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An arbitrage-free interest rate model consistent with economic constraints for long-term asset liability management

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Abstract

This paper proposes an Heath-Jarrow-Morton model of the yield curve that can fit the particular requirements of long-term asset and liability management (ALM). In particular, the proposed HJM model can reproduce expected long-term statistical properties of any two interest rates, while still satisfying the no-arbitrage constraints. We describe the methodology to calibrate the model in this particular constrainted setting, i.e. to find the model parameters as a function of the expected statistical properties. We precisely give the constraints on these expectations to ensure the existence of a solution.

Keywords: HJM modeling, Assets and Liability Management, Calibration.

JEL Classification: G12; G23. **AMS Classification**: 91B28.

1 Introduction

We propose an interest rate model that can fit the particular requirements of long-term asset and liability management(ALM). As Cairns [2] pointed out, there has been recently an increase in research effort to build interest rate model that would fit both short term arbitrage-free constraints and long-term realistic dynamics. Before that, the development of interest rate models for ALM was focused on reproducing long-term statistical properties with discrete time series methods. Examples of this modeling methodology is given by Wilkie's seminal work on UK pension fund assessment [15, 16] and the TY model by Yakoubov et al [17]. But, Cairns' requirements have become legitimate, since quantitative models have been more and more used also to perform risk management analysis and strategic asset allocation. This is the case for many ALM models designed around the world [18], and in particular for Towers Perrin ALM model [8, 9] and for the Russel-Yasuda Kasai model [3, 4]. Nevertheless, developing an ALM model market consistent and statistically plausible is not sufficient. Long-term asset and liability management models are designed for financial institutions (insurance companies, pension funds...), governemental institutions or private corporates facing financial commitments on maturities for which markets gives little information (50 years to a century). And, as Mulvey et al [8] stresses, the parameter estimation procedure of ALM quantitative models does not solely rely upon pure statistical calibration techniques such as those that can be developped in fixed-income desks for derivatives pricing. They are the result between statistical calibration methods and expertise considerations. Indeed, since stakeholders of financial institutions dealing

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with long-term commitment are aware of the impact of economical hypothesis on risk perception, they prefer to discuss them with economists rather than letting a numerical procedure dertermining some strategic parameters. Hence, decision-makers want to have the ability to introduce their views in the long-term behaviour of the model. These views can be seen either as long-term economic expectations for the most probable forecast given by a chief-economist, or the decision-maker's way to introduce its risk aversion in the model ("I want to see what is going to happen for the value of our portofolio if the short-term interest rate is around 3% in the long-term").

This third requirement gives rise to a difficult problem. Generally, it is not an easy task to build a model that would fit observed past statistical properties of prices, be consistent with market hypothesis (arbitrage free prices) and allow for the exogenous introduction of long-term economic expectations. Quantitative models for ALM studies involves generally the modeling of inflation, bonds, equity and the three main currencies (USD, EUR, YEN). It gives a system of at least fifteen equations with each time at least three parameters and a correlation matrix of more than a hundred coefficients. But, stakeholders will only discuss a fraction of these parameters: average value for short-term interest rate and its volatility, equity risk premia, purchasing power parity long-term value, and maybe the correlation between long-term and short term interest rate. Hence, it is not obvious that the model will still be generating consistent prices once some of these paramaters have been fixed exogenously, while the others are the results of a statistical estimation. In particular, one important issue comes from the interest rate term structure. This term structure has a strategic importance in stochastic ALM model since it is generaly at the beginning of the cascade modeling (see [4, 9, 17]), and has an paramount impact on the discount rate for liabilities. For instance, in Mulvey et al. [9], an interest rate model is designed based on a mean-reverting process for two maturities. This model allows a straightforward and easy computation for any other interest rate maturity by a simple interpolation, and the introduction of economists views by means of bounds for different parameters. Unfortunatly, there is no guarantee that the resulting dynamics will still be arbitrage free.

We propose here a model that would complete the preceeding works to allow both market consistency and exogenous economic predictions. Starting from the Heath-Jarrow-Morton (HJM) framework stating the general no-arbitrage conditions for interest rate dynamics [7], we build an interest rate model with the given requirements. More precisely, our model allows to fix economic predictions (expected returns, volatilities, correlation) for the interest rates of two different maturities. In addition, the model is able to admit an exogeneous correlation between bond yields of the same maturities.

This paper is structured as follow. The problem is presented in section 2 with successively the description of the modeling requirements (section 2.1), the set of exogeneous contraints (section 2.2) and the justification for our modeling choices for the HJM framework amongst other possible alternatives like CIR [5] or Duffie & Kan's [6] models (section 2.3). Then, in section 3, we present a way to solve the non-linear calibration problem and provide an exhaustive description of the feasibility domain. Indeed, we show that predictions for different economic parameters such as the long-term expected short-term return, volatilities and so on can not be choosen in an arbitrary way. Section 4 provides first, some non-trivial numerical consequences and second, an illustration of the different dynamics obtained with and without respect of the non-arbitrage condition.

2 Problem formulation

In an ALM study, the term structure of interest rates plays an essential role since it links asset returns and discount factor of interest rates. Bonds returns must be related to interest rates variations while cash return must be related to short-term interest rate and the evolution of the discount factor must be coherent with the evolution of some long-term interest rate. These examples stress the existence of a strong consistency between all financial variables and raise the difficulty of reproducing this consistency in a stochastic model with a limited number of parameters. The difficulties of designing such a stochastic interest rate model can be summarized in two main points: ensuring the absence of arbitrage opportunities (either static or dynamic) and allowing for a calibration procedure of all the parameters. In this section, we first describe the general requirements (2.1) an interest rate model has to satisfy in long-term ALM applications, then, we provide the specific exogenous constraints (2.2) the interest rate model has to satisfy. Finally, we justify the use of the HJM framework and formulate the calibration problem (2.3) in this setting.

2.1 Modeling requirements

Following the reference book on ALM modeling by Ziemba & Mulvey [18], we can summarize as follow the general properties an interest rate model should satisty for long-term ALM applications to be able to provide both sound risk management analysis and allow for the introduction of decision-makers economic predictions:

(i) *no arbitrage condition.* An arbitrage free model is necessary for derivatives pricing or short-term dynamic hedging. It could be argued that this property is less relevant for long-term ALM decision problems, since hedging derivatives is not the core of ALM problems. As seen in the introduction, some ALM models do not make it a strong requirement (see Mulvey et al [9]). Nevertheless, an arbitrage-free model is more natural and makes decision-makers more comfortable about its use.

