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René Carmona Max Fehr Juri Hinz Arnaud Porchet

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MARKET DESIGN FOR EMISSION TRADING SCHEMES

RENÉ CARMONA *, MAX FEHR †, JURI HINZ ‡, AND ARNAUD PORCHET §

Abstract. This paper is concerned with the mathematical analysis of emissions markets. We review the existing quantitative analyses on the subject and introduce some of the mathematical challenges posed by the implementation of the new phase of the European Union Emissions Trading Scheme as well as the cap-and-trade schemes touted by the US, Canada, Australia and Japan. From a practical point of view, the main thrust of the paper is the design and numerical analysis of new cap-and-trade schemes for the control and the reduction of atmospheric pollution. We develop tools intended to help policy makers and regulators understand the pros and cons of the emissions markets. We propose a model for an economy where risk neutral firms produce goods to satisfy an inelastic demand and are endowed with permits in order to offset their pollution at compliance time and avoid having to pay a penalty. Firms that can easily reduce emissions do so, while those for which it is harder buy permits from firms which anticipate they will not need all their permits, creating a financial market for pollution credits. Our equilibrium model elucidates the joint price formation for goods and pollution allowances, capturing most of the features of the first phase of the European Union Emissions Trading Scheme. We show existence of an equilibrium and uniqueness of emissions credit prices. We also characterize the equilibrium prices of goods and the optimal production and trading strategies of the firms. We use the electricity market in Texas to numerically illustrate the qualitative properties of these cap-and-trade schemes. Comparing the numerical implications of cap-and-trade schemes to the Business-As-Usual benchmark, we show that our numerical results match those observed during the implementation of the first phase of the European Union cap-and-trade CO_2 emissions scheme. In particular, we confirm the presence of windfall profits criticized by the opponents of these markets. We also demonstrate the shortcomings of tax and subsidy alternatives. Finally we introduce a relative allocation scheme which, while easy to implement, leads to smaller windfall profits than the standard scheme.

1. Introduction.

Emission trading schemes, also known as cap-and-trade systems, have been designed to reduce pollution by introducing appropriate market mechanisms. At the time of the writing of the first version of the paper, the two most prominent examples of existing cap-and-trade systems are the EU-ETS (European Union Emission Trading Scheme) and the US Sulfur Dioxide Trading System. In such systems, a central authority sets a limit (cap) on the total amount of pollutant that can be emitted within a pre-determined period. To ensure that this target is complied with, a certain number of credits are allocated to appropriate installations and a penalty is applied as a charge per unit of pollutant emitted outside the limits. Firms may either reduce their own pollution or purchase emission credits in anticipation of potentially significant penalties. This transfer of allowances by trading is considered to be the core principle leading to the minimization of the costs caused by regulation; companies that can easily reduce emissions will do so, while those for which it is harder buy credits.

*Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544. Also with the Bendheim Center for Finance and the Applied and Computational Mathematics Program. (rcarmona@princeton.edu), Partially supported by NSF: DMS-0806591

†Institute for Operations Research, ETH Zurich, CH-8092 Zurich, Switzerland (maxfehr@ifor.math.ethz.ch).

‡National University of Singapore, Department of Mathematics, 2 Science Drive, 117543 Singapore (mathj@nus.edu.sg), Partially supported by WBS R-703-000-020-720 / C703000 of the Risk Management Institute at the National University of Singapore

§ CREST and EDF R&D, Crest, Timbre J320, 15 Boulevard Gabriel Péri, 92245 Malakoff Cedex, France, (arnaud.porchet@gmail.com).

In a cap-and-trade system, the initial allocation (i.e. the total number of allowances issued by the regulator) should be chosen in order for the scheme to reach a given emissions level. This total initial allocation is indeed a crucial parameter that the regulator can use to control the emission level. While the value of the total initial allocation is dictated by the emissions target, the regulator can choose the specific distribution of these allowances among various producers and market participants in order to create incentives to design and build cleaner and more efficient production units. The initial distribution of the allowances among the market participants does not affect the environmental efficiency of the cap-and-trade scheme. It only affects the distribution of the costs of the legislation and the quantitative analysis of this cost distribution is possibly the main thrust of this paper. The fact that in a competitive equilibrium model, the final level of emissions only depends upon the total initial allocation and not on the gory details on how the allowances are distributed among the participants, was already noted in Montgomery's groundbreaking paper [19] introducing the cap-and-trade schemes for the first time. An informative discussion of several of the important issues raised by the initial allowance distribution can be found in the economic analysis proposed by Stavins in [25] which appeared after the first version of this paper was completed.

An important feature of our analysis is the assumption that at the end of the compliance period, unused allowance certificates expire and cannot be used in subsequent periods. The fact that certificates cannot be banked for later use was a rule in force during the implementation of the first phase of the EU-ETS. In the present study, it turns out to be a mathematical convenience that makes the analysis of one compliance period easier. The results presented in this paper have been extended to the multiperiod case with banking in several subsequent works. These models are more realistic renditions of the second phase of the EU-ETS. See [5, 7, 9, 8] for example.

Naturally, any emission reduction policy increases the cost of the goods whose production causes those emissions. These costs are passed on to the end consumer and are partly responsible for the occurrence of substantial windfall profits. Based on an empirical analysis of power generation profitability in the context of the first phase of the EU-ETS, strong empirical evidence for the existence of such profits is given in [24]. The authors of this study conclude that power companies realize substantial profits since allowances are received for free while they are always priced into electrical power at a rate that depends upon the emissions rate of the marginal production unit: producers seem to take advantage of the trading scheme to make extra profits. We believe that the free distribution of allowances is not the only reason for producers' windfall profits. This phenomenon is more subtle: it can happen in a competitive setting even in simple deterministic models. For the sake of illustration, we present a simple instance of these profits.

Let us consider a set of firms that must satisfy a demand of $D = 1$ MWh of electricity at each time $t = 0, 1, \cdots, T-1$, and let us assume that there are only two possible technologies to produce electricity: gas technology which has unit cost 2 \$ and emits 1 ton of CO_2 per MWh produced, and coal technology which has unit cost 1 \$ and emits 2 tons of CO_2 per MWh. In this simple model, the total capacity of gas is 1 MWh and the total capacity of coal is also 1 MWh. We also suppose that producers face a penalty of \$ π per ton of CO_2 not offset by credits, and that a total of $T-1$ credits are distributed to the firms, allowing them to offset altogether $T-1$ tons of CO_2. Here π is a number strictly greater than 1. In this situation, we arrive at two conclusions. First, as demand needs to be met, total emissions will be greater than or equal to T tons, even if all firms use the clean technology (gas). Second, firms are always better off reducing emissions than paying the penalty. As a consequence, the optimal generation strategy is to only use gas technology and emit T tons of CO_2. At least one firm has to pay the penalty and the price of emission credits is necessarily equal to π at each time. The missing credit has a value π for both the buyer and the seller, so the price of electricity is $2 + \pi$ because a marginal decrease in demand will induce a marginal gain in generation cost and a marginal decrease in penalty paid. The total profit for the producers is $\pi(T-1)$, the penalty paid by the producers to the regulator is π, and the total cost for the customers is $(2+\pi)T$. Consider now, still in the competitive equilibrium framework, the Business As Usual (BAU) scenario: the demand is met by using coal technology, the price of electricity is 1, the total profit for producers is 0 and the total cost for the customers is T. In this simple example the producers cost induced by the trading scheme is $T + \pi$; producers must buy more expensive fuel, so a profit T is made by the fuel supplier and the producers have to pay the penalty π. The increase in fuel price, or switching cost, is a marginal cost that must factor into the electricity price. The penalty is a fixed cost paid at the end, but we see that in this trading scheme, this fixed cost is rolled over the entire period and paid by the customers at each time, inducing a windfall profit for the producers. This windfall profit is exactly equal to the market value of the $T-1$ credits. However, notice that if we increase the demand to 2 MWh at each time $t = 0, 1, \cdots, T-1$ then the windfall profits exceed the market value of the allowances.

Another feature of emissions trading schemes is the risk of non compliance faced by the producers and the regulator. The EU-ETS was introduced as a way to comply with the targets set by the Kyoto Protocol. Phase 1 of the Kyoto Protocol sets a fixed cap for annual emissions of CO_2 by year 2012 to all industrialized countries that ratified the protocol (Annex I countries). This reduction should guarantee on average a level of emissions of no more than 95 % of what it was in year 1990. All countries are free to adopt the emission reduction policy of their choice, but in case of non-compliance in 2012, they face a penalty (payment of 1.3 emission allowances for each ton not offset in Phase 1). The EU-ETS was designed to ensure compliance for the whole EU zone. However, in an uncertain environment, there is a distinct possibility that the scheme will eventually fall short of its goal and that producers will exceed the fixed cap set at the beginning of the compliance period. In this case, it is the regulator's responsibility either to pay a penalty, or to comply with the target by relying on one or more of three flexible mechanisms: International Emission Trading, Clean Development Mechanism or CDM, and Joint Implementation, or JI for short. International Emission Trading allows Annex 1 countries to exchange allowances to meet their Kyoto targets, while CDM and JI allow Annex 1 countries or companies settled in Annex 1 countries to generate allowances by financing emission reduction projects in other Annex 1 countries or in less developed countries. These allowances are called Certificates of Emission Reduction or CERs and are also worth one ton of CO_2 equivalent. The design of emission trading schemes must also address this question and possibly limit or control the use of these CERs. The reader is referred to [5] for a mathematical discussion of CER price formation.

In the present work, we give a precise mathematical foundation to the analysis of emission trading schemes and quantitatively investigate the impact of emission regulation on consumer costs and firm profits. Based on a model for perfect competition, we show that in equilibrium, a standard emission trading scheme combines two contrasting aspects. On the one hand, the system reduces pollution at the lowest cost for the society, as expected. On the other hand, it forces a notable transfer of wealth from consumers to producers, which in general exceeds the social costs of pollution reduction.

In a perfect economy where all customers are shareholders, windfall profits are redistributed, at least partially through dividends payments. However, this situation is not always the case, and the impact of regulation on prices needs to be addressed. There are several other ways to return part of the windfall profits to the consumers. The most prominent ones are taxation and charging for the initial allowance distribution. Beyond the political risks associated with the levy of new taxes, we will show that one of the main disadvantages of this first method is its poor control of the final level of emissions under random demand for goods and stochastic abatement costs.

Concerning auctioning, it is important to notice that, for the second phase of the EU-ETS, individual countries did not have to give away the totality of their credit allowances for free. They could choose to auction up to 10% of their total allowances. Strangely enough, except for Denmark, none of them exercised this option. On the other hand auctioning the entire set of initial allowances as a way to abolish windfall profits oses one of the main features of cap-and-trade schemes, namely the mechanism which controls incentives to invest in and develop cleaner production technologies. Indeed, a significant reduction of windfall profits through auctioning, if at all possible, requires to auction a huge amount of allowances, even possibly the total initial allocation. Further it involves a significant risk for companies since the capital invested to procure allowances at the auction may be higher than the income later recovered from allowances prices. For a non-technical discussion of the pros and the cons of auctioning of EU-ETS phase II allowances, the interested reader is referred to [15]. A quantitative analysis of some of these issues is provided in [4].

In this work, we argue that cap-and-trade schemes can work, even in the form implemented in the first phase of EU-ETS, at least as long as allowance allocation is properly implemented. Moreover, we prove that it is possible to design emission trading schemes that overcome most of the problems documented so far. We show how to establish trading schemes that reduce windfall profits while exhibiting the same emission reduction performance as the generic cap-and-trade system used in the first phase of the EU-ETS. These schemes also have the nice feature that a significant amount of the allowances can be allocated as initial allocation to encourage cleaner technologies.

