_F]$, the second firm can finally bear the regulator preference for first firm and be indifferent between *followership* and a decision of the regulator. Here, $1 \leq P_1(y) \leq P_2(y)$ and according to the facts stated in subsection 2.2, equilibrium exists when both firms exercise. The continuity at point Y_F with region (e) spares us from detailing.

Few comments are in order. On the right of point Y_L, the asymptotic probabilities of outcomes given by equation (3.15) tend to the fair distribution of [4]. This is a direct consequence of a_S vanishing to 0, and thus less intervention from the regulator. At point Y_1, there is a strong discontinuity in the optimal behaviour of the second firm. For $y < Y_1$, the mixed strategy used by the latter tends toward a pure strategy with systematic investment. However at the point itself, the second firm differs investment and becomes the follower. We propose the following interpretation. As y tends to Y_1 from left, firm one is less refrained from investing, although her probability to act is still lower than the one of firm two. For firm two, this is the signal that she should invest and try to pre-empt firm one, otherwise she will clearly lose interest in the pre-emption situation when Y_1 is reach. For firm one, being a follower is better than letting the regulator decide before Y_1. At the point however, she is indifferent between these two positions so that she will suddenly seek for the leader's position. One last word. Starting from the initial framework where sharing the market is the only outcome of a simultaneous investment, we can see that the progressive introduction of alternatives in the regulator's range of decisions reduces the synchronization game on $[Y_L, Y_1]$, and intervention of the regulator comes at point Y_2 earlier than at Y_F. In the case where $q_1 = q_2 =: q$, and $Y_1 = Y_2 =: Y_S$, the intermediate advantage given to the first firm reduces to nothing. Assuming this is the case, we obtain two limits

$$\lim_{q \uparrow 1/2} Y_S = Y_L \quad \text{and} \quad \lim_{q \downarrow 0} Y_S = Y_F \ .$$

Therefore, our setting encompasses in a continuous manner the two usual types of games described in the previous subsection.

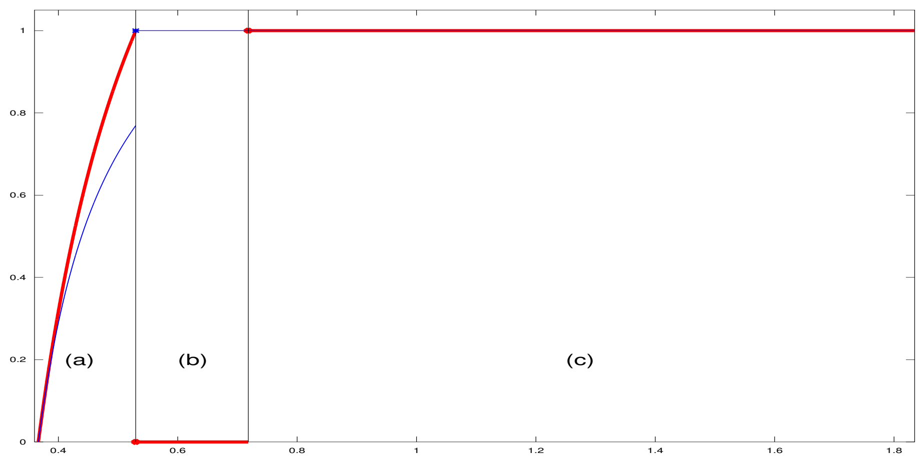

Figure 1: Values of equilibrium mixed strategies $p_1(y)$ (blue) and $p_2(y)$ (red) in the asymmetrical case $(q_1, q_2, q_S) = (0.5, 0.2, 0.3)$. Areas (a), (b) and (c) are separated by vertical lines at $Y_1 = 0.53$ and $Y_2 = 0.72$ on $[Y_L, Y_F] = [0.37, 1.83]$. Area (d) is then at the left of the graph and (e) at the right of it. Note that p_1 and p_2 are right-continuous. Same parameters as in Figure 2.