(ii) *Term structure realism.* The model must be able to generate all the possible forms of interest rate term structure since on the long-run, all possible economic situations may occur. This excludes for instant, the use of a pure Vasicek interest rate model based on only the instanteneous short-term rate [14] (as in Munk et al [10] for instance), since these models can not reproduce shifts from contango to backwardation term structure. Generally, a two-factor model will be at least needed to provide a compromise between tractability and description of possible dynamics of the interest rate term structure.

(iii) *Integration capacity.* Since the interest rate term structure is at the beginning of the cascade modeling of many other asset prices (bond price, equity returns), its modeling should be compliant with the efficient simulation of all the others depending asset prices.

(iv) *Simplicity.* Long-term ALM risk management studies imply the computation of a very large number of scenarios on a period of sometimes a century. Hence, it is an important feature of a model to be able to perform this intensive computation in an efficient way. Closed-form solutions for bond prices and interest rates, at any time, a good way to ensure this efficiency requirement.

(v) *Calibration on historical data.* Calibration on historical data must be feasible to measure the capacity of the model to reproduce past situations.

(vi) *Exogenous economic constraints.* The model must also be able to satisfy specific characteristics given by experts and which are different from what has been observed in the past. This degree of freedom can also be used to study different characteristics of the future in order to assess "what if" scenarios.

These requirements can be classified into two categories. The first set of desired properties refers to the compromise between realism and tractability (requirements (i) to (iv)). They are intuitive. The second set of properties deals with the calibration capabilities of the model (requirements (v) and (vi)). The calibration problem is essential in generating realistic scenarios of the future. Indeed, in a short-term study, it is natural to calibrate the model mainly on historical data, according to the intuition that "what will happen in the near future is close to what happened in the near past" (requirement (v)). But, in long-term studies, the global model must be able to be sufficiently close to long-term target values provided by economists and financial experts who can convert qualitative information on the evolution of financial institutions and regulation on the long-run into quantitative variables. The main financial variables for which target values are generaly given are expected stock return and volatilities, inflation trend and volatility and expected returns, volatility and correlation for a small set of bonds.

2.2 Exogenous constraints

In this paper, we are looking for an interest rate model with the capacity to take as exogenous the following values:

- long-term expectation of two interest rates of maturity m and m',
- long-term volatility of two interest rates of maturity m and m',

- correlation between these two interest rates.

Hence, there are 5 exogenous constraints based on experts forecasts. We will see in section 2.3 that a sixth constraint will have to be introduced to ensure consistency with the other variables modeled in the global scenario generation model.

2.3 Modeling choice

Now, our objective is to build an interest rate term structure model that has the characteristics (i) to (vi), and that can be calibrated in order to respect the constraints defined in section 2.2. We have chosen to formulate and solve this problem in the HJM framework, and more precisely, using multi-factor Gaussian models with Vasicek type volatilities. To our knowledge, there were four main modeling alternatives for long-term applications that would ensure arbitrage-free interest rate dynamics: the CIR model [5], Cairns' model [2] and Duffie & Kan's model [6]. Two-factor CIR model is the intuitive direct extension of our modeling choice. However, the calibration constraints expressed in the CIR setting are more intricate than the calibration procedure in the HJM framework. Moreover, there are difficulties to easily simulate both bond prices and fixed maturity interest rates.

Cairns' model is a recent and attractive since it allows to reproduce the main expected properties of the interest rates term structure. But, it is also not straightforward to see how parameters can be modified to fit the constraints above. Moreover, the computation of bond prices is not analytical and requires a numerical approximation of a one-dimensional integral. Hence, the simulation under the historical probability becomes quickly difficult. In particular, in this setting, since only the consideration of constant risk premia will lead to Gaussian distribution and simple simulation process, it restricts the possible models intances to a set which is close to our modeling choice.

Duffie & Kan [6] proposed a multi-factor model that can reproduce given characteristics of interest rates. Their framework is more general than the HJM one since it makes possible to consider an arbitrary number of non-necessarily Gaussian factors. Each factor can represent a bond return at a given maturity and is modelled by a stochastic process. Therefore, this type of model can easily monitor the evolution of bonds returns. However, the authors aknowledge themselves the complexity of the relations linking parameters to the interest rates. This gives very little hope to find an analytical results on the existence of a solution that would respect some given constraints

For these reasons, we focused our effort on an Heath-Jarrow-Morton (HJM) type modeling. Even basic instances of HJM models are very rich [1, 12] and allow to represent a lot of the term-structure very specific observed dynamics. The counterparty for this richness is that the calibration of an HJM model is a well known difficult problem. But, we propose here a particular instance which will make the calibration process tractable and allow to simulate both bond prices and fixed maturity interest rates with analytical closed formulae.

We consider a Gaussian HJM model where the volatility of zero-coupon bonds $B_t(T)$ is a deterministic function of time $\Gamma_t(T)$. The absence of arbitrage opportunity between zero-coupon bonds of different maturities implies the existence of a vector of risk premia λ_t which allows to write the relation between the different zero-coupon bond yields as:

$$\frac{dB_t(T)}{B_t(T)} = r_t dt + \Gamma_t(T) \cdot (dW_t + \lambda_t dt), \tag{1}$$

where $\cdot$ is the scalar product and r_t represents the instantaneous rate. One can deduce the expression of $B_t(T)$ by writing the same equation for $T = t$ and using the fact that $B_t(t) = 1$. Moreover, it is possible to avoid the use of the unobserved variable r_t by computing the ratio $B_t(T)$ and $B_t(t)$. Then:

$$B_t(T) = \frac{B_0(T)}{B_0(t)} \exp\left\{ \int_0^t \left[\Gamma_s(T) - \Gamma_s(t)\right] \cdot (dW_s + \lambda_s ds) \right. \\ \left. - \frac{1}{2}\int_0^t \left|\Gamma_s(T)\right|^2 - \left|\Gamma_s(t)\right|^2 ds \right\}.$$

The associated discount rate $R_t(T)$ is deduced using:

$$B_t(T) = e^{-(T-t)R_t(T)}, \tag{2}$$

which leads to:

$$\begin{aligned} R_t(T) = \frac{TR_0(T) - tR_0(t)}{T-t} &- \frac{1}{T-t}\int_0^t \left[\Gamma_s(T) - \Gamma_s(t)\right] \cdot (dW_s + \lambda_s ds) \\ &+ \frac{1}{2(T-t)}\int_0^t \left|\Gamma_s(T)\right|^2 - \left|\Gamma_s(t)\right|^2 ds \end{aligned} \tag{3}$$