Despite frequent articles in the popular press and numerous speculative debates in specialized magazines and talk-shows, the scientific literature on cap-and-trade systems is rather limited. We briefly mention a few related works chosen because of their relevance to our agenda. The authors of [11] and [19] proposed a market model for the public good *environment* introduced by tradable emission credits. Using a static model for a perfect market with pollution certificates, [19] shows that there exists a minimum cost equilibrium for companies facing a given environmental target. The conceptual basis for dynamic permit trading is, among others, addressed in [10], [26], [21], [17], [22] and [23]. Meanwhile, the recent work [23] suggests also a continuous-time model for carbon price formation. Beyond these examples, there exists a vast literature on several related topics, including equilibrium [2], empirical evidence from already existing markets [16], [24], and uncertainty and risk [13], [18], [27]. The model we present below follows the baseline suggested in [12].

1.1. European Union Emissions Trading Scheme (EU-ETS). In January 2005 the European Union Greenhouse Gas Emission Trading Scheme (EU-ETS) began operation as the largest (more than $10,000$ installations from various sectors) multi-country (15 EU members), multi-sector (power generation, oil refining, coke ovens, metal industry, production of cement and lime, ...) Greenhouse Gas (GHG) emissions trading scheme world-wide. The scheme is based on Directive 2003/87/EC from October 25, 2003 [1]. It follows a *downstream* approach in the sense that fuels end users are liable under the scheme instead of the fuel producers. We review the salient features of its implementation in order to motivate some of the assumptions made in the paper, and to illustrate some of our technical results.

The total number of emission rights actually allocated at the beginning of the program was chosen at the beginning of the program on the basis of National Allocation Plans (NAPs) presented by each of the 15 participating countries. As we will show in our analysis, the total number of emission rights injected in the system plays a crucial role in the price of the allowances (see also the comments following Figure 1.1) and obviously the level of emission reduction at the end of the program. We will also show that the way in which this total allocation of emission rights is distributed among the participant installations is much less relevant, playing essentially no role in the final level of emission reduction, and only impacting the distribution of the costs of the climate policy.

During the first phase, a penalty of $\pi = 40$€ was levied for each ton of emitted CO_2 not covered by a surrendered allowance certificate. This penalty was raised to $\pi = 100$€ in the second phase of the EU-ETS. In a typical cap-and-trade scheme, at the end of each compliance period, an independent authority tallies the emissions of each installation, and those installations which cannot surrender as many allowances as the total number of tons of CO_2 emitted will pay a fine computed as the number of uncovered tons of CO_2 times the penalty level. Moreover, they will have to purchase and surrender as many certificates from those to be issued for the next compliance period. So de facto, the actual cost for each ton of $CO_2$2 not covered by a surrendered emission permit is equal to the penalty plus the spot price of one certificate from the next period. This fact is incorporated in the models used in [7, 9, 8]. However, for the sake of simplicity, the equilibrium models introduced in this paper cover only a single one-year period, and they do not include this carry-over feature of the penalty.

Figure 1.1 illustrates what happened in April 2006, 9 months before the end of the first phase of the EU-ETS, and right after the public announcement of the level of emissions for the year 2005. Installations realized that the level of emissions had been over-estimated and that emission rights were going to be in over supply. After the announcement, all the futures prices dropped dramatically, but because the European Union Allowances (EUAs for short) issued for the first phase could not be banked, the price of the first phase allowances did not recover. In fact, these prices ended up essentially converging to 0. Not only is this phenomenon intuitively clear, but we will show that it happens in great generality. Indeed, our theoretical analysis shows that, in a cap-and-trade without banking like the first phase of the EU-ETS, the price of an allowance certificate converges to a two-valued random variable, these two values being 0 or the penalty level imposed by the regulator. The price converges toward 0 in the case of over supply of certificates, and this is exactly what happened. Should this be regarded as a flaw of the scheme, the criticism should be directed toward the regulator who should not have flooded the market with an over supply of certificates.

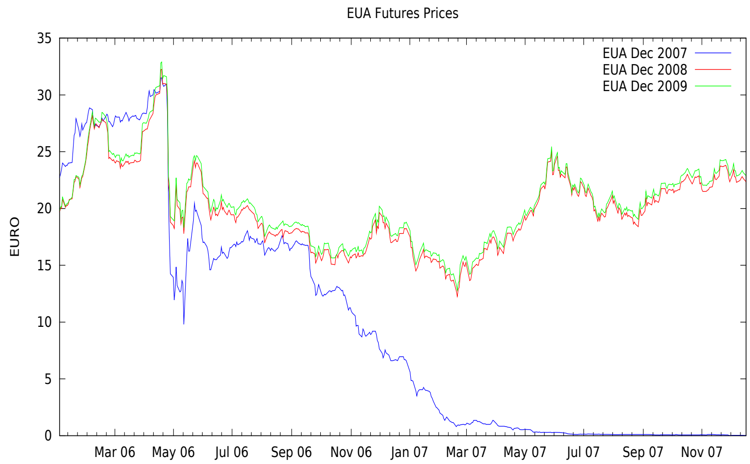

FIGURE 1.1. *Price drop before the end of the first phase of the EU-ETS.*

As transparency in price formation increases and carbon price signals become reliable, the market for futures contracts on EUAs gained in liquidity and options on these futures contracts are already traded in significant numbers. While equilibrium models like those presented in this paper are of great value to understand the inner workings of these markets, they have little to offer when it comes to pricing and hedging these derivatives. These tasks are better performed with simpler reduced form models designed for their tractability and the ease with which they can be calibrated to price data. Such a model is proposed in [8], and we expect that more research in this direction will develop in the near future.

Another interesting phenomenon is illustrated in Figure 1.2. The price of CERs is not equal to the price of an EUA even though a CER, like an EUA, is a certificate which can be used to offset one ton-equivalent of CO_2 emissions. An equilibrium theory in the spirit of the theory presented in this paper is possible, prices of goods, EUAs and CERs result jointly from such an equilibrium, and price spreads like the one illustrated in Figure 1.2 can be explained. However these models are more involved, their notations rather tedious, and in order to avoid distracting the reader from the main thrust of the paper with mathematical technicalities, we chose not to present these results in this survey. Instead, we refer the interested reader to [5].

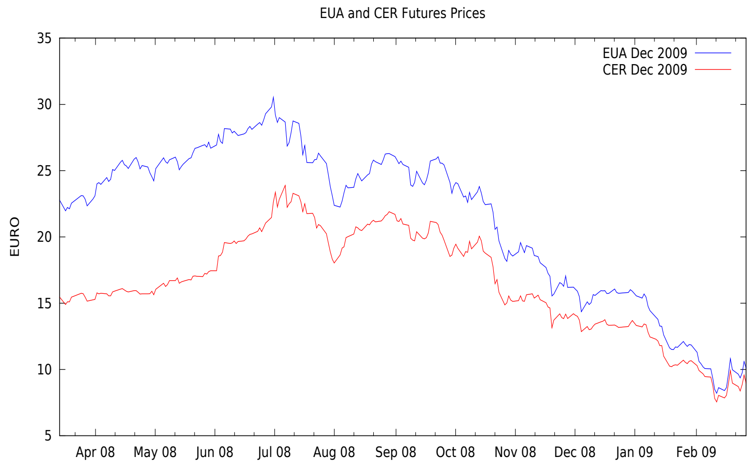

FIGURE 1.2. *Prices of the December 2012 EUA futures contract (EU-ETS second phase), together with the price of the corresponding CER futures contract.*

1.2. Summary. We close this rather lengthy introduction with a quick summary of the contents of the paper.

Section 2 gives the details of the mathematical model used to capture the dynamic features of a cap-and-trade system in an uncertain environment where demand for goods and costs of production are random. We introduce the notation needed to describe the production of goods and the profit mechanisms in a competitive economy. Exogenous demand for goods is modeled by means of adapted stochastic processes. We assume that demand is inelastic in the sense that it has to be met exactly by supply. This assumption could be viewed as restrictive, but we argue that it is quite realistic in the case of the electricity markets. We also introduce the emissions allowance allocations and the rules of trading in these allowances.

Section 3 defines the notion of competitive equilibrium for risk neutral firms involved in our cap-and-trade scheme. Preliminary work shows that most of the theoretical results of this paper still hold for risk averse firms if preferences are modeled with exponential utility. However, in order to avoid *muddying the water* with unnecessary technical issues which could distract the reader from the important issues of pollution abatement, we restrict ourselves to the less technical case of risk neutral firms. For the sake of completeness, we solve the equilibrium problem in the Business As Usual (BAU from now on) case corresponding to the absence of a market for emissions permits. In this case, as expected, the prices of goods are given by the standard *merit order* pricing typical of deregulated markets. The section closes with the proof of a couple of enlightening necessary conditions for the existence of an equilibrium in our framework. These mathematical results show that at compliance time, the equilibrium price of an emission certificate can only be equal to 0 or the penalty level chosen by the regulator. The second important necessary condition proven in this section shows that in equilibrium, the prices of the goods are still given by a merit order pricing provided that the production costs are adjusted for the cost of emissions. This result is important as it shows exactly how the price of pollution gets incorporated in the prices of goods in the presence of a cap-and-trade scheme. The following Section 4 is devoted to the rigorous proof of the existence of an equilibrium. The proof uses classical functional analysis results from optimization theory in infinite dimensional spaces. It follows the lines of a standard argument based on the analysis of what an informed central planner (*representative agent*) would do in order to minimize the social cost of meeting the demand for goods.

Section 5 is devoted to the analysis of the standard cap-and-trade scheme featured in the implementation of the first phase of the EU-ETS. By comparison with BAU scenarios, we show that properly chosen levels of penalty and pollution certificate allocations lead to desired emissions targets. However, our numerical experiments with data from the electricity market in Texas show the existence of excessive *windfall profits*. Our choice of Texas was made for several compelling reasons. This region known as ERCOT, is one of three electrical interconnections in the United States. It is essentially isolated from the others in the sense that there are no synchronous electrical interconnections between ERCOT and the rest of the US or Mexico, and only one direct high voltage connection. This electrical isolation, together with the information readily available concerning power generation capacity and fuel types, made Texas a natural choice for our case study. It is important to emphasize that the results of our numerical experiments are typical of many electricity markets. The interested reader can consult [6] for a similar case study devoted to the analysis of the potential impacts of cap-and-trade schemes on the Japanese electric sector. As explained earlier in our literature review, windfall profits have been observed in the first phase of EU-ETS, giving credibility to the critics of cap-and-trade systems. Section 6 can be viewed as the main thrust of the paper beyond the theoretical results proven up to that point. We propose a general framework including taxes and subsidies along the standard cap-and-trade schemes. We demonstrate the shortcomings of the tax systems which suffer from poor control of the windfall profits and unexpected expensive reduction policies when it comes to emissions reduction targets under stochastic abatement costs. We concentrate our analysis on several new alternative cap-and-trade schemes and we show numerically that a relative allocation scheme can resolve most of the issues with the other schemes. Such a relative allocation scheme is easy to describe and implement as pollution allowances are distributed proportionally to production. It is reminiscent of the alleged *output based* allocation schemes touted in some of the non-technical economic studies of cap-and-trade schemes. See for example [3] and [14] for discussions in deterministic settings of the impact of such schemes on the so-called *leakage* effect. Even though the number of permits is random in a relative scheme, and hence cannot be known exactly in advance, its statistical distribution is well understood as it is merely a scaled version of the distribution of the demands for goods. Consequently, setting up caps to meet pollution targets is not much different from the standard cap-and-trade schemes. Moreover, the coefficient of proportionality providing the number of permits is an extra parameter which should make calibration more efficient. Indeed, one shows that properly calibrated, the relative schemes reach the same pollution targets as the standard schemes while at the same time, keeping consumer burden and windfall profits in control.

