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Rapport de Recherche
RR-FiME-10-11
Mai 2010

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No marginal arbitrage of the second kind for high production regimes in discrete time production-investment models with proportional transaction costs

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May 27, 2010

Abstract

We consider a class of production-investment models in discrete time with proportional transaction costs. For linear production functions, we study a natural extension of the no-arbitrage of the second kind condition introduced by M. Rásonyi [13]. We show that this condition implies the closedness of the set of attainable claims and is equivalent to the existence of a strictly consistent price system under which the evaluation of future production profits are strictly negative. This allows to discuss the closedness of the set of terminal wealth in models with non-linear production functions which may admit arbitrages of the second kind for low production regimes but not marginally for high production regimes.

Key words : financial markets with transaction costs, non-linear returns, no-arbitrage of the second kind, consistent price systems.

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[*] This research is part of the Chair *Finance and Sustainable Development* sponsored by EDF and Calyon, and the Chair *Les particuliers face au risque* sponsored by Groupama.

[†] Adrien Nguyen Huu expresses his gratitude to EDF R&D and FiME *Laboratoire de Finance des Marchés d'Énergies* for its financial support.

1 Introduction

This paper is motivated by applications in optimal hedging of electricity derivatives for electricity producers. Electricity producers sell derivative contracts that allow to buy electricity at different periods and at a price fixed in advance. In practice, the producer can deliver the required quantities of electricity either by producing it or by buying it on the spot market. He can also try to cover itself through future contracts, but the granularity of the maturities available on the market is in general not sufficient.

It is a typical situation where a financial agent can manage a portfolio by either trading on a financial market or by producing itself a good. Such models have already been studied in the literature, in particular by Bouchard and Pham [1] who discussed the questions of no-arbitrage, super-hedging and expected utility maximization in a discrete time model with proportional transaction costs, see also Kabanov and Kijima [6] and the references therein.

In the above paper, the assets are divided in two classes. The first class corresponds to pure *financial assets*, e.g. bonds, stocks, options, etc... The second class corresponds to *industrial assets*, e.g. plants or buildings. Contrary to pure financial assets, industrial assets can not be short-sold. Moreover, they produce at each period a (random) return, labeled in terms of pure financial assets, which depend on the current inventory in industrial assets.

This model is well-adapted to industrial investment problems but not to production issues, since the production regime does not appear as a control.

In this paper, we consider another approach. As in [1], we work in a discrete time model with proportional transaction costs. Although it does not need to be explicit in the model, we have in mind that the assets are divided in two classes: the pure financial assets and the ones that are used for production purposes. Both can be traded on the market but some of them can be consumed in order to produce other assets. For instance, coal can be traded on the market but is also used to produce electricity that can then be sold so as to provide currencies. The quantity used for production on the time period $[t, t+1]$ is chosen at time t. It gets out of the portfolio and enters a production process. Depending on the quantity used, a (random) return enters the portfolio at time $t+1$. Therefore, the main difference with [1] is that we explicitly decide at each time what should be the regime of production, rather than letting it be determined just by inventories.

Obviously, both approaches could be combined. We refrain from doing this in this paper in order to isolate the effect of our production model and to avoid too many unnecessary complexities.

As in [1], we first discuss the absence of arbitrage opportunity and its dual characterization. In [1], the authors adapt the notion of robust no-arbitrage introduced by Schachermayer

[14]. It essentially means that there is still no-arbitrage even if transaction costs are slightly reduced and production returns are slightly increased. It should be noted that the arguments used in [1] could be easily adapted to our context. However, we prefer to adopt the (more natural) notion of no-arbitrage of the second kind, which was recently introduced in the context of financial markets with transactions costs by Rásonyi [13] under the name of *no-sure gain in liquidation value*, see also [4] for a continuous time version. It says that we can not turn a position which is not solvent at time t into a position which is a.s. solvent at a later time T by trading on the market. In models without transaction costs, this corresponds to the usual notion of no-arbitrage.

Another difference with Bouchard and Pham [1] is that we allow for reasonable arbitrages due to the production possibilities. Here, reasonable means that it may be possible to have a.s. *positive* net returns for low production regimes. However, they should be limited. Otherwise stated *marginal* arbitrages for high production regimes are not possible. The way we model this consists in assuming that the production function $\beta \to R(\beta)$, which is typically concave, admits an affine upper bound $\beta \to c + L\beta$, and that the linear model in which R is replaced by L admits no arbitrage of the second kind. We have in mind that it should hold for L defined by $\lim_{\alpha\to\infty} R(\alpha\beta)/\alpha = L\beta$, i.e. no-arbitrage holds in a marginal way for large regimes β. From the economic point of view, this means that gains can be made from the production in reasonable situations, but that it becomes (marginally) risky when the regime of production is pushed too high.

From the mathematical point of view, it allows to reduce at first to a linear model for which a nice dual formulation of the no-arbitrage condition is available, in the sense that the set of dual variables can be fully described in terms of martingales evolving in appropriate sets. This is not the case for non-linear models, compare with [1]. They are constructed by following the arguments of Rásonyi [13] which do not require to prove the closure of the set of attainable claims a-priori. Once they are constructed, one can then easily show that the set of attainable claims is indeed closed in probability (more precisely, Fatou-closed) in the linear and in the original models. As usual this leads to a dual formulation of these sets, and can also be used to prove existence for expected utility maximization problems, which, in particular, opens the door to the study of indifference prices.

We refer to [9] for a wide overview of models with proportional transaction costs. See also [11] and [12] for some more recent results in discrete time, and [5] for the continuous time setting.

The rest of the paper is organized as follows. We first describe our model, state the dual characterization of our no-arbitrage condition and important closure properties in Section 2. Section 3 discusses applications to super-hedging and utility maximization problems. The proofs are collected in Section 4.

Notations: If nothing else is specified, any element $x \in \mathbb{R}^d$ will be viewed as a column vector with entries x^i, $i \le d$, and transposition is denoted by x' so that $x'y$ stands for the natural scalar product. We write $\mathbb{M}^d$ to denote the set of square matrices M of dimension d with entries M^{ij}, $i,j \le d$. The identity matrix is denoted by I_d. The unit ball of $\mathbb{R}^d$ is denoted by B_1. As usual, $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ stand for $[0,\infty)^d$ and $(-\infty,0]^d$. The closure of a set $\Theta \subset \mathbb{R}^n$ is denoted by $\bar{\Theta}$, $n \ge 1$. We write cone(Θ) (resp. conv(Θ)) to denote the cone (resp. convex cone) generated by Θ. Given a filtrations $\mathbb{F}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a set-valued $\mathcal{F}$-measurable family $A = (A_t)_{t \le T}$, we denote by $L^0(A, \mathbb{F})$ the set of adapted processes $X = (X_t)_{t \le T}$ such that $X_t \in A_t$ $\mathbb{P} -$ a.s. for all $t \le T$. For a σ-algebra $\mathcal{G}$ and a $\mathcal{G}$-measurable random set A, we write $L^0(A, \mathcal{G})$ for the collection of $\mathcal{G}$-measurable random variables that take values in A $\mathbb{P} -$ a.s. We similarly define the notations $L^p(A, \mathcal{G})$ for $p \in \mathbb{N} \cup \infty$, and simply write L^p if A and $\mathcal{G}$ are clearly given by the context. If nothing else is specified, inequalities between random variables or inclusion between random sets have to be understood in the a.s. sense.