From now on, we will consider the class of Gaussian two-factor models with Vasicek type volatilities. In this case, the 2 stochastic factors impacting the evolution of the yield curve are modelled by the $\mathbb{R}^2$-valued Brownian motion $W_t = (W_t^{(1)}, W_t^{(2)})$. The volatily functions have the following form:

$$\Gamma_t^{(i)}(T) = \sigma_i \frac{1 - e^{-a_i(T-t)}}{a_i}, \quad i = 1, 2 \tag{4}$$

and we will consider that the risk premia are constant in time $\lambda_t^{(i)} = \lambda_i$. In this context, it is possible to compute the long-term expectation and covariances of two different interest rates of constant time to maturity m et m', i.e. $R_t(t+m)$ and $R_t(t+m')$. Considering Vasicek type volatility functions and constant risk premia, focusing on a constant time to maturity $m = T - t$ and $m' = T' - t$, and taking the limit $t \to \infty$, we find the following relations for the limit values:

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}\left\{R_t(t+m)\right\} =& R_0^\infty - \frac{1}{m}\sum_{i=1}^2 \frac{\sigma_i \lambda_i}{a_i^2}(1 - e^{-a_i m}) \\ &+ \frac{1}{2m}\sum_{i=1}^2 \frac{\sigma_i^2}{a_i^3}\left[1 - e^{-a_i m} + \frac{1}{2}(1 - e^{-a_i m})^2\right], \end{aligned} \tag{5}$$

$$\lim_{t\to\infty} \mathbb{C}ov\left\{R_t(t+m), R_t(t+m')\right\} = \sum_{i=1}^2 \frac{\sigma_i^2}{2a_i^3}\frac{(1 - e^{-a_i m})(1 - e^{-a_i m'})}{m\, m'} \tag{6}$$

where R_0^∞ is the value for the interest rate of "infinite" maturity observed on the initial yield curve. In practice, it corresponds to a maturity of 30 years. Notice that the expression for the variances are obtained by taking $m = m'$ in relation (6).

It is well known that one can consider that the two Brownian factors are independent without loss of generality. The only restriction of this model is the choice of the volatility function. In the following we then assume that the correlation between the two Brownian factors are independent. But, as stated by requirements (iii) (see section 2.1) the yield curve simulation is integrated in a global generation process of all the financial variables. The generation of coherent scenarios of all the variables is done through the correlation between the yield curves points and the other assets prices. Therefore, to avoid potential inconsistency between historical estimation of the correlation matrice and the parameter calibration of the interest model, the correlation between the two bond yields has also to be considered as an input of the calibration process. Hence, we introduce a sixth relation to assess this global coherence requirement. This sixth relation is the correlation between the bond yields of maturities m and m'. The expression of this correlation as a function of the HJM model parameters is given by the following expressions:

$$\begin{aligned} \mathbb{C}ov\left(\frac{dB_t}{B_t}(t+m), \frac{dB_t}{B_t}(t+m')\right) &= \mathbb{V}ar\left(r_t dt\right) + \Gamma_t(t+m)\Gamma_t^T(t+m')dt \\ &+ \mathbb{C}ov\left(r_t dt, \Gamma_t(t+m)dW_t\right) \\ &+ \mathbb{C}ov\left(r_t dt, \Gamma_t(t+m')dW_t\right) \end{aligned}$$

This expression involves terms of different orders and unknown values like the variance of the instantaneous rate or the covariance between HJM factors and the instantaneous rate. However, these terms are of order dt^α with $\alpha > 1$. We propose an approximation of these expression keeping only terms of order dt and then neglecting the terms depending on the instantaneous rate. Noting ρ_{dB} the correlation between two bond yields of maturity m and m' and using the previous expression with Vasicek type volatilites, we obtain:

$$\rho_{dB} = \frac{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})(1-e^{-a_i m'})}{\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})^2}\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m'})^2}} \tag{7}$$

We are now in a position to state the non-linear system corresponding to calibration problem. We use the following notations for the contraints values of the models defined in section 2.2:

- μ and μ' the long-term expectations of the two interest rate of maturity m and m'.
- Σ and Σ' the long-term variances of the two interest rate of maturity m and m'.
- ρ the long-term correlation between these two interest rates.
- ρ_{dB} the instantaneous correlation between the two bond yields of maturities m and m'.

Our calibration problem is to find, for given values of μ, μ', Σ, Σ' and ρ, in which conditions there exists a set of values σ_i, λ_i and a_i, $i = 1, 2$ satisfying the following non-linear system:

$$\sum_{i=1}^{2}\left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i\lambda_i}{a_i^2}\right)(1-e^{-a_i m}) = 2m(\mu - R_0^\infty) - m^2\Sigma^2 \tag{8a}$$

$$\sum_{i=1}^{2}\left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i\lambda_i}{a_i^2}\right)(1-e^{-a_i m'}) = 2m'(\mu' - R_0^\infty) - m'^2\Sigma'^2 \tag{8b}$$

$$\sum_{i=1}^{2}\frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m})^2 = 2m^2\Sigma^2 \tag{8c}$$

$$\sum_{i=1}^{2}\frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m})(1-e^{-a_i m'}) = 2mm'\rho\Sigma\Sigma' \tag{8d}$$

$$\sum_{i=1}^{2}\frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m'})^2 = 2m'^2\Sigma'^2 \tag{8e}$$

$$\frac{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})(1-e^{-a_i m'})}{\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})^2}\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m'})^2}} = \rho_{dB} \tag{8f}$$

We can already point out the specific following constraint on the correlations:

$$0 < \rho < 1, \tag{9}$$
$$0 < \rho_{dB} < 1, \tag{10}$$

which comes from the fact that left-hand size of equation (8d) must be positive. This necessary condition for a solution to exist is related to the particular class of models we have choosen. Generaly, it is not verified. But, in practice, the correlation between the two interest rates that we will have to deal with, is observed to be highly positive. Hence, this constraint will always be satisfied.

3 Calibration method

This section focuses on both describing the resolution method (section 3.1) and precisely determining the constraints that the parameters have to satisfy in order to guaranty the existence of a solution (section 3.2). In particular we will see that the main difficulties appear on that latter part of the problem.