Section 7 gathers more mathematical properties of the generalized cap-and-trade schemes introduced in the previous section. Our results demonstrate the versatility and the flexibility of such a generalized framework. They show that regulators can control cap-and-trade schemes in order to reach pre-assigned pollution targets with zero windfall profits and *reasonably small* consumer costs, or even to force equilibrium electricity prices to be equal to target prices. However, because of the level of complexity of their implementations, it is unlikely that the schemes identified there will be used by policy makers or regulators. The paper concludes with Section 8 which reviews the main results of the paper, recasting them in the perspective of the public policy challenging issues uncovered by the results of the paper.

Acknowledgements. The authors would like to thank the referees for insightful comments and suggestions that lead to significant improvements in the presentation of the paper.

2. Mathematical Model for a Standard Cap-and-Trade Scheme. In this section we present the elements of our mathematical analysis. We consider an economy where firms produce and supply goods to end-consumers over a period $[0, T]$. The production of these goods is a source of pollutant emissions. In order to reduce this externality, a regulator distributes emissions allowances to the firms at time 0, allows them to trade the allowances on an organized market between times 0 and T, and at the end of this compliance period, levies penalties proportionally to their net cumulative emissions. As explained in the introduction above, our stylized scheme does not allow for banking in the sense that unused allowances cannot be used in future compliance periods. We also ignore the need to *borrow* allowances from the next period allocation in order to cover the emissions which could not be offset by the redemption of allowance certificates from the current compliance period. This simplifying assumption allows us to treat the compliance period in isolation. However, as can be seen from the analysis of more realistic models, our quantitative results still hold in more general set-ups without these simplifying assumptions. See for example [9, 8], and [5]

In what follows $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t,\ t \in \{0, 1, \ldots, T\}\}, \mathbb{P})$ is a filtered probability space. We denote by $\mathbb{E}[.]$ the expectation operator under the probability $\mathbb{P}$ and by $\mathbb{E}_t[.]$ the expectation operator conditional to $\mathcal{F}_t$. The σ-field $\mathcal{F}_t$ represents the information available at time t. We will also make use of the notation $\mathbb{P}_t(.) := \mathbb{E}_t[\mathbf{1}_{\{.\}}]$ for the conditional probability with respect to $\mathcal{F}_t$.

2.1. Production of Goods. A finite set I of firms produce and sell a set K of different goods at times $0, 1, \ldots, T-1$. Each firm $i \in I$ has access to a set $J^{i,k}$ of different technologies to produce good $k \in K$, that are sources of emissions (e.g. greenhouse gases). Each technology $j \in J^{i,k}$ is characterized by:

- a marginal cost $\widetilde{C}_t^{i,j,k}$ of producing one unit of good k at time t;
- an emission factor $e^{i,j,k} \geq 0$ measuring the volume of pollutants emitted per unit of good k produced by firm i with technology j;
- a production capacity $\kappa^{i,j,k}$.

The fact that $\kappa^{i,j,k}$ is not-random and constant over time rules out the possibility to take plants off line for maintenance or emergency repairs. Doing so would significantly increase the complexity of the notations already cumbersome enough, without changing much to the nature of the results, so we decided to refrain from doing it. For the sake of notation we introduce the index sets

$$M_i = \{(j,k) : k \in K, j \in J^{i,k}\}, \qquad i \in I\ ,$$
$$M = \{(i,j,k) : i \in I, k \in K, j \in J^{i,k}\}\ .$$

Since electricity production from fossil fuels is the main source of CO_2 emissions, our main example of produced good is electric power. We make the assumption that the production costs are non-negative, adapted and integrable processes.

For each $0 \leq t \leq T-1$, firm $i \in I$ produces the amount $\xi_t^{i,j,k}$ of good $k \in K$, throughout the period $[t, t+1)$, using the technology $j \in J^{i,k}$. Since the choice of the production level $\xi_t^{i,j,k}$ is based only on present and past observations, the processes $\xi^{i,j,k}$ are also supposed to be adapted and, since production cannot exceed capacity, we require that the inequalities

$$0 \leq \xi_t^{i,j,k} \leq \kappa^{i,j,k}, \qquad i \in I,\ k \in K,\ j \in J^{i,k},\ t = 0, 1, \cdots, T-1, \tag{2.1}$$

hold almost surely. Our market is driven by an exogenous and *inelastic* demand for goods. Since electricity production is a significant proportion of the emissions covered by the existing schemes, this inelasticity assumption is reasonable. We denote by D_t^k the demand at time t for good $k \in K$. This demand process is supposed to be adapted to the filtration $\{\mathcal{F}_t\}_t$. For each good $k \in K$, we assume that the demand is always smaller than the total production capacity for this good, namely that:

$$0 \leq D_t^k \leq \sum_{i \in I} \sum_{j \in J^{i,k}} \kappa^{i,j,k} \quad \text{almost surely, } k \in K. \tag{2.2}$$

This assumption is a natural extension of the assumption of inelasticity of the demand as it will conveniently discard issues such as blackouts which would only be a distraction given the purposes of the paper.

2.2. Emission Trading. We denote by $\pi \in [0, \infty)$ the *penalty per unit of pollutant*. As reported in the introduction, in the original design of the European Union Emissions Trading Scheme (EU-ETS) π was set to 40€ per metric ton of *Carbon Dioxyde equivalent* (tCO_2e). For each firm, the net cumulative emission is the amount of emissions which have not been offset by *allowances* at the end of the compliance period. It is computed at time T as the difference between the total amount of pollutants emitted over the entire period $[0, T]$ minus the number of allowances held by the firm at time T and redeemed for the purpose of emissions abatement. The net cumulative emission is this difference whenever positive, and 0 otherwise.

For the sake of simplicity we assume that the entire period $[0, T]$ corresponds to one simple compliance period. We explained the far-reaching consequences of this assumption in the previous subsection. In particular, at maturity T, all the firms have to cover their emissions by allowances or pay a penalty. Moreover, certificates become worthless if not used as we do not allow *banking* from one phase to the next. So in this economy, operators of installations that emit pollutants will have two fundamental choices in order to avoid unwanted penalties: *reduce* emissions by producing with cleaner technologies or *buy* allowances.

At time $t = 0$, each firm $i \in I$ is endowed with $\Lambda^i = \Lambda_0^i$ pollution permits called allowances. So if it were to hold on to this initial allowance endowment until the end, it would be able to offset up to Λ^i units of emissions, and start paying only if its actual cumulative emissions exceeded that level. This is the *cap* part of a cap-and-trade scheme. Depending upon their views on the demands for the various products and their risk appetites, firms may choose production schedules leading to cumulative emissions in excess of their caps. In order to offset expected penalties, they may engage in buying allowances from firms which expect to meet demand with less emissions than their own cap. This is the *trade* part of a cap-and-trade schemes.

In the simplest form of a cap-and-trade scheme, allowances are distributed at the beginning of each compliance period. However, there is no real reason to force the allocation of allowances to take place at one single timepoint $t = 0$. For example, in the 2007-2012 phase of the EU-ETS, allowances are allocated in March each year, while the 5 year compliance period starts in January. We deal with more general set-ups in Section 6 where we assume that the distribution of pollution permits is given by adapted stochastic processes $\{\Lambda_t^i\}_{t=0,1,\cdots,T-1}$. For the sake of consistency, we will denote by

$$\Lambda^i = \sum_{t=0}^{T-1} \Lambda_t^i \tag{2.3}$$

the total allowance endowment of firm i before the end of the compliance period, and we will use the notation

$$\Lambda = \sum_{i\in I} \Lambda^i \tag{2.4}$$

for the cap (i.e. the total allocation). We make the following assumption throughout the paper:

Assumption 0.

$$\mathbb{P}\{\Lambda > 0\} = 1. \tag{2.5}$$

In other words, we do not consider scenarios where the regulator does not give away any permit. A cap-and-trade scheme does not make sense without allocation of pollution permits!

This leads the way to further generalizations of the allowance distribution scheme. In this paper, we will consider the following. The regulator rewards the firms with allocations which, at each time $t = 0, 1, \cdots, T-1$, depend in a specific algorithmic way, on their production ξ_t^i. In other words, we will consider cases where $\Lambda_t^i = \Lambda_t^i(\xi_t^i)$ for some agreed upon function form for $\Lambda_t^i(\xi_t^i)$. For the sake of simplicity, we shall consider only affine functions (see for example Proposition 6.1).

As our analysis will show, existence, uniqueness and characterization of some of the equilibrium price processes depend only upon the total number of emission permits issued during the compliance period, not on the way the permits are distributed over time and among the various participating installations. However as we will demonstrate, the statistical properties of consumer costs and windfall profits depend strongly upon the way permits are allocated. The challenge faced by policy makers is to optimally design these allocation schemes to minimize consumer costs while satisfying emissions reduction targets, controlling producers windfall profits and setting incentives for the development of cleaner production technologies. We shall concentrate on these special allocation procedures in Sections 6 and 7.

Allowances are physical in nature, since they are certificates which can be redeemed at time T to offset measured emissions. But, firms do not have to wait for the time of compliance to buy and sell allowances, and the latter can change hands at each time $t = 0, 1, \cdots, T$. Because compliance takes place at time T, and only at that time, and because entering a forward contract does not require any initial capital (not even owning the commodity), it is natural to allow the trading at time $t = 0, 1, \cdots, T-1$ of emission allowances to be done via forward contracts settled at time T. Under these conditions, and without any loss of generality, one can also restrict trading of the actual allowance certificates to time $t = T$. At times $t = 0, 1, \cdots, T-1$, each firm can take very large long and short positions, irrespective of the actual number of allowance certificates actually existing. However, except for settling financially the outstanding forward contracts, trading at time $t = T$ can only be done with existing allowances that can be delivered, hence the specific set of restrictions which we will have to impose at that time. Allowing trading in forward contracts provides a more flexible setting: it is more general than considering only spot trading, since it allows for trading pollution permits even before these allowances are issued and allocated. This turns out to be an important feature when dealing with general allocation schemes. In our mathematical model, a simple no-arbitrage argument implies that for $t < T$, the forward and spot allowance prices differ only by a discounting factor, such that trading allowances or forwards gives the same expected discounted payoff at time T. Therefore under the equilibrium definition that will be introduced in Section 3, considering only forward trading yields no loss of generality.

We denote by A_t the price at time t of a forward contract guaranteeing either physical or financial settlement of one allowance certificate at maturity T and by $A = \{A_t\}_{t=0,1,\cdots,T}$ its time evolution. As we just mentioned, A_T is the spot price at time T. The terminology *forward price at time t* is misleading as there is no exchange of funds at time t. A_t is better seen as a *strike* than a price: it is the price (in time T currency) at which the buyer at time t of the forward contract agrees to purchase the allowance certificate at time T.