2 Definitions and main results

2.1 Model description

From now on we denote by $T \in \mathbb{N} \setminus \{0\}$ a fixed time horizon and set $\mathbb{T} := \{0, 1, \dots, T\}$. The complete filtration of the investor, $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$, is supported by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ is trivial.

As in [14], we model exchange prices by an adapted process $\pi = (\pi_t)_{t \in \mathbb{T}}$ taking values in the set $\mathbb{M}^d$ of square d-dimensional matrices, for some $d \ge 1$, satisfying the following conditions for all $t \le T$ and $i, j, k \le d$:

$$\text{(i) } \pi_t^{ij} > 0, \text{ (ii) } \pi_t^{ii} = 1, \text{ (iii) } \pi_t^{ij} \pi_t^{jk} \ge \pi_t^{ik} \, .$$

Here, π_t^{ij} should be interpreted as the number of units of asset i required to obtain one unit of asset j at time t. The conditions (i) and (ii) need no comment. The third condition is also natural, it means that it is always cheaper to buy directly units of asset k from units of asset i rather then going through the asset j. Note that, combined with (ii), it implies that $\pi_t^{ij} \pi_t^{ji} \ge 1$, which means that the ask price is always greater than the bid price. The case where $\pi_t^{ij} \pi_t^{ji} = 1$ corresponds to the situation where the ask and bid prices are the same, i.e. there is no friction.

All over this paper, we shall consider the so-called *efficient friction* case:

Standing Assumption (EF) : $\pi_t^{ij} \pi_t^{ji} > 1$ for all $i \neq j \le d$ and $t \in \mathbb{T}$.

It means that ask prices are always strictly greater than bid prices.

As in [7] and [8], we model portfolios as d-dimensional processes, each component i corresponding to the number of units of asset i held. The composition of a portfolio holding V_t at time t can be changed by acting on the financial market. If ξ_t denotes the net number of additional units of each asset in the portfolio after trading at time t, it should satisfy the standard self-financing condition. In our context, this means that $\xi_t \in -K_t$, whenever we allow to throw away a non-negative number of the holdings, where, for each $\omega \in \Omega$,

$$K_t(\omega) := \mathrm{conv}\left(\pi^{ij}(\omega)e_i - e_j, e_i \; ; \; i, j \leq d\right) ,$$

where e_i stands for the i-th unit vector of $\mathbb{R}^d$ defined by $e_i^k = 1_{i=k}$.

Note that $V_t \in K_t$ means that there exists $\xi_t \in -K_t$ such that $V_t + \xi_t = 0$. This explains why K_t is usually referred to as the *solvency cone*, i.e. the set of positions that can be turned into positions with non-negative entries by immediately trading on the market.

As in Bouchard and Pham [1], we also allow for production. In [1], the production regime depends only on the inventories in some production assets. Here, we consider a different approach based on a full control of the production regimes. Namely, we consider a family of random maps $(R_t)_{t\in\mathbb{T}}$ from $\mathbb{R}^d_+$ into $\mathbb{R}^d$ which corresponds to production functions. It turns β_t units of assets taken from the portfolio at time t into $R_{t+1}(\beta_t)$ additional units of assets in the portfolio at time $t+1$. For the moment, we only assume that R_{t+1} is $\mathcal{F}_{t+1}$ measurable, in the sense that $R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$ for all $\beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$. The control β_t can be associated to a *regime of production*. The greater, component by component, β_t is and the more the producer is putting inputs into the production system.

All together, a strategy is a pair of adapted processes

$$(\xi, \beta) \in \mathcal{A}_0 := L^0((-K) \times \mathbb{R}^d_+),$$

i.e. such that $(\xi_t, \beta_t) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$ for all $0 \leq t \leq T$. The corresponding portfolio process, starting from 0, can be written as $V^{\xi,\beta} = (V_t^{\xi,\beta})_{t\in\mathbb{T}}$ where

$$V_t^{\xi,\beta} := \sum_{s=0}^{t} \left(\xi_s - \beta_s + R_s(\beta_{s-1})1_{s\geq 1}\right) \; . \tag{2.1}$$

Example 2.1 *Let us consider the case of a firm which produces electricity from coal. It has two plants, one in the United-States, the other one in Europe. The first asset, asset* 1, *is dollars. The second one, asset* 2, *is euros. The coal can be bought and delivered in the United-State, this is the third asset, or in Europe, this is the fourth one. Eventually, it produces electricity which can be sold in the area where it is produced. The random return depends typically on the spot price, but additional randomness du to the production process* *itself can be added. Since the production regime depends on the amount of coal used for production, the production function should only depend on the regime* β *through* (β^3, β^4)*, i.e.* $R_t(\beta) = R_t(\tilde{\beta})$ *whenever* $(\beta^3, \beta^4) = (\tilde{\beta}^3, \tilde{\beta}^4)$*. The electricity produced in the United-States (resp. Europe) is converted into dollars (resp. euros), which implies* $R_t^i(\beta) = R_t^i(\tilde{\beta})$ *whenever* $\beta^{2+i} = \tilde{\beta}^{2+i}$ *for* $i = 1, 2$*. Due to operational costs, paid in dollars (resp. euros) for the plant located in the United-States (resp. Europe), we may have* $R_t^i(\beta) \leq 0$ *for* $i = 1, 2$*. However, electricity production does not transform into coal, so that* $R^i \equiv 0$ *for* $i = 3, 4$*. Note that, although the model allows strategies* β *such that* $\beta^1 \neq 0$ *or* $\beta^2 \neq 0$*, they would never be optimal. It would therefore be equivalent, from no-arbitrage or optimal investment points of view, to impose the constraint* $\beta^1 \equiv \beta^2 \equiv 0$*.*

Remark 2.1 Observe that we do not impose constraints on portfolio processes. In particular, one can consume some asset for production purposes although we do not hold them. This means that one can borrow some units of assets to use them in the production system. As usual additional convex constraints could be introduced without much difficulty.

In the following, we shall denote by

$$A_t^R(T) := \left\{ \sum_{s=t}^{T} \xi_s - \beta_s + R_s(\beta_{s-1}) 1_{s \geq t+1} \, , \; (\xi, \beta) \in \mathcal{A}_0 \right\} \; , \; t \leq T \; ,$$

the set of portfolio holdings that are attainable at time T by trading from time t with a zero initial holding.

Remark 2.2 The sequence of random cones $K = (K_t)_{t \in \mathbb{T}}$ is defined here through the bid-ask process π. However, it should be clear that all our analysis would remain true in a more abstract framework. Namely, one could only consider that K is a sequence of closed convex cones such that K_t is $\mathcal{F}_t$-measurable, $\mathbb{R}_+^d \subset K_t$ and $K_t \cap (-K_t) = \{0\}$ for all $t \leq T$.

2.2 The no-arbitrage condition

In a model without production, i.e. $R \equiv 0$, it was recently proposed by Rásonyi [13] to consider the following *no-arbitrage of the second kind condition*, also called *no-sure gain in liquidation value*, **NGV** in short:

$\mathbf{NA2^0}$: $(\zeta + A_t^0(T)) \cap L^0(K_T, \mathcal{F}) \neq \{0\} \Rightarrow \zeta \in L^0(K_t, \mathcal{F})$, for all $\zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $t \leq T$.