3.1 Solving method

The resolution method proposed is based on two remarks: (i) the risk premia λ_1 and λ_2 appear only in equations (8a) and (8b), and (ii) conditionnally to the other parameters, the system relating the risk premia to the predictions of expectations μ and μ' is linear. Therefore the global system (8) can be solved in two steps:

- Solving equations (8c-8f):

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m})^2 = 2m^2\Sigma^2 \tag{11a}$$

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m})(1-e^{-a_i m'}) = 2mm'\rho\Sigma\Sigma' \tag{11b}$$

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m'})^2 = 2m'^2\Sigma'^2 \tag{11c}$$

$$\frac{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})(1-e^{-a_i m'})}{\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})^2}\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m'})^2}} = \rho_{dB} \tag{11d}$$

The resolution of this first set of equations will provide a solution for $(a_1, a_2, \sigma_1, \sigma_2)$.

- Once a solution of the previous problem is found, we solve the subsystem composed of equations (8a-8b):

$$\sum_{i=1}^{2} \left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i\lambda_i}{a_i^2}\right)(1-e^{-a_i m}) = 2m(\mu - R_0^\infty) - m^2\Sigma^2, \tag{12a}$$

$$\sum_{i=1}^{2} \left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i\lambda_i}{a_i^2}\right)(1-e^{-a_i m'}) = 2m'(\mu' - R_0^\infty) - m'^2\Sigma'^2, \tag{12b}$$

  where the remaining unknowns are only λ_1 and λ_2. This last step is very easy since the system is linear w.r.t. λ_1 and λ_2.

3.1.1 Calibration of the long-term covariances

We set the following notations:

$$s_i = \frac{\sigma_i^2}{a_i^3}, \qquad i = 1,2$$

$$A = 2m^2\Sigma^2, \qquad B = 2mm'\rho\Sigma\Sigma', \qquad C = 2m'^2\Sigma'^2.$$

The equations (11a), (11b) and (11c) can be rewritten in the equivalent following hierarchical structure:

$$s_2 = \frac{A - s_1(1-e^{-a_1 m})^2}{(1-e^{-a_2 m})^2} \tag{13a}$$

$$\frac{1-e^{-a_2 m'}}{1-e^{-a_2 m}} = \frac{B - s_1(1-e^{-a_1 m})(1-e^{-a_1 m'})}{A - s_1(1-e^{-a_1 m})^2} \tag{13b}$$

$$s_1 = \frac{AC - B^2}{A(1-e^{-a_1 m'})^2 + C(1-e^{-a_1 m})^2 - 2B(1-e^{-a_1 m})(1-e^{-a_1 m'})} \tag{13c}$$

Equation (13c) gives the expression of the set of solutions represented by a relation between a_1 and s_1. Any couple (a_1, s_1) verifying (13c) is an admissible solution of the subsystem. The choice of one particular value may be based on the resolution of equation (11d). The resolution of this equation is not possible analytically.

However, as we will show in section 3.2, the set of admissible solutions for a_1 is a bounded interval. Therefore it is possible to determine an approximation of a solution by an appropriate algorithm. It is well known that two-factors models produce significant correlation between two bond yields (cf. [12], p. 31), sometimes higher than what is observed in the market. Therefore, the exact equality of equation (11d) is very restrictive and may even lead to infeasibilities. That is why we propose to relax this equality and to choose the solution of the minimization problem:

$$a_1 = \arg\min_{a_1 \in \mathscr{A}_1} \left\{ \| \rho_{dB} - h(a_1) \| \right\} \tag{14}$$

where $\mathscr{A}_1$ is the set of admissible values for a_1 and $h(a_1)$ is the expression of the left hand side of equation (11d) where a_2, σ_1 and σ_2 are replaced by their value as a function of a_1, given expressions (13a)-(13c). The fact that $\mathscr{A}_1$ will be a bounded interval (see section 3.2) and that $h(a_1)$ is sufficiently regular imply that an exhaustive research of the minimum is numerically tractable.

Once the solution a_1 has been computed, we can deduce the associated value of s_1, a_2 and s_2 using respectively equations (13c), (13b) and (13a). Note that the determination of a_2 needs the inversion of a particular function:

$$\phi(a) = \frac{(1 - e^{-am'})}{(1 - e^{-am})} \tag{15}$$

This function is not analytically invertible but has the particular advantage of being constantly decreasing from $\frac{m'}{m} > 1$. Indeed, computing the differential of ϕ and, for example, noting that $\frac{m'e^{-am'}}{1-e^{-am'}} < \frac{me^{-am}}{1-e^{-am}}$ for any $a > 0$ and $m' > m$, we can show that ϕ is strictly decreasing. Calculating the limit at 0 and $+\infty$ the set of possible values for ϕ is $]1\ ;\ \frac{m'}{m}[$. Therefore, the inversion of equation (13b) can be numerically done with, for example, a dichotomic algorithm. The existence of a solution is conditionned by the fact that the right hand side of equation (13b) must be in $]1\ ;\ \frac{m'}{m}[$.

3.1.2 Calibration of the long-term expectations

Once we have determined the parameters (a_1, a_2, s_1, s_2), it remains to estimate the two risk premia λ_1 and λ_2 using equations (12a) and (12b). This subsystem of two equations is linear with respect to λ_1 and λ_2. Hence the estimation of theses parameters is trivial once we have ensured that the system is invertible. To see that it is always the case, we rewrite the system as follows:

$$\begin{bmatrix} \psi(a_1,\sigma_1,m) & \psi(a_2,\sigma_2,m) \\ \psi(a_1,\sigma_1,m') & \psi(a_2,\sigma_2,m') \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \varphi(a_1,a_2,\sigma_1,\sigma_2,R_0^\infty) \\ \varphi'(a_1,a_2,\sigma_1,\sigma_2,R_0^\infty) \end{pmatrix} \tag{16}$$

with

$$\psi(a,\sigma,m) = \frac{\sigma}{a^2}(1 - e^{-am}) \tag{17}$$

and φ and φ' depending also on the value of the predictions. It is trivial to show that this system is always invertible once, for example, $a_2 > a_1$ and $m' > m$ calculating the determinant of the matrix of equation (16) and using the fact that the function $\phi(\cdot)$ is strictly decreasing.

