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Numerical methods for the quadratic hedging problem in Markov models with jumps*

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Abstract

We develop algorithms for the numerical computation of the quadratic hedging strategy in incomplete markets modeled by pure jump Markov process. Using the Hamilton-Jacobi-Bellman approach, the value function of the quadratic hedging problem can be related to a triangular system of parabolic partial integro-differential equations (PIDE), which can be shown to possess unique smooth solutions in our setting. The first equation is non-linear, but does not depend on the pay-off of the option to hedge (the pure investment problem), while the other two equations are linear. We propose convergent finite difference schemes for the numerical solution of these PIDEs and illustrate our results with an application to electricity markets, where time-inhomogeneous pure jump Markov processes appear in a natural manner.

Key words: Quadratic hedging, electricity markets, Markov jump processes, Partial integro-differential equation, Hamilton-Jacobi-Bellman equation, Hölder spaces, discretization schemes for PIDE.

1 Introduction

In an incomplete market setting, where exact replication of contingent claims is not possible, quadratic hedging is the most common approach, among both academics and practitioners. This method consists in minimizing the $\mathbb{L}^2$ distance between the hedging portfolio and the claim. Its popularity is due to the fact that the strategy is linear with respect to the claim, and is relatively easy to compute in a variety of settings.

In its most general form, the quadratic hedging problem can be formulated as follows. Consider a random variable $H \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{P})$ (which stands for the option one wants to hedge) and a set of admissible strategies, which has to be carefully specified, $\theta \in \mathcal{X}$, where $\mathcal{X}$ is a subset of adapted processes with caglad paths. The quadratic hedging problem becomes

$$\text{minimize } \mathbb{E}^{\mathbb{P}}\left[\left(x + \int_0^T \theta_t dS_t - H\right)^2\right] \text{ over } x \in \mathbb{R} \text{ and } \theta \in \mathcal{X} \tag{1.1}$$

where S is a semimartingale modeling the stock price. If (x^*, θ^*) is a minimizer, we call θ^* the optimal mean-variance hedging strategy and x^* its price. This problem has been extensively studied in the literature, starting with the seminal works of Föllmer and Sondermann (1986) and Föllmer and Schweizer (1991) and until the complete theoretical solution in the general semimartingale setting was given in Černỳ and Kallsen (2007). The case when S is a $\mathbb{P}$-square integrable martingale is particularly simple and can be solved using the well known Galtchouk-Kunita-Watanabe decomposition. The general case is much more involved, and has only been solved in Černỳ and Kallsen

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(2007) by means of introducing a specific non martingale change of measure (the opportunity neutral measure).

The problem of numerical computation of the hedging strategy is an important issue in its own right, since various objects appearing in the theoretical solution (opportunity neutral measure, Galtchouk-Kunita-Watanabe decomposition, Föllmer-Schweizer decomposition) are often not known in explicit form. When the underlying asset is modeled by a Lévy process, a complete semi-explicit solution was obtained in Hubalek et al. (2006) using Fourier methods. Their approach was extended to additive processes in Goutte et al. (2011). Laurent and Pham (1999) and Heath et al. (2001) characterize the optimal strategy via an HJB equation in continuous Markovian stochastic volatility models while Černỳ and Kallsen (2008) and Kallsen and Vierthauer (2009) treat affine stochastic volatility models using Fourier methods.

In this paper, we propose algorithms for the numerical computation of the quadratic hedging strategy in general Markovian models with jumps. We first review the HJB characterization of the value function, obtained in De Franco (2012). We only give a brief review, referring the readers to De Franco (2012) for full details and proofs because in this paper, we are interested in the numerical schemes for the computation of the hedging strategies and in applications to electricity markets. The value function of the quadratic hedging problem can be related to a triangular system of parabolic PIDEs, which can be shown to possess unique smooth solutions in our setting. The first equation is a non-linear PIDE of HJB type, but does not depend on the pay-off of the option to hedge (the pure investment problem), while the other two equations are linear. We next propose two finite difference schemes for the numerical solution of the linear and the nonlinear PIDEs, which are shown to converge to the unique solutions of the respective equations. For the numerical schemes, we concentrate on the infinite variation case, which is more relevant in applications.

Our main motivation comes from hedging problems in electricity markets. These markets are structurally incomplete and often illiquid, owing to a relatively small number of market participants and the particular nature of electricity, which is a non-storable commodity. As pointed out in Geman and Roncoroni (2006) and Meyer-Brandis and Tankov (2008), due to these features, the electricity prices exhibit highly non-Gaussian behavior with jumps and spikes (upward movements followed by quick return to the initial level) and several authors have therefore suggested to model electricity prices by pure jump processes (Benth et al., 2007; Deng and Jiang, 2005).

On the other hand, since the spot electricity is non storable, the main hedging instruments in electricity markets are futures. A typical future contract with maturity T and duration d guarantees to its holder continuous delivery of electricity during the period $[T, T + d]$. Maturities, durations and amounts of electricity are standardized for listed contracts. This continuous delivery feature implies that even if the spot electricity follows a simple model, such as the exponential of a Lévy process, the price of the future contract will be a general Markov process with jumps, non-homogeneous in time and space.[1] Therefore, Fourier methods such as the ones developed in Hubalek et al. (2006) and Goutte et al. (2011) cannot be applied in this setting. For this reason, in Section 5, after introducing a model for the futures prices, where the spot price is described by the exponential of the Normal Inverse Gaussian (NIG) process, we derive the associated HJB equations and use the finite difference schemes to compute the hedging strategies and analyze their behavior. The numerical results illustrate the performance of our method and show in particular that the computation of the hedging strategies under the true historical probability (as opposed to the martingale probability, which does not require solving non-linear HJB equations) leads to a considerable improvement in the efficiency of the hedge.

The paper is structured as follows. After introducing the model and the quadratic hedging problem in Section 2, we review the HJB characterization of the solution and the regularity results in Section 3. The finite difference schemes for the solution of the HJB equations, which are the main results of this paper, are presented in Section 4. In Section 5 these results are applied to a concrete hedging problem in electricity markets. The proofs of the convergence results are postponed to Section 6.

[1] If a single delivery length is fixed, it is possible to model the future price directly as the exponential of a process with independent increments, as in Goutte et al. (2011). However this approach does not allow to treat problems involving futures of different durations, say, hedging a product with specified duration with the listed contracts.

2 The model and the quadratic hedging problem

Let J be a Poisson random measure on $[0,+\infty) \times \mathbb{R}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $\mathcal{F}_t$ being the natural filtration of J. We suppose that $\mathcal{F}_0$ contains the null sets and also $\mathcal{F} = \mathcal{F}_T$ where $T > 0$ is given. Let also $dt \times \nu(dy)$ be the intensity measure of J where ν satisfies the standard integrability condition $\int_{\mathbb{R}} \left(1 \wedge |y|^2\right) \nu(dy) < \infty$. We denote

$$\tilde{J}(dt \times dy) := J(dt \times dy) - dt \times \nu(dy)$$

the compensated martingale jump measure. On this probability space we introduce the family of $\mathbb{R}$-valued Markov pure jump process as the solution of the following:

$$dZ_r^{t,z} := \mu\left(r, Z_r^{t,z}\right) dr + \int_{\mathbb{R}} \gamma\left(r, Z_{r-}^{t,z}, y\right) \tilde{J}\left(dr \times dy\right), \; Z_t^{t,z} = z \tag{2.1}$$

for $t \in [0,T)$ and $z \in \mathbb{R}$. The stock price process S is given by $S_u^{t,z} = \exp(Z_u^{t,z})$. We make the following assumptions:

Assumption 2.1.
[C]- The coefficients.

i). There exists $\overline{\mu} \geq 0$ such that $\|\mu\|_\infty \leq \overline{\mu}$.

ii). For all $t \in [0,T]$ and $y \in \mathbb{R}$ the functions $z \to \mu(t,z)$ and $z \to \gamma(t,z,y)$ belong to $\mathcal{C}^1(\mathbb{R},\mathbb{R})$.

iii). There exist $K_{lip}^c \geq 0$, $K_{lip}^d \geq 0$ and a positive locally bounded function $\rho : \mathbb{R} \to \mathbb{R}^+$ such that for all $y \in \mathbb{R}$ and all $t \in [0,T]$ we have

$$|\mu(t,z) - \mu(t,z')| \leq K_{lip}^c |z - z'|$$
$$|\gamma(t,z,y) - \gamma(t,z',y)| \leq K_{lip}^d \rho(y) |z - z'|$$

[L]- The Lévy measure. *The Lévy measures $\nu(dy)$ verifies $\nu(dy) = \nu(y)dy$ where $\nu(y) := g(y)|y|^{-(1+\alpha)}$ for some $\alpha \in (1,2)$, where g is a measurable function verifying $0 < m_g \leq g(y) \leq M_g$, $\forall y \in \mathbb{R}$, for some positive constants m_g, M_g. We also assume that*

$$\lim_{y \to 0^-} g(y) = g(0^-) \quad \text{and} \quad \lim_{y \to 0^+} g(y) = g(0^+) \quad \text{with } g(0^+), g(0^-) > 0.$$

[I]- Integrability conditions. *The function*

$$\tau(y) := \max\left(\sup_{t,z} \left(|\gamma(t,z,y)|, e^{|\gamma(t,z,y)|} - 1 \right), \; \rho(y) \right)$$

verifies, for some $y_0 \in (0,1)$

$$\sup_{0 < |y| \leq y_0} \frac{\tau(y)}{|y|} \leq M \qquad \text{and} \qquad \tau \in \mathbb{L}^4(\{|y| \geq y_0\}, \nu(dy))$$

[ND]- No degeneracy. *The function*

$$\Gamma(y) := \inf_{t,z} \left(e^{\gamma(t,z,y)} - 1 \right)^2 \qquad \text{verifies} \qquad |\Gamma| := \int_{\mathbb{R}} \Gamma^2(y) \nu(dy) > 0$$

[RG]- Regularity of the γ function.

i). For any t, z the mapping $y \to \gamma(t,z,y)$ is twice continuously differentiable around zero and there exists two positive constants m_1, m_2 such that

$$0 < m_1 \leq \inf_{t,z,|y| \leq y_0} |\gamma_y(t,z,y)| \text{and} \qquad \sup_{t,z,|y| \leq y_0} |\gamma_{yy}(t,z,y)| \leq m_2$$

In particular γ is invertible in $(-y_0, y_0)$: we call $\gamma^{-1}(t,z,y)$ its inverse.

ii). For all $t, z \in [0, T) \times \mathbb{R}$ $\gamma_y(t, z, 0) = 1$

iii). The function γ_y *is Lipschitz continuous in the variable z:*

$$\sup_{t,z,|y|\leq y_0} |\gamma_y(t, z+h, y) - \gamma_y(t, z, y)| \leq m_2|h|$$

We denote $K_{max} := \max(K^c_{lip}, K^d_{lip})$,

$$\tilde{\mu} := \mu + \int_{\mathbb{R}} (e^{\gamma} - 1 - \gamma)\nu(dy) \text{ and } \|\tilde{\mu}\| := \sup_{t,z} |\tilde{\mu}(t, z)| \tag{2.2}$$

In the rest of the paper we denote $\|\tau\|_{1,\nu} := \int_{|y|\geq 1} \tau(y)\nu(dy)$ *whereas* $\|\tau\|^2_{2,\nu} := \int_{\mathbb{R}} \tau^2(y)\nu(dy)$.

It is well known that there exists a unique semimartingale Z which solves the SDE defined above (Jacod and Shiryaev, 2003).