It means that we can not end-up at time T with a solvable position without taking any risk if the initial position was not already solvable.

In this paper, we shall impose a similar condition on the pure financial part of the model, i.e. there is no-arbitrage of the second kind for strategies of the form $(\xi, 0) \in \mathcal{A}_0$. Contrary to [1], we do not exclude arbitrages coming from the production whenever the production regime is small. We only exclude marginal arbitrages for high regimes of production in the following sense:

Definition 2.1 1. *We say that there is* no marginal arbitrage of the second kind for high production regimes, *in short* **NMA2** *holds, if there exists* $(c, L) \in L^\infty(\mathbb{R}^d \times \mathbb{M}^d, \mathbb{F})$ *such that* $\mathbf{NA2}^L$ *holds and*

$$c_{t+1} + L_{t+1}\beta - R_{t+1}(\beta) \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \text{ for all } \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \text{ and } t < T. \tag{2.2}$$

2. *We say that there is* no arbitrage of the second kind *for L, in short* $\mathbf{NA2}^L$ *holds, if*

(i) $\zeta - \beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \zeta \in K_t$,

(ii) $-\beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \beta = 0$,

for all $(\zeta, \beta) \in L^0(\mathbb{R}^d \times \mathbb{R}^d_+, \mathcal{F}_t)$ *and* $t < T$

The condition (2.2) means that the production function R_t admits an affine upper-bound. In most production models, the map R_t is concave and therefore typically admits such a bound. In the second assertion, we focus on the production model where R is replaced with the linear map associated to L. The fact that we consider the production map $\beta \mapsto L_{t+1}\beta$ instead of $\beta \mapsto c_{t+1} + L_{t+1}\beta$ coincides with the idea that we only want to avoid arbitrages for high production regimes: for large values of $|L_{t+1}\beta|$, c_{t+1} becomes negligible.
For $L \equiv 0$, the condition (i) is equivalent to the **NGV** condition of [13], this follows from a simple induction under the standing assumption **(EF)** above. Our version is a simple extension to the production-investement model. The condition (i) means that, even if we produce, we can not have for sure a solvable position at time $t + 1$ if the position was not already solvable at time t. The condition (ii) means that producing may lead to net *losses*.

Remark 2.3 If $\operatorname{esssup}\{|R_{t+1}(\beta)|, \beta \in L^0(\mathbb{R}^d_+, \mathcal{F})\} \in L^\infty$ for all $t < T$, then one can choose $L \equiv 0$. In this case, **NMA2** coincides with the **NGV** condition of [13] on the pure financial part, i.e. the no-arbitrage condition is set only on strategies of the form $(\xi, 0)$.

We conclude this section with an example.

Example 2.2 *Consider the electricity production model of Example 2.1. Recall that* $R^j_t \equiv 0$ *for* $j \geq 3$ *and that* R^i_t *depends on* β *only through* β^{2+i} *for* $i = 1, 2$. *If* R^1_t *and* R^2_t *are* $\mathbb{P}-a.s.$ *concave and non-decreasing, component by component, then* $R_t(\alpha\beta)/\alpha$ *admits* $\mathbb{P}-a.s.$ *a limit* $L_t(\beta)$ *as* $\alpha \to \infty$, *where the map* $\beta \mapsto L_t(\beta)$ *is* $\mathbb{P}-a.s.$ *linear and can thus be associated to a random matrix* L_t *of dimension* 4. *Clearly,* $L^{ij}_t \equiv 0$ *if* $i \in \{3, 4\}$ *or* $j \in \{1, 2\}$. *Moreover, we clearly can find* $c_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ *such that* (2.2) *holds.*

2.3 Dual characterization of the no-arbitrage condition and closure properties

Before to state our main results, let us introduce some additional notations and definitions. We first define the positive dual cone process $K^* = (K^*_t)_{t\in\mathbb{T}}$ associated to K by

$$K^*_t(\omega) := \left\{ z \in \mathbb{R}^d \ : \ x'z \geq 0 \ \text{ for all } x \in K_t(\omega) \right\} \ , \ \omega \in \Omega \ .$$

For $t \leq \tau \leq T$, we denote by $\mathcal{M}^\tau_t(\text{int}K^*)$ the set of martingales Z with positive components satisfying $Z_s \in L^0(\text{int}K^*_s, \mathcal{F}_s)$ for all $t \leq s \leq \tau$.
Elements of $\mathcal{M}^T_t(\text{int}K^*)$ were called *strictly consistent price systems*, on $[t, T]$, in [14]. They have the standard interpretation to be associated to a system of prices in a fictitious market without transaction costs that admits a martingale measure, and such that the relative prices evolve in the interior of the corresponding bid-ask intervals of the original model induced by π, i.e. are more favorable for the financial agent. Indeed, one easily checks that

$$K^*_t(\omega) := \left\{ z \in \text{int}\mathbb{R}^d_+ \ : \ z^j/z^i \leq \pi^{ij}_t(\omega) \ \text{ for all } i \neq j \leq d \right\} \ . \tag{2.3}$$

Otherwise stated, given $Z \in \mathcal{M}^T_t(\text{int}K^*)$, the process $\bar{Z}$, defined by $\bar{Z}^i_s := Z^i_s/Z^1_s$ for $t \leq s \leq T$, i.e. where the first asset is taken as a numéraire, is a martingale on $[t, T]$ under the measure $\mathbb{Q}$ induced by the conditional density process $(Z^1_s/Z^1_t)_{t\leq s\leq T}$ and satisfies $\bar{Z}^j_s/\bar{Z}^i_s < \pi^{ij}_s$ for $t \leq s \leq T$.

Remark 2.4 Note that the condition **(EF)** above implies that, and is actually equivalent to, $\text{int}K^*_t \neq \emptyset$ for all $t \leq T$. This follows from (2.3).

Altogether, elements of $\mathcal{M}^T_0(\text{int}K^*)$ play a similar role as equivalent martingale measures in frictionless markets, see e.g. [14] and the references therein. In particular, it was shown in [13] that, for $L \equiv 0$, the no-arbitrage condition $\mathbf{NA2}^0$ is equivalent to:

$\mathbf{PCE}^0$: for each $0 \leq t \leq T$ and $X \in L^1(\text{int}K^*_t, \mathcal{F}_t)$, there exists a process $Z \in \mathcal{M}^T_t(\text{int}K^*)$ satisfying $Z_t = X$.

This not only means that the no-arbitrage condition $\mathbf{NA2}^0$ implies the existence of a strictly consistent price system, but that strictly consistent price systems defined on any subinterval $[t, \tau]$ can also be extended consistently on $[t, T]$: for $Z \in \mathcal{M}^\tau_t(\text{int}K^*)$, one can find a strictly consistent price system $\tilde{Z} \in \mathcal{M}^T_t(\text{int}K^*)$ such that $\tilde{Z} = Z$ on $[t, \tau]$.
Such a property is obvious in frictionless markets but in general not true in our multivariate setting where the geometry of the cones $(K^*_t)_{t\in\mathbb{T}}$ is non-trivial.