3.2 Feasibilty set

The existence of a solution of this subsystem is ensured only under some constraints on the value of the parameters. In the following we show that these constraints all result in constraints on a_1. After having expressed these constraints on a_1 of the form $a_1 \in \mathscr{A}_1$, we show that ensuring $\mathscr{A}_1$ to be non-empty implies constraints on the prediction values of Σ, Σ' and ρ. Recalling equations (13a), (13b) and (13c) we directly obtain the following four constraints:

- In equation (13a), since s_2 must be positive, we must have:

$$A - s_1(1 - e^{-a_1 m})^2 > 0 \tag{18}$$

- In equation (13b), since $\phi(a_2)$ must be in $]1\ ;\ \frac{m'}{m}[$, we must have the two restrictions:

$$\frac{B - s_1(1 - e^{-a_1 m})(1 - e^{-a_1 m'})}{A - s_1(1 - e^{-a_1 m})^2} > 1 \tag{19}$$

$$\frac{B - s_1(1 - e^{-a_1 m})(1 - e^{-a_1 m'})}{A - s_1(1 - e^{-a_1 m})^2} < \frac{m'}{m} \tag{20}$$

- In equation (13c), since s_1 must be positive, we must have:

$$\frac{AC - B^2}{(\sqrt{A}(1 - e^{-a_1 m'}) - \sqrt{C}(1 - e^{-a_1 m}))^2 + 2(\sqrt{AC} - B)(1 - e^{-a_1 m})(1 - e^{-a_1 m'})} > 0 \tag{21}$$

Actually we have only two real constraints: if the inequations (19) and (20) are repected then there exists a solution of the subsystem. Indeed replacing s_1 by its value defined on (13c) we show that $A - s_1(1 - e^{-a_1 m})^2$ is always strictly positive. Also the inequation (21) is always verified, since $AC - B^2 > 0$ because of the definition of A, B and C.

Separating the parameters and the prediction values in the constraints (19) and (20) and replacing s_1 using equation (13c), we obtain these two expressions:

$$f(a_1) < B - A, \tag{22}$$

$$g(a_1) < \frac{m'}{m}A - B, \tag{23}$$

with

$$f(a_1) = \frac{(AC - B^2)\left((1 - e^{-a_1 m})(1 - e^{-a_1 m'}) - (1 - e^{-a_1 m})^2\right)}{A(1 - e^{-a_1 m'})^2 + C(1 - e^{-a_1 m})^2 - 2B(1 - e^{-a_1 m})(1 - e^{-a_1 m'})}$$

$$g(a_1) = \frac{(AC - B^2)\left(\frac{m'}{m}(1 - e^{-a_1 m})^2 - (1 - e^{-a_1 m})(1 - e^{-a_1 m'})\right)}{A(1 - e^{-a_1 m'})^2 + C(1 - e^{-a_1 m})^2 - 2B(1 - e^{-a_1 m})(1 - e^{-a_1 m'})}$$

In Appendix A we study the functions f and g and the associated set of admissible values for a_1. More precisely, the constraints (22) and (23) result in a set of admissible values for a_1 depending on the prediction values. Indeed the set of admissible values for a_1 is not empty if and only if:

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m + m')\Sigma\Sigma'} \tag{24}$$

and, in this case, the restricted area of admissible values for a_1 is:

$$a_1 \in \left]0\ ;\ \phi^{-1}\left(\frac{C - B}{B - A}\right)\right[\tag{25}$$

3.3 Consequences for possible economic prediction values

Since f and g are always strictly positive, we must necessarily have $B - A > 0$ and $\frac{m'}{m}A - B > 0$ (cf. eq. (22) and (23)). Adding the constraints (24) of existence of a solution, we can summarize the constraints on the exogenous prediction values as follows:

$$\rho > \frac{m\Sigma}{m'\Sigma'}, \tag{26}$$

$$\rho < \frac{\Sigma}{\Sigma'}, \tag{27}$$

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}, \tag{28}$$

In particular, this imply that $\frac{m\Sigma}{m'\Sigma'} < 1$ and then $\Sigma' > \frac{m}{m'}\Sigma$. Under this assumption the resulting restriction of ρ is:

$$\rho \in \left] \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \; ; \; \frac{\Sigma}{\Sigma'} \right[\tag{29}$$

Actually it is trivial to show that this interval is non-empty only when $\Sigma' < \Sigma$. Finally, the constraints of existence of a solution, in terms of restriction of acceptable values of the exogenous predictions are:

$$\frac{m}{m'}\Sigma < \Sigma' < \Sigma \tag{30}$$

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \tag{31}$$

These constraints have non-trivial consequences for exogenous prediction values. However we can deduce some necessary properties of the prediction values. First, the volatility of interest rates must be decreasing with the maturity. This property has long been observed on the market and hence, this constraint is always consistent with the observations. Second, the closer m and m' are, the closer the volatilities of the corresponding interest rates are. Moreover, their correlation gets closer to one. However, even if m and m' are significantly different, the model imposes a high value for the correlation. Actually this is observed in the market. For example, the empirical correlation between rates of maturity 1 month and 10 year, computed on historical data from january 1999 to may 2008, is about 80%.

3.4 Calibration solution synthesis

The parameter estimation procedure constructed above can be summerized in the following way. First, the only constraints that have to be satisfied by the predictions are the following:

$$\frac{m}{m'}\Sigma < \Sigma' < \Sigma$$

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}$$

And, when these constraints are satisfied, the estimation algorithm is the following:

$$a_1 = \arg\min_{a_1 \in \mathscr{A}_f \cap \mathscr{A}_g} \{\|\rho_{dB} - h(a_1)\|\} \qquad \text{cf. eq.(14)}$$

$$\sigma_1^2 = \frac{a_1^2(AC - B^2)}{A(1-e^{-a_1 m'})^2 + C(1-e^{-a_1 m})^2 - 2B(1-e^{-a_1 m})(1-e^{-a_1 m'})} \qquad \text{cf. eq.(13c)}$$

$$a_2 = \phi^{-1}\left(\frac{B - s_1(1-e^{-a_1 m})(1-e^{-a_1 m'})}{A - s_1(1-e^{-a_1 m})^2}\right) \qquad \text{cf. eq.(13b)}$$

$$\sigma_2^2 = \frac{a_2^2\left(A - s_1(1-e^{-a_1 m})^2\right)}{(1-e^{-a_2 m})^2} \qquad \text{cf. eq. (13a)}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{bmatrix} \psi(a_1,\sigma_1,m) & \psi(a_2,\sigma_2,m) \\ \psi(a_1,\sigma_1,m') & \psi(a_2,\sigma_2,m') \end{bmatrix}^{-1} \begin{pmatrix} \varphi(a_1,a_2,\sigma_1,\sigma_2,R_0^\infty) \\ \varphi'(a_1,a_2,\sigma_1,\sigma_2,R_0^\infty) \end{pmatrix} \qquad \text{cf. eq.(16)}$$

4 Numerical tests

In this section the numerical calibration process described previously is exemplified on a case where the two interest rate maturities for which long-term economic expectation are provided, are the 1 month and 10 years maturity euro interest rates. Their respective long-term expectation μ and μ', volatility Σ and Σ' and their correlation ρ are given in Table 1.