On the Assumption $[RG] - ii)$

Among the assumptions listed above, undoubtedly $[RG] - ii)$ seems to be the most restrictive one: if for example the jump function is of the form $\gamma(t, z, y) = \hat{\gamma}(t, z)y$ then the only possible choice would be $\hat{\gamma}(t, z) = 1$ for all t, z. In this paragraph we prove that this assumption could be relaxed by making a special change of variable. More precisely, we look at some process L: $L_t = \phi(t, Z_t)$ for some smooth function ϕ such that

$$dL^{t,l}_s := \mu^L(s, L^{t,l}_s)ds + \int \gamma^L(s, L^{t,l}_{s-}, y)\bar{J}(dyds) \tag{2.3}$$

with μ^L and γ^L verifying Assumptions 2.1, with especially $\partial_y \gamma^L(t, l, 0) = 1$ for all t, l. If for such ϕ, the function $z \to \phi(t, z)$ is invertible and smooth enough to apply Itô's formula then

$$\gamma^L(t, l, y) := \phi(t, \phi^{-1}(t, l) + \gamma(t, \phi^{-1}(t, l), y)) - l \tag{2.4}$$

$$\begin{aligned} \mu^L(t, l) :=& \frac{\partial \phi}{\partial t}(t, \phi^{-1}(t, l)) + \mu(t, \phi^{-1}(t, l))\frac{\partial \phi}{\partial z}(t, \phi^{-1}(t, l)) \\ &+ \int_{|y|\leq 1} \left(\gamma^L(t, l, y) - \gamma(t, \phi^{-1}(t, l), y)\frac{\partial \phi}{\partial z}(t, \phi^{-1}(t, l))\right)\nu(dy) \end{aligned} \tag{2.5}$$

In particular one has

$$\gamma^L_y(t, l, 0) = \frac{\partial \phi}{\partial z}(t, \phi^{-1}(t, l)))\gamma_y(t, \phi^{-1}(t, l), 0)$$

If we select for example

$$\phi(t, z) := \int_0^z \frac{ds}{\gamma_y(t, s, 0)} \tag{2.6}$$

then trivially $\gamma^L_y(t, l, 0) = 1$ for all t, l. The following Lemma shows that this choice guarantees that the coefficients μ^L and γ^L verify Assumptions 2.1

Lemma 2.2. *Assume that there exist some positive constants* $0 < m_1, m_2$ *such that*

i). For all $t, z \in [0, T] \times \mathbb{R}$ *the mapping* $y \to \gamma(t, z, y)$ *is differentiable at* $y = 0$ *and*

$$0 < m_1 \leq |\gamma_y(t, z, 0)| \leq m_2 \text{ for all } t, z \in [0, T] \times \mathbb{R}$$

ii). The function $(t, z) \to \gamma_y(t, z, 0)$ *is differentiable and*

$$\left|\frac{d}{dt}\gamma_y(t, z, 0)\right| + \left|\frac{d}{dz}\gamma_y(t, z, 0)\right| \leq m_2 \text{ for all } t, z \in [0, T] \times \mathbb{R}$$

iii). The function $z \to \frac{d}{dt}\gamma_y(t,z,0)$ *is Lipschitz continuous:*

$$\left|\frac{d}{dt}\gamma_y(t,z,0) - \frac{d}{dt}\gamma_y(t,z',0)\right| \leq m_2|z-z'| \text{ for all } t \in [0,T],\ z,z' \in \mathbb{R}$$

Then the functions μ^L *and* γ^L *defined in* (2.4)–(2.5) *with the choice of* ϕ *given by* (2.6) *verify Assumptions 2.1.*

The proof of this Lemma can be found in De Franco (2012), Lemma 7.16.
The message coming from this Lemma is that, up to some regularity of the function γ_y at $y = 0$, it is possible to remove the assumption $[RG]-ii)$: we could work with the process L instead of Z and derive all the results for L. By applying ϕ we will obtain their analogous for the process Z. We refer to Chapter 7 in De Franco (2012) for further details.

Nevertheless, we prefer here to work with assumption $[RG]-ii)$ because it will make all computations easier to handle.

Admissible strategies and the value functions

To describe the set of admissible strategies in the quadratic hedging problem we follow the ideas developed in Černý and Kallsen (2007): we first introduce the sets of simple strategies:

$$\begin{aligned}
&\mathcal{D} := \left\{\theta := \sum_i Y_i \mathbf{1}_{[\varsigma_i,\varsigma_{i+1})},\ Y_i \in \mathbb{L}^\infty(\mathcal{F}_{\varsigma_i}),\ \varsigma_i \leq \varsigma_{i+1} \text{ stopping times}\right\} \\
&\mathcal{D}_t := \left\{\theta \mathbf{1}_{(t,T]} \mid \theta \in \mathcal{D}\right\}
\end{aligned} \tag{2.7}$$

The set of admissible strategies is a subset of the $\mathbb{L}^2(\mathbb{P})$-closure of $\mathcal{D}$:

$$\mathcal{X} := \left\{\theta \in \bar{\mathcal{D}} : \int \theta dS \in \mathbb{L}^2(\mathbb{P})\right\} \tag{2.8}$$

We define the wealth process for all t, z, x simply by

$$dX_r^{t,z,x,\theta} := \theta_{r-} dS_r^{t,z}, \quad X_t^{t,z,x,\theta} := x \tag{2.9}$$

where θ represents the number of shares in the portfolio at time t. The set of admissible controls is then given by

$$\mathcal{X}(t,z,x) := \left\{\theta \mathbf{1}_{(t,T]} \mid \theta \in \mathcal{X} \quad x + \int_t^\cdot \theta_{r-} dS_r^{t,z} \in \mathbb{L}^2(\mathbb{P})\right\} \tag{2.10}$$

Consider a European option of the form $f(Z_T)$ where f is, for the moment, a bounded and measurable function. The quadratic hedging problem can be formulated as follows:

$$\begin{aligned}
\mathbf{QH}: \qquad & \text{minimize } \mathbb{E}^{\mathbb{P}}\left[\left(f\left(Z_T^{0,z}\right) - X_T^{0,z,x,\theta}\right)^2\right] \\
& \text{over } \theta \in \mathcal{X}(0,z,x)
\end{aligned}$$

The dynamic version of **QH** gives us the value function of the problem:

$$\begin{aligned}
&v^f(t,z,x) := \inf_{\theta \in \mathcal{X}(t,z,x)} \mathbb{E}^{\mathbb{P}}\left[\left(f\left(Z_T^{t,z}\right) - X_T^{t,z,x,\theta}\right)^2\right] \\
&v^f(T,z,x) = (f(z) - x)^2
\end{aligned} \tag{2.11}$$

As remarked by several authors, the function v_f has the following structure

$$v_f(t,z,x) = a(t,z)x^2 + b(t,z)x + c(t,z) \tag{2.12}$$

In particular, by taking $f = 0$, one has

$$v_0(t,z,x) := x^2 \inf_{\theta \in \mathcal{X}(t,z,x)} \mathbb{E}\left[\left(1 + \int_t^T \theta_{r-} dS_r^{t,z}\right)^2\right] \tag{2.13}$$

because the set $\mathcal{X}(t,z,x)$ is a cone. Consequently

$$a(t,z) := \inf_{\theta \in \mathcal{X}(t,z,1)} \mathbb{E}\left[\left(1 + \int_t^T \theta_{r-} dS_r^{t,z}\right)^2\right] \tag{2.14}$$

This problem is known in the literature as the *pure investment problem.* The dual formulation of this problem relates the function a to the so called variance optimal martingale measure (Černỳ and Kallsen, 2007). We recall here some fundamental properties on the function a, whose proof can be found in Chapter 5 of De Franco (2012).

Theorem 2.3. *Under Assumptions 2.1-*[$\mathbf{C}, \mathbf{I}, \mathbf{ND}$] *the function a verifies*

$$e^{-C(T-t)} \leq a(t,z) \leq 1 \qquad \text{where} \qquad C := \frac{\|\tilde{\mu}\|^2}{|\Gamma|}$$

Furthermore, there exists $T^* > 0$ and $K_{lip}^a \geq 0$ such that if $T < T^*$ then

$$|a(t,z') - a(t,z)| \leq K_{lip}^a |z - z'|$$

for all $t \in [0,T]$ and $z, z' \in \mathbb{R}$. T^* depends on $\overline{\mu}$, τ, C and K_{max} defined in Assumptions 2.1. Moreover $T^* \to +\infty$ when $K_{max} \to 0$ and the other constants remain fixed.

Remark that these results hold true without assuming any particular structure of the Lévy measure $\nu(dy)$. The next goal is to characterize the functions a, b and c as the solutions of certain PIDEs.

3 HJB formulation and main regularity results

Remarks on notation For a function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ we denote $\|f\|_\infty := \sup_{t \leq T, x \in \mathbb{R}} |f(t,x)|$. M, M_1, M_2, $\cdots$ denote positive constants which may change from line to line. For a function φ defined on $[0,T] \times \mathbb{R}$ and $k \in \mathbb{N}$ we denote $D^k \varphi := \partial^k \varphi / \partial^k x$ whereas $\partial_t \varphi$ denotes the derivative in the time variable. We adopt the following convention: for any $l \in \mathbb{R}^+$

$$l = \lfloor l \rfloor + \{l\}^-, \text{ where } \{l\}^- \in [0,1)$$
$$l = \lceil l \rceil + \{l\}^+, \text{ where } \{l\}^+ \in (0,1]$$

Let us first introduce the functional spaces in which we will work: for $\beta \in (0,1]$ we define

$$\langle \psi \rangle^{(\beta)} := \sup_{x, 0 < |h| \leq 1} \frac{|\psi(x+h) - \psi(x)|}{|h|^\beta}, \qquad \langle \varphi \rangle_{Q_T}^{(\beta)} := \sup_{t, x, 0 < |h| \leq 1} \frac{|\varphi(t,x+h) - \varphi(t,x)|}{|h|^\beta}$$

The elliptic Hölder space of order l, $H^l(\mathbb{R}^n)$, is defined as the space of continuously differentiable functions ψ for all order $j \leq \lceil l \rceil$ with finite norm

$$\|\psi\|_l := \sum_{j=0}^{\lceil l \rceil} \sum_{(j)} \left\| D_x^j \psi \right\|_\infty + \sum_{(\lceil l \rceil)} \langle D_x^{\lceil l \rceil} \psi \rangle^{(\{l\}^+)} \tag{3.1}$$

where $\sum_{(j)}$ represents the summation over all possible derivative of order j. The parabolic Hölder space $H^l([0,T] \times \mathbb{R}^n)$ space is defined as the set of measurable functions $\varphi : [0,T] \to H^l(\mathbb{R}^n)$ with finite norm

$$\|\varphi\|_l := \sum_{j=0}^{\lceil l \rceil} \sum_{(j)} \left\| D_x^j \varphi \right\|_\infty + \sum_{(\lceil l \rceil)} \langle D_x^{\lceil l \rceil} \varphi \rangle_{Q_T}^{(\{l\}^+)} \tag{3.2}$$

The spaces defined above are all Banach spaces equipped with their respective norms. For a complete description see for example Chapter I in Adams and Fournier (2009).

In the spirit of HJB approach we now introduce the operators associated to the process Z:

Definition 3.1. *For a real valued function $\varphi \in H^{\alpha+\delta}([0,T]\times\mathbb{R})$, $\delta>0$, we define the following linear operators*

$$\begin{aligned}
\mathcal{A}\varphi(t,z) &:= -\mu\frac{\partial\varphi}{\partial z}(t,z)\\
\mathcal{B}_t\varphi(u,z) &:= \int_{\mathbb{R}}\left(\varphi(t,z+\gamma(t,z,y))-\varphi(t,z)-\gamma(t,z,y)\frac{\partial\varphi}{\partial z}(t,z)\right)\nu(dy)\\
\mathcal{Q}\varphi(t,z) &:= \tilde{\mu}\varphi(t,z)+\int_{\mathbb{R}}(e^{\gamma}-1)\left(\varphi(t,z+\gamma(t,z,y))-\varphi(t,z)\right)\nu(dy)\\
\mathcal{G}\varphi(t,z) &:= \int_{\mathbb{R}}(e^{\gamma}-1)^2\varphi(t,z+\gamma(t,z,y))\nu(dy)
\end{aligned}$$

where μ stands for $\mu(t,z)$ and so on. In addition, $\mathcal{H}$ denotes the nonlinear operator

$$\mathcal{H}_t[\varphi](z) := \inf_{|\pi|\le\bar{\Pi}}\left[2\pi\mathcal{Q}_t\varphi(t,z)+\pi^2\int_{\mathbb{R}}(e^{\gamma(t,z,y)}-1)^2\varphi(t,z+\gamma(t,z,y))\nu(dy)\right]$$

where

$$\bar{\Pi} := \frac{e^{CT}}{|\Gamma|}\max\left(\|\tilde{\mu}\|_{\infty}, 2\left(\|\tau\|^4_{4,\nu}+\|\tau\|^2_{2,\nu}\right)\right)\left(1+K^a_{lip}\right). \tag{3.3}$$

The main result concerning the functions a is:

Theorem 3.2. *Let Assumptions 2.1 hold true and consider $T<T^*$ as in Theorem 2.3. The function a is the unique solution of*

$$0=-\frac{\partial a}{\partial t}+\mathcal{A}_t a-\mathcal{B}_t-\mathcal{H}_t[a], \qquad a(T,z)=1 \tag{3.4}$$

in the Hölder space $H^{\alpha+\delta}([0,T]\times\mathbb{R})$ for $0<\delta<\alpha-1$. The function $t\mapsto a(t,z)$ is also differentiable on $(0,T)$. The optimal strategy for the stochastic control problem (2.13) is

$$\theta^*_t = e^{-Z_{t-}}\pi^*(t,Z_{t-})X^{\theta^*}_{t-}, \qquad X^{\theta^*}_t := x+\int_0^t\theta^*_{r-}dS_r$$

where

$$\pi^*(t,z) := -\frac{\mathcal{Q}a(t,z)}{\mathcal{G}a(t,z)} \tag{3.5}$$

□

For the general value function v^f we have

Theorem 3.3. *Let $T<T^*$ as in Theorem 2.3. Let also Assumptions 2.1 hold true and $f\in H^{\alpha+\delta}(\mathbb{R})$ for some $0<\delta<\alpha-1$. The function v^f in (2.11) admits the decomposition*

$$v^f(t,z,x)=a(t,z)x^2+b(t,z)x+c(t,z)$$

where a is defined in (2.14), so it does not depend on f, and it is the unique solution in $H^{\alpha+\delta}([0,T]\times\mathbb{R})$ of 3.4, whereas b and c are the unique solutions of the following linear parabolic PIDEs

$$0=-\frac{\partial b}{\partial t}+\mathcal{A}b-\mathcal{B}b-\pi^*\mathcal{Q}b, \qquad b(T,.)=-2f; \tag{3.6}$$

$$0=-\frac{\partial c}{\partial t}+\mathcal{A}c-\mathcal{B}c+\frac{1}{4}\frac{(\mathcal{Q}b)^2}{\mathcal{G}a}, \qquad c(T,.)=f^2 \tag{3.7}$$

in the Hölder space $H^{\alpha+\delta}([0,T]\times\mathbb{R})$, where π^* is defined in (3.5)*. The functions $t\mapsto a(t,.),b(t,.),c(t,.)$ are also differentiable on $(0,T)$.*
Furthermore the optimal policy in the control problem (2.11) *is given by*

$$\theta^*_t := e^{-Z_{t-}}\left(\pi^*(t,Z_{t-})X^{\theta^*}_{t-} - \frac{1}{2}\frac{\mathcal{Q}b(t,Z_{t-})}{\mathcal{G}a(t,Z_{t-})}\right),\; X^{\theta^*}_t := x + \int_0^t \theta^*_{r-}dS_r \tag{3.8}$$

□

The proof of these results can be found in Chapter 7 of De Franco (2012). From the decomposition (2.12) we also obtain the optimal price in (2.11):

$$x^*(f) := \arg\inf_{x\in\mathbb{R}} v^f(t,z,x) = -\frac{b^f(t,z)}{2a(t,z)} \tag{3.9}$$

which is a linear function of the payoff f since b^f is.

Non smooth payoff Theorem 3.3 allows us to characterize the value function v^f when the payoff function f is sufficiently smooth, i.e. $f\in H^{\alpha+\delta}(\mathbb{R})$. However, in most cases of interest (for example put options, straddles or bear spreads) this function is not even continuously differentiable. The following lemma proves the stability of the optimal price $x^*(f)$ and the optimal hedging strategy under small perturbations of the function f:

Lemma 3.4. *Let f_1,f_2 be two measurable functions with $f_i(Z^{t,z}_T)\in\mathbb{L}^2(\mathbb{P})$ for all t,z, $i=1,2$. Then for any $t<T$ and $z\in\mathbb{R}$*

$$|x^*(f_1)(t,z)-x^*(f_2)(t,z)| \le a(t,z)^{-1/2}\left\|(f_1-f_2)(Z^{t,z}_T)\right\|_{\mathbb{L}^2(\mathbb{P})}$$

$$\left|\left(v^{f_1}-v^{f_2}\right)(t,z,x)\right| \le 2\left(x+\left\|(f_1+f_2)(Z^{t,z}_T)\right\|_{\mathbb{L}^2(\mathbb{P})}\right)\left\|(f_1-f_2)(Z^{t,z}_T)\right\|_{\mathbb{L}^2(\mathbb{P})}$$

Fix now (t,z,x) and let f_n such that $\left\|(f_n-f)(Z^{t,z}_T)\right\|_2\to 0$, $n\to\infty$. If θ^n is the optimal control in the problem (2.11) *with payoff function f_n then, for all $\varepsilon>0$, there exists some $N>0$ such that for any $n\ge N$ one has*

$$\left|v^f(t,z,x)-\mathbb{E}^{\mathbb{P}}\left[\left(f(Z^{t,z}_T)-x-\int_t^T\theta^n_{r-}dS^{t,z}_r\right)^2\right]\right|\le\varepsilon$$

The proof of this Lemma can be found in Chapter 5 of De Franco (2012). One can thus approximate a non-smooth payoff function f with smooth functions f_n, controlling the error on the value function and the cost of the hedging strategy with $\|f-f_n\|_2$. Furthermore the corresponding strategies (θ_n) become ε-optimal for the pay-off f starting from sufficiently large n.

4 Numerical solution schemes

We now present a numerical scheme to solve the PIDE introduced in Section 3 when the Lévy measure ν verifies the Assumption 2.1-[**L**]. From (3.8) and (3.9) we remark that in order to solve the problem (2.11), i.e. to find the optimal strategy θ^* and the optimal price x^*, we only need to compute the functions a and b, solutions, respectively, of PIDEs (3.4) and (3.6). We first focus on the PIDE (3.4), which does not depend on the particular choice of the option one wants to hedge and propose a monotone, stable and consistent scheme. We then propose a stable and consistent scheme to solve the linear equation for b.

First of all, let us reverse the time by substituting $t\mapsto T-t$ in the coefficients of the PIDEs (3.4) and (3.6) and, with an abuse of notation, change the notation into $\mu(t,.,.)\mapsto\mu(T-t,.,.)$, $\gamma(t,z,y)\mapsto\gamma(T-t,z,y)$. The PIDE will be solved on the truncated domain $[0,T]\times[-\underline{Z},\underline{Z}]$. Due to the presence of the integral terms, the boundary conditions must be imposed not only at the boundary of the domain, but also outside, on the set $[0,T]\times[-\hat{Z},-\underline{Z}]\cup[\underline{Z},\hat{Z}]$, where $\underline{Z}<\hat{Z}$. The integrals appearing in the coefficients and the operators of the above PIDEs will be truncated at the value $\hat{Y}$. $\hat{Y}$ and $\underline{Z}$ are chosen such that $\hat{Y}$ is "large enough" and

$$\underline{Z}+\max_{z\in\mathbb{R},y\in[-\hat{Y},\hat{Y}]}\gamma(t,z,y)\le\hat{Z},\quad -\underline{Z}+\min_{z\in\mathbb{R},y\in[-\hat{Y},\hat{Y}]}\gamma(t,z,y)\ge-\hat{Z}\text{ for all }t\in[0,T]$$

If we introduce

$$\begin{aligned}
\mathcal{B}^{tr}\varphi(t,z) &:= \int_{-\hat{Y}}^{\hat{Y}}\left(\varphi(t,z+\gamma(t,z,y))-\varphi(t,z,y)-\gamma(t,z,y)\frac{\partial\varphi}{\partial z}(t,z)\right)\nu(dy)\\
\mathcal{Q}^{tr}\varphi(t,z) &:= \tilde{\mu}(t,z)\varphi(t,z)+\int_{-\hat{Y}}^{\hat{Y}}\left(e^{\gamma(t,z,y)}-1\right)\left(\varphi(t,z+\gamma(t,z,y))-\varphi(t,z)\right)\nu(dy)\\
\mathcal{G}^{tr}\varphi(t,z) &:= \int_{-\hat{Y}}^{\hat{Y}}\left(e^{\gamma(t,z,y)}-1\right)^2\varphi(t,z+\gamma(t,z,y)\nu(dy)\\
\pi^{tr}[\varphi](t,z) &:= -\,\mathcal{Q}^{tr}\varphi(t,z)\left(\mathcal{G}^{tr}\varphi(t,z)\right)^{-1}
\end{aligned}$$

then, after truncating, the PIDEs for a and b take the following form:

$$\begin{cases}
0 = \dfrac{\partial a}{\partial t}+\mathcal{A}a-\mathcal{B}^{tr}a+\sup_{|\pi|\le\bar{\Pi}}\left\{-2\pi\mathcal{Q}^{tr}a-\pi^2\mathcal{G}^{tr}a\right\}\\
\qquad\text{for }(t,z)\in(0,T]\times[-\underline{Z},\underline{Z}]\\
a(0,z) = 1,\ z\in[-\underline{Z},\underline{Z}]\\
a(t,z) = 1,\ (t,z)\in[0,T]\times[-\hat{Z},-\underline{Z}]\cup[\underline{Z},\hat{Z}]
\end{cases}\tag{4.1}$$

and, after transforming $\tilde{b}(t,z):=b(t,z)e^{-\eta t}$,

$$\begin{cases}
0 = \dfrac{\partial\tilde{b}}{\partial t}-\mu\dfrac{\partial\tilde{b}}{\partial z}-\mathcal{B}^{tr}\tilde{b}+\eta\tilde{b}-\pi^{tr}[a]\mathcal{Q}^{tr}b\\
\qquad\text{for }(t,z)\in(0,T]\times[-\underline{Z},\underline{Z}]\\
\tilde{b}(0,z) = -2f(z),\ z\in[-\underline{Z},\underline{Z}]\\
\tilde{b}(t,z) = -2f(z)a(t,z)e^{-\eta t},\ (t,z)\in[0,T]\times[-\hat{Z},-\underline{Z}]\cup[\underline{Z},\hat{Z}]
\end{cases}\tag{4.2}$$

Remark 4.1. *The effect of truncating the integral in a PIDE has been studied by Jakobsen and Karlsen (2005) (See theorem 6.1 for error estimations for source problems).*

Remark 4.2. *Taking $a=1$ as boundary condition can be justified by the fact that if S is a martingale, this is indeed the exact solution, and in the non-martingale case, the effect of drift should not be too strong. Alternatively, one could approximate the process Z with a Lévy process and use the resulting explicit solution.*

Remark 4.3. *The choice of the boundary condition for $\tilde{b}$ outside the domain is justified as follows. The value $-b(t,z)/2a(t,z)$ (the minimizer of the value function, (3.9)) can be interpreted as the cost of hedging the payoff f, that is, the wealth at time t which leads to the minimal hedging error at maturity. In the regions far from the money (and under the assumption of zero interest rate), the cost of hedging can be approximated by the option's payoff, whence the boundary condition for b.*

4.1 Numerical scheme for the function a

In order to solve PIDE (4.1) numerically, we adapt the methodology proposed by Forsyth et al. (2007) to our case of time and space dependent jumps. In addition to Assumptions 2.1, we impose the following technical conditions in this section:

- The Lévy density ν is twice continuously differentiable with bounded derivatives outside any neighborhood of zero;
- The function g has bounded derivatives near zero;
- Assumption 2.1-[**RG**] $-$ **i**) holds true with $y_0 = +\infty$;
- The function $\gamma(t, z, y)$ is 3 times continuously differentiable with respect to y with bounded derivatives;
- The data of the problem are such that the derivatives up to order 4 of the functions a and b with respect to z are bounded.

We calculate the values of a on a regular grid $z_j = j\Delta z$ for $j \in (-N, N)$. Introduce the integers $j_{-\hat{Z}}$ and $j_{\hat{Z}}$ such that $-\hat{Z} < z_j < \hat{Z}$ if and only if $j_{-\hat{Z}} < j < j_{\hat{Z}}$.

We now detail the computation of the integral terms of the integro-differential operators in (4.1). In order to avoid interpolating the values of a, we use a space and time dependent grid for discretizing the Lévy density. We select the discretization point $y_i(t, z)$, which corresponds to the center of i-th discretization interval, as the unique solution of the equation $\gamma(t, z, y_i(t, z)) = i\Delta z$. The boundaries of the discretization intervals will then correspond to half-integer values of i. Although these discretization points depend on t and z, we will sometimes omit this dependence to simplify notation.