In our production-investment setting, such price systems should also take into account the production function. When it is linear, given by the random matrix process L, the cost in

units at time t of a return (in units) $L_{t+1}\beta$ at time $t+1$ is $\beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$. For the price system $\bar{Z}$ and the pricing measure $\mathbb{Q}$, at time t, the value of the return is $\mathbb{E}^{\mathbb{Q}}[\bar{Z}'_{t+1}L_{t+1}\beta|\mathcal{F}_t]$ and the value of the cost is $\bar{Z}'_t\beta$. If no-arbitrage holds in this fictitious market, one should then be able to find $\mathbb{Q}$ such that $\mathbb{E}^{\mathbb{Q}}[\bar{Z}'_{t+1}L_{t+1}\beta|\mathcal{F}_t] \le \bar{Z}'_t\beta$ for all $\beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$.
If the fictitious price system is strictly more favorable than the original one, one should actually be able to choose it in such a way that $\mathbb{E}^{\mathbb{Q}}[\bar{Z}'_{t+1}L_{t+1}\beta|\mathcal{F}_t] < \bar{Z}'_t\beta$ for all $\beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$.

The above discussion naturally leads to the introduction of the set $\mathcal{L}^T_t(\mathrm{int}\mathbb{R}^d_-)$ of martingales Z with positive components satisfying $\mathbb{E}\left[Z'_{s+1}(L_{s+1} - I_d)|\mathcal{F}_s\right] \in L^0(\mathrm{int}\mathbb{R}^d_-)$ for all $t \le s < T$, $t \le T$.

Our first main result extends the property $\mathbf{NA2}^0 \Leftrightarrow \mathbf{PCE}^0$ to $\mathbf{NA2}^L \Leftrightarrow \mathbf{PCE}^L$ where

$\mathbf{PCE}^L$: for each $0 \le t \le T$ and $X \in L^1(\mathrm{int}K^*_t, \mathcal{F}_t)$, there exists a process $Z \in \mathcal{M}^T_t(\mathrm{int}K^*) \cap \mathcal{L}^T_t(\mathrm{int}\mathbb{R}^d_-)$ satisfying $Z_t = X$.

Theorem 2.1 $\mathbf{NA2}^L \Leftrightarrow \mathbf{PCE}^L$*.*

Following arguments used in [2], this easily implies that the sets

$$A^L_0(T) := \left\{ \sum_{s=0}^{T} \left(\xi_s - \beta_s + L_s(\beta_{s-1})1_{s\ge 1}\right),\ (\xi, \beta) \in \mathcal{A}_0 \right\},$$

and $A^R_0(T)$ of attainable claims in the linear and original models are Fatou-closed, in the following sense.

Definition 2.2 *We say that a set $A \subset L^0(\mathbb{R}^d, \mathcal{F})$ is Fatou-closed if for any sequence $(g^n)_{n\ge 1} \subset A$ which converges $\mathbb{P}$ − a.s. to some $g \in L^0(\mathbb{R}^d, \mathcal{F})$ and such that, for some $\kappa \in \mathbb{R}^d$, $g^n + \kappa \in K_T$ for all $n \ge 1$, then $g \in A$.*

In order to prove the closure property for $A^R_0(T)$, we shall need an additional upper-semicontinuity assumption:

$$(\mathbf{USC}) : \limsup_{\beta \in \mathbb{R}^d_+, \beta \to \beta^0} R_t(\beta) - R_t(\beta^0) \in -K_t \ \text{ for all } \beta^0 \in \mathbb{R}^d_+ ,$$

where the limsup is taken component by component.

Theorem 2.2 *$A^L_0(T)$ is Fatou-closed under* $\mathbf{NA2}^L$*. The same holds for $A^R_0(T)$ under* **NMA2** *and* (**USC**)*.*

3 Applications

3.1 Super-hedging theorems

As usual, the closure property allows to provide dual formulations for the set of attainable claims. We first formulate it in the linear model.

Proposition 3.1 *Assume that* $\mathbf{NA2}^L$ *holds. Let* $V \in L^0(\mathbb{R}^d, \mathcal{F})$ *be such that* $V + \kappa \in L^0(K_T, \mathcal{F})$ *for some* $\kappa \in \mathbb{R}^d$. *Then, the following are equivalent:*
(i) $V \in A_0^L(T)$,
(ii) $\mathbb{E}\left[Z_T' V\right] \le 0$ *for all* $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-)$.

In the original non-linear model, an abstract dual formulation is also available. However, due to the non-linearity of the set of attainable terminal claims, it requires the introduction of the following support function:

$$\alpha^R(Z) := \sup\left\{\mathbb{E}\left[Z_T' V\right],\ V \in A_{0b}^R(T)\right\}\ ,\ Z \in \mathcal{M}_0^T(K^*)\ ,$$

where

$$A_{0b}^R(T) := \left\{V \in A_0^R(T) \text{ s.t. } V + \kappa \in K_T \ \text{ for some } \ \kappa \in \mathbb{R}^d\right\}.$$

Moreover, as usual, we shall need the set $A_0^R(T)$ to be convex, which is easily checked under the following additional assumption:

$$\begin{aligned}(\mathbf{R}):\quad &\text{(a)}\quad \alpha R_t(\beta_1) + (1-\alpha)R_t(\beta_2) - R_t(\alpha\beta_1 + (1-\alpha)\beta_2) \in -K_t \\ &\qquad\qquad \text{for all } \alpha \in L^0([0,1], \mathcal{F}), \beta_1, \beta_2 \in L^0(\mathbb{R}^d_+, \mathcal{F})\ ,\ t \le T. \\ &\text{(b)}\quad R_t(\beta) \in L^\infty(\mathbb{R}^d, \mathcal{F}) \quad \text{for all } t \le T \text{ and } \beta \in L^\infty(\mathbb{R}^d, \mathcal{F})\ .\end{aligned}$$

The last assumption implies that bounded strategies lead to bounded terminal wealth, which will be of important use in our proofs. For ease of notations, we now set

$$C_t^T := \sum_{s=t+1}^{T} c_s\ \ ,\ \ t < T\ . \tag{3.1}$$

Proposition 3.2 *Assume that* **NMA2** , **(USC)** *and* **(R)** *hold. Let* $V \in L^0(\mathbb{R}^d, \mathcal{F})$ *be such that* $V + \kappa \in L^0(K_T, \mathcal{F})$ *for some* $\kappa \in \mathbb{R}^d$. *Then, the following are equivalent:*
(i) $V \in A_0^R(T)$
(ii) $\mathbb{E}\left[Z_T' V\right] \le \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(\mathrm{int}K^*)$.
Moreover, $\alpha^R(Z) \le \mathbb{E}\left[Z_T' C_0^T\right] < \infty$ *for all* $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-)$.

In the case where the linear map L coincides with the asymptotic behavior of R, i.e.

$$(\mathbf{RL}):\ \lim_{\eta\to\infty} R_t(\eta\beta)/\eta = L_t\beta \quad \text{for all } \beta \in \mathbb{R}^d_+\ ,$$

one can restrict to elements in $\mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-)$ in the above dual formulation.

Proposition 3.3 *Let the conditions of Proposition 3.2 hold. Assume further that* **(RL)** *is satisfied. Let* $V \in L^0(\mathbb{R}^d, \mathcal{F})$ *be such that* $V + \kappa \in L^0(K_T, \mathcal{F})$ *for some* $\kappa \in \mathbb{R}^d$*. Then, the following are equivalent:*
(i) $V \in A_0^R(T)$
(ii) $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}_-^d)$.