The value of being preferred. We continue the case study with $q_1 = 1 - q_2 = 1$. We intend to draw the reader's attention on the fact that the leader's role is not decided exogenously. The advantage of firm one is conditioned to the simultaneity of investment decision of firms. If firm two pre-empt firm one, this advantage is useless. For firm two also, this is a privilege in the case of simultaneous investment since it gives her the choice to postpone investment. In that case $P_1 = 1$,

and firm one has always interest to exercise the option for $y \geq Y_L$. Knowing that, firm two shall wait the trigger Y_F to exercise her option. To sum up, the equilibrium behaviour of firms is $(p_1, p_2) = (1, 0)$ on $[Y_L, Y_F)$, which corresponds to case (b) exclusively.
For curiosity, we compare here this situation to the case $q_S = 1$. We do that because [4] already compared the situation $q_S = 1$ to the *ex-ante* attribution of roles, where firm two is the follower and must wait for firm one to exercise her option. The difference of value for the leader is designed to be a *priority option*, i.e., the value of being designated leader. Here, we introduce a weaker option, which only gives the value of being preferred by the regulator. The option allows to shift (q_1, q_S) from $(0, 1)$ to $(1, 0)$. We compare in Figure 2 the two option values. Following definition (2.4), we denote $E_1^{(q_1,q_S)}(t, x)$ the expected payoff of firm one when $Q = 0$, with q_1 and q_S explicitly given. Let us define now the value of being preferred by $\pi^0(y) := E_1^{(1,0)}(y) - E_1^{(0,1)}(y)$. Then π^0 is given by

$$\pi^0(y) = (L(y) - F(y))^+ \quad \text{for all } y \geq 0.$$

For $y \geq Y_L$, $E_1^{(1,0)}(t, y) = L(y)$. By putting (2.6) in formula (2.4), we retrieve the rent equalization principle which says that $E_1^{(0,1)}(t, y) = F(y)$. If $y < Y_L$, by continuity of the demand process, firm one should exercise at time $\tau(Y_L)$ where $L(Y_L) = F(Y_L)$ to avoid pre-emption of her opponent. Her expected additional value is thus 0. This option has three interesting properties. Firstly, it gives an advantage to its owner without penalizing the other firm, who can still expect the payoff F. Secondly, the advantage given by this option to its buyer is the same if its purchase is *hidden*. Consider the incomplete information assumption where firm one knows that $(q_1, q_S) = (1, 0)$ and firm two assumes wrongly that $(q_1, q_S) = (0, 1)$. Then the behaviour of firm two will change to a mixed strategy $P_2(y)$ on $[Y_L, Y_F)$. Firm one's best behaviour is thus to invest with probability one on this interval, which is exactly what is already recommended with the option in hand in the perfect information setting. Therefore, no advantage is given by hiding the regulator's decision to the penalized firm. It is also remarkable that there is no advantage of being preferred if the follower's option value is greater or equal to the leader's value function, even with the consideration of evolution of the latter. This is due to the continuity of the state variable Y, so a question that naturally arises is whether the introduction of jumps in the state variable dynamics has a pertinent effect on the value of being preferred. We skip this one as many others to focus on the situation with possible simultaneous exercise.

Two valid equilibrium strategies. One mathematical situation is left to explore, i.e. $q_2 = 0$ and $q_1 \in (0, 1)$, but reminding the introduction of the present subsection, we think that it is also economically relevant to study the exclusion by the regulator of second firm. Imagine for example that firm one has a definitive advantage, like safety and health standards in the case of a new drug product, but that simultaneous investment is not literally forbidden and the regulator shall publicly prove his fairness if simultaneity is acknowledged. We desire to keep the clearness of our approach. We then don't study a marginal advantage value provided by a shift in the value of q_1 or q_2. We only study the equilibrium evolution when q_2 goes to 0. We simply observe from equations (3.12) and (3.16) that

$$\lim_{q_2 \downarrow 0} Y_2 = Y_F \quad \text{and} \quad Y_1^* := \lim_{q_2 \downarrow 0} Y_1 \text{ verifies } F(Y_1^*) = q_1 L(Y_1^*) + (1 - q_1) S(Y_1^*) \, .$$

The consequence is straightforward: the interval $[Y_1, Y_2)$ expands to $[Y_1^*, Y_F)$. The fact that $Y_1^* > Y_L$ for $q_1 < 1$ implies also a specific situation. Consider the equilibrium strategy $(p_1, p_2) = (P_1(y), P_2(y))$,