Each firm can take positions on the forward market, and we denote by θ^i_t the number of forward contracts held by firm i at time t. As usual, $\theta^i_t > 0$ when the firm is long θ^i_t contracts, and $\theta^i_t < 0$ when it is short $|\theta^i_t|$ contracts. A trading strategy of firm i in the forward allowance market is understood as an adapted process $\{\theta^i_t\}_{t=0,\cdots,T-1}$. The net cash position at time T resulting from trading the forward contracts is given by:

$$R^A_T(\theta) \;=\; \sum_{t=0}^{T-1} \theta^i_t(A_{t+1} - A_t). \tag{2.6}$$

because for each $t = 0, \ldots, T-1$, holding a position θ_t in a forward contract through the time interval $[t, t+1]$ yields the cash-flow

$$\theta_t(A_{t+1} - A_t) = \theta_t(A_T - A_t) - \theta_t(A_T - A_{t+1})$$

at maturity T. This follows from the very concept of forward trading: if the position θ_t is taken at time t and held until maturity, then the agent receives at time T the difference $\theta_t(A_T - A_t)$. Thus, holding a position θ_t through $[t, t+1]$, is equivalent to opening a position at time t by purchasing θ_t units and closing this position at time $t+1$ by purchasing $-\theta_t$ units, and holding both until maturity. Notice that eventhough firms can take very large long/short positions, each sale must be offset by a purchase and vice-versa, so that the clearing constraint

$$\sum_{i\in I} \theta^i_t = 0 \tag{2.7}$$

must hold at each time $t = 0, \ldots, T-1$. Following such a forward trading strategy, the agent still holds at time $T-1$ a number Λ^i of physical allowances. However, to comply with emission targets, the physical position can be adjusted just before the compliance date T. Let us denote the change in physical allowances due to trading at time T by θ^i_T. With this notation, the physical position of firm i at compliance time T is $\Lambda^i + \theta^i_T$. Since the total number of physical allowances remains constant, all physical position changes must sum up to zero so that

$$\sum_{i\in I} \theta^i_T = 0.$$

Furthermore, since any given firm cannot sell more than what it actually owns, we must have $\theta_T^i + \Lambda^i \geq 0$ for all $i \in I$. Consequently, an allowance trading strategy $(\theta_t)_{t=0}^T$ consists of forward positions (held in the form of futures contracts for $t = 0, 1, \cdots, T-1$) and spot positions (held in the form of physical alowances at time T), and yields the net cash position at time T

$$R_T^A(\theta) = \sum_{t=0}^{T-1} \theta_t^i (A_{t+1} - A_t) - \theta_T^i A_T, \tag{2.8}$$

and satisfies the clearing conditions

$$\sum_{i \in I} \theta_t^i = 0, \qquad t = 0, 1, \cdots, T, \tag{2.9}$$

and the compliance constraints

$$-\Lambda^i \leq \theta_T^i, \qquad i \in I. \tag{2.10}$$

Note that $\Lambda^i + \theta_T^i$ is the number of (physical) allowances surrendered by firm i for compliance, and is used for the computation of the potential penalty given by formula below (2.15).

REMARK 1. *In the same way any given firm cannot sell more than what it actually owns, it cannot buy more than what is actually available. In other words, one must also have:*

$$\theta_T^i \leq \sum_{j \in I \setminus \{i\}} \Lambda^j = \Lambda - \Lambda^i \qquad \textit{for all } i \in I. \tag{2.11}$$

We shall refrain from using this upper bound because it is not as convenient as the lower bound for it requires the knowledge of what the other firms hold, or at least the total supply of allowances, and in some of the schemes considered in this paper, this total supply can depend upon the actual production of these other firms. In any case, we do not really need this upper bound because it is clearly redundant. Indeed, if the lower bound is satisfied for every $i \in I$, then the clearing condition (2.9) implies that the upper bound (2.11) is automatically satisfied for every $i \in I$.

REMARK 2. *In our mathematical model, monetization of emissions is done via allowance forwards, although real-world markets show a significant volume of trades on spot and futures contracts. From a modeling perspective, our focus on forward contracts can be justified by the following considerations: emission permits are considered as financial assets allowing for short and long positions whose cost of carry can be neglected. For standard financial assets, the differences between spot forward and futures prices can be explained by the term structure of interest rates. So, modeling allowance forward price is sufficient since given forward prices, spot and futures prices can be derived by simple no-arbitrage arguments once a model for the term structure of interest rates is specified.*

2.3. Profits. As we argued earlier, it is natural to work with T-forward allowance contracts because compliance takes place at time T. By consistency, it is convenient to express all cash flows, position values, firm wealth, and goods values in time T-currency. As a side fringe benefit, this will help us avoid having to discount cash flows in the computations to come. So we use for numéraire the price $B_t(T)$

at time t of a Treasury (i.e. non defaultable) zero coupon bond maturing at T. We denote by $\{\tilde{S}_t^k\}_{t=0,1,\cdots,T}$ the adapted spot price process of good $k \in K$, and according to the convention stated above, we shall find it convenient to work at each time t with the T-forward price

$$S_t^k = \tilde{S}_t^k / B_t(T)$$

and we skip the dependence in T from the notation of the T-forward price as T is the only maturity we are considering.

Hence, a cash flow X_t at time t is equivalently valued as a cash flow $X_t/B_t(T)$ at maturity T. So if firm i follows the production policy $\xi^i = \{(\xi_t^{i,j,k})_{k\in K\, j\in J^{i,k}}\}_{t=0}^{T-1}$ its instantaneous revenues at time t from goods production is given by

$$\sum_{(j,k)\in M_i} (\tilde{S}_t^k - \widetilde{C}_t^{i,j,k})\xi_t^{i,j,k}$$

and its time T-forward value is given by:

$$\sum_{(j,k)\in M_i} (S_t^k - C_t^{i,j,k})\xi_t^{i,j,k}$$

provided we set $C_t^{i,j,k} = \widetilde{C}_t^{i,j,k}/B_t(T)$. The total net gains from producing and selling goods are thus:

$$\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S_t^k - C_t^{i,j,k})\xi_t^{i,j,k} \ . \tag{2.12}$$

In order to hedge their production decisions, firms trade on the emissions market by adjusting their forward positions in allowances. In addition, at maturity T, each firm i redeems allowances to cover its emissions and/or pay a penalty. Let

$$\Pi^i(\xi^i) := \sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} e^{i,j,k}\xi_t^{i,j,k} \tag{2.13}$$

be the actual cumulative emissions of firm i when it uses production strategy ξ^i. We also suppose that there exist others source of emissions on which firm i has no control, denoted by $\Delta^i \geq 0$, and supposed to be an $\mathcal{F}_T$-measurable random variable. If we think of electricity as one of the produced goods for example, the presence of this uncontrolled source of emissions can easily be explained. Usually electricity producers are required to hold a reserve margin in order to respond to short time demand changes and to protect against sudden outages or unexpectedly rapid ramps in demand. When scheduling their plants it is not yet known how much of this reserve margin will be used. Therefore in most markets there is an uncertainty on the exact emission level when a production decision is made.

In a first reading Δ^i can be thought of as being 0 for the sake of simplicity. We shall see later in the paper that its presence helps characterizing the equilibrium of the economy and that it is a useful tool for modeling several variations of the model. Introducing the net amount Γ^i of allowances that producer $i \in I$ can use to offset the scheduled emissions by

$$\Gamma^i = \Delta^i - \Lambda^i = \Delta^i - \sum_{t=0}^{T-1} \Lambda_t^i \tag{2.14}$$

the total penalty paid by firm i at time T is:

$$\pi(\Gamma^i + \Pi^i(\xi^i) - \theta^i_T)^+ \ . \tag{2.15}$$

Combining (2.12) and (2.15) together with (2.8), we obtain the expression for the terminal wealth (profits and losses at time T) of firm i:

$$\begin{aligned} L^{A,S,i}(\theta^i, \xi^i) := \sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S^k_t - C^{i,j,k}_t)\xi^{i,j,k}_t \\ + \sum_{t=0}^{T-1} \theta^i_t (A_{t+1} - A_t) - \theta^i_T A_T \\ - \pi(\Gamma^i + \Pi^i(\xi^i) - \theta^i_T)^+ \ . \end{aligned} \tag{2.16}$$

To emphasize the mathematical technicalities of the model, we underline the fact that demands and production costs change with time in a stochastic manner. The statistical properties of these processes are given exogenously, and are known at time 0 by all firms. Moreover, we always assume that these processes satisfy the constraints (2.1) and (2.2) almost surely. Firms adjust their production and trading strategies in a non-anticipative manner to their observations of the fluctuations in demand and production costs. In turn, the production and trading strategies ξ^i and θ^i become respectively adapted stochastic processes on the stochastic base of the demand and production costs.

3. Market Equilibrium. In this section, we follow the common apprehension that a realistic market state is described by prices which correspond to a so-called market equilibrium, a situation, where the demand for each product is covered, all financial positions are in zero net supply, and each firm is satisfied by its own strategy. We define such an equilibrium and provide necessary conditions for its existence.

3.1. Definition of Equilibrium. For any $1 \le p \le \infty$ and for any normed vector space F, we introduce the following space of adapted processes:

$$\mathcal{L}^p_t(F) := \left\{ (X_s)^t_{s=0};\ \mathbb{F} - \text{adapted},\ F\text{-valued}, \|X_s\| \in L^p(\mathcal{F}_s), s = 0, \dots, t \right\}. \tag{3.1}$$

We also introduce the spaces of admissible production strategies:

$$\begin{aligned} \mathcal{U}^i &:= \left\{ (\xi^i_t)^{T-1}_{t=0} \in \mathcal{L}^\infty_{T-1}(\mathbb{R}^{M_i});\ 0 \le \xi^{i,j,k}_t \le \kappa^{i,j,k}\ \mathbb{P}.a.s.\ , \quad t = 0, \dots, T-1 \right\}, \\ \mathcal{U} &:= \left\{ \xi \in \prod_{i\in I} \mathcal{U}^i;\ \sum_{i\in I} \sum_{j\in J^{i,k}} \xi^{i,j,k}_t = D^k_t\ \mathbb{P}.a.s.\ ,\ \ k \in K,\ \ t = 0, \cdots, T-1 \right\} \end{aligned}$$

and the spaces of admissible trading strategies:

$$\begin{aligned} \mathcal{V}^i(A, \Lambda^i) &:= \left\{ (\theta^i_t)^T_{t=0},\ \mathbb{F} - \text{adapted},\ R^A_T(\theta^i) \in L^1(\mathcal{F}_T),\ \theta^i_T \ge -\Lambda^i\ \mathbb{P} - a.s. \right\} \\ \mathcal{V}(A, \Lambda) &:= \prod_{i\in I} \mathcal{V}^i(A, \Lambda^i). \end{aligned}$$

In order to avoid possible problems with existence of the expected values of the random variables appearing in (2.16), we assume that the uncontrolled emissions and the production costs are integrable:

ASSUMPTION 1.

$$\Delta^i \in L^1(\mathcal{F}_T), \quad \Lambda^i \in L^1(\mathcal{F}_T), \quad \{C^i_t = (C^{i,j,k}_t)_{(j,k)\in M_i}\}_{t=0}^{T-1} \in \mathcal{L}^1_{T-1}(\mathbb{R}^{M_i}) \qquad i \in I, \tag{3.2}$$

In what follows, we also use a technical assumption on the nature of the uncontrolled emissions. Even though this assumption is not needed for most of the equilibrium existence results, it will help us characterize the prices in equilibrium by ruling out pathological situations. This technical assumption states that up until the end of the compliance period, there is always uncertainty about the expected pollution level due to unpredictable events as described in Section 2.3 in the sense that, conditionally on the information available at time $T-1$, the sum of all the Γ^is has a continuous distribution. More precisely, we shall assume that

ASSUMPTION 2. *The $\mathcal{F}_{T-1}$-conditional distribution of the total uncontrolled emissions $\Delta := \sum_{i\in I}\Delta^i$ possesses almost surely no point mass, or equivalently, for all $\mathcal{F}_{T-1}$-measurable random variables Z*

$$\mathbb{P}\{\Delta = Z\} = 0. \tag{3.3}$$

As we already pointed out, this technical assumption will help us refine the statements of some of the results in equilibrium. Following the intuition that given price processes $A = \{A_t\}_{t=0}^T$ and $S = \{(S^k_t)_{k\in K}\}_{t=0}^{T-1}$ each firm aims at increasing its own wealth by maximizing

$$(\theta^i, \xi^i) \mapsto \mathbb{E}[L^{A,S,i}(\theta^i, \xi^i)], \tag{3.4}$$

over its admissible investment and production strategies, we are led to define equilibrium in the following way:

DEFINITION 1. *The pair of price processes $(A^*, S^*) \in \mathcal{L}^1_T(\mathbb{R}) \times \mathcal{L}^1_{T-1}(\mathbb{R}^{|K|})$ form an equilibrium of the market if for each $i \in I$ there exists $(\theta^{*i}, \xi^{*i}) \in \mathcal{V}^i(A^*, \Lambda^i) \times \mathcal{U}^i$ such that:*

(i) All financial positions are in zero net supply, i.e.

$$\sum_{i\in I} \theta^{*i}_t = 0, \qquad t = 0, \dots, T, \ \mathbb{P}.a.s. \tag{3.5}$$

(ii) Supply meets demand for each good

$$\sum_{i\in I}\sum_{j\in J^{i,k}} \xi^{*i,j,k}_t = D^k_t, \qquad k \in K, \ \ t = 0, \dots, T-1, \ \ \mathbb{P}.a.s. \tag{3.6}$$

(iii) Each firm $i \in I$ is satisfied with its own strategies in the sense that

$$\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i}, \xi^{*i})] \geq \mathbb{E}[L^{A^*,S^*,i}(\theta^i, \xi^i)] \quad \textit{for all } (\theta^i, \xi^i) \in \mathcal{V}^i(A^*, \Lambda^i) \times \mathcal{U}^i. \tag{3.7}$$

3.2. Equilibrium in the Business-As-Usual Scenario. When the penalty π is equal to zero, in other words in the absence of a climate policy and an emission market, our equilibrium theory should recover what is known as the Business-As-Usual (BAU for short) scenario. As we explain below, it is characterized by the classical merit order production strategy which can be described in the following way. At each time t, and for each good k, all the production means of the economy are ranked by increasing production costs $C_t^{i,j,k}$. Demand is met by producing from the cheapest production means and good k's equilibrium spot price is the marginal cost of production of the most expensive production means used to meet demand D_t^k. In order to prove this claim we notice that, if (A^*, S^*) is an equilibrium when $\pi = 0$, the optimization problem (3.4) of firm i becomes

$$\sup_{(\theta^i,\xi^i)\in\mathcal{V}^i(A^*,\Lambda^i)\times\mathcal{U}^i} \mathbb{E}\left[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t^{*k} - C_t^{i,j,k})\xi_t^{i,j,k} + \sum_{t=0}^{T-1}\theta_t^i(A_{t+1}^* - A_t^*) - \theta_T^i A_T^*\right] .$$

The objective function to maximize is the sum of a function of the production strategies and a function of the investment strategy. Moreover, since the feasible set is a product, trading and production strategies are *decoupled* and we can maximize the two terms separately. Production strategies and the prices of the goods are determined by the first (and only meaningful) optimization problem which reads like a classical competitive equilibrium problem where each firm maximizes

$$\sup_{\xi^i\in\mathcal{U}^i} \mathbb{E}\left[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t^{*k} - C_t^{i,j,k})\xi_t^{i,j,k}\right] , \tag{3.8}$$

and the equilibrium prices S^* are set so that supply meets demand. The solution of this equilibrium problem is given by the following linear program. For each good $k \in K$:

$$\begin{aligned}((\xi_t^{*i,j,k})_{j\in J^{i,k}})_{i\in I} = \operatorname{argmax}_{((\xi_t^{i,j,k})_{j\in J^{i,k}})_{i\in I}} & \sum_{i\in I}\sum_{j\in J^{i,k}} -C_t^{i,j,k}\xi_t^{i,j,k} \\ \text{s.t. } & \sum_{i\in I}\sum_{j\in J^{i,k}} \xi_t^{i,j,k} = D_t^k \\ & 0 \le \xi_t^{i,j,k} \le \kappa^{i,j,k} \quad \text{for } i\in I, j\in J^{i,k},\end{aligned} \tag{3.9}$$

for all times t. The associated equilibrium prices are

$$S_t^{*k} = \max_{i\in I,\, j\in J^{i,k}} C_t^{i,j,k}\mathbf{1}_{\{\xi_t^{*i,j,k}>0\}}, \tag{3.10}$$

This is exactly the merit order pricing mechanism described earlier. In the case of electricity, it can be observed in most deregulated electricity markets without emission trading scheme. Conversely, it is easily seen that the above prices together with the above strategies define an equilibrium. In Section 4 we will see that even under an emission trading scheme, the dispatching of production among producers is still a merit order-like dispatching with costs adjusted to take into account the mark-to-market value of emissions.

3.3. Necessary Conditions for the Existence of an Equilibrium. Before turning to the full characterization of the equilibriums, we present some necessary conditions that will provide interesting insight.

Proposition 3.1 (Necessary Conditions). *Let (A^*, S^*) be an equilibrium and (θ^*, ξ^*) an associated set of optimal strategies. Then the following conditions hold:*

(i) The allowance price A^* is a bounded martingale with values in $[0,\pi]$ satisfying

$$\{A^*_T = 0\} \supseteq \{\sum_{i\in I}(\Gamma^i + \Pi^i(\xi^{*i})) < 0\} \tag{3.11}$$

$$\{A^*_T = \pi\} \supseteq \{\sum_{i\in I}(\Gamma^i + \Pi^i(\xi^{*i}) > 0\} \tag{3.12}$$

up to sets of probability zero.
(ii) Under Assumption 2, the price process A^* is almost surely given by

$$A^*_t = \pi\mathbb{P}\{\sum_{i\in I}(\Gamma^i + \Pi^i(\xi^{*i})) \geq 0|\mathcal{F}_t\} \tag{3.13}$$

for all $t = 0,\dots,T$.
(iii) The spot prices S^{*k} and the optimal production strategy ξ^{*i} correspond to a merit order-type equilibrium with adjusted costs $\tilde{C}^{i,j,k}_t = C^{i,j,k}_t + e^{i,j,k}A^*_t$.

Proof. We first show that A^* has to be a martingale. This is seen as follows: if not, there exists a time $t \in \{0,\dots,T-1\}$ and a set $\mathcal{A} \in \mathcal{F}_t$ of non-zero probability such that $\mathbb{E}_t\{A^*_{t+1}\mathbf{1}_{\mathcal{A}}\} > \mathbf{1}_{\mathcal{A}}A^*_t$ (resp. $<$). Then for each agent $i \in I$ the trading strategy given by $\bar{\theta}^i_s = \theta^{*i}_s$ for all $s \neq t$ and $\bar{\theta}^i_t = \theta^{*i}_t + 1_{\mathcal{A}}$ (resp $\bar{\theta}^i_t = \theta^{*i}_t - 1_{\mathcal{A}}$) outperforms the strategy θ^{*i}, contradicting the third property of an equilibrium.
To prove (3.11) and (3.12), notice that according to the definition of the equilibrium, $\theta^{*i}_T(\omega)$ coincides for almost all $\omega \in \Omega$ with the maximizer of

$$z \to \varphi^{*i}(z) = -A^*_T(\omega)z - \pi(\Gamma^i(\omega) + \Pi^i(\xi^{*i})(\omega) - z)^+ \tag{3.14}$$

over the interval $[-\Lambda^i(\omega),\infty)$. Note that the function φ^{*i} is either affine, or continuous and piecewise affine with two linear pieces. As a consequence, we conclude that $A^*_T \in [0,\pi]$ almost surely. Indeed, if $A^*_T(\omega) < 0$ then all the linear pieces have positive slopes and there is no finite maximizer to (3.14). On the other hand, if $A^*_T(\omega) > \pi$ then for every $i \in I$, the maximizer of φ^{*i} is $z = -\Lambda^i(\omega)$ which does not give an equilibrium either since $\sum_{i\in I}\theta^{*i}(\omega) = -\sum_{i\in I}\Lambda^i(\omega) = -\Lambda(\omega)$ which is different from 0 with strictly positive probability.

Further, observe that $\Delta^i + \Pi^i(\xi^{*i}) \geq 0$. Hence, if $A^*_T(\omega) \in (0,\pi]$ then the maximizer is less than or equal to $\Gamma^i(\omega) + \Pi^i(\xi^{*i})(\omega) = \Delta^i(\omega) - \Lambda^i(\omega) + \Pi^i(\xi^{*i})(\omega)$. Since this holds almost surely for each $i \in I$, the following inclusions are satisfied almost surely as well:

$$\{A^*_T \in (0,\pi]\} \subseteq \cap_{i\in I}\{\theta^{*i}_T \leq \Gamma^i + \Pi^i(\xi^{*i})\} \subseteq \{\sum_{i\in I}\theta^{*i}_T \leq \sum_{i\in I}(\Gamma^i + \Pi^i(\xi^{*i}))\} \tag{3.15}$$

and consequently

$$\{A^*_T \in (0,\pi]\} \cap \{\sum_{i\in I}(\Gamma^i + \Pi^i(\xi^{*i})) < 0\} \subseteq \left\{\sum_{i\in I}\theta^{*i}_T < 0\right\} \tag{3.16}$$

Since $\sum_{i\in I}\theta^{*i}_T = 0$ because of the first equilibrium condition, the event on the right hand sides of (3.16) is a set of probability zero. This implies inclusion (3.11).

Now if $A^*_T(\omega) \in [0,\pi)$, then the maximizer is greater than or equal to $\Gamma^i(\omega) + \Pi^i(\xi^{*i})(\omega)$. Consequently we have

$$\{A^*_T \in [0,\pi)\} \subseteq \cap_{i\in I}\{\theta^{*i}_T \geq \Gamma^i + \Pi^i(\xi^{*i}) \subseteq \{\sum_{i\in I}\theta^{*i}_T \geq \sum_{i\in I}(\Gamma^i + \Pi^i(\xi^{*i}))\} \tag{3.17}$$

Consequently, using again the fact that $\sum_{i\in I}\theta_T^{*i}=0$, we get the desired inclusion (3.12)

$$\{A_T^*\in[0,\pi)\}\subseteq\{\sum_{i\in I}(\Gamma^i+\Pi^i(\xi^{*i}))\le 0\} \tag{3.18}$$

Condition (ii) is a direct consequence of (i) and Assumption 2.

In order to prove (iii), we first notice that the optimization problem for firm i reads

$$\sup_{\xi^i\in\mathcal{U}^i}\sup_{\theta_T^i\in\mathcal{V}(A,\Lambda^i)}\mathbb{E}[L^{A^*,S^*,i}(\theta^i,\xi^i)]. \tag{3.19}$$

and we argue as before that for every $\xi^i\in\mathcal{U}^i$ the inner supremum can be attained by maximizing pointwise inside the expectation. This inner supremum is attained at $\theta^i(\xi^i)$ with $\theta_t^i(\xi^i)=0$ for all $t=0,\dots,T-1$ and a $\theta_T^i(\xi^i)$ which is up to null sets the maximizer of

$$z\hookrightarrow\varphi^i(z)=-A_T^*(\omega)z-\pi(\Gamma^i(\omega)+\Pi^i(\xi^i)(\omega)-z)^+ \tag{3.20}$$

over $[-\Lambda^i(\omega),\infty)$ for all $\omega\in\Omega$. We use essentially the same argument as before together with the fact that we now know that A_T^* can only take the values 0 or π almost surely. Fix any $\xi^i\in\mathcal{U}^i$. On $\{A_T^*=0\}$ the optimal value of (3.20) is 0. On the other hand, on $\{A_T^*=\pi\}$, it follows from $\Delta^i\ge 0$ that the optimal value of (3.20) is $-(\Gamma^i+\Pi^i(\xi^i))\pi$. In both cases, the optimal value of (3.20) is almost surely $-(\Gamma^i+\Pi^i(\xi^i))A_T^*$.