3.2 Utility maximization

In order to avoid technical difficulties, we shall only discuss here the case of a (possibly) random utility function defined on $\mathbb{R}^d$ and essentially bounded from above. More general cases could be discussed by following the line of arguments of [1].
We therefore let U be a $\mathbb{P}-$ a.s.-upper semi-continuous concave random map from $\mathbb{R}^d$ to $[-\infty, 1]$ such that $U(V) = -\infty$ on $\{V \notin K_T\}$ for $V \in L^0(\mathbb{R}^d, \mathcal{F})$. Given an initial holding $x_0 \in \mathbb{R}^d$, we assume that

$$\mathcal{U}(x_0) := \left\{V \in A_0^R(T) \ : \ \mathbb{E}\left[|U(x_0 + V)|\right] < \infty\right\} \neq \emptyset.$$

Then, existence holds for the associated expected utility maximization problem under the conditions of Proposition 3.2.

Proposition 3.4 *Assume that* **NMA2** *,* **(USC)** *and* **(R)** *hold. Assume further that* $\mathcal{U}(x_0) \neq \emptyset$*. Then, there exists* $V(x_0) \in A_0^R(T)$ *such that*

$$\mathbb{E}\left[U(x_0 + V(x_0))\right] = \sup_{V \in \mathcal{U}(x_0)} \mathbb{E}\left[U(x_0 + V)\right] \ .$$

4 Proofs

4.1 No-arbitrage of the second kind in the linear model and (K, L)-strictly consistent price systems

In this section, we first prove that the no-arbitrage of the second kind assumption $\mathbf{NA2}^L$ implies the existence of an element $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}_-^d)$, which we call (K, L)-*strictly consistent price system.*

The arguments used in the proof of Proposition 4.1 below are inspired by [13], up to non-trivial modifications. This proposition readily implies that $\mathbf{NA2}^L \Rightarrow \mathbf{PCE}^L$ up to an obvious induction argument. Before to state it, we recall the following technical result that will be used in its proof, see Lemma 4.3 in [12].

Lemma 4.1 *Let $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$ be σ-algebras. Let $C \subset B_1$ be a $\mathcal{H}$-measurable random convex compact set. Then, there exists a $\mathcal{G}$-measurable random convex compact set $\mathbb{E}[C|\mathcal{G}] \subset B_1$ satisfying*

$$L^0(\mathbb{E}[C|\mathcal{G}], \mathcal{G}) = \{\mathbb{E}[\vartheta|\mathcal{G}] : \vartheta \in L^0(C, \mathcal{H})\}.$$

Proposition 4.1 *Assume that $\mathbf{NA2}^L$ holds. Then, for all $t < T$ and $X \in L^1(\text{int}K_t^*, \mathcal{F}_t)$, there exist $Z \in L^1(\text{int}K_{t+1}^*, \mathcal{F}_{t+1})$ such that $X = \mathbb{E}[Z \mid \mathcal{F}_t]$ and $\mathbb{E}[Z'(L_{t+1} - I_d) \mid \mathcal{F}_t] \in \text{int}\mathbb{R}^d_-$.*

Proof We fix $t < T$. For ease of notation, we set $M_{t+1} := L_{t+1} - I_d$. Recall that $L_{t+1} \in L^\infty(\mathbb{M}^d, \mathcal{F}_{t+1})$ so that M_{t+1} is essentially bounded.

1. We first show that $\text{int}\mathbb{R}^d_- \subset \text{cone}(\text{int}\mathbb{E}[\Theta|\mathcal{F}_t]) =: H$, where

$$\Theta := \left\{M'_{t+1}y + r,\ (y, r) \in (K^*_{t+1} \cap B_1) \times [0,1]^d\right\},$$

recall that B_1 is the unit ball of $\mathbb{R}^d$. For later use, observe that, since M_{t+1} is essentially bounded, Lemma 4.1 applies to Θ up to an obvious scaling argument.

If $\text{int}\mathbb{R}^d_- \not\subset H$, then $\mathbb{R}^d_- \not\subset \overline{H}$ on a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$. For each $\omega \in A$, $\bar{H}(\omega)$ being a closed convex cone, we can then find $p(\omega) \in \mathbb{R}^d_-$ and $\beta(\omega) \in \mathbb{R}^d$ such that

$$p(\omega)'\beta(\omega) < 0 \le q'\beta(\omega) \text{ for all } q \in \bar{H}(\omega) \quad \text{for } \omega \in A\,. \tag{4.1}$$

By a standard measurable selection argument, see e.g. [3, III-45], one can assume that p and β are $\mathcal{F}_t$-measurable. The right-hand side of (4.1), Lemma 4.1 and the fact that K^*_{t+1} is a cone then imply that

$$(Y'M_{t+1} + \rho')\beta\mathbf{1}_A \ge 0 \ \text{ for all } (Y, \rho) \in L^\infty(K^*_{t+1} \times \mathbb{R}^d_+, \mathcal{F}_{t+1}),$$

which leads to $\beta\mathbf{1}_A \in \mathbb{R}^d_+$ and $M_{t+1}\beta\mathbf{1}_A \in K_{t+1}$. In view of $\mathbf{NA2}^L$, this implies that $\beta\mathbf{1}_A = 0$, which contradicts the left-hand side of (4.1).

2. We next show that there exists $\tilde{Y} \in L^\infty(\text{int}K^*_{t+1}, \mathcal{F}_{t+1})$ such that $\mathbb{E}\left[\tilde{Y}'M_{t+1} \mid \mathcal{F}_t\right] \in \text{int}(\mathbb{R}^d_-)$.

To see this, fix $\eta \in L^\infty(\text{int}\mathbb{R}^d_-, \mathcal{F}_t)$ and $Z \in L^\infty(\text{int}K^*_{t+1}, \mathcal{F}_{t+1})$. Set $\bar{Z} := \mathbb{E}[Z'M_{t+1} \mid \mathcal{F}_t]$. We can then find $\varepsilon \in L^\infty((0,1], \mathcal{F}_t)$ such that $\eta - \varepsilon\bar{Z} \in L^\infty(\text{int}\mathbb{R}^d_-, \mathcal{F}_t)$. In view of step 1 and Lemma 4.1, there exists $(Y, \rho) \in L^\infty(K^*_{t+1} \times \mathbb{R}^d_+, \mathcal{F}_{t+1})$ and $\alpha \in L^0(\text{int}\mathbb{R}_+, \mathcal{F}_t)$ such that $\eta - \varepsilon\bar{Z} = \alpha\mathbb{E}[Y'M_{t+1} + \rho \mid \mathcal{F}_t]$ or, equivalently, $\eta - \alpha\mathbb{E}[\rho \mid \mathcal{F}_t] = \mathbb{E}[(\alpha Y + \varepsilon Z)'M_{t+1} \mid \mathcal{F}_t]$. Clearly, $\eta - \alpha\mathbb{E}[\rho \mid \mathcal{F}_t] \in \text{int}(\mathbb{R}^d_-)$ and $\alpha Y + \varepsilon Z \in L^0(\text{int}(K^*_{t+1}), \mathcal{F}_{t+1})$. The required result is thus obtained for $\tilde{Y} := (\alpha Y + \varepsilon Z)/(1 + \alpha)$.