To treat the singularity of the Lévy density at zero, for $k \geq 1$ we divide the segment $[-\hat{Y}, \hat{Y}]$ into three disjoint regions

$$\begin{aligned}
\hat{\Omega}_0(t,z) &= \left\{ y | y_{-k-\frac{1}{2}}(t,z) \leq y \leq y_{k+\frac{1}{2}}(t,z) \right\}, \\
\hat{\Omega}_1(t,z) &= \left\{ y | y_{k+\frac{1}{2}}(t,z) < y < 1 \text{ or } -1 < y < y_{-k-\frac{1}{2}}(t,z) \right\}, \\
\hat{\Omega}_2(t,z) &= \left\{ y | 1 \leq |y| \leq \hat{Y} \right\},
\end{aligned}$$

Set $\Delta y_i(t, z) = y_{i+\frac{1}{2}}(t, z) - y_{i-\frac{1}{2}}(t, z)$ and introduce the weights :

$$\omega(t, z, y_i) = \begin{cases} \dfrac{1}{y_i^2(t,z)} \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} y^2 \nu(y) dy & \text{if } |i| > k \text{ and } y_{i-1/2}, y_{i+1/2} \in \hat{\Omega}_1 \\ 0 & \text{if } |i| \leq k \\ \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} \nu(y) dy & \text{otherwise} \end{cases}$$

The integrals $\omega(t, z, y_i)$ are calculated numerically with the trapezoidal rule over 5 points, e.g., in the third case above we have

$$\hat{\omega}(t, z, y_i(z)) = \frac{\Delta y_i}{4} \left(\frac{1}{2}\nu(y_i - \frac{\Delta y_i}{2}) + \nu(y_i - \frac{\Delta y_i}{4}) + \nu(y_i) + \nu(y_i + \frac{\Delta y_i}{4}) + \frac{1}{2}\nu(y_i + \frac{\Delta y_i}{2}) \right) \tag{4.3}$$

Since the Lévy density is assumed to be smooth enough, we obtain $\omega(t, z, y_i) = \hat{\omega}(t, z, y_i) + O((\Delta y_i)^3)$.

Remark 4.4. *From the definition of the integration points y_i we have*

$$\Delta z = \gamma(t, z, y_{i+\frac{1}{2}}(t, z)) - \gamma(t, z, y_{i-\frac{1}{2}}(t, z)) = \Delta y_i \int_0^1 \partial_y \gamma(t, z, y_{i-\frac{1}{2}}(t, z) + r\Delta y_i) dr$$

and from Assumption 2.1-[**RG**] $-$ **i**)*, which is supposed to hold on all $\mathbb{R}$, we obtain*

$$\Delta z \geq m_1 \Delta y_i, \textit{ for all } i$$

It follows, in particular,

$$\sup_{i,t,z} |\omega(t, z, y_i) - \hat{\omega}(t, z, y_i)| = O(\Delta z^3) \tag{4.4}$$

□

We start with the operator $\mathcal{B}^{tr}$. With the notation

$$F(t,z,y) := a(t,z+\gamma(t,z,y)) - a(t,z) - \gamma(t,z,y)\frac{\partial a}{\partial z}(t,z)$$

this operator is given by

$$\mathcal{B}^{tr}a(t,z) := \int_{\hat\Omega_0(z)} F(t,z,y)\nu(y)dy + \int_{\hat\Omega_1(z)} F(t,z,y)\nu(y)dy + \int_{\hat\Omega_2(z)} F(t,z,y)\nu(y)dy.$$

Let us first consider the integral over $\hat\Omega_0(z)$: with the notation

$$D(t,z) := \int_{\hat\Omega_0} \gamma(t,z,y)^2\nu(y)dy, \tag{4.5}$$

this integral satisfies

$$\int_{\hat\Omega_0(z)} F(t,z,y)\nu(y)dy = \frac{D(t,z)}{2}\frac{\partial^2 a}{\partial z^2} + \int_{\hat\Omega_0} O(\gamma^3)\nu(y)dy$$

and since $|\gamma(t,z,y)| \le M|y|$ around zero (Assumption 2.1-[**I**]) we can write

$$\int_{\hat\Omega_0(z)} F(t,z,y)\nu(y)dy = \frac{D(t,z)}{2}\frac{\partial^2 a}{\partial z^2} + O(\Delta z^{3-\alpha}) \tag{4.6}$$

In the region $\hat\Omega_2$, away from zero, the Lévy density is non-singular and on each integration interval around the point $y_i(t,z)$ the integral can be computed as follows:

$$\begin{aligned}
\int_{y_{i-1/2}}^{y_{i+1/2}} F(t,z,y)\nu(y)dy &= F(t,z,y_i)\int_{y_{i-1/2}}^{y_{i+1/2}} \nu(y)dy + \int_{y_{i-1/2}}^{y_{i+1/2}} F_y(t,z,y_i)(y-y_i)\nu(y)dy \\
&\quad + \frac{1}{2}\int_{y_{i-1/2}}^{y_{i+1/2}} F_{yy}(t,z,y^*)(y-y_i)^2\nu(y)dy \\
&= F(t,z,y_i)\omega(t,z,y_i) + F_y(t,z,y_i)\nu(y_i)\int_{y_{i-1/2}}^{y_{i+1/2}}(y-y_i)dy \\
&\quad + (F_y(t,z,y_i)\nu'(y^{**}) + \frac{1}{2}F_{yy}(t,z,y^*)\nu(y^{***}))\int_{y_{i-1/2}}^{y_{i+1/2}}(y-y_i)^2dy \\
&= F(t,z,y_i)\omega(t,z,y_i) + \frac{1}{2}F_y(t,z,y_i)\nu(y_i)\Delta y_i(y_{i+1/2}+y_{i-1/2}-2y_i) \\
&\quad + (F_y(t,z,y_i)\nu'(y^{**}) + \frac{1}{2}F_{yy}(t,z,y^*)\nu(y^{***}))\int_{y_{i-1/2}}^{y_{i+1/2}}(y-y_i)^2dy,
\end{aligned}$$

where y^*, y^{**} are points in the interval $[y_{i-1/2}, y_{i+1/2}]$ (appearing in the residual term of the Taylor formula) and y^{***} is the point where $\nu(y)$ achieves its maximum in this interval. Clearly, $\int_{y_{i-1/2}}^{y_{i+1/2}}(y-y_i)^2dy \le \Delta y_i^3$. Also, since γ_y is bounded from below, and γ_{yy} is bounded from above (Assumption 2.1-[**RG**] − **i**)),

$$y_{i+1/2} + y_{i-1/2} - 2y_i = O(\Delta z^2).$$

With the additional technical assumptions listed in the beginning of this section, we then obtain that

$$\int_{y_{i-1/2}}^{y_{i+1/2}} F(t,z,y)\nu(y)dy = F(t,z,y_i)\omega(t,z,y_i) + O(\Delta y_i\Delta z^2)$$

and so, since the integration is carried out over a finite domain,

$$\begin{aligned}
\int_{\hat\Omega_2(z)} F(t,z,y)\nu(y)dy &= \sum_{i\, y_i\in\hat\Omega_2} \omega(t,z,y_i)F(t,z,y_i) + O(\Delta z^2) \\
&= \sum_{i;\, y_i\in\hat\Omega_2} \omega(t,z,y_i)\left(a(t,z+i\Delta z) - a(t,z) - i\Delta z\frac{\partial a}{\partial z}\right) + O(\Delta z^2)
\end{aligned} \tag{4.7}$$

In the region $\hat{\Omega}_1$, we adapt the argument in paragraph 3.1 and Appendix A of Forsyth et al. (2007). We introduce the transformed Lévy density $\tilde{\nu}(dy) := \nu(y)y^2$ and the transformed integrand

$$\tilde{F}(t,z,y) = y^{-2}\left(a(t,z+\gamma(t,z,y)) - a(t,z) - \gamma(t,z,y)\frac{\partial a}{\partial z}(t,z)\right)$$

Under the additional assumptions listed in the beginning of this section, it is easy to see that $\tilde{F}_y$ and $\tilde{F}_{yy}$ are bounded. The error on a single discretization interval can then be estimated as follows:

$$\int_{y_{i-1/2}}^{y_{i+1/2}} \tilde{F}(t,z,y)\tilde{\nu}(y)dy = \tilde{F}(t,z,y_i)y_i^2\omega(t,z,y_i) + \int_{y_{i-1/2}}^{y_{i+1/2}} \tilde{F}_y(t,z,y_i)(y-y_i)\tilde{\nu}(y)dy$$
$$+ \frac{1}{2}\tilde{F}_{yy}(t,z,y^*)\int_{y_{i-1/2}}^{y_{i+1/2}} (y-y_i)^2\tilde{\nu}(y)dy$$

By integration by parts,

$$\int_{y_{i-1/2}}^{y_{i+1/2}} (y-y_i)\tilde{\nu}(y)dy = \frac{1}{2}(y_{i+1/2}+y_{i-1/2}-2y_i)\int_{y_{i-1/2}}^{y_{i+1/2}} \tilde{\nu}(y)dy$$
$$+ \frac{1}{2}\int_{y_{i-1/2}}^{y_{i+1/2}} \left(\frac{\Delta y_i^2}{4} - \left(y - \frac{y_{i+1/2}+y_{i-1/2}}{2}\right)^2\right)\tilde{\nu}'(y)dy$$

This implies that there exists a constant $C < \infty$ such that

$$\left|\int_{y_{i-1/2}}^{y_{i+1/2}} \tilde{F}(t,z,y)\tilde{\nu}(y)dy - \tilde{F}(t,z,y_i)y_i^2\omega(t,z,y_i)\right| \le C\Delta z^2\left\{\int_{y_{i-1/2}}^{y_{i+1/2}} \tilde{\nu}(y)dy + \int_{y_{i-1/2}}^{y_{i+1/2}} |\tilde{\nu}'(y)|dy\right\}$$

and so

$$\left|\int_{\hat{\Omega}_1} \tilde{F}(t,z,y)\tilde{\nu}(y)dy - \sum_{i:y_i\in\hat{\Omega}_1} \tilde{F}(t,z,y_i)y_i^2\omega(t,z,y_i)\right| \le C\Delta z^2\left\{\int_{\hat{\Omega}_1} \tilde{\nu}(y)dy + \int_{\hat{\Omega}_1} |\tilde{\nu}'(y)|dy\right\}$$
$$= O\left(\Delta z^{\min(2-\epsilon,3-\alpha)}\right)$$

for any $\epsilon > 0$ arbitrary small, so that

$$\int_{\hat{\Omega}_1}\left(a(t,z+\gamma(t,z,y)) - a(t,z) - \gamma(t,z,y)\frac{\partial a}{\partial z}(t,z)\right)\nu(y)dy \tag{4.8}$$
$$= \sum_{i:\, y_i\in\hat{\Omega}_1} \omega(z,y_i)\left(a(t,z+i\Delta z) - a(t,z) - i\Delta z\frac{\partial a}{\partial z}(t,z)\right) + O\left(\Delta z^{\min(2-\epsilon,3-\alpha)}\right)$$

Adding up (4.6), (4.7) and (4.8), together with the numerical integration of the weights $\omega(t,z,y_i)$ and (4.4), we can finally write

$$\mathcal{B}^{tr}a(t,z) := \frac{D(t,z)}{2}\frac{\partial^2 a}{\partial z^2}$$
$$+ \sum_{y_i,\,|i|>k} \hat{\omega}(t,z,y_i)\left(a(t,z+i\Delta z) - a(t,z) - i\Delta z\frac{\partial a}{\partial z}(t,z)\right) + O\left(\Delta z^{\min(2-\epsilon,3-\alpha)}\right)$$

for ϵ arbitrary small. In the same way, we approximate the operators $\mathcal{Q}^{tr}$ and $\mathcal{G}^{tr}$, obtaining

$$\mathcal{Q}^{tr}a(t,z) := \tilde{\mu}a(t,z) + D(t,z)\frac{\partial a}{\partial z}$$
$$+ \sum_{y_i,\,|i|>k} I^{\mathcal{Q}}(t,z,y_i)\left(a(t,z+i\Delta z) - a(t,z)\right) + O\left(\Delta z^{\min(2-\epsilon,3-\alpha)}\right)$$
$$\mathcal{G}^{tr}a(t,z) := D(t,z)a(t,z) + \sum_{y_i,\,|i|>k} I^{\mathcal{G}}(t,z,y_i)a(t,z+i\Delta z) + O\left(\Delta z^{\min(2-\epsilon,3-\alpha)}\right)$$

where $\hat{I}^{\mathcal{Q}}$ and $\hat{I}^{\mathcal{G}}$ are five point approximations of, respectively,

$$I^{\mathcal{Q}}(t,z,y_i) := \begin{cases} \dfrac{1}{y_i^2(t,z)} \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} \left(e^{\gamma(t,z,y)} - 1\right) |y|^2 \nu(dy) & \text{if} \quad y_{i+1/2}, y_{i-1/2} \in \hat{\Omega}_1 \\ \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} \left(e^{\gamma(t,z,y)} - 1\right) \nu(dy) & \text{otherwise} \end{cases}$$

and

$$I^{\mathcal{G}}(t,z,y_i) := \begin{cases} \dfrac{1}{y_i^2(t,z)} \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} \left(e^{\gamma(t,z,y)} - 1\right)^2 |y|^2 \nu(dy) & \text{if} \quad y_{i+1/2}, y_{i-1/2} \in \hat{\Omega}_1 \\ \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} \left(e^{\gamma(t,z,y)} - 1\right)^2 \nu(dy) & \text{otherwise} \end{cases}$$