Hence the outer optimization problem reduces to

$$\begin{aligned}&\sup_{\xi^i\in\mathcal{U}^i}\mathbb{E}\left[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t^{*k}-C_t^{i,j,k})\xi_t^{i,j,k}-(\Gamma^i+\Pi^i(\xi^i))A_T^*\right]\\&=\sup_{\xi^i\in\mathcal{U}^i}\mathbb{E}\left[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t^{*k}-C_t^{i,j,k}-e^{i,j,k}A_t^*)\xi_t^{i,j,k}-\Gamma^iA_T^*\right].\end{aligned} \tag{3.21}$$

where we used again the martingale property of A^*. Comparing the above optimization problem with (3.8), we observe that the equilibrium can be seen as a competitive production equilibrium with adjusted costs $\tilde{C}_t^{i,j,k}=C_t^{i,j,k}+e^{i,j,k}A_t^*$. This concludes the proof. □

REMARK 3. *The proof of inclusions (i)-(ii) was carried out under fixed ξ^*. Hence it is straightforward to extend Proposition 3.1 (i)-(ii) to the case where the allocation depends on production strategies.*

The above results provide a better understanding of what a potential equilibrium should be. The allowance price must always be in $[0,\pi]$, which is very intuitive since buying an extra allowance at time t will result in a gain of at most π at time T. As highlighted in the previous subsection, the equilibrium in the BAU scenario can be related to a global cost minimization problem. We shall see in the next section that the equilibrium in the presence of a trading scheme enjoys the property of social optimality in the sense that any equilibrium corresponds to the solution of a certain global optimization problem, where the total pollution is reduced at minimal overall costs. We call this optimization problem the *representative agent* problem. Note that it is sometimes called the *informed central planner* problem. Beyond the economic interpretation of social-optimality, the importance of the global optimization problem is that its solution helps calculate the allowance prices in equilibrium. We now explore this connection in detail.

4. Equilibrium and Global Optimality. In this section, we show rigorously the existence of an equilibrium as defined in Definition 1. We do so by framing the problem as an equivalent global optimization problem involving a hypothetical *informed central planner* whom we call the representative agent. We prove the equivalence of the two approaches, and as a by-product of the necessary condition proven in the previous section, we derive the uniqueness of the allowance price process.

4.1. The Representative Agent Problem. For each admissible production strategy $\xi = \{\xi^i\}_{i \in I} \in \mathcal{U}$, the overall production costs are defined as

$$C(\xi) := \sum_{t=0}^{T-1} \sum_{(i,j,k) \in M} \xi_t^{i,j,k} C_t^{i,j,k},$$

and the overall cumulated emissions as

$$\Pi(\xi) := \sum_{t=0}^{T-1} \sum_{(i,j,k) \in M} e^{i,j,k} \xi_t^{i,j,k} = \sum_{i \in I} \Pi^i(\xi^i). \tag{4.1}$$

Using the notation

$$\Gamma := \Delta - \Lambda = \sum_{i \in I} (\Delta^i - \Lambda^i) = \sum_{i \in I} \Gamma^i \tag{4.2}$$

for the aggregate uncontrolled emissions minus the allowance endowments, the societal costs from production and penalty payments can be defined as

$$G(\xi) := C(\xi) + \pi(\Gamma + \Pi(\xi))^+, \qquad \xi \in \mathcal{U} \ . \tag{4.3}$$

We introduce the global optimization problem

$$\inf_{\xi \in \mathcal{U}} \mathbb{E}[G(\xi)] \tag{4.4}$$

which corresponds to the objective of an informed central planner trying to minimize overall expected societal costs. Recall that ξ is admissible if $\xi \in \mathcal{U}$, i.e. if the demand is met and the capacity constraints are satisfied. The reason for the introduction of this global optimization problem is contained in the second necessary condition for the existence of equilibrium.

PROPOSITION 4.1. *If (A^*, S^*) is an equilibrium with associated strategies (θ^*, ξ^*), then ξ^* is a solution of the global optimization problem (4.4) .*

Proof. Obviously, it suffices to show that

$$\mathbb{E}[G(\xi^*)] \leq \mathbb{E}[G(\xi)] \quad \text{ for all } \xi \in \mathcal{U}. \tag{4.5}$$

In order to do so we notice that:

$$
\begin{aligned}
\sum_{i\in I}\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i},\xi^{*i})] &= \mathbb{E}\Big[\sum_{t=0}^{T-1}\sum_{i,j,k\in M}(S_t^{*k}-C_t^{i,j,k})\xi_t^{*i,j,k} \\
&\quad + \sum_{t=0}^{T-1}\Big(\sum_{i\in I}\theta_t^{*i}\Big)(A_{t+1}^*-A_t^*) - \Big(\sum_{i\in I}\theta_T^{*i}\Big)A_T^* \\
&\quad - \pi\sum_{i\in I}(\Gamma^i+\Pi^i(\xi^{*i})-\theta_T^{*i})^+\Big] \\
&= \mathbb{E}\Big[\sum_{t=0}^{T-1}\sum_{k\in K}S_t^{*k}\big(\sum_{i\in I}\sum_{j\in J^{i,k}}\xi_t^{*i,j,k}\big)-C(\xi^*) \\
&\quad - \pi\sum_{i\in I}(\Gamma^i+\Pi^i(\xi^{*i})-\theta_T^{*i})^+\Big]
\end{aligned}
$$

where we used the fact that in equilibrium, $\sum_{i\in I}\theta_t^{*i}=0$ holds for all $t=0,\dots,T$ due to the clearing condition (i) of Definition 1. Next we use the convexity inequality

$$
\sum_{i\in I}x_i^+ \geq \big(\sum_{i\in I}x_i\big)^+
$$

and once more the fact that the financial positions are in zero net supply to conclude that

$$
\begin{aligned}
\sum_{i\in I}\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i},\xi^{*i})] &\leq \sum_{t=0}^{T-1}\sum_{k\in K}\mathbb{E}[S_t^{*k}D_t^k]-\mathbb{E}[C(\xi^*)] \\
&\quad - \pi\mathbb{E}\Big[\big(\sum_{i\in I}\Gamma^i+\sum_{i\in I}\Pi^i(\xi^{*i})\big)^+\Big] \\
&= \sum_{t=0}^{T-1}\sum_{k\in K}\mathbb{E}[S_t^{*k}D_t^k]-\mathbb{E}[C(\xi^*)]-\pi\mathbb{E}[(\Gamma+\Pi(\xi^*))^+] \\
&= \sum_{t=0}^{T-1}\sum_{k\in K}\mathbb{E}[S_t^{*k}D_t^k]-\mathbb{E}[G(\xi^*)].
\end{aligned}
$$

Now, for each $\xi\in\mathcal{U}$ we define $\theta(\xi)$ as

$$
\theta_t^i(\xi)=0 \quad \text{for all } i=1,\dots,N,\ t=0,\dots,T-1, \tag{4.6}
$$

$$
\theta_T^i(\xi)=\Delta^i-\Lambda^i+\Pi^i(\xi^i)-\frac{\sum_{i\in I}(\Delta^i-\Lambda^i+\Pi^i(\xi^i))}{\sum_{i\in I}(\Delta^i+\Pi^i(\xi^i))}(\Delta^i+\Pi^i(\xi^i))\ . \tag{4.7}
$$

The above denominator does not vanish by assumption and clearly, $\theta_T^i(\xi)$ satisfies $-\Lambda^i\leq\theta_T^i(\xi)\leq\Lambda-\Lambda^i$. Moreover, repeating the above argument for $(\theta(\xi),\xi)$ yields

$$
\sum_{i\in I}\mathbb{E}[L^{A^*,S^*,i}(\theta^i(\xi),\xi^i)]=\sum_{t,k}\mathbb{E}[S^{*k}_{\ t}D_t^k]-\mathbb{E}[G(\xi)]. \tag{4.8}
$$

Applying the third property (*each agent is satisfied with its own strategy*) of the (A^*,S^*) equilibrium to the optimal investment and production strategies (θ^{*i},ξ^{*i})

and $(\theta^i(\xi), \xi^i)$ yields

$$\mathbb{E}[G(\xi^*)] \le \sum_{t,k} \mathbb{E}[S_t^{*k} D_t^k] - \sum_{i\in I} \mathbb{E}[L^{A^*,S^*,i}(\theta^{*i}, \xi^{*i})]$$
$$\le \sum_{t,k} \mathbb{E}[S_t^{*k} D_t^k] - \sum_{i\in I} \mathbb{E}[L^{A^*,S^*,i}(\theta^i(\xi), \xi)] = \mathbb{E}[G(\xi)].$$

This holds for all $\xi \in \mathcal{U}$ completing the proof. □

The existence of an optimal $\bar{\xi}$ for the global optimization problem (4.4) follows from standard functional analytic arguments.

PROPOSITION 4.2. *Under Assumption 1, there exists a solution* $\bar{\xi} \in \mathcal{U}$ *of the global optimal control problem* (4.4).

Our proof relies on two simple properties which we state and prove as lemmas for the sake of clarity. First, we note that $\mathcal{L}^1 := \prod_{i\in I} \mathcal{L}^1_{T-1}(\mathbb{R}^M)$, equipped with the norm

$$\|X\| = \sum_{t=0}^{T-1} \sum_{(i,j,k)\in M} \mathbb{E}[|X_t^{i,j,k}|]$$

is a Banach space with dual $\mathcal{L}^\infty := \prod_{i\in I} \mathcal{L}^\infty_{T-1}(\mathbb{R}^M)$, the duality form being given by

$$\langle X, \xi\rangle := \sum_{t=0}^{T-1} \sum_{(i,j,k)\in M} \mathbb{E}[X_t^{i,j,k} \xi_t^{i,j,k}], \qquad X \in \mathcal{L}^1, \xi \in \mathcal{L}^\infty.$$

Next, we consider the weak* topology $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$ on $\mathcal{L}^\infty$ (see [20]), namely the weakest topology for which all the linear forms

$$\mathcal{L}^\infty \ni \xi \longmapsto \langle X, \xi\rangle \in \mathbb{R}, \tag{4.9}$$

for $X \in \mathcal{L}^1$ are continuous.

LEMMA 4.3. *The real valued function*

$$\mathcal{L}^\infty \ni \xi \longmapsto \mathbb{E}[G(\xi)] = \mathbb{E}[C(\xi)] + \pi\mathbb{E}[(\Gamma + \Pi(\xi))^+] \tag{4.10}$$

is lower semi-continuous for the weak topology.*

Proof. Obviously, the real valued function

$$\mathcal{L}^\infty \ni \xi \longmapsto \mathbb{E}[C(\xi)]$$

is continuous for the weak* topology since it is of the form $\xi \longmapsto \langle X, \xi\rangle$ for some $X \in \mathcal{L}^1$ since $X = C = \{C_t^{i,j,k}\}$ is a fixed element in $\mathcal{L}^1$ by assumption. So we only need to prove that the real valued function

$$\mathcal{L}^\infty \ni \xi \longmapsto \mathbb{E}[(\Gamma + \Pi(\xi))^+] \tag{4.11}$$

is lower semi-continuous. Using the fact that for any integrable random variable X one has

$$\mathbb{E}[X^+] = \sup_{0\le Y\le 1} \mathbb{E}[XY]$$

one sees that

$$\mathbb{E}[(\Gamma + \Pi(\xi))^+] = \sup_{0 \le Y \le 1} (\mathbb{E}[\Gamma Y] + \mathbb{E}[Y\Pi(\xi)])$$

and hence that the function (4.11) is the supremum of a family of continuous function since for fixed Y, $\mathbb{E}[\Gamma Y]$ is a constant and $\xi \longmapsto \mathbb{E}[Y\Pi(\xi)]$ is continuous for the weak* topology by the very definition of this topology. Since the supremum of any family of continuous functions is lower semi-continuous, this concludes the proof that (4.10) is lower semi-continuous. □

LEMMA 4.4. *For the convex subset* $\mathcal{U}$ *of* $\mathcal{L}^\infty$ *it holds that:*
(i) $\mathcal{U}$ *is norm-closed in* $\mathcal{L}^1$
(ii) $\mathcal{U}$ *is weakly** *closed in* $\mathcal{L}^\infty$.