3. We now show that $K^*_t \times \{0\} \subset \text{cone}(\mathbb{E}[\Gamma|\mathcal{F}_t]) =: E$ where

$$\Gamma := \left\{(y, M'_{t+1}y + r),\ (y, r) \in (K^*_{t+1} \cap B_1) \times [0,1]^d\right\}\ .$$

Since $\mathbb{E}\left[\Gamma|\mathcal{F}_t\right]$ is a.s. convex and compact, see Lemma 4.1, it follows that E is $\mathbb{P}-$a.s. convex and closed. Thus, if $K_t^* \times \{0\} \not\subset E$ on a set $A \in \mathcal{F}_t$, with $\mathbb{P}[A] > 0$, the same arguments as in step 1 imply that we can find $(p, 0) \in L^0(K_t^* \times \{0\}, \mathcal{F}_t)$ and $(\zeta, \beta) \in L^0(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{F}_t)$ such that

$$p'\zeta < 0 \text{ on } A \quad \text{and} \quad 0 \leq Y'(\zeta + M_{t+1}\beta) + \rho'\beta \text{ for all } (Y, \rho) \in L^\infty(K_{t+1}^* \times \mathbb{R}_+^d, \mathcal{F}_{t+1}) \; .$$

The right-hand side implies that $\beta \in \mathbb{R}_+^d$ and $\zeta + M_{t+1}\beta \in K_{t+1}$. In view of $\mathbf{NA2}^L$, this implies that $\zeta \in K_t$, thus leading to a contradiction with the left-hand side, since $p \in K_t^*$.

4. We can now conclude the proof. Fix $X \in L^1(\text{int}K_t^*, \mathcal{F}_t)$, let $\tilde{Y}$ be as in step 2 and fix $\varepsilon \in L^1((0, 1], \mathcal{F}_t)$ such that $\tilde{X} := X - \varepsilon\mathbb{E}\left[\tilde{Y} \mid \mathcal{F}_t\right] \in L^1(K_t^*, \mathcal{F}_t)$. It then follows from step 3 and Lemma 4.1 that we can find $Y \in L^0(K_{t+1}^*, \mathcal{F}_{t+1})$ such that $\tilde{X} = \mathbb{E}\left[Y \mid \mathcal{F}_t\right]$ and $\mathbb{E}\left[Y'M_{t+1} \mid \mathcal{F}_t\right] \in \mathbb{R}_-^d$. This implies that $X = \mathbb{E}\left[Y + \varepsilon\tilde{Y} \mid \mathcal{F}_t\right]$ and $\mathbb{E}\left[(Y + \varepsilon\tilde{Y})'M_{t+1} \mid \mathcal{F}_t\right] \in \text{int}\mathbb{R}_-^d$ where $Y + \varepsilon\tilde{Y} \in \text{int}K_{t+1}^*$. Since $X \in L^1$ and $K^* \subset \mathbb{R}_+^d$, we must have $Z := Y + \varepsilon\tilde{Y} \in L^1$. This shows the required result. □

It remains to prove the opposite inclusion of Theorem 2.1.

Proposition 4.2 $\mathbf{PCE}^L \Rightarrow \mathbf{NA2}^L$.

Proof We fix $t < T$.

1. We first assume that we can find $(\zeta, \beta) \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+^d, \mathcal{F}_t)$ satisfying

$$\zeta - \beta + L_{t+1}\beta \in K_{t+1}, \tag{4.2}$$

and such that $\zeta \notin K_t$ on a set $A \in \mathcal{F}_t$ of positive measure. This implies that we can find $Z_t \in L^1(\text{int}K_t^*, \mathcal{F}_t)$ such that

$$Z_t'\zeta < 0 \;\text{ on } A \; . \tag{4.3}$$

In view of $\mathbf{PCE}^L$, we can then find $Z_{t+1} \in L^1(\text{int}K_{t+1}^*, \mathcal{F}_{t+1})$ such that $\mathbb{E}\left[Z_{t+1}|\mathcal{F}_t\right] = Z_t$ and $\mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \text{int}\mathbb{R}_-^d$. By (4.2), we have $Z_{t+1}'\zeta + Z_{t+1}'(L_{t+1} - I_d)\beta \geq 0$ which, by taking conditional expectations, leads to $Z_t'\zeta + \mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right]\beta \geq 0$. Since $\mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \text{int}\mathbb{R}_-^d$ and $\beta \in \mathbb{R}_+^d$, this leads to a contradiction with (4.3).

2. We now assume that $\beta \in L^0(\mathbb{R}_+^d, \mathcal{F}_t)$ is such that $(L_{t+1} - I_d)\beta \in K_{t+1}$. For Z_{t+1} defined as above, we obtain $Z_{t+1}'(L_{t+1} - I_d)\beta \geq 0$ while $\mathbb{E}\left[Z_{t+1}'(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \text{int}\mathbb{R}_-^d$. This implies that $\beta = 0$. □

4.2 The closure properties

In this section, we prove that the sets $A_0^L(T)$ and $A_0^R(T)$ are Fatou-closed whenever there exists a (K, L)-strictly consistent price system, i.e. $\mathcal{M}_0^T(\text{int}K^*) \cap \mathcal{L}_0^T(\text{int}\mathbb{R}_-^d) \neq \emptyset$. In view of Theorem 2.1, Theorem 2.2 is a direct consequence of Corollary 4.1 below. We start with the following key lemma which is inspired from Lemma 12 in [2].

Lemma 4.2 *Assume that there exists* $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-)$. *Then, there exists* $\mathbb{Q} \sim \mathbb{P}$ *and a constant* $\alpha \geq 0$, *such that, for all* $\kappa \in \mathbb{R}^d$ *and* $(\xi, \beta) \in \mathcal{A}_0$ *satisfying* $V_T^{\xi,\beta} + \kappa \in K_T$, *one has:*

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{t\in\mathbb{T}} (|\xi_t| + |\beta_t|)\right] \leq \alpha \left(\mathbb{E}\left[Z_T' C_0^T\right] + Z_0'\kappa\right) \ .$$

Proof In this proof, we set $M_{t+1} := L_{t+1} - I_d$ and $\bar{Z}_t := \mathbb{E}\left[Z_{t+1}' M_{t+1} | \mathcal{F}_t\right]$, for $t < T$, in order to alleviate notations. We first observe that $(Z_t, \bar{Z}_t) \in \mathrm{int}K_t^* \times \mathrm{int}\mathbb{R}^d_-$ implies:

$$Z_t'\xi \leq -\varepsilon|\xi| \text{ and } \bar{Z}_t'\beta \leq -\varepsilon|\beta| \text{ for all } (\xi, \beta) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t) \ , \ t \leq T \ , \tag{4.4}$$

for some $\varepsilon \in L^0((0,1), \mathcal{F})$, compare with Lemma 11 in [2].
We next deduce from (2.1)-(2.2) that

$$V_T^{\xi,\beta} = X_T \text{ where } X_t := \sum_{s\leq t} \xi_s + \zeta_s + c_s + M_s\beta_{s-1}\mathbf{1}_{s\geq 1} \ \text{ for some } \zeta \in L^0(-K). \tag{4.5}$$