The above computation allows us to rewrite PIDE (4.1) on $[-\underline{Z}, \underline{Z}]$ in the following form:

$$\begin{aligned} 0 = \frac{\partial a}{\partial t}(t,z) - \frac{D(t,z)}{2}\frac{\partial^2 a}{\partial z^2} - \mu(t,z)\frac{\partial a}{\partial z} + \sup_{\pi \in [-\bar{\Pi},\bar{\Pi}]} \Big\{ \Big(\tilde{V}(t,z) \ -2\pi D(t,z) \Big) \frac{\partial a}{\partial z} \\ - \sum_{|i| \geq k} \tilde{W}(t,z,y_i,\pi) a(t, z+i\Delta z) + \tilde{R}(t,z,\pi) a(t,z) \Big\} \\ a(0,z) = 1 \end{aligned} \tag{4.9}$$

where

$$\tilde{W}(t,z,y_i,\pi) := \hat{\omega}(t,z,y_i) + 2\pi \hat{I}^{\mathcal{Q}}(t,z,y_i) + \pi^2 \hat{I}^{\mathcal{G}}(t,z,y_i)$$

$$\tilde{V}(t,z) := \Delta z \sum_{|i| \geq k} i \hat{\omega}(t,z,y_i)$$

$$\tilde{R}(t,z,\pi) := \sum_{|i|>k} \Big(\hat{\omega}(t,z,y_i) + 2\pi \hat{I}^{\mathcal{G}}(t,z,y_i) \Big) - 2\pi\tilde{\mu} - \pi^2 D(t,z)$$

To solve Equation (4.9), we use an implicit scheme for the convection diffusion part and an explicit one for the integral part. We use time step Δt and $a^n(.)$ stands for $a(n\Delta t, .)$:

$$\begin{aligned} \frac{a^{n+1} - a^n}{\Delta t} - \frac{D(t^{n+1}, z)}{2} \frac{\partial^2 a^{n+1}}{\partial z^2} - \mu(t^{n+1}, z) \frac{\partial a^{n+1}}{\partial z} \\ + \sup_{\pi \in [-\bar{\Pi}, \bar{\Pi}]} \Bigg[(\tilde{V}(t^n, z) - 2\pi D(t^n, z)) \frac{\partial a^n}{\partial z} + \\ \tilde{R}(t^n, z, \pi) a^n - \sum_{|i|>k} \tilde{W}(t^n, z, y_i, \pi) a^n(t^n, z + i\Delta z) \Bigg] = 0 \end{aligned} \tag{4.10}$$

For the implicit term a classical central difference scheme (order two) is used for the first order differential term coupled with forward/backward differencing when matrix coefficients due to convection and diffusion are negative (see for example Forsyth et al. (2005)). The explicit first order differential term is treated to have monotonicity of the scheme. Equation (4.10) is then approximated by

$$
\begin{aligned}
0 = a_j^{n+1}(1+\Delta t(\alpha_j(t^{n+1})+\beta_j(t^{n+1})) - \Delta t\alpha_j(t^{n+1})a_{j-1}^{n+1} - \Delta t\beta_j(t^{n+1})a_{j+1}^{n+1}\\
+ \sup_{\pi\in[-\bar\Pi,\bar\Pi]}\Bigg[a_j^n(-1+\Delta t(\tilde R(t^n,z_j,\pi)+\frac{|\tilde V(t^n,z_j)-2\pi D(t^n,z_j)|}{\Delta z})\\
-a_{j-1}^n\Delta t\frac{(\tilde V(t^n,z_j)-2\pi D(t^n,z_j))^+}{\Delta z} - a_{j+1}^n\Delta t\frac{(\tilde V(t^n,z_j)-2\pi D(t^n,z_j))^-}{\Delta z}\\
-\Delta t\sum_{|i|>k}\tilde W(t^n,z_j,y_i,\pi)a_{j+i}^n\Bigg]
\end{aligned}
$$

where a_j^{n+1} stands for an approximation of $a^{n+1}(z_j)$ calculated at point z_j and α_j and β_j are positive weights (see for example Forsyth and Labahn (2007)) given by :

$$
\begin{aligned}
\alpha_{j,central}(t) &= \frac{D(t,z_j)}{2\Delta z^2} - \frac{\mu(t,z_j)}{2\Delta z}\\
\beta_{j,central}(t) &= \frac{D(t,z_j)}{2\Delta z^2} + \frac{\mu(t,z_j)}{2\Delta z}
\end{aligned}
$$

if $\alpha_{j,central}(t)$ or $\beta_{j,central}(t)$ is negative, we use

$$
\begin{aligned}
\alpha_{j,forward/backward}(t) &= \frac{D(t,z_j)}{2\Delta z^2} + max(0,\frac{-\mu(t,z_j)}{\Delta z})\\
\beta_{j,forward/backward}(t) &= \frac{D(t,z_j)}{2\Delta z^2} + max(0,\frac{\mu(t,z_j)}{\Delta z})
\end{aligned}
$$

Remark 4.5. *As explained in Forsyth et al. (2007) the results would be more accurate if we used an order two characteristic scheme in order to avoid a transport dominated problem. Here the coefficient μ is time dependent so in order to integrate the characteristics we would have to solve numerically a non linear time dependent equation, which would strongly increase the computational cost of the algorithm.*

Defining the tridiagonal matrix $\mathcal{M}$ such that

$$
\begin{aligned}
\mathcal{M}_{j,j}(t^{n+1}) &= \alpha_j(t^{n+1})+\beta_j(t^{n+1}),\\
\mathcal{M}_{j,j+1}(t^{n+1}) &= \beta_j(t^{n+1}),\\
\mathcal{M}_{j,j-1}(t^{n+1}) &= \alpha_j(t^{n+1}),
\end{aligned}
$$

and the matrix B such that

$$
\begin{aligned}
B_{j,j}(t^n,\pi) &= \tilde R(t^n,\pi,z_j) + \frac{|\tilde V(t^n,z_j)-2\pi D(t^n,z_j)|}{\Delta z},\\
B_{j,j-1}(t^n,\pi) &= -(\tilde V(t^n,z_j)-2\pi D(t^n,z_j))^+,\\
B_{j,j+1}(t^n,\pi) &= -(\tilde V(t^n,z_j)-2\pi D(t^n,z_j))^-,\\
B_{j,j+i}(t^n,\pi) &= -\tilde W(t^n,\pi,z_j,y_i),\ \text{if } |i|>k,\\
B_{j,j+i}(t^n,\pi) &= 0,\ \text{for } 1<|i|\le k,
\end{aligned}
$$

the system can be written :

$$
(I+\Delta t\mathcal{M}(t^{n+1}))a^{n+1} + \sup_{\pi\in[-\bar\Pi,\bar\Pi]}(-I+\Delta tB(t^n,\pi))a^n = 0 \tag{4.11}
$$

Proposition 4.6. *Under the CFL condition*

$$
\sup_{t,j}\Bigg[\frac{\tilde V(t,z_j)}{\Delta z} + 2\bar\Pi(|\mu(t,z_j)|+D(t,z_j)) + \sum_{|i|>k}(2\bar\Pi I^{\mathbb{Q}}(z_j,y_i)+\omega(z_j,y_i)) + \bar\Pi^2D(t,z_j))\Bigg]\Delta t<1, \tag{4.12}
$$

the scheme (4.11) *is consistent, monotone, L_∞ stable, and converges to the viscosity solution of equation* (4.1).

The proof is postponed to paragraph 6.1

Remark 4.7. *A scheme for finite variation processes can be easily derived from the one presented by suppressing the Ω_1 domain.*

4.2 Numerical scheme for the function b

For the truncated PIDE (4.2), using the same discretization as before and we get the following equation to solve on $[-\underline{Z}, \underline{Z}]$:

$$
\begin{aligned}
0 = \frac{\partial \tilde{b}}{\partial t} - \frac{D(t,z)}{2}\frac{\partial^2 \tilde{b}}{\partial z^2} + \frac{\partial \tilde{b}}{\partial z}\Big((\tilde{V}(t,z) - \mu(t,z) - \pi^{tr}[a]D(t,z)\Big) \\
+\tilde{b}\Big(\eta + \hat{R}(t,z,\pi^{tr}[a])\Big) - \sum_{|i|\geq k}\hat{W}(t,z,y_i,\pi^{tr}[a])\tilde{b}(t,z+i\Delta z) \\
\tilde{b}(0,z) = -2f(z),\ z \in [-\underline{Z}, \underline{Z}] \\
\tilde{b}(t,z) = -2f(z)a(t,z)e^{-\eta t},\ (t,z) \in [0,T] \times [-\hat{Z}, -\underline{Z}] \cup [\underline{Z}, \hat{Z}]
\end{aligned}
\tag{4.13}
$$

where

$$\hat{W}(t,z,y_i,\pi^{tr}[a]) := \hat{\omega}(t,z,y_i) + \pi^{tr}[a]\hat{I}^{\mathbb{Q}}(t,z,y_i)$$

$$\hat{R}(t,z,\pi^{tr}[a]) := \sum_{|i|>k}\Big(\hat{\omega}(t,z,y_i) + \pi^{tr}[a]\hat{I}^{\mathbb{Q}}(t,z,y_i)\Big) - \pi^{tr}[a]\mu(t,z)$$

and $\tilde{V}$ and $D(t,z)$ are the function introduced in paragraph 4.1. We propose the following time discretization scheme :

$$
\begin{aligned}
\frac{\tilde{b}^{n+1} - \tilde{b}^n}{\Delta t} - \frac{D(t^{n+1},z)}{2}\frac{\partial^2 \tilde{b}^{n+1}}{\partial z^2} + (\hat{R}(t^{n+1},z,\pi^{tr}[a]^{n+1}) + \eta)\tilde{b}^{n+1} + \\
(\tilde{V}(t^{n+1},z) - \pi[a]^{n+1}D(t^{n+1},z) - \mu(t^{n+1},z))\frac{\partial \tilde{b}^{n+1}}{\partial z} - \\
\sum_{|i|>k}(\hat{W}(t^{n+1},z,y_i,\pi^{tr}[a]^{n+1}))^+ b^n(z+i\Delta z) + \\
\sum_{|i|>k}(\hat{W}(t^{n+1},z,y_i,\pi^{tr}[a]^{n+1}))^- b^n(z+i\Delta z) = 0
\end{aligned}
\tag{4.14}
$$

Using the same kind of discretization as in previous paragraph, we get

$$
\begin{aligned}
\tilde{b}_j^{n+1}(1 + \Delta t(\alpha_j(t^{n+1}) + \beta_j(t^{n+1}) + \hat{R}(t^{n+1},.,\pi^{tr}[a]^{n+1}) + \eta) - \\
\Delta t \alpha_j(t^{n+1})\tilde{b}_{j-1}^{n+1} - \Delta t \beta_j(t^{n+1})\tilde{b}_{j+1}^{n+1} \\
+\Delta t \sum_{|i|>k}(\hat{W}(t^{n+1},z_j,y_i,\pi^{tr}[a]^{n+1}))^- \tilde{b}_{j+i}^n \\
-\Delta t \sum_{|i|>k}(\hat{W}(t^{n+1},z_j,y_i,\pi^{tr}[a]^{n+1}))^+ \tilde{b}_{j+i}^n = b_j^n
\end{aligned}
\tag{4.15}
$$

with α_j, β_j positive weights given by :

$$
\begin{aligned}
\alpha_{j,central}(t,\pi) &= \frac{D(t,z_j)}{2\Delta z^2} + \frac{\tilde{V}(t,z_j) - \pi D(t,z) - \mu(t,z_j)}{2\Delta z} \\
\beta_{j,central}(t,\pi) &= \frac{D(t,z_j)}{2\Delta z^2} - \frac{\tilde{V}(t,z_j) - \pi D(t,z) - \mu(t,z_j)}{2\Delta z}
\end{aligned}
$$

if $\alpha_{j,central}(t,\pi)$ or $\beta_{j,central}(t,\pi)$ is negative, we use

$$\alpha_{j,forward/backward}(t,\pi) = \frac{D(t,z_j)}{2\Delta z^2} + \left(\frac{\tilde{V}(t,z_j) - \pi D(t,z) - \mu(t,z_j)}{\Delta z}\right)^+$$

$$\beta_{j,forward/backward}(t,\pi) = \frac{D(t,z_j)}{2\Delta z^2} + \left(\frac{\tilde{V}(t,z_j) - \pi D(t,z) - \mu(t,z_j)}{\Delta z}\right)^-$$

Proposition 4.8. *For a space discretization accurate enough (Δz small enough), taking*

$$\eta = (\Pi + \epsilon)||\bar{\mu}|| + 2\int_{\Omega_2} (1 + \bar{\Pi}|e^y - 1|)\nu(dy),$$

the scheme (4.15) *is consistent and stable so it converges to the smooth solution of the PIDE* (4.2).