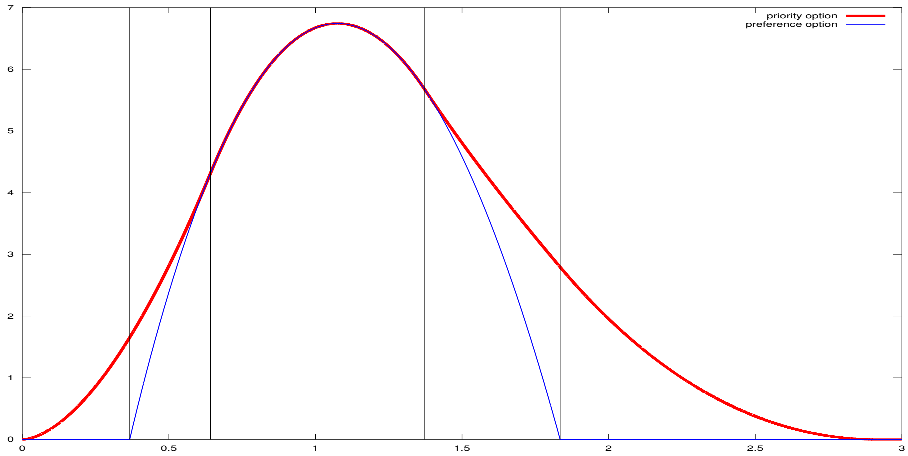

Figure 2: Priority option value (red) and Preference option value (blue) in function of y. Vertical lines at $Y_L = 0.37$, $Y_1 = 0.64$, $Y_2 = 1.37$ and $Y_F = 1.83$. Option values are equal on $[Y_1, Y_2]$. Parameters set at $(K, \nu, \eta, \mu, \sigma, r, D_1, D_2) = (10, 0.01, 0.2, 0.04, 0.3, 0.03, 1, 0.35)$.

where firm two still attempt to pre-empt the other on the interval $[Y_L, Y_1^*)$ with a positive probability, although she has clearly no advantage of a simultaneous exercise with firm one. Firm one, in that strategic situation fears the probability $(1 - q_1)$ of sharing the market, and thus uses a mixed strategy as defined in Section 2. The expected value of return is given by (3.14) for both firms. We intuit the equilibrium $(p_1, p_2) = (1, 0)$ to be more relevant as the previous one. Indeed, if firm one invests systematically on $[Y_1^*, Y_F]$, then firm two has no chance of being the leader (even by a choice of the regulator), and therefore finds satisfying the follower's position. In opposition of the trembling-hand equilibrium, this pure strategy can be well figured by being called a *steady-hand* equilibrium. In that case

$$(E_1(y), E_2(y)) = (L(y), F(y)) \text{ for } y \in [Y_L, Y_1^*) \ . \tag{3.17}$$

Comparing (3.14) to (3.17), it appears more interesting for firm one to push player two to this strategy, but indifference for firm two. There is thus a definitive advantage to play this strategy rather than the former. Notice also the connexion with the preference option we described in the last paragraph. We leaves to the reader the final choice of the most relevant equilibrium definition, if for example she has in mind the *aggressive* dimension of pre-emption by the advantaged firm, or on the contrary an hesitating behaviour.

4 Aversion for confrontation

This section can be considered as an addendum because most ideas regarding asymmetry in front of a regulator's decision has been presented in the previous section. We will essentially focus on the effect of the random decision of the regulator and the mixed strategy by means of risk-aversion of the firms. We study the combination of both effects in what we called the *aversion for confrontation*. We emphasize the intricate relation between asymmetry and risk-aversion.

Risk profile in complete market. In this section, we do not change the setting of the market, nor the project's value. We assume the market complete. However, we now endow each firm with the same utility function U designed to be strictly concave. To avoid initial wealth dependency, we focus on the CARA utility function given by

$$U(x) = -\exp(-\gamma x)$$

where $\gamma > 0$ is the risk-aversion of firms. Since the market is complete, both firms still price the leader, the follower and sharing positions with the unique risk-neutral probability $\mathbb{Q}$. For each of them, a firm receives the corresponding utility. We denote $l(y) := U(L(y))$, $f(y) := U(F(y))$ and $s(y) := U(S(y))$, and index variables with γ to make the dependence explicit. In the case where both firms simultaneously declare their desire to exercise the option, the regulator comes into play. In that case, he randomly attributes a payoff to each player, which receives a utility from it. The regulator is then defined via

$$(A_1, A_2)(y) := \begin{cases} (l,f)(y) & \text{with probability } q_1 \\ (f,l)(y) & \text{with probability } q_2 \\ (s,s)(y) & \text{with probability } q_S := 1 - q_1 - q_2 \end{cases} .$$