Since $X_T + \kappa = V_T^{\xi,\beta} + \kappa \in K_T$, we have $Z_T' X_T \geq -Z_T'\kappa$ so that $\mathbb{E}\left[Z_T' X_T | \mathcal{F}_{T-1}\right]$ is well-defined since $Z_T \in L^1$. It then follows from the martingale property of Z, (4.4) and (4.5) that

$$-Z_{T-1}'\kappa \ \ \leq \ \ \mathbb{E}\left[Z_T' X_T | \mathcal{F}_{T-1}\right] \leq Z_{T-1}' X_{T-1} + \mathbb{E}\left[Z_T' C_{T-1}^T - \varepsilon\left(|\xi_T| + |\zeta_T| + |\beta_{T-1}|\right) | \mathcal{F}_{T-1}\right] \ .$$

Iterating this procedure leads to

$$-Z_0'\kappa \ \ \leq \ \ \mathbb{E}\left[Z_T' X_T\right] \leq \mathbb{E}\left[Z_T' C_0^T - \varepsilon \sum_{t\in\mathbb{T}} (|\xi_t| + |\zeta_t| + |\beta_{t-1}|\mathbf{1}_{t\geq 1})\right] \tag{4.6}$$

which implies the required result for $\mathbb{Q} \sim \mathbb{P}$ defined by $d\mathbb{Q}/d\mathbb{P} := \varepsilon\alpha$ with $\alpha := 1/\mathbb{E}[\varepsilon]$. □

We can now prove the closure properties.

Corollary 4.1 *Let the conditions of Lemma 4.2 hold. Then,* $A_0^L(T)$ *is Fatou-closed. If moreover* **(USC)** *holds, then* $A_0^R(T)$ *is Fatou-closed.*

Proof We only provide the proof for $A_0^R(T)$. Considering the case $R = L$ will then allow to conclude for $A_0^L(T)$ as well. Let $(V^n)_{n\geq 1} \subset A_0^R(T)$ be such that

$$V^n + \kappa \in K_T \text{ for all } n \geq 1 \text{ and } V^n \to V \in L^0(\mathbb{R}^d, \mathcal{F}) \ \mathbb{P} - \text{a.s.} \tag{4.7}$$

Let $(\xi^n, \beta^n)_{n\geq 1} \subset \mathcal{A}_0$ be such that

$$V_T^{\xi^n,\beta^n} = V^n \ \text{ for all } n \geq 1. \tag{4.8}$$

It then follows from Lemma 4.2 and the left-hand side of (4.7) that there exists $\mathbb{Q} \sim \mathbb{P}$ such that

$$\sup_{n\geq 1} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t\in\mathbb{T}} \left(|\xi_t^n| + |\beta_t^n| \right) \right] < \infty \, .$$

Using Lemma 4.3 below and a standard induction argument for $t = 0$ to $t = T$, one can then assume, after possibly passing to ($\mathcal{F}_t$-measurable random) subsequences, that $(\xi_t^n, \beta_t^n)_{n\geq 1}$ converges $\mathbb{P}-$ a.s. to some $(\xi_t, \beta_t) \in L^0((-K_t) \times \mathbb{R}_+^d, \mathcal{F}_t)$, for all $t \leq T$. Passing to the limit sup in (4.8), recall (2.1), and using the semi-continuity assumption **(USC)** then shows that $V_T^{\xi,\beta} + \sum_{t\in\mathbb{T}} \zeta_t = V$ where $(\xi, \beta) := (\xi_t, \beta_t)_{t\in\mathbb{T}}$ and $\zeta := (\zeta_t)_{t\in T} \in L^0(-K)$. Hence, $V_T^{\xi+\zeta,\beta} = V \in A_0^R(T)$. □

We conclude this section with the statement we used in the above proof, see [10].

Lemma 4.3 *Fix $t \leq T$ and $(\eta^n)_{n\geq 1} \subset L^0(\mathbb{R}^d, \mathcal{F}_t)$ be such that $\liminf_{n\to\infty} |\eta^n| < \infty$. Then, there exists a $\mathbb{P}-$ a.s.-increasing sequence $(\sigma(n))_{n\geq 1} \subset L^0(\mathbb{N}, \mathcal{F}_t)$ converging $\mathbb{P}-$ a.s. to ∞ such that $(\eta^{\sigma(n)})_{n\geq 1}$ converges $\mathbb{P}-$ a.s.*

4.3 Super-hedging theorems

We now turn to the proof of the super-hedging theorems, Propositions 3.1, 3.2 and 3.3.

Proof of Proposition 3.2: For ease of notations, we write M for $L - I_d$.
1. The fact that $\alpha^R(Z) \leq \mathbb{E}\left[Z_T' C_0^T\right]$ for all $Z \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}_-^d)$ is a consequence of the right-hand side inequality of (4.6) in the proof of Lemma 4.2 above.
2. Clearly, $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T(\mathrm{int}K^*)$ and $V \in A_{0b}^R(T)$, by definition of α^R. We now prove the converse implication. Fix $V \in L^0(\mathbb{R}^d, \mathcal{F})$ such that $V + \kappa \in K_T$ for some $\kappa \in \mathbb{R}^d$, and assume that $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T(\mathrm{int}K^*)$, but that $V \notin A_{0b}^R(T)$. Then, $A_{0b}^R(T)$ being Fatou-closed by Theorem 2.2, if follows that, for k large enough, $V^k := V\mathbf{1}_{|V|\leq k} - \kappa \mathbf{1}_{|V|>k}$ does not belong to $A_{0b}^R(T)$ either but satisfies

$$\mathbb{E}\left[Z_T' V^k\right] \leq \mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z) \;\text{ for all } Z \in \mathcal{M}_0^T(\mathrm{int}K^*). \tag{4.9}$$

Since $A_{0b}^R(T)$ is Fatou-closed, it follows from the Krein-Smulian theorem that $A_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F})$ is $\sigma(L^\infty(\mathbb{R}^d, \mathcal{F}), L^1(\mathbb{R}^d, \mathcal{F}))$-closed. The later being convex under **(R)**, we deduce from the Hahn-Banach theorem that we can find $Y \in L^1(\mathbb{R}^d, \mathcal{F})$ and $r \in \mathbb{R}$ such that

$$\mathbb{E}\left[Y' X\right] \leq r < \mathbb{E}\left[Y' V^k\right] \;\text{ for all } X \in A_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}) \, .$$

Set $Z_t^Y := \mathbb{E}\left[Y | \mathcal{F}_t\right]$. Recalling that $R(0)$ is essentially bounded under **(R)** and considering strategies of the form $(\xi \mathbf{1}_{t=s}, 0)_{t\in\mathbb{T}}$, with $\xi \in L^\infty(-K_s, \mathcal{F}_s)$, easily leads to $Z_s^Y \in K_s^*$ for

$s \leq T$. Fix $\tilde{Z} \in \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-) \neq \emptyset$, see Theorem 2.1, and $\varepsilon \in (0,1)$, so that $\check{Z} := \varepsilon\tilde{Z} + (1-\varepsilon)Z^Y \in \mathcal{M}_0^T(\mathrm{int}K^*)$ and

$$\mathbb{E}\left[\check{Z}_T' X\right] \leq (1-\varepsilon)r + \varepsilon\alpha^R(\tilde{Z}) < \mathbb{E}\left[\check{Z}_T' V^k\right] \quad \forall\, X \in A_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}), \tag{4.10}$$

where we recall from step 1 that $\alpha^R(\tilde{Z}) < \infty$. In order to conclude the proof, it suffices to show that

$$\alpha^R(Z) = \sup\left\{\mathbb{E}\left[Z_T' X\right],\ X \in A_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F})\right\}\ ,\ Z \in \mathcal{M}_0^T(K^*)\ , \tag{4.11}$$

which, combined with (4.10), would imply that $\alpha^R(\check{Z}) < \mathbb{E}\left[\check{Z}_T' V^k\right]$, thus leading to a contradiction to (4.9) since $\check{Z} \in \mathcal{M}_0^T(\mathrm{int}K^*)$.