Proof. (i)If $(\xi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{U}$ converging in $\mathcal{L}^1$ to an element $\xi \in \mathcal{L}^1$, then $(\xi^{i,j,k}_{t,n})_{n\in\mathbb{N}}$ converges in mean for each $(i,j,k) \in M$ and $t = 0,\ldots,T-1$, and extracting a subsequence if necessary, one concludes that $(\xi^{i,j,k}_{t,n})_{n\in\mathbb{N}}$ and $(\sum_{i\in I}\sum_{j\in J^{i,k}} \xi^{i,j,k}_{t,n})_{n\in\mathbb{N}}$ converge almost surely to $\xi^{i,j,k}_t$ and $\sum_{i\in I}\sum_{j\in J^{i,k}} \xi^{i,j,k}_t$ respectively, showing that the constraints defining $\mathcal{U}$ are satisfied in the limit, implying that $\xi \in \mathcal{U}$.

(ii) Since $\mathcal{U}$ is a convex norm-closed subset of $\mathcal{L}^1$, it follows from the Hahn-Banach Theorem that $\mathcal{U}$ is the intersection of halfspaces containing $\mathcal{U}$, i.e. of the form $H_{\xi,c} = \{X \in \mathcal{L}^1 | \langle X, \xi\rangle \le c\}$ with $\xi \in \mathcal{L}^\infty$ and $c \in \mathbb{R}$ such that $\mathcal{U} \subseteq H$. Since $\mathcal{L}^\infty \subseteq \mathcal{L}^1$ it holds for each of these halfspaces $H_{\xi,c}$ that $\xi \in \mathcal{L}^1$. Thus we conclude that $H_{\xi,c} \cap \mathcal{L}^\infty = \{X \in \mathcal{L}^\infty | \langle X, \xi\rangle \le c\}$ is closed in $(\mathcal{L}^\infty, \sigma(\mathcal{L}^\infty, \mathcal{L}^1))$. Since by definition it holds that $\mathcal{U} \subseteq \mathcal{L}^\infty$, it follows that $\mathcal{U}$ is given by the intersection of the sets $H_{\xi,c} \cap \mathcal{L}^\infty$. Since any intersection of closed sets is closed we conclude that $\mathcal{U}$ is weakly* closed in $\mathcal{L}^\infty$. □

Proof of Proposition 4.2. Since $\mathcal{U}$ is bounded and weakly* closed due to Lemma 4.4, it follows from the theorem of Banach-Alaoglu that $\mathcal{U}$ is weakly* compact. We then conclude using Lemma 4.3 and the fact that any lower semi-continuous function attains its minimum on a compact set. □

REMARK 4. *Proposition 4.2 applied to allocations depending upon the production as long as* $\Lambda^i(\xi)$ *is an affine function of* ξ. *As already mentioned, this case will be considered extensively in Sections 6 and 7.*

4.2. Relation with the Original Equilibrium Problem. As a consequence of Assumption 2, for each production policy $\xi \in \mathcal{U}$, no point masses occur in the $\mathcal{F}_{T-1}$-conditional distribution of $\Gamma - \Pi(\xi)$. Hence, for all $t = 0,\ldots,T-1$ we have:

$$\mathbb{P}_t(\Gamma + \Pi(\xi) \ge 0) = \mathbb{P}_t(\Gamma + \Pi(\xi) > 0). \tag{4.12}$$

In the next theorem, we show that the value of the conditional probability in (4.12) characterizes the equilibrium allowance price at time t. To prepare for the proof of this important result, we first prove a technical lemma.

LEMMA 4.5. *Let* $\overline{\xi}$ *be any argument of the social optimization problem* (4.4) *whose existence is guaranteed by Proposition 4.2. It holds:*
(i) For fixed $t \in \{0,\ldots,T-1\}$ *and any* $\xi \in \mathcal{U}$ *with* $\xi_s = \overline{\xi}_s$ *for all* $s = 0,\ldots,t-1$

$$\mathbb{E}_t[G(\xi)] \ge \mathbb{E}_t[G(\overline{\xi})] \tag{4.13}$$

holds almost surely.
(ii) If Assumption 2 is satisfied, then for each $k \in K$ *and* $i, i' \in I$, $j \in J^{i,k}, j' \in J^{i',k}$ *it holds that*

$$\{\overline{\xi}_t^{i,j,k} \in [0, \kappa^{i,j,k})\} \cap \{\overline{\xi}_t^{i',j',k} \in (0, \kappa^{i',j',k}]\} \\ \subseteq \{C_t^{i,j,k} + e^{i,j,k}\overline{A}_t \geq C_t^{i',j',k} + e^{i',j',k}\overline{A}_t\} \tag{4.14}$$

for all $t = 0, \ldots, T-1$ *where* $\overline{A}_t = \pi \mathbb{P}_t(\Gamma + \Pi(\overline{\xi}) \geq 0)$.

Proof. (i) Assertion (4.13) is proven in the following way. If it does not hold, the $\mathbb{F}_t$–measurable set

$$\mathcal{O} := \{E_t(G(\xi)) < E_t(G(\overline{\xi})\},$$

has positive measure, i.e. $\mathbb{P}(\mathcal{O}) > 0$, and can be used to outperform $\overline{\xi}$ by ξ' defined as

$$\xi'_s = 1_{\mathcal{O}}\xi_s + 1_{\Omega\setminus\mathcal{O}}\overline{\xi}_s \text{ for all } s = 0, \ldots, T-1. \tag{4.15}$$

Note that since ξ and ξ' coincide at times $0, \ldots, t-1$, this definition yields an adapted process $\xi' \in \mathcal{U}$. With (4.15), we have the decomposition

$$G(\xi') = \mathbf{1}_{\mathcal{O}}G(\xi) + \mathbf{1}_{\Omega\setminus\mathcal{O}}G(\overline{\xi}),$$

which gives a contradiction to the optimality of $\overline{\xi}$ because:

$$\begin{aligned} \mathbb{E}[G(\xi')] &= \mathbb{E}[\mathbb{E}_t[1_{\mathcal{O}}G(\xi) + \mathbf{1}_{\Omega\setminus\mathcal{O}}G(\overline{\xi})]] \\ &= \mathbb{E}[\mathbf{1}_{\mathcal{O}}\mathbb{E}_t[G(\xi)] + \mathbf{1}_{\Omega\setminus\mathcal{O}}\mathbb{E}_t[G(\overline{\xi})] \\ &< \mathbb{E}[\mathbf{1}_{\mathcal{O}}\mathbb{E}_t[G(\overline{\xi})] + \mathbf{1}_{\Omega\setminus\mathcal{O}}\mathbb{E}_t[G(\overline{\xi}))] = \mathbb{E}[G(\overline{\xi})]. \end{aligned}$$

(ii) So let us now assume that $\overline{\xi}_t^{i,j,k} \in [0, \kappa^{i,j,k})$ and $\overline{\xi}_t^{i',j',k} \in (0, \kappa^{i',j',k}]$, and let us consider at time t, a small deviation from the global optimal strategy $\overline{\xi}$ consisting in a shift in production of $h_t > 0$ units of the good $k \in K$ whereby the firms $i \in I$ and $i' \in I$ increase and decrease their outputs from technologies $j \in \mathcal{J}^{i,k}$ and $j' \in \mathcal{J}^{i',k}$ respectively, in such a way that the new policy $\overline{\xi} + \chi$ with

$$\chi_t^{i,j,k} = h_t, \quad \chi_t^{i',j',k} = -h_t, \quad \text{and} \quad \chi_t^{i,j,k} = 0 \quad \text{if } s \neq t.$$

is still in $\mathcal{U}$. Take for example $h_t = \min\{\kappa^{i,j,k} - \overline{\xi}_t^{i,j,k}, \overline{\xi}_t^{i',j',k}\}$. Consider

$$D(\overline{\xi}, \lambda) = \frac{\mathbb{E}_t[G(\overline{\xi} + \lambda\chi)] - \mathbb{E}_t[G(\overline{\xi})]}{\lambda}, \quad \text{for} \quad \lambda \in (0, 1],$$

and let $\lambda \searrow 0$ along a countable set $(0, 1] \cap \mathbb{Q}$. By dominated convergence, we have

$$\lim_{\lambda\to 0} D(\overline{\xi}, \lambda) = -\left((C_t^{i,j,k} - C_t^{i',j',k}) - \pi\mathbb{P}_t[\Gamma + \Pi(\overline{\xi}) > 0](e^{i,j,k} - e^{i',j',k})\right) h_t$$

if $e^{i,j,k} \leq e^{i',j',k'}$ and

$$\lim_{\lambda\to 0} D(\overline{\xi}, \lambda) = -\left((C_t^{i,j,k} - C_t^{i',j',k}) - \pi\mathbb{P}_t[\Gamma + \Pi(\overline{\xi}) \geq 0](e^{i,j,k} - e^{i',j',k})\right) h_t.$$

if $e^{i,j,k} \geq e^{i',j',k'}$. So if Assumption 2 holds, both limits are given by

$$\lim_{\lambda\to 0} D(\overline{\xi}, \lambda) = -\left((C_t^{i,j,k} - C_t^{i',j',k}) - \overline{A}_t(e^{i,j,k} - e^{i',j',k})\right) h_t, \tag{4.16}$$

and since both limits should be non-positive by part (i), we conclude that

$$-\Big((C_t^{i,j,k} - C_t^{i',j',k}) - \mathbb{P}_t(\Gamma + \Pi(\overline{\xi}) > 0)(e^{i,j,k} - e^{i',j',k})\Big)\, h_t \le 0$$

almost surely. Hence the inclusion

$$\{h_t > 0\} \subseteq \{-\Big((C_t^{i,j,k} - C_t^{i',j',k}) - \overline{A}_t(e^{i,j,k} - e^{i',j',k})\Big) \le 0\}$$

holds almost surely, which is equivalent to (4.14). □

We can now turn to the main result of this section.

THEOREM 4.6. *If Assumptions 1 and 2 hold and $\overline{\xi}$ is any argument of the social optimization problem* (4.4) *whose existence is guaranteed by Proposition 4.2, we have:*
(i) the processes $(\overline{A}, \overline{S})$ defined by

$$\overline{A}_t = \pi \mathbb{P}_t[\Gamma + \Pi(\overline{\xi}) \ge 0], \qquad t = 0, \dots, T \tag{4.17}$$

and

$$\overline{S}_t^k = \max_{i \in I,\, j \in J^{i,k}} (C_t^{i,j,k} + e^{i,j,k}\overline{A}_t) 1_{\{\overline{\xi}_t^{i,j,k} > 0\}}, \qquad t = 0, \dots, T-1 \;\; k \in K, \tag{4.18}$$

form a market equilibrium in the sense of Definition 1, for which the associated production strategy is $\overline{\xi}$.
(ii) The equilibrium allowance price process is unique almost surely.
(iii) For each good $k \in K$, the price $\overline{S}^k$ is the smallest equilibrium price for good k in the sense that for any other equilibrium price process S^{*k}, we have $\overline{S}^k \le S^{*k}$ almost surely.