As the reader is now aware of, the regulator only intervenes in one outcome of the coordination game of subsection 2.2. Each firm uses a mixed strategy p_i^γ, in order to maximise the expected utility, as in (2.4):

$$E_1^\gamma(y) = (a_1^\gamma + a_s^\gamma q_1) l(y) + (a_2^\gamma + a_S^\gamma q_2) f(y) + a_s^\gamma q_S s(y) \tag{4.1}$$

We then understand that all the calculus of Section 3 hold in this setting by just changing (L, F, S) for (l, f, s). It follows that in that case

$$P_{i,\gamma}(y) = \frac{l(y) - f(y)}{q_i(l(y) - f(y)) - q_S(l(y) - s(y))} \text{ with } i \in \{1, 2\}, \tag{4.2}$$

and that, as in section 3, the behaviour of firms can be characterized by four strategic interaction types depending on the value of y on four intervals (d), (a), (b) and (c)+(e).

Influence of aversion on regions (a),(b) and (c). The question we address in this section is how risk-aversion influences the different intervals. Notice that the definition of regions (d) and (e) do not change since they are independent from risk-profile of players: prices are sufficiently explicit under the complete market hypothesis. Let's come back to equation (4.2) for the analysis of regions (a), (b) and (c). To that purpose, we first consider the case $q_S = 1$ of [4], where $P_{1,\gamma}(y) = P_{2,\gamma}(y) =: p_\gamma(y)$. Consider a fixed value $y \in (Y_L, Y_F)$ to avoid putting it in the notations. From (4.2) we get

$$p_\gamma = \frac{l - f}{l - s} = \frac{-e^{\gamma L} + e^{\gamma F}}{-e^{\gamma L} + e^{\gamma S}} = \frac{e^{\gamma(L-F)} - 1}{e^{\gamma(L-S)} - 1} .$$

Since $u(x) := -1 - U(-x) = e^{\gamma x} - 1$ is a positive strictly convex function on $(0, \infty)$ with $u(0) = 0$, we have that

$$p_\gamma = \frac{u(L - F)}{u(L - S)} < \frac{L - F}{L - S} =: p_0 \tag{4.3}$$

This expresses an aversion for confrontation. For γ going to zero, we apply l'Hôpital's rule to obtain that $\lim_{\gamma\downarrow 0} p_\gamma = p_0$. By augmenting the risk aversion γ, the probability to act in the infinite game of section 2 reduces to an asymptotic limit:

$$\lim_{\gamma\uparrow\infty} p_\gamma = \lim_{\gamma\uparrow\infty} e^{-\gamma(F-S)} = 0 \ .$$

This is true for all $y \in [Y_L, Y_F)$. It is clear from (4.3) that p_γ is monotonous for $\gamma \in (0, \infty)$, and then according to the above limit, it is convex decreasing with γ. We now take $q_2 \leq q_1 < 1 - q_2$. First, notice that Proposition 3.4 allows to retrieve the $P_{i,\gamma}$:

$$P_{i,\gamma} = \frac{1}{q_i} \frac{p_\gamma}{p_\gamma + \alpha_i} \ , \ i \in \{1, 2\}, \tag{4.4}$$

with $\alpha_i \in (0, 1)$ defined in (3.11). The $P_{i,\gamma}$ are concave non-decreasing functions of p_γ. They are then decreasing functions of γ, but we cannot state convexity or concavity of the function, see Figure 3. Now recalling that the regions are separated via conditions $P_{2,\gamma} = 1$ and $P_{1,\gamma} = 1$, we can say the following.

(a) Since $P_{2,\gamma}(y)$ is decreasing with γ, Y_1 is an increasing function of γ: the region (a) spreads on the left with γ. Adapting (3.12) to the present values, Y_1 shall verify:

$$q_1(1 - e^{\gamma(L(Y_1)-F(Y_1))}) + q_S(1 - e^{\gamma(S(Y_1)-F(Y_1))}) = 0$$

and when γ goes to ∞, we need $L(Y_1) - F(Y_1)$ to go to 0, so that Y_1 tends toward Y_F.