To see that the above claim holds, first observe that, for $X \in A_0^R(T)$ such that $X + \rho \in K_T$ for some $\rho \in \mathbb{R}^d$, one can always construct an essentially bounded sequence, $X^n := X\mathbf{1}_{|X| \leq n} - \rho\mathbf{1}_{|X|>n}$ for $n \geq 1$, which Fatou-converges to X. Using Fatou's Lemma, one then obtains $\liminf_{n\to\infty} \mathbb{E}\left[Z_T' X^n\right] \geq \mathbb{E}\left[Z_T' X\right]$ for all $Z \in \mathcal{M}_0^T(K^*)$. Moreover, $X + \rho \in K_T$ implies $X - X^n \in K_T$ so that $X^n \in A_0^R(T)$ for all $n \geq 1$. This proves (4.11). □

Proof of Proposition 3.3: It suffices to repeat the argument of the above proof and to show that one can choose Z^Y such that $\mathbb{E}\left[Z_t^{Y\prime}(L_{t+1} - I_d)|\mathcal{F}_t\right] \in \mathbb{R}^d_-$ for all $t \leq T$. To see this, recall from the above arguments that Z^Y is a martingale and that it satisfies

$$\mathbb{E}\left[Z_T^{Y\prime} X\right] \leq r \ \text{ for all } X \in A_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F})\ ,$$

for some $r \in \mathbb{R}$. Recalling the second assertion in **(R)**, we see that any strategy of the form $\nu := (0, \beta_s\mathbf{1}_{t=s})_{t\in\mathbb{T}}$, with $\beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)$ is such that $V_T^\nu \in A_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F})$. Given $s \leq T-1$, we thus have

$$\mathbb{E}\left[Z_{s+1}^{Y}{}'(R_{s+1}^0(\beta_s) - \beta_s)\right] + \ell \leq r \quad \text{for all } \beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)\ , \tag{4.12}$$

where

$$R^0 := R - R(0) \ \text{ and } \ \ell := \mathbb{E}\left[Z_T^{Y\prime} \sum_{1\leq t\leq T} R_t(0)\right]\ .$$

Using the first assertion in **(R)**, we then deduce that, for $\eta \geq 1$ and $\beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)$,

$$\begin{aligned}
&R_{s+1}(\beta_s) - \eta^{-1}R_{s+1}(\eta\beta) - (1-\eta^{-1})R_{s+1}(0) \\
&= R_{s+1}(\eta^{-1}\eta\beta_s + (1-\eta^{-1})0) - \eta^{-1}R_{s+1}(\eta\beta) - (1-\eta^{-1})R_{s+1}(0) \in K_{s+1}\ .
\end{aligned}$$

This shows that, for all $\beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)$, the sequence $(Z_{s+1}^{Y}{}'R_{s+1}^0(n\beta_s)/n)_{n\geq 1}$ is non-increasing and that, by (4.12),

$$\mathbb{E}\left[Z_{s+1}^{Y}{}'(R_{s+1}^0(n\beta_s)/n - \beta_s)\right] \leq (r - \ell)/n\ .$$

Sending $n \to \infty$, using the monotone convergence theorem and recalling **(RL)** leads to

$$\mathbb{E}\left[{Z^Y_{s+1}}'(L_{s+1}\beta_s - \beta_s)\right] \le 0 \; .$$

By arbitrariness of $\beta_s \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_s)$, this readily implies that $\mathbb{E}\left[{Z^Y_{s+1}}'(L_{s+1} - I_d)|\mathcal{F}_s\right] \in \mathbb{R}^d_-$. □

Proof of Proposition 3.1: The result follows from Proposition 3.3 applied to $R = L$. In particular, since $0 \in A^L_0(T)$, taking $C^T_0 = 0$ in Proposition 3.2 implies $\alpha^L = 0$, where α^L is defined as α^R for $R = L$. □

4.4 Utility maximization

Proof of Proposition 3.4. Let $(V^n)_{n\ge1}$ be a maximizing sequence. Since $U(V) = -\infty$ on $\{V \notin K_T\}$, it must satisfy $V^n + x_0 \in K_T$ for all $n \ge 1$. It then follows from Proposition 3.2 that $\mathbb{E}\left[Z'_T V^n\right] \le \alpha^R(Z) < \infty$ for all $n \ge 1$, for a given $Z \in \mathcal{M}^T_0(\mathrm{int}K^*) \cap \mathcal{L}^T_0(\mathrm{int}\mathbb{R}^d_-)$. Hence, for $n \ge 1$, we have $\mathbb{E}\left[Z'_T(V^n + x_0)\right] \le \alpha^R(Z) + Z'_0 x_0 =: r < \infty$ where $V^n + x_0 \in K_T$ and $Z_T \in \mathrm{int}K^*_T$. This implies that $\mathbb{E}\left[\varepsilon|V^n + x_0|\right] \le r$ for some $\varepsilon \in L^0((0,1], \mathcal{F})$, compare with Lemma 11 in [2]. By Komlos Lemma, one can then find a sequence $(\tilde{V}^n)_{n\ge1}$ such that $\tilde{V}^n \in \mathrm{conv}(V^k,\ k \ge n)$ for all $n \ge 1$, and $(\tilde{V}^n)_{n\ge1}$ converges $\mathbb{P}-$a.s. to some $V(x_0) \in L^0(\mathbb{R}^d, \mathcal{F})$. Since $A^R_0(T)$ is convex under **(R)**, $(\tilde{V}^n)_{n\ge1} \subset A^R_0(T)$. Since $\tilde{V}^n + x_0 \in K_T$, for all $n \ge 1$, and $A^R_0(T)$ is Fatou-closed, see Theorem 2.2, we have $V(x_0) \in A^R_0(T)$. Moreover, the random map U being $\mathbb{P}-$a.s. concave, $(\tilde{V}^n)_{n\ge1}$ is also a maximizing sequence. Since $U(x_0 + \tilde{V}^n)^+ \le 1$ for each $n \ge 1$, we finally deduce from Fatou's Lemma and the $\mathbb{P}-$a.s. upper semi-continuity of U that

$$\sup_{V \in \mathcal{U}(x_0)} \mathbb{E}\left[U(x_0 + V)\right] = \limsup_{n\to\infty} \mathbb{E}\left[U(x_0 + \tilde{V}^n)\right] \le \mathbb{E}\left[U(x_0 + V(x_0))\right] \; .$$

□

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