	1 month	10 years
expectation (μ and μ')	3%	4%
volatility (Σ and Σ')	1.5%	1%
correlation between rates (ρ)	80 %	
correlation between yields (ρ_{dB})	30%	

Table 1: Long term expected economic value for the 1 month and 10 years euro interest rate.

4.1 Interpolation of the initial rate curve

We consider the initial yield curve in euro as the one observed at 2008/10/01 issued from Datastream [1]. In order to obtain the initial rate curve at any point, we choose a Nelson-Siegel type interpolation method [11]:

$$R_0(T) = \beta_0 + \beta_1 \left(\frac{1 - e^{\frac{T}{\tau}}}{\frac{T}{\tau}} \right) + \beta_2 \left(\frac{1 - e^{\frac{T}{\tau}}}{\frac{T}{\tau}} - e^{\frac{T}{\tau}} \right)$$

with the parameter τ set *a priori* to 1.5, following the practice of Banque de France [13] and β_0, β_1 and β_2 are obtained by linear regression from historical data. This interpolation is made necessary because the simulation of the interest rate curve requires the initial rates for all maturities, (see the term $R_0(t)$ in equation (3)). The interpolation leads to the limit value $R_0^\infty = \beta_0$, see eq. (8).

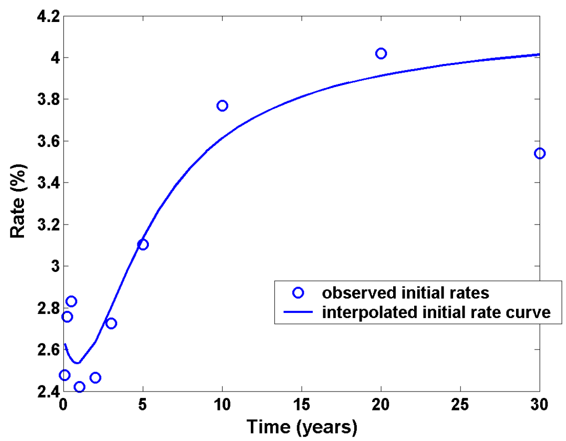

Figure 1: Initial rate curve: the observed initial rates are 1, 3 and 6 months, 1, 2, 3, 5, 10, 20 and 30 years. The interpolation method is of Nelson-Siegel type with parameter $\tau = 1.5$.

4.2 Calibration

The solution of the non-linear system (8) obtained by the algorithm summarized in Section 3.4 is given in Table 2.

a_1	0.0852	σ_1	0.0049	λ_1	0.0895
a_2	9.4853	σ_2	0.0580	λ_2	2.0583

Table 2: Model parameter values for associated economic expectations.

We stress that the set of prediction values satisfies the existence constraints given by (30) and (31), which means that the resulting parameters of Table 2 allow us to produce interest rates with the prediction values defined in Table 1. Indeed, the economic expectations given in Table 1 imply that the correlation between rates, ρ, must be up to 55%. The set $\mathscr{A}_1$ of possible values of a_1 is therefore]0 ; 0.086[, and Figure 2 shows all the admissible values for the bonds yield correlation, ρ_{dB}. Because $\rho_{dB} = 30\%$ can be reached, the obtained parameters can exactly respect all the expected predictions. Actually the historical correlation ρ_{dB} between yields is lower: for example, historical correlation between euro yields of maturity 1 month and 10 years, observed from 03/2000 to 08/2008 is about 7.5%, which cannot be reached. This is essentially due to our modeling restriction to only two factors. This observation has already be done in [12].

[1] www.datastream.com

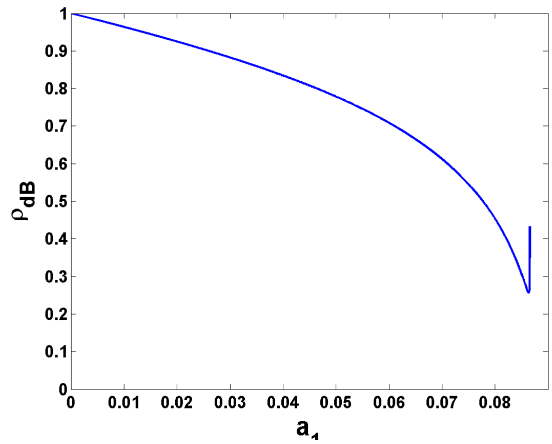

Figure 2: Admissible values of ρ_{dB} as a function of the value of a_1.

4.3 Simulation of interest rates

We have simulated 2,000 trajectories of 1-month and 10-years interest rates during one century, with a discretization step of one week. From these trajectories we determined the mean values and the quantiles 5% and 95% to define a confidence interval at 90% along time.

(a) 1 month-interest rate

(b) 10 years-interest rate

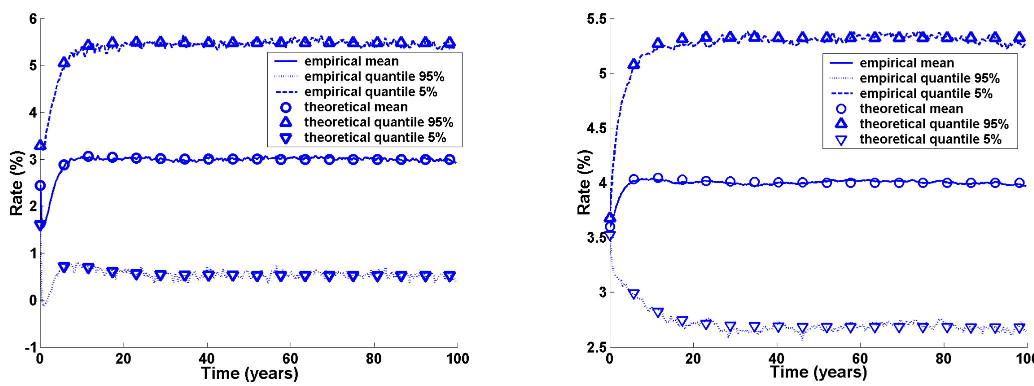

Figure 3: Simulation result of (a) 1-month and (b) 10-years interest rates. The dotted and solid lines represent empirical statistics of the rates (means and confidence interval at 90%) while the different symbols represent theoretical statistics.