(b) The width of region (b) is not monotonous in γ. From numerical simulation, we observe that the region is increasing and then decreasing, converging to zero according to (a).

(c) Since $P_{1,\gamma}(y)$ is decreasing with γ, Y_2 is an increasing function of γ: the regions (c) reduces from the right with γ, until disappearance.

Highlight on region (a). With risk aversion, the region (a) takes more importance. Let us denote $(a_1^\gamma, a_2^\gamma, a_S^\gamma)$ the probabilities of outcomes with risk averse firms who uses the mixed strategy $p_i^\gamma := P_{i,\gamma}$, and (a_1^0, a_2^0, a_S^0) the probabilities of outcomes by using probability p_i of Section 3. From above $p_i^\gamma < p_i$ and since a_S^γ is increasing in both variables (p_1^γ, p_2^γ),

$$a_S^\gamma < a_S^0 \ . \tag{4.5}$$

Then we can see that

$$\frac{a_1^\gamma}{a_2^\gamma} = \frac{p_1^\gamma - p_1^\gamma p_2^\gamma}{p_2^\gamma - p_1^\gamma p_2^\gamma} = \frac{q_S - (1-q_2)p_\gamma}{q_S - (1-q_1)p_\gamma} \tag{4.6}$$

and at the limit or using (3.13),

$$\frac{a_1^0}{a_2^0} = \frac{p_1 - p_1 p_2}{p_2 - p_1 p_2} = \frac{q_S - (1-q_2)p_0}{q_S - (1-q_1)p_0} = \frac{F - S_1}{F - S_2} \ .$$

Notice that this last term is lower than 1 on region (a). Differentiating equation (4.6) in p_γ, we obtain that a_1^γ / a_2^γ is decreasing in that variable, and therefore increasing in γ. As a corollary,

$$\frac{a_1^0}{a_2^0} < \frac{a_1^\gamma}{a_2^\gamma} < \lim_{\gamma\uparrow\infty} \frac{a_1^\gamma}{a_2^\gamma} = 1 \ . \tag{4.7}$$

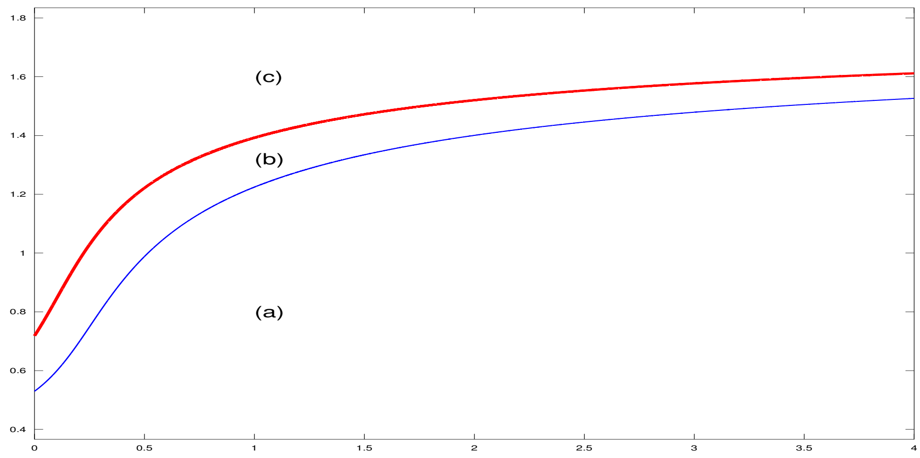

Figure 3: Values of Y_1 (blue) and Y_2 (red) as a function of risk aversion γ. Y-axis limited to $[Y_L, Y_F] = [0.37, 1.83]$. Limit values of (Y_1, Y_2) for γ going to 0 corresponds to $(0.53, 0.72)$ of Figure 1. Same parameters as previous figures.