The proof is postponed to paragraph 6.2.

5 Application to the electricity market

On the energy market, the forward curve at date t for one MW delivered at date T is often modeled with a HJM model as in Clewlow and Strickland (2000). Many studies have shown that spikes on the electricity and gas markets are incompatible with Gaussian factors (Geman and Roncoroni, 2006; Meyer-Brandis and Tankov, 2008) and many models have been developed using Lévy processes to try to fit the observed fat tails (Deng and Jiang, 2005; Benth et al., 2007). Most of them directly model the price under the martingale measure, or make some assumption on the change of probability resulting in a similar model under the martingale measure (Benth et al., 2007).

We propose here a model for the forward curve deformation which satisfies Assumptions 2.1 so that we can apply Theorems 3.2–3.3. We start by introducing a Lévy process $\hat{L}$ as follows

$$\hat{L}_s = \zeta s + \int_0^s \int_{\mathbb{R}} y\tilde{J}(ds \times dy) \tag{5.1}$$

where $\zeta \in \mathbb{R}$ and $\tilde{J}$ is a compensated Poisson random measure, whose Lévy measure is denoted by $\nu(dy)$. Fix $c \in \mathbb{R}^+$, $l(s) = e^{-cs}$ and

$$A_t := \int_0^t e^{cs} d\hat{L}_s \tag{5.2}$$

If $T \to \psi(0,T)$ denotes the price at time 0 of a future contract with maturity T and instantaneous delivery (which is supposed to be known), then we will model the price at time t of the same future contract as a random perturbation of the forward curve ψ: using the above notation we have

$$\bar{F}_{0,T,t} = \psi(0,T)e^{l(T)A_t}$$

By no arbitrage in the futures market, the price at time t of a future contract with duration d is equal to the average over the time period $[T, T+d]$ of the future contract prices with instantaneous delivery. We therefore model the price at time t of a future contract with delivery time T and duration $d > 0$ by

$$F_{d,T,t} = \frac{1}{d}\int_T^{T+d} \bar{F}_{0,s,t} ds = \frac{1}{d}\int_T^{T+d} \psi(0,s)e^{l(s)A_t} ds$$

For reasons which will become clear in the sequel, we prefer the following notation:

$$F_{d,T,t} := \exp(\Phi(A_t)) \quad \text{where} \quad \Phi(A) := \log\left(\frac{1}{d}\int_T^{T+d} \psi(0,s)e^{l(s)A} ds\right) \tag{5.3}$$

The model (5.3) essentially states that the price of a future contract $F_{d,T,t}$ is the average price on the interval $[T, T+d]$ of the future contract with instantaneous delivery up to the random perturbation $e^{l(s)A}$. In this context, the problem of hedging a European option on $F_{d,T,t}$ with the quadratic hedging approach becomes

$$\textbf{minimize} \quad \mathbb{E}\left[\left(H(F_{d,T,t}) - x - \int_t^T \theta_{u-} dF_{d,T,u}\right)^2\right] \quad \text{over } \theta \text{ and } x \in \mathbb{R} \tag{5.4}$$

for a given map H. The process $F_{d,T,t}$ corresponds to S in the formulation (1.1). The following results proves that $Z = \log(F)$ is a Markov jump process satisfying our assumptions.

Lemma 5.1. *The process* $Z_t := \log(F_{d,T,t})$ *verifies*

$$dZ_t = \mu(t, Z_t)dt + \int \gamma(t, Z_{t-}, y)\tilde{J}(dydt)$$

where

$$\begin{aligned} \gamma(t, z, y) :=& \Phi(\Phi^{-1}(z) + ye^{ct}) - z \\ \mu(t, z) :=& \zeta e^{ct}\Phi'(\Phi^{-1}(z)) + \int_{\mathbb{R}} \left(\gamma(t, z, y) - ye^{ct}\Phi'(\Phi^{-1}(z))\right) \nu(dy) \end{aligned}$$

Assume that the Lévy measure $\nu(dy)$ *is given by* $\nu(dy) = g(y)|y|^{-(1+\alpha)}$*, for some* $\alpha \in (1, 2)$ *and a bounded, strictly positive and measurable* g *such that the following condition hold true:*

i). there exists some positive $m \geq 0$ *such that for all* $y, y' \in (-y_0, 0) \cup (0, y_0)$ *with* $yy' > 0$*,* $|g(y) - g(y')| \leq m|y - y'|$

ii). $\lim_{y \to 0^-} g(y) = g(0^-)$ *and* $\lim_{y \to 0^+} g(y) = g(0^+)$ *with* $g(0^+), g(0^-) > 0$

iii). $\int_{y \leq -1} y^4 \nu(dy) + \int_{1 < y} e^{4y} \nu(dy) < +\infty$

then the functions μ *and* γ *verify the Assumptions 2.1-*$[\mathbf{C}, \mathbf{L}, \mathbf{I}, \mathbf{ND}, \mathbf{RG_i}, \mathbf{RG_{iii}}]$*, where the function* τ *is given by*

$$\tau(y) = \max\left(|y|, |e^y - 1|\right)$$

The proof is postponed to Section 6.3.

In order to apply our results (Theorems 3.2 and 3.3) we also need to verify the Assumption 2.1-$[\mathbf{RG_{ii}}]$ and it is easy to see that the function γ does not verify it: however, as we have already said in Section 2, this can be avoided by making a change of variable $L_t = \phi(t, Z_t)$. We refer to Chapter 7 and 8 in De Franco (2012) for further details.

We can transform the problem (5.4) by using the process Z to obtain

$$v^f(t, z, x) = \inf_{\theta \in \mathcal{X}(t,z,x)} \mathbb{E}\left[\left(f(Z_T^{t,z}) - x - \int_t^T \theta_{u-} d\exp(Z_u^{t,z})\right)^2\right] \tag{5.5}$$

where $\mathcal{X}(t, z, x)$ is defined in (2.10) and $f(z) = H(e^z)$.

We now describe a special class of options one may want to use in problem (5.5). First define, for some $G > 0$ the function $p(x) : (G - x)^+$, the usual put function, and

$$h(A) := \frac{1}{d'} \int_T^{T+d'} \psi(0, s)e^{l(s)A} ds$$

for some $d' \neq d$. From (5.3) it follows that $h \circ \Phi^{-1}(Z_t) = F_{d',T,t}$, and then, by defining $f := p(h \circ \Phi^{-1})$, we obtain $f(Z_t) = (G - F_{d',T,t})^+$, which is a put option written on a future contract with different duration d'. Using this particular option we can rewrite problem (5.5) as follows

$$\inf_{\theta \in \mathcal{X}(t,z,x)} \mathbb{E}^{t,z,x}\left[\left((G - F_{d',T,t})^+ - x - \int_t^T \theta_{u-} dF_{d,T,u}\right)^2\right]$$

The financial meaning of the above problem is particularly interesting: one tries to hedge (in the quadratic sense) a put option written on a future contract with duration $d' \neq d$ using as hedging instrument the future contract with duration d. This may be useful when, for example, one sells a future contract with a non-standardized duration in the OTC market and hedges the resulting position using instruments which are liquidly traded.

5.1 A degenerate case

In this last section we will study the problem (5.5) when $\hat{L}$ in (5.1) is a Normal Inverse Gaussian process with parameters α, β, δ, u: $\hat{L}_t \sim NIG(\alpha, \beta, \delta t, ut)$.

Remark 5.2. *The parameter α should not be mistaken for the parameter in Lemma 5.1. We use this notation because it is standard in the literature.*

We can write then

$$\hat{L}_t = \left(u + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}}\right)t + \int_0^t \int_{\mathbb{R}} y\tilde{J}(dyds)$$

where $\tilde{J}$ is a compensated Poisson random measure with intensity

$$\nu(dy) = \nu(y)dy, \quad \nu(y) = \frac{\alpha\delta}{\pi|y|}K_1(\alpha|y|)e^{\beta y},$$

where K_1 is the modified Bessel function of the second kind (paragraph 4.4.3 in Cont and Tankov (2004)). The Lévy density $\nu(y)$ satisfies

$$\nu(y) \overset{y\to 0}{\sim} \frac{\delta}{\pi|y|^2}, \qquad \nu(y) \overset{y\to+\infty}{\sim} \frac{1}{|y|^{3/2}}e^{-(\alpha-\beta)y}, \qquad \nu(y) \overset{y\to-\infty}{\sim} \frac{1}{|y|^{3/2}}e^{-(\alpha+\beta)|y|}.$$

Remark 5.3. *The NIG is a infinite variation Lévy process with stable-like behavior of small jumps, and since the Blumenthal-Getoor index is equal to 1, we cannot formally apply Lemma 5.1 and Theorems 3.2–3.3. It is nevertheless a case of interest because the NIG model is popular among practitioners, and we shall see in the sequel that our numerical schemes yield good results for this model.*

The goal of this paragraph is to solve numerically the PIDEs (4.1)–(4.2) where the model for Z is given in Lemma 5.1 and the source of randomness $\hat{L}$ in (5.1) is given by the NIG process introduced above. We first apply scheme (4.11) for the function a with maturity $T = 1$. First notice that it is necessary to compute numerically the coefficients μ and γ given in Lemma 5.1:

$$\begin{aligned}\mu(t,z) =& \Phi'(\Phi^{-1}(z))\left(u + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}\right)e^{ct} + \int_{\mathbb{R}}\left[\Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z - ye^{ct}\Phi'(\Phi^{-1}(z))\right]\nu(dy)\\ \gamma(t,z,y) =& \Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z\end{aligned}$$

and the function $D(t,z)$ introduced in (4.5): for this, we compute the integration points y_i, as in Section 4.1, such that $\gamma(t,z,y_i(t,z)) = i\Delta z$, or equivalently

$$y_i(t,z) := e^{-ct}\left(\Phi^{-1}(z + i\Delta z) - \Phi^{-1}(z)\right)$$

By expanding γ around zero we obtain

$$D(t,z) := \int_{y_{-k-1/2}(t,z)}^{y_{k+1/2}(t,z)} \gamma(t,z,y)^2 \nu(dy) \simeq e^{2ct}(\Phi^{'}(\Phi^{-1}(z)))^2 \int_{y_{-k-1/2}(t,z)}^{y_{k+1/2}(t,z)} y^2 \nu(dy)$$
$$\simeq e^{ct}\left(\Phi^{-1}(z+(k+\frac{1}{2})\Delta z) - \Phi^{-1}(z-(k+\frac{1}{2})\Delta z)\right)(\Phi^{'}(\Phi^{-1}(z)))^2 \frac{\delta}{\pi}$$

since, around zero, we have $y^2\nu(dy) \simeq \frac{\delta}{\pi} + \frac{\delta\beta}{\pi} y + O(y^2)$. (See for example Raible (2000)).

By using a method similar to the ones of Section 4.1, we can estimate

$$\int_{\mathbb{R}} \left[\Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z - ye^{ct}\Phi^{'}(\Phi^{-1}(z))\right] \nu(dy)$$

with

$$\frac{1}{2}\Delta z(\Phi^{'}(\Phi^{-1}(z))e^{ct})^2\frac{\delta}{\pi} + \sum_{y_i,\,|i|>k} \hat{\omega}(t,z,y_i)\left[\Phi\left(\Phi^{-1}(z)+y_ie^{ct}\right) - z - y_ie^{ct}\Phi^{'}(\Phi^{-1}(z))\right]$$
$$=\frac{1}{2}\Delta z(\Phi^{'}(\Phi^{-1}(z))e^{ct})^2\frac{\delta}{\pi} + \sum_{y_i,\,|i|>k} \hat{\omega}(t,z,y_i)\left[i\Delta z - \left(\Phi^{-1}(z+i\Delta z) - \Phi^{-1}(z)\right)\Phi^{'}(\Phi^{-1}(z))\right]$$

where the weights $\hat{\omega}(t,z,y_i)$ are given in (4.3), relatively to the Lévy measure ν. The effect of approximating the coefficient of the PIDE on the solution of the source problem, appearing for example in the pricing of European options, has been studied in Jakobsen and Karlsen (2005).