Figure 3 shows the results of these simulations with the parameters defined in Table 2. In each graph the dotted and solid lines represent the empirical statistics of 1-month and 10-years interest rates by the mean value and the confidence interval at 90%. The lines drawn by symbols "o", "▵" and "▿" represent the theoretical statistics of these rates. The correspondence between theoretical and empirical results shows that the model produces the right expected predictions in terms of mean and volatility. It can be seen also that it takes about 30 years for the interest rate to reach its expected value. The way interest rates converges to their long-term expected value depends crucialy on the initial rate curve. Indeed the evolution of both rates follows the pattern given by

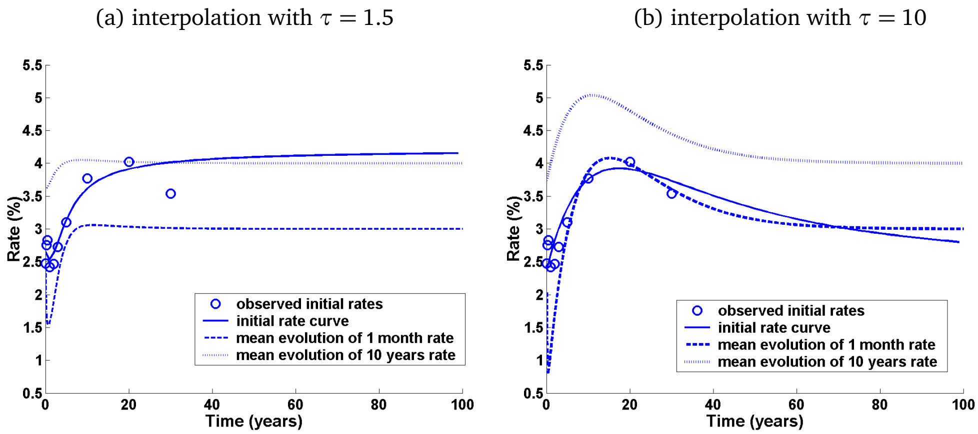

Figure 4: Comparison between initial rate curve and mean evolution of 1 month and 10 years interest rates.

the initial rate curve. During the first 30 years, the interest rates evolves according to the information provided by the initial rate curve. And after 30 years the rates tend to the only available information which is the set of expected value to be reproduced.

This result shows that the initial rate curve plays an essential role in the dynamic evolution of the interest rates. Therefore the importance of the interpolation method can not be understressed. Figure 4 highlights theses points. Mean statistics are presented for the 10-years interest rates when the initial rate curve is obtained by Nelson-Siegel type interpolation with parameter $\tau = 10$. We can see that the convergence time to the expected value as well as the way it reaches this value significantly depend on the form of the initial rate curve.

Finally we can see a positive probability to have negative interet rates, in particular for the 1-month rate. Indeed, the proposed model does not ensure the positivity of the interest rates. Because the case of negative rates appears mostly in a short time, particularly in the beginning of the period, the impact of these negative values is negligible in long-term ALM studies. We leave to the resolution of this problem directly with a model ensuring positivity.

5 Conclusion

In this paper we proposed a model of the yield curve within the HJM framework that can fit the specific requirements of long-term asset and liability management. In particular, the proposed HJM model can reproduce expected long-term statistical properties of any two interest rates, while still satisfying the no-arbitrage constraints. Moreover, the choice of a two factor model with Vasicek type volatility functions leads to an easy computation of interest rate derivatives. This simplicity allows to build a quasi-analytical procedure to calibrate the model, i. e. to find the model parameters as a function of the expected statistical properties. Precise constraints on these expected values were given so as to ensure the existence of a solution. The numerical example highlights the essential role of the interpolation method to reconstruct the initial yield curve, in the dynamics of the two interest rates to be controlled.

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A Appendix: Admissible sets for a_1

This appendix is focused on the study of the set of admissible values for parameter a_1. Recalling the expression of the two constraints of existence (22) and (23) we have to study the following functions:

$$\begin{aligned} f(a_1) &= \frac{(AC-B^2)\left((1-e^{-a_1 m})(1-e^{-a_1 m'})-(1-e^{-a_1 m})^2\right)}{A(1-e^{-a_1 m'})^2+C(1-e^{-a_1 m})^2-2B(1-e^{-a_1 m})(1-e^{-a_1 m'})}, \\ g(a_1) &= \frac{(AC-B^2)\left(\frac{m'}{m}(1-e^{-a_1 m})^2-(1-e^{-a_1 m})(1-e^{-a_1 m'})\right)}{A(1-e^{-a_1 m'})^2+C(1-e^{-a_1 m})^2-2B(1-e^{-a_1 m})(1-e^{-a_1 m'})}, \end{aligned}$$

with $A=2m^2\Sigma^2$ and $B=2mm'\rho\Sigma\Sigma'$, in order to determine the values of a_1 that respect the constraints:

$$\begin{aligned} f(a_1) &< B-A, \\ g(a_1) &< \frac{m'}{m}A-B, \end{aligned}$$

In the following we study independently the two function f and g to define the respective sets $\mathscr{A}_f$ and $\mathscr{A}_g$ of admissible solution for a_1 and finally we define the intersection $\mathscr{A}_1=\mathscr{A}_f\cap\mathscr{A}_g$ which defines the set of values for a_1 that ensure the existence of the global solution. Each function study follows the same steps: we begin by defining the values of a_1 that reach the bound ($B-A$ for f and $\frac{m'}{m}A-B$ for g), and then, we compute the limit values of the function to find the set of a_1 that respects the constraint.