From inequality (4.5) we shall conclude the following. Aversion for confrontation instinctively leads to a lower probability of simultaneity. This is due to the case that both firms increase their utility by reducing the risk of their decision. Consider the two reasonable directions : both firms act more steadily by increasing p_1^γ and p_2^γ, or on the contrary are more hesitant and reduce these quantities. By augmenting their probability to act, a_S^γ grows up. However, the intervention of the regulator makes leader and follower positions remaining alternatives. Therefore the risk keeps high. If both firms reduce their instantaneous probability to exercise, a_S^γ reduces and the risk reduces to the two positions of leader and follower by pre-emption. When the risk aversion augments, this is the most desirable alternative. Equation (4.7) confirms that conclusion. The lower the probability of a regulator intervention, the lower is the impact of the asymmetry. When the regulator is fair, only the simultaneous exercise probability a_S^γ is affected.

Indifference prices. How does γ impact the real outcome of the game? To compare homogeneously the expected values of options L, F and S to the expected utility provided by (4.1), we inverse the utility and compute indifference prices e_i:

$$e_{i,\gamma}(y) := U^{-1}(E_1^\gamma(y)) = -\frac{1}{\gamma} \log\left(-a_1^\gamma l(y) - a_2^\gamma f(y) - a_S^\gamma s_i(y)\right) .$$

It is clear that this computation is relevant only in the region (a). Applying Jensen's inequality, we get

$$e_{i,\gamma}(y) \geq (a_1^\gamma + a_s^\gamma q_1) L(y) + (a_2^\gamma + a_S^\gamma q_2) F(y) + a_s^\gamma q_S S(y)$$

and from the last paragraph, $a_S^\gamma < a_S^0$ and $(a_1^\gamma, a_2^\gamma) > (a_1^0, a_2^0)$ so that $e_{i,\gamma}(y) \geq E_i(y)$. This is evident from the fact that now the price shall include a risk-premium which is growing with γ. One can also compute the limit when γ goes to 0 and find out that $e_{i,\gamma}(y)$ converges to $E_i(y) = F(y)$. Here, asymmetry plays a minor role into valuation.

5 Conclusion

We have focused our attention on the classical real option game revisited with an arbitrator. Starting from the simple desire to formalize the coin toss procedure introduced in the framework forbidding simultaneous investment, we end with an intriguing economical situation, where a regulator comes into play and may prefer a firm from another in the duopoly game. The formal game theoretical procedure of [5] allows to derive mathematically all Nash equilibria. After that, using the well-acknowledged value functions in the complete market setting [4], we study several situations that can follow from the possible parametrizations of the regulator. We come to many interesting conclusions.
First, we formalize the coin toss procedure and realize that any parametrization of the form $(q_1, 1 - q_1, 0)$ with $q_1 \in (0, 1)$ conducts to the same strategic behaviour. The exclusion of sharing the projects is also a limit case of the general parametrization, and so for the situation of [4] where simultaneous investment systematically leads to sharing the project. Our framework thus encompass in a formal way the two different situations.
Secondly, we shed more light on the asymmetrical situation. We derive as the price of a financial option the value of being totally preferred by the regulator compared to a case where the regulator split the project fairly for the two firms. This comes as a real option in a pre-emptive situation, with a weaker advantage than the Stackelberg advantage studied in [4]. We also study the multiple variations of the regulator's preferences, and their impact on the strategic behaviour of firms. A complex situation emerge with three subregions, discontinuous behaviour and serious advantage for the preferred firm. The quantification of such an advantage can be precious in numerous financial situations. Indeed regulators often intervene for important projects in energy, territorial acquisition and highly sensitive products such as drugs. The behaviour of economical agents in competitive real markets can have a strategic impact on valuation, and we hope that the proposed framework inspires some consideration for regulator's influence.
Finally, we extended the setting to risk-averse firms, but still in complete market. This allows to focus risk-aversion on the coordination game and the arbitrator's intervention, which we called aversion for confrontation. If it has few relevance for the symmetric situation, we obtain interesting influence of this concept on the asymmetry and the strategic regions of coordination. The fear of the arbitrator's judgement and the uncertainty of the outcome of the coordination game refrain investment decision, and thus the advantage of one player on the other reduces. For the extreme averse situation, it appears as if players desire to agree between themselves, *with a coin toss*, who will take the role of leader. Given the insights risk-aversion brings to the model, we hope that this procedure spreads out in studies of asymmetrical real option games.

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