We want to solve problem (5.5) for European options f, with maturity one week and delivery for the 7 days of the week. We recall that the future contract in this case is given by

$$F_{7days,1week,t} = \frac{1}{7}\int_7^{14} \psi(0,s)e^{l(s)A_t}ds \tag{5.6}$$

and A_t is given in (5.2) with $\hat{L}$ being the NIG process defined above. The forward curve for the seven days of delivery is given in Table 1, indicating that prices are lower for the week end. The

Day	s	Price ($\psi(0,s)$)
Monday	$s \in [7,8)$	80
Tuesday	$s \in [8,9)$	90
Wednesday	$s \in [9,10)$	70
Thursday	$s \in [10,11)$	90
Friday	$s \in [11,12)$	80
Saturday	$s \in [12,13)$	70
Sunday	$s \in [13,14]$	60

Table 1: The forward curve. Prices are given in Eur.

parameters of the NIG process are $u = 0.08$, $\alpha = 6.23$, $\beta = 0.06$, $\delta = 0.1027$. The mean reverting coefficient c is taken equal to 0.19. This case with continuous long delivery corresponds to a non stationary process where the hedge cannot be calculated efficiently as in Hubalek et al. (2006) or Goutte et al. (2011).
We use the scheme (4.11) to obtain a numerical approximation of the function a and the optimal control π^*. Figure 1 shows the function a whereas the optimal control π^* is given in Figure 2. For all numerical experiments we suppose that $\hat{Z} = 12$, $\underline{Z} = 8$, take the mesh size equal to 800 and the number of time steps equal to 800. The value k for defining the diffusion zone in the PIDE is equal to 3. Figure 3 shows the optimal price b for an at-the-money call option and Figure 4 shows the function b for an at-the-money put option.

Practitioners usually price this kind of option and calculate the hedging strategy by assuming that the underlying process F is a martingale. It is therefore interesting to evaluate the loss of

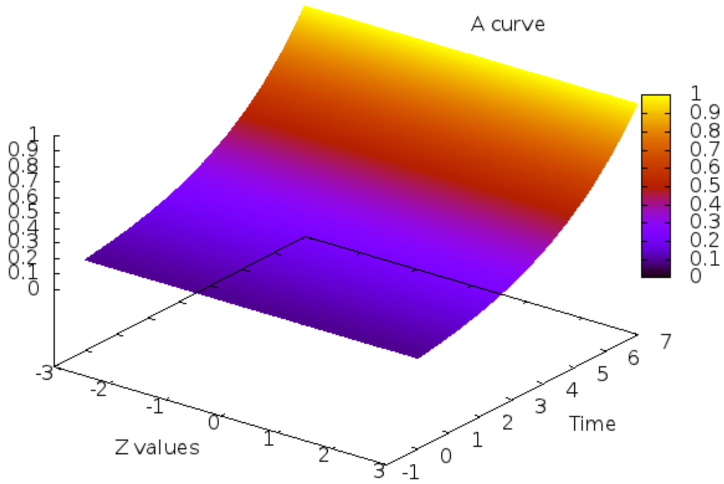

Figure 1: The value function $(t, z) \mapsto a(t, z)$ for the NIG process.

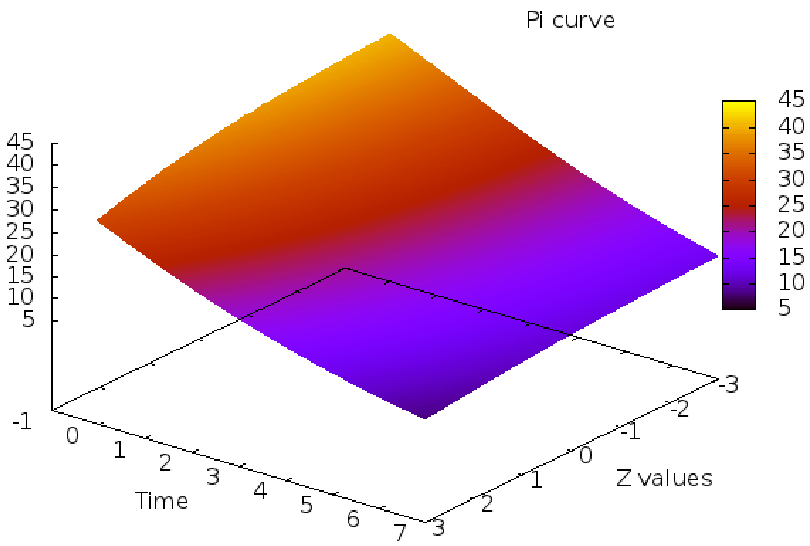

Figure 2: The optimal control $(t, z) \mapsto \pi^*(t, z)$ for the NIG process.

efficiency when using the hedging strategy computed in the martingale model. Assuming that F is a martingale means that we should have

$$F_{d,T,t} := \frac{1}{d} \int_T^{T+d} \psi(0, s) \exp\left(M(s, t) + l(s) A_t\right) ds$$

for some M that makes F a martingale under the historical probability $\mathbb{P}$. By using Lemma 15.1

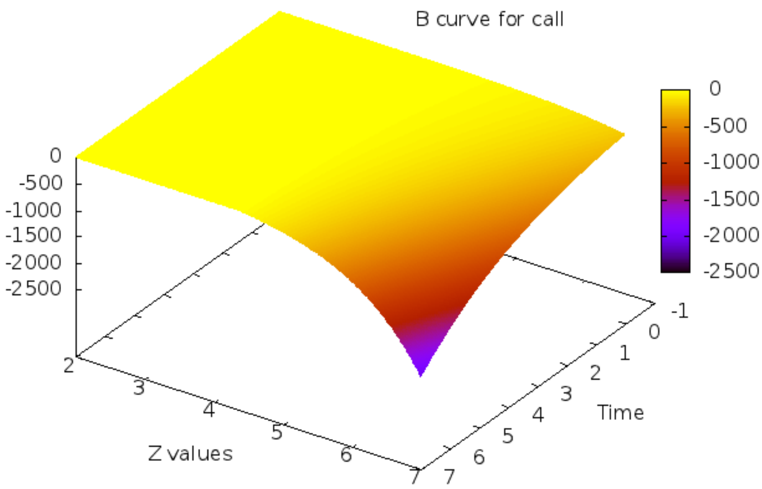

Figure 3: The value function $b(t, z)$ for an at-the-money call option.

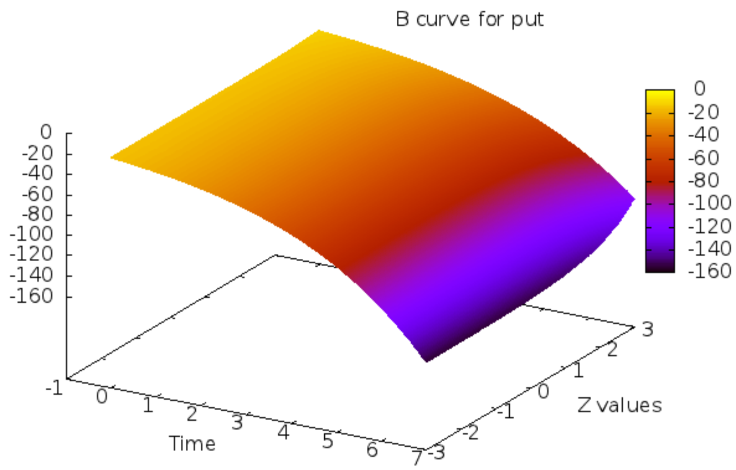

Figure 4: The value function $b(t, z)$ for an at-the-money put option.

in Cont and Tankov (2004) we obtain

$$M(t,s) = -\int_0^t \left(\left(u + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}} \right) e^{-c(s-r)} + \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + e^{-c(s-r)})^2} \right) \right) dr$$

First remark that $a = 1$ when the underlying process F is a martingale: indeed, from PIDE (3.4), we have

$$0 = -\frac{\partial a}{\partial t} - \mu \frac{\partial a}{\partial z} - \int_{\mathbb{R}} \left(a(t, z+\gamma) - a(t,z) - \gamma \frac{\partial a}{\partial z}(t,z) \right) \nu_L(dy) - \inf_{|\pi| \le \bar{\Pi}} \left\{ 2\pi \mathcal{Q} a(z) + \pi^2 \mathcal{G} a \right\}$$
$$a(T, z) = 1$$

Option H	Moneyness	Option value (x^{true})	efficiency(θ^{true})	efficiency(θ^{mart})
Call	1	4.199	1.085	1.316
Put	1	4.213	1.087	1.315
Call	1.5	0.120	0.168	0.212
Put	1.5	38.80	0.175	0.34

Table 2: Pricing and standard deviation of hedged portfolio

On the other hand, from Definition 3.1, we have

$$\mathcal{Q}a(z) := \int_{\mathbb{R}} (e^{\gamma} - 1)\left(a(t, z + \gamma(t, z, y)) - a(t, z)\right)\nu(dy)$$

since $\tilde{\mu}$, given in (2.2), is equal to zero (it is the drift of the process F which is now a martingale). From this, it is straightforward to deduce that the function $a = 1$ is the unique solution of PIDE (3.4). So that, when F is a martingale, one only needs to compute the function b.

We now evaluate the loss of efficiency when using the martingale hedging strategies compared to the quadratic hedging strategies. Our efficiency comparison criterion is the following: if H is the option (call/put) and $\theta^{true}, \theta^{mart}$ are, respectively, the optimal quadratic hedging strategy and the martingale strategy, then the efficiency is measured in terms of the standard deviation of the hedged portfolios:

$$\text{efficiency}(\theta^{true})^2 := \text{Var}\left(H(F_{d,T,t}) - x^{true} - \int_t^T \theta^{true}_{r-} dF_{d,T,r}\right)$$

where x^{true} is the true optimal price given in (3.9). Similarly

$$\text{efficiency}(\theta^{mart})^2 := \text{Var}\left(H(F_{d,T,t}) - x^{mart} - \int_t^T \theta^{mart}_{r-} dF_{d,T,r}\right)$$

where x^{mart} is the price given in (3.9) when one uses the functions a and b computed in the martingale model., i.e. x^{mart} is the risk neutral price of H. The variances are computed by Monte Carlo over 2 million paths, with 800 rebalancing dates in each path. The trajectory of $F_{d,T,t}$ is simulated using the true model in both cases.

Table 2 summarizes the results of simulations, for $t = 0$. The numerical experiment proves that one loses efficiency when using the martingale hedging strategy. This is consistent with the fact that θ^{true} achieves the minimum in problem (5.5) and outperforms the strategy θ^{mart}.

6 Proofs

6.1 Proof of Proposition 4.6

Proof. To prove consistency, suppose that the solution a is regular, use the fact that differential scheme used is of order 1 and the Taylor expansions calculated previously are of order at least 1. Define

$$\mathcal{L}_\pi[a](t,z) = \mathcal{A}[a](t,z) - \mathcal{B}[a](t,z) - [2\pi(t,z)\mathcal{Q}[a](t,z) + \pi^2\mathcal{G}[a](t,z)]$$

Then calculate the error with a Taylor expansion:

$$|(\frac{\partial a}{\partial t})^n_j + \sup_{\pi\in(-\Pi,\Pi)} \mathcal{L}_\pi[a](t,z)^n_j - \frac{a^{n+1}_j - a^n_j}{\Delta t} - \sup_{\pi\in(-\Pi,\Pi)} [B(t^n,\pi)a^n]_j - [M(t^{n+1})a^{n+1}]_j|$$

$$\leq |(\frac{\partial a}{\partial t})^n_j - \frac{a^{n+1}_j - a^n_j}{\Delta t}| + \sup_{\pi\in(-\Pi,\Pi)} |(\mathcal{L}_\pi[a](t,z))^n_j - [B(t^n,\pi))a^n]_j - [M(t^{n+1})a^{n+1}]_j|$$

$$\leq O(\Delta t) + \sup_{\pi\in(-\Pi,\Pi)} |(\mathcal{L}_\pi[a](t,z))^n_j - [B(t^n,\pi)a^n]_j - [M(t^n)a^n]_j| + ||M(t^n)a^n - M(t^{n+1})a^{n+1}||$$

$$\leq O(\Delta t) + O(\Delta z) + ||M(t^{n+1})(a^{n+1} - a^n)|| + ||(M(t^{n+1}) - M(t^n))a^n||$$

$$\leq O(\Delta t) + O(\Delta z) \quad (6.1)$$

As for the monotony, rewrite equation (4.11) as:

$$T(a_j^{n+1}, a_{i\neq j}^{n+1}, a_i^n) = [(I + \Delta t M(t^{n+1}))a^{n+1} + \sup_{\pi\in(-\Pi,\Pi)} (-I + \Delta t B(t^n, \pi))a^n]_j$$

First notice that

$$\tilde{W}(t, z, y_i, \pi) = \begin{cases} \dfrac{1}{y_i^2}\displaystyle\int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} y^2(\pi(e^\gamma - 1) + 1)^2\nu(y)dy, & \text{if } y_i(z) \in \Omega_1(z), \\ \displaystyle\int_{y_i-\frac{1}{2}}^{y_i+\frac{1}{2}} (\pi(e^\gamma - 1) + 1)^2\nu(y)dy, & \text{if } y_i(z) \in \Omega_2(z), \\ 0, & \text{if } y_i(z) \in \Omega_0(z) \end{cases}$$

so $\tilde{W} \geq 0$ and $B_{i,j} \leq 0$ for $j \neq i$. Because of equation (4.12) , $-1 + \Delta t B_{i,i} \leq 0$. So we get that T is decreasing with respect to a_i^n for all j. Besides α, β are positive, so T is decreasing with respect to $a_{i\neq j}^{n+1}$ and increasing with respect to a_j^{n+1}. We conclude that the scheme is monotone (Barles and Souganidis, 1991).