A.1 Study of f

Here we study the first constraint with the resolution of:

$$f(a_1)-(B-A)=0. \tag{32}$$

This equation leads to a polynomial of degree 2 on the variable $\phi(a_1)$ defined in equation (15). Indeed equation (32) is equivalent to:

$$A(A-B)\phi(a_1)^2+\left[B(B-A)+A(C-B)\right]\phi(a_1)+B(B-C)=0.$$

The solutions of this polynom leads to the final result:

$$f(a_1)=(B-A)\Leftrightarrow\phi(a_1)=\frac{B}{A}\quad\text{or}\quad\phi(a_1)=\frac{C-B}{B-A}$$

Since the range of ϕ is $]1\,;\,\frac{m'}{m}[$, these solutions exist only if $\frac{B}{A}$ and $\frac{C-B}{B-A}$ belong to this interval. Recalling the definition of A, B and C we have:

$$\begin{aligned} \frac{B}{A} &= \frac{\rho m'\Sigma'}{m\Sigma} \\ \frac{C-B}{B-A} &= \frac{m'\Sigma'(m'\Sigma'-\rho m\Sigma)}{m\Sigma(\rho m'\Sigma'-m\Sigma)} \end{aligned}$$

Actually as the constraints $B-A>0$ and $\frac{m'}{m}A-B>0$ leads to the relation:

$$\frac{m\Sigma}{m'\Sigma'}<\rho<\frac{\Sigma}{\Sigma'},$$

it becomes trivial that under this constraint, the ratio $\frac{B}{A}$ is always in $]1\,;\,\frac{m'}{m}[$. Also it is trivial to see that:

$$\frac{C-B}{B-A}>\frac{B}{A},$$

by direct computing, for example the difference between these two parts and remembering that $AC - B^2$ and $B - A$ are positive. Therefore we have $\frac{C-B}{B-A} > 1$. We must then define the constraint ensuring that $\frac{C-B}{B-A} < \frac{m'}{m}$, which leads to the final result:

$$\frac{C - B}{B - A} < \frac{m'}{m} \Leftrightarrow \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}$$

In addition, we can compute the limiting value of f:

$$\lim_{a_1 \to +\infty} f(a_1) = 0$$

By the fact that f is continuous and ϕ is decreasing, we deduce the set of a_1 that respects the constraint (22) depending on the prediction values:

$$\begin{aligned}
\mathscr{A}_f &= \left]\phi^{-1}\left(\frac{B}{A}\right) \ ; \ +\infty\right[&& \text{if } \rho \leq \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \\
&= \left]0 \ ; \ \phi^{-1}\left(\frac{C-B}{B-A}\right)\right[\bigcup \left]\phi^{-1}\left(\frac{B}{A}\right) \ ; \ +\infty\right[&& \text{if } \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}
\end{aligned}$$

A.2 Study of g

We apply the same reasoning as above to the function g. We begin by solve the equation:

$$g(a_1) - \left(\frac{m'}{m}A - B\right) = 0. \tag{33}$$

This equation leads also to a polynom of degree 2 on the variable $\phi(a_1)$:

$$\left[AB - \frac{m'}{m}A^2\right]\phi(a_1)^2 + \left[2\frac{m'}{m}AB - AC - B^2\right]\phi(a_1) + \left[BC - \frac{m'}{m}B^2\right] = 0.$$

The solutions of this polynom leads to the final result:

$$g(a_1) = (B - A) \Leftrightarrow \phi(a_1) = \frac{B}{A} \quad \text{or} \quad \phi(a_1) = \frac{\frac{m'}{m}B - C}{\frac{m'}{m}A - B}$$

The study of f showed that $\frac{B}{A} \in]1 \ ; \ \frac{m'}{m}[$ under the existing constraints on the predictions. It remains to study the value of the second solution. First it is trivial to show that:

$$\frac{\frac{m'}{m}B - C}{\frac{m'}{m}A - B} < \frac{B}{A},$$

calculating the difference between the two parts and remembering that $AC - B^2$ and $\frac{m'}{m}A - B$ are positive. Therefore, the existence of this solution is only conditionned by the lower bound of the interval, which leads to the final result:

$$\frac{\frac{m'}{m}B - C}{\frac{m'}{m}A - B} > 1 \Leftrightarrow \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}$$

In addition, we can compute the limiting value of g:

$$\lim_{a_1 \to 0} g(a_1) = 0$$

By the fact that g is continuous and ϕ is decreasing, we deduce the set $\mathscr{A}_g$ of a_1 that respects the constraint (23) depending on the value of the predictions:

$$\begin{aligned}\mathscr{A}_g &= \left]0\ ;\ \phi^{-1}\left(\frac{B}{A}\right)\right[&& \text{if } \rho \leq \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}\\ &= \left]0\ ;\ \phi^{-1}\left(\frac{B}{A}\right)\right[\bigcup \left]\phi^{-1}\left(\frac{\frac{m'}{m}B - C}{\frac{m'}{m}B - A}\right)\ ;\ +\infty\right[&& \text{if } \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}\end{aligned}$$

A.3 Admissible set for a_1

The remarkable result of the two previous studies is that a common condition appears on the prediction values for determining the set of admissible values of a_1. From these previous result we deduce that the only way for having a non-empty set of admissible values is when:

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}$$

and, in this case, $a_1 \in \mathscr{A}_f \cap \mathscr{A}_g$. Since $\frac{\frac{m'}{m}B-C}{\frac{m'}{m}B-A} < \frac{B}{A} < \frac{C-B}{B-A}$ and ϕ is decreasing, we obtain:

$$a_1 \in \left]0\ ;\ \phi^{-1}\left(\frac{C-B}{B-A}\right)\right[\bigcup \left]\phi^{-1}\left(\frac{\frac{m'}{m}B - C}{\frac{m'}{m}B - A}\right)\ ;\ +\infty\right[\tag{34}$$

In addition, we can show the specific relation:

$$\phi(a_2) = \frac{B\phi(a_1) - C}{A\phi(a_1) - B},$$

Therefore if a_1 belongs to the first part of $\mathscr{A}_f \cap \mathscr{A}_g$, i.e. $a_1 < \phi^{-1}\left(\frac{C-B}{B-A}\right)$ then a_2 necessarily belongs to the second part, i.e. $a_2 > \phi^{-1}\left(\frac{\frac{m'}{m}B-C}{\frac{m'}{m}B-A}\right)$. By the symetric definition of a_1 and a_2 we can restrict the set of admissible values for a_1 and assume:

$$a_1 \in \left]0\ ;\ \phi^{-1}\left(\frac{C-B}{B-A}\right)\right[$$

In particular, the inversion of the system will then lead to a short-term factor and a long-term factor since a_1 and a_2 will belong to distinct intervals.

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