As for the L_∞ stability we follow the methodology described in Forsyth et al. (2005). Let $\tilde{a}$ be the discretized solution of (4.11) and suppose that we begin the scheme with a small perturbation e^0 :

$$\hat{a}^0 = \tilde{a}^0 + e^0$$

with $e^n = (e_0^n, ..., e_N^n)$ the perturbation error at time step n. Because we impose Dirichlet conditions, we may assume that $e_j^n = 0$ for $j \leq j_{-\hat{Z}}$ and $j \geq j_{\hat{Z}}$. In the sequel π is the discrete strategy obtained by solving equation (4.11). For $j_{-\hat{Z}} < j < j_{\hat{Z}}$, the error evolves as follows :

$$e_j^{n+1}(1 + \Delta t(\alpha_j(t^{n+1}) + \beta_j(t^{n+1}))) - \Delta t\alpha_j(t^{n+1})e_{j-1}^{n+1} - \Delta t\beta_j(t^{n+1})e_{j+1}^{n+1} = e_j^n(1 - \Delta t B_{j,j}(t^n, \pi)) - \Delta t\sum_{i\neq j} B_{j,i}(t^n, \pi)e_i^n$$

We denote $||e^n|| = \sup_j |e_j^n|$. Because all coefficients are positive under the CFL condition (4.12),

$$e_j^{n+1}(1 + \Delta t(\alpha_j(t^{n+1}) + \beta_j(t^{n+1}))) \leq ||e^n|| \left[1 - \Delta t(B_{j,j}(t^n, \pi) + \sum_{i\neq j} B_{j,i}(t^n, \pi))\right] + \Delta t(\alpha_j(t^{n+1}) + \beta_j(t^{n+1}))||e^{n+1}||$$

Let j^* be the index such that $e_{j^*}^{n+1} = ||e^{n+1}||$. Then,

$$||e^{n+1}||(1 + \Delta t(\alpha_{j^*}(t^{n+1}) + \beta_{j^*}(t^{n+1})) \leq ||e^n|| + \Delta t(\alpha_{j^*}(t^{n+1}) + \beta_{j^*}(t^{n+1}))||e^{n+1}||$$

It follows that

$$||e^{n+1}|| \leq ||e^n||$$

and we get the stability

Remark 6.1. *The use of a θ scheme or an implicit scheme for the integral terms is not theoretically possible because it doesn't allow us to have a stable scheme. On the other hand, numerically, π and $\pi\mu$ turn out to be negative so using an implicit scheme does not break stability.*

As noted in Biswas et al. (2010) in the proof of theorem 3.1, the convergence towards viscosity solution is only an adaptation of Barles-Souganidis argument (Barles and Souganidis, 1991). □

6.2 Proof of Proposition 4.8

Proof. The consistence is an obvious result for the differential operator used. As for stability, we calculate as previously the perturbation error propagation coming from an initial error $e^0 = (e^0_0, ..., e^0_N)$. Notice once again that the error is zero at the boundary due to the Dirichlet condition we impose. The error for $j_{-\hat{Z}} < j < j_{\hat{Z}}$ evolves as follows :

$$e_j^{n+1}(1 + \Delta t(\alpha_j(t^{n+1}) + \beta_j(t^{n+1}) + \hat{R}(t^{n+1}, \pi[a]^{n+1}, z_j) + \eta) - \Delta t\alpha_j(t^{n+1})e_{j-1}^{n+1} - \Delta t\beta_j(t^{n+1})e_{j+1}^{n+1} + \Delta t \sum_{|i|>k} (\hat{W}(t^{n+1}, \pi[a]^{n+1}, z_j, y_i))^- e_{j+i}^{n+1} = (1 + \Delta t \sum_{|i|>k} (\hat{W}(t^{n+1}, \pi[a]^{n+1}, z_j, y_i))^+)e_{j+i}^n$$

An easy calculation shows that

$$\hat{R}(t^{n+1}, \pi[a]^{n+1}, z_j) + \eta \geq \epsilon||\hat{\mu}|| + \int_{\Omega_2} (1 + \Pi|e^y - 1|\nu(dy) + O(\Delta z). \tag{6.2}$$

Letting j^* be the index such that $e_{j^*}^{n+1} = ||e^{n+1}||$, we get

$$e_{j^*}^{n+1}(1 + \Delta t(\alpha_{j^*}(t^{n+1}) + \beta_{j^*}(t^{n+1}) + \hat{R}(t^{n+1}, \pi[a]^{n+1}, z_j) + \eta) \leq \Delta t(\alpha_{j^*}(t^{n+1}) + \beta_{j^*}(t^{n+1}))||e^{n+1}|| + \Delta t \sum_{|i|>k} (\hat{W}(t^{n+1}, \pi[a]^{n+1}, z_{j^*}, y_i))^- ||e^{n+1}|| + (1 + \Delta t \sum_{|i|>k} (\hat{W}(t^{n+1}, \pi[a]^{n+1}, z_{j^*}, y_i))^+)||e^n||,$$

and so

$$||e^{n+1}||(1 + \Delta t(\hat{R}(t^{n+1}, \pi[a]^{n+1}, z_{j^*}) + \eta - \sum_{|i|>k} \hat{W}(t^{n+1}, \pi[a]^{n+1}, z_{j^*}, y_i)^-)) \leq (1 + \Delta t \sum_{|i|>k} (\hat{W}(t^{n+1}, \pi[a]^{n+1}, z_{j^*}, y_i))^+)||e^n||.$$

Using equation (6.2) we then get that

$$||e^{n+1}|| \leq ||e^n||.$$

□

6.3 Proof of Proposition 5.1

Proof. Before we start, remark that the function $A \mapsto F_{d,T}(A)$ is strictly increasing, so invertible, and infinitely differentiable: in particular

$$\Phi'(A) = \frac{\int_T^{T+d} \psi(0, s)l(s)e^{l(s)A}ds}{\int_T^{T+d} \psi(0, s)e^{l(s)A}ds}$$

$$\Phi''(A) = \frac{\left(\int_T^{T+d} \psi(0, s)l^2(s)e^{l(s)A}ds\right)\left(\int_T^{T+d} \psi(0, s)e^{l(s)A}ds\right) - \left(\int_T^{T+d} \psi(0, s)l(s)e^{l(s)A}ds\right)^2}{\left(\int_T^{T+d} \psi(0, s)e^{l(s)A}ds\right)^2}$$

from which we deduce

$$e^{-c(T+d)} \leq \Phi'(A) \leq e^{-cT} \qquad \text{and} \qquad e^{-2c(T+d)} - e^{-2cT} \leq \Phi''(A) \leq e^{-2cT} - e^{-2c(T+d)}$$

From Itô's formula, we obtain

$$dZ_t = \left(\Phi'(A_t)e^{ct}\zeta + \int_{\mathbb{R}} \left(\Phi(A_{t-} + e^{ct}y) - \Phi(A_{t-}) - ye^{ct}\Phi'(A_{t-})\right)\nu(dy)\right)dt + \int_{\mathbb{R}} \left(\Phi(A_{t-} + e^{ct}y) - \Phi(A_{t-})\right)\tilde{J}(dydt)$$

or equivalently

$$dZ_t = \mu(t, Z_t)dt + \int \gamma(t, Z_{t-}, y)\tilde{J}(dydt)$$

We can now prove that μ and γ verify the Assumptions 2.1. We detail the computations only for the function γ, since similar computations can be done for μ. First we remark that $z \mapsto \gamma(t, z, y)$ is differentiable and we can compute this derivative to obtain

$$\begin{aligned} \partial_z \gamma(t, z, y) =& -1 + (\Phi'(\Phi^{-1}(z)))^{-1}\Phi'(\Phi^{-1}(z) + ye^{ct}) \\ =& e^{ct} y(\Phi'(\Phi^{-1}(z)))^{-1} \int_0^1 \Phi''(\Phi^{-1}(z) + re^{ct})dr \end{aligned}$$

so that

$$\begin{aligned} |\partial_z \gamma(t, z, y)| =& \left| e^{ct} y(\Phi'(\Phi^{-1}(z)))^{-1} \int_0^1 \Phi''(\Phi^{-1}(z) + re^{ct})dr \right| \\ \leq& |y| e^{cT} (\inf_A |\Phi(A)|)^{-1} \left\| \Phi'' \right\|_\infty \leq e^{cT} e^{-c(T+d)} \left\| \Phi'' \right\|_\infty |y| \\ \leq& |y| e^{cT} e^{c(T+d)} \left(e^{-2cT} - e^{-2c(T+d)} \right) \leq e^{cd} |y| \end{aligned}$$

From the bounds on the first and second derivative of Φ we obtain $\sup_{t,z} |\partial_z \gamma(t, z, y)| \leq e^{cd}|y|$, which gives us the function ρ introduced in Assumptions 2.1. Again by the definition of Φ in (5.3) we have

$$\exp(e^{-c(T+d)}y) - 1 \leq e^{\gamma(t,z,y)} - 1 \leq e^y - 1$$

if $y > 0$ and the inverse inequality stands in force if $y < 0$ which yield $\sup_{t,z} |e^{\gamma(t,z,y)} - 1| \leq |e^y - 1|$. According to the definition of the function τ given in Assumptions 2.1 and the estimations above we deduce that

$$\tau(y) := \max \left(\sup_{t,z} \left(|\gamma(t, z, y)|, \left| e^{\gamma(t,z,y)} - 1 \right| \right), \rho(y) \right) = e^{cd} \max(|y|, |e^y - 1|)$$

If follows then that Assumptions 2.1-[$\mathbf{C}, \mathbf{I}, \mathbf{L}$] hold true. For Assumption 2.1-[**ND**] we have, from the definition of γ

$$\left(e^{\gamma(t,z,y)} - 1 \right)^2 \geq \left(\exp(e^{-c(T+d)}y) - 1 \right)^2$$

so then, for some positive $M > 0$ we have

$$\begin{aligned} \Gamma(y) :=& \int_{\mathbb{R}} \inf_{t,z} \left(e^{\gamma(t,z,y)} - 1 \right)^2 \nu(dy) \geq \int_{\mathbb{R}} \inf_{t,z} \left(\exp(e^{-c(T+d)}y) - 1 \right)^2 \nu(dy) \\ \geq& M \int_{|y| \leq \epsilon} |y|^{1-\alpha} g(y)dy > 0 \end{aligned}$$

since $g(0^+)$ and $g(0^-)$ are strictly positive, we can select ϵ small enough and obtain

$$\Gamma(y) \geq M \int_{|y| \leq \epsilon} |y|^{1-\alpha} dy > 0$$

We can derive γ w.r.t y to obtain

$$\begin{aligned} \gamma_y(t, z, y) =& e^{ct} \Phi'(\Phi^{-1}(z) + e^{ct}y) \\ \gamma_{yy}(t, z, y) =& e^{2ct} \Phi''(\Phi^{-1}(z) + e^{ct}y) \end{aligned}$$

so then $e^{-c(T+d)} \leq |\gamma_y(t, z, y)| \leq e^{cT}$ and $|\gamma_{yy}(t, z, y)| \leq e^{2cT}$, which proves that Assumptions 2.1-[$\mathbf{RG_i}$] holds true. For Assumption 2.1-[$\mathbf{RG_{iii}}$], one can differentiate γ_y w.r.t. z and give for it an upper bound to prove that indeed $z \to \gamma_y(t, z, y)$ is Lipschitz continuous uniformly in t, y. The Assumption 2.1-[$\mathbf{RG_{ii}}$] does not hold true since trivially $\gamma_y(t, z, y) = e^{ct}\Phi'(\Phi^{-1}(z) + e^{ct}y) \neq 1$.

□

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