Proof. (i) We show that $\overline{A}$ and $\overline{S}$ so defined form an equilibrium by an explicit construction of firm investment strategies $\overline{\theta} = \theta(\overline{\xi})$ such that $(\overline{\theta}^i, \overline{\xi}^i)$ satisfies (3.5), (3.6) and (3.7). For the sake of convenience we recall the definition of $\overline{\theta} = \theta(\overline{\xi})$ from (4.6) and (4.7).

$$\overline{\theta}_t^i = 0 \quad \text{for all } i = 1, \dots, N,\ t = 1, \dots, T-1,$$
$$\overline{\theta}_T^i = \Delta^i - \Lambda^i + \Pi^i(\overline{\xi}^i) - \frac{\sum_{i \in I}(\Delta^i - \Lambda^i + \Pi^i(\overline{\xi}^i))}{\sum_{i \in I}(\Delta^i + \Pi^i(\overline{\xi}^i))}(\Delta^i + \Pi^i(\overline{\xi}^i))\ .$$

Since conditions (3.5) and (3.6) are obviously fulfilled, we focus on (3.7). We first show that $\mathbb{E}[L^{\overline{A},\overline{S},i}(\overline{\theta}^i(\xi^i), \xi^i)] \ge \mathbb{E}[L^{\overline{A},\overline{S},i}(\theta^i, \xi^i)]$ for all $(\theta^i, \xi^i) \in \mathcal{V}^i(\overline{A}, \Lambda^i) \times \mathcal{U}^i$, where $\overline{\theta}^i(\xi^i)$ is constant equal to 0 until time $T-1$ and

$$\overline{\theta}_T^i(\xi^i) := \Delta^i - \Lambda^i + \Pi^i(\xi^i) - \frac{\sum_{i \in I}(\Delta^i - \Lambda^i + \Pi^i(\overline{\xi}^i))}{\sum_{i \in I}(\Delta^i + \Pi^i(\overline{\xi}^i))}(\Delta^i + \Pi^i(\xi^i)).$$

Notice that $\overline{\theta}_T^i(\xi^i) \le \Gamma^i + \Pi^i(\xi^i)$ when $\Gamma + \Pi(\xi) \ge 0$ and $\overline{\theta}_T^i(\xi^i) < \Gamma^i + \Pi^i(\xi^i)$ otherwise. We have:

$$\mathbb{E}[L^{\overline{A},\overline{S},i}(\theta^i, \xi^i)] = \mathbb{E}\left[\sum_{t=0}^{T-1} \sum_{(j,k) \in M_i} (\overline{S}_t^k - C_t^{i,j,k})\xi_t^{i,j,k} - \theta_T^i \overline{A}_T - \pi(\Gamma^i + \Pi^i(\xi^i) - \theta_T^i)^+\right]$$

since $\overline{A}$ defined by (4.17) is a bounded martingale. For each $\xi^i \in \mathcal{U}^i$, we show that we can maximize the above quantity by computing the maximum pointwise in θ^i inside the expectation. In view of (4.17), when $\omega \in \{\Gamma + \Pi(\overline{\xi}) < 0\}$ we have $\overline{A}_T(\omega) = 0$ and the maximum of the function

$$z \hookrightarrow \overline{\varphi}^i(z) = -z\overline{A}_T(\omega) - \pi(\Gamma^i(\omega) + \Pi^i(\xi^i)(\omega) - z)^+ \qquad z \geq -\Lambda^i(\omega) \tag{4.19}$$

is attained at any point $z \in [\Gamma^i(\omega) + \Pi^i(\xi^i)(\omega), \infty)$ showing that $\overline{\theta}^i_T(\xi^i)(\omega)$ is a maximizer. On the other hand, when $\omega \in \{\Gamma + \Pi(\overline{\xi}) \geq 0\}$, we have $\overline{A}_T(\omega) = \pi$, the maximum of (4.19) is attained at any point $z \in [-\Lambda^i(\omega), \Gamma^i(\omega) + \Pi^i(\xi^i)(\omega)]$ and once again, $\overline{\theta}^i(\xi^i)$ is a maximizer. Notice for later reference that in both cases, the value of the maximum of (4.19) is almost surely $-(\Gamma^i + \Pi^i(\xi^i))\overline{A}_T$.
To finish the proof, we prove that $\mathbb{E}[L^{\overline{A},\overline{S},i}(\overline{\theta}^i, \overline{\xi}^i)] \geq \mathbb{E}[L^{\overline{A},\overline{S},i}(\overline{\theta}^i(\xi^i), \xi^i)]$ for all $\xi^i \in \mathcal{U}^i$. According to the above computation, we have:

$$\begin{aligned}
&\mathbb{E}[L^{\overline{A},\overline{S},i}(\overline{\theta}^i(\xi^i), \xi^i)] \\
&\quad = \mathbb{E}\left[\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (\overline{S}^k_t - C^{i,j,k}_t)\xi^{i,j,k}_t - (\Gamma^i + \Pi^i(\xi^i))\overline{A}_T\right] \\
&\quad = \mathbb{E}\left[\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (\overline{S}^k_t - C^{i,j,k}_t - e^{i,j,k}\overline{A}_T)\xi^{i,j,k}_t - \Gamma^i\overline{A}_T\right] \\
&\quad = \mathbb{E}\left[\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (\overline{S}^k_t - C^{i,j,k}_t - e^{i,j,k}\overline{A}_t)\xi^{i,j,k}_t - \Gamma^i\overline{A}_T\right] .
\end{aligned} \tag{4.20}$$

We now show that the following inclusions hold almost surely:

$$\{\overline{S}^k_t - C^{i,j,k}_t - e^{i,j,k}\overline{A}_t > 0\} \subseteq \{\overline{\xi}^{i,j,k}_t = \kappa^{i,j,k}\}, \tag{4.21}$$

$$\{\overline{S}^k_t - C^{i,j,k}_t - e^{i,j,k}\overline{A}_t < 0\} \subseteq \{\overline{\xi}^{i,j,k}_t = 0\}. \tag{4.22}$$

Inclusion (4.22) is a direct consequence of Definition (4.18) of the price process $\overline{S}$. Using this same Definition (4.18) and Lemma 4.5 we see that:

$$\begin{aligned}
&\{\overline{S}^k_t > C^{i,j,k}_t + e^{i,j,k}\overline{A}_t\} \\
&\quad \subset \bigcup_{i'\in I, j'\in J^{i',k}} \{C^{i',j',k}_t + e^{i',j',k}\overline{A}_t > C^{i,j,k}_t + e^{i,j,k}\overline{A}_t\} \cap \{\overline{\xi}^{i',j',k}_t > 0\} \\
&\quad \subset \bigcup_{i'\in I, j'\in J^{i',k}} \left(\{\overline{\xi}^{i,j,k}_t = \kappa^{i,j,k}\} \cup \{\overline{\xi}^{i',j',k}_t = 0\}\right) \cap \{\overline{\xi}^{i',j',k}_t > 0\} \\
&\quad \subset \{\overline{\xi}^{i,j,k}_t = \kappa^{i,j,k}\} .
\end{aligned}$$

These inclusions allow us to show that $\mathbb{E}[L^{\overline{A},\overline{S},i}(\overline{\theta}^i(\xi^i), \xi^i)] \leq \mathbb{E}[L^{\overline{A},\overline{S},i}(\overline{\theta}^i, \overline{\xi}^i)]$, thus completing the proof of (i).
(ii) Proposition 3.1 gives the form of an equilibrium price. Due to part (i) of Proposition 3.1 and Proposition 4.1 to prove almost sure uniqueness of the allowance price process, it is sufficient to prove that for any two solutions $\hat{\xi}$, $\tilde{\xi}$ of the global optimization problem (4.4) we have:

$$\mathbb{P}\left[\left(\{\Gamma+\Pi(\hat{\xi}) > 0\}\cap\{\Gamma+\Pi(\tilde{\xi}) > 0\}\right)\bigcup\left(\{\Gamma+\Pi(\hat{\xi}) < 0\}\cap\{\Gamma+\Pi(\tilde{\xi}) < 0\}\right)\right] = 1 \tag{4.23}$$

We know that these production strategies are solution of the global problem (4.4), that we rewrite as a linear programming problem:

$$\inf_{\substack{\xi\in\mathcal{U},\ Z\in L^1(\mathcal{F}_T)\\ Z\geq\Gamma+\Pi(\xi),\ Z\geq 0}} \mathbb{E}[C(\xi)+\pi Z]\ . \tag{4.24}$$

Each solution $(\xi^\star, Z^\star)$ of (4.24) satisfies

$$Z^\star = (\Gamma+\Pi(\xi^\star))^+ \tag{4.25}$$

almost surely. Assume now that there are two optimal solutions $(\hat{\xi},\hat{Z})$ and $(\tilde{\xi},\tilde{Z})$ of the above linear programming problem. Due to the linearity of (4.24) it follows that any convex linear combination

$$(\lambda\hat{\xi}+(1-\lambda)\tilde{\xi}, \lambda\hat{Z}+(1-\lambda)\tilde{Z}) \tag{4.26}$$

is also a solution to (4.24) for all $\lambda\in[0,1]$. In view of (4.25), we conclude that for each $\lambda\in[0,1]$

$$\lambda(\Gamma+\Pi(\hat{\xi}))^+ + (1-\lambda)(\Gamma+\Pi(\tilde{\xi}))^+ = \Big(\lambda(\Gamma+\Pi(\hat{\xi}))+(1-\lambda)(\Gamma+\Pi(\tilde{\xi}))\Big)^+$$

holds almost surely. Since the above assertion is obviously violated on

$$\{\Gamma+\Pi(\hat{\xi})<0<\Gamma+\Pi(\tilde{\xi})\}\cup\{\Gamma+\Pi(\hat{\xi})>0>\Gamma+\Pi(\tilde{\xi})\}$$

this union must have a probability 0, which together with Assumption 2, yields (4.23).
(iii) Assume on the contrary that there exists an equilibrium price process S^* with

$$S_t^{*k}(\omega)<\bar{S}_t^k(\omega) \text{ for all } \omega\in B \tag{4.27}$$

for some $t\in\{0,1,\dots,T-1\}$, $B\in\mathcal{F}_t$, $\mathbb{P}(B)>0$ and $k\in K$. Let ξ^* be the corresponding equilibrium strategies. Since the equilibrium allowance price $\bar{A}$ is unique, it follows from (4.22) that

$$\{S_t^{*k}-C_t^{i,j,k}-e^{i,j,k}\bar{A}_t<0\}\subseteq\{\xi_t^{*i,j,k}=0\}$$

up to sets of probability zero. Consequently we obtain

$$\begin{aligned}\sum_{i\in I}\sum_{j\in J^{i,k}}\xi_t^{*i,j,k} &= \sum_{i\in I}\sum_{j\in J^{i,k}}\xi_t^{*i,j,k}1_{\{S_t^{*k}\geq C_t^{i,j,k}+e^{i,j,k}\bar{A}_t\}} \\ &\leq \sum_{i\in I}\sum_{j\in J^{i,k}}\kappa^{i,j,k}1_{\{S_t^{*k}\geq C_t^{i,j,k}+e^{i,j,k}\bar{A}_t\}}\end{aligned} \tag{4.28}$$

almost surely. Moreover it follows from (4.22) and (4.21) that

$$\begin{aligned}&\sum_{i\in I}\sum_{j\in J^{i,k}}\kappa^{i,j,k}1_{\{\bar{S}_t>C_t^{i,j,k}+e^{i,j,k}\bar{A}_t\}} \\ &\quad=\sum_{i\in I}\sum_{j\in J^{i,k}}\bar{\xi}_t^{i,j,k}-\sum_{i\in I}\sum_{j\in J^{i,k}}\bar{\xi}_t^{i,j,k}1_{\{\bar{S}_t^k=C_t^{i,j,k}+e^{i,j,k}\bar{A}_t\}} \\ &\quad<\sum_{i\in I}\sum_{j\in J^{i,k}}\bar{\xi}_t^{i,j,k}\end{aligned} \tag{4.29}$$

holds almost surely. In the last equality we used the fact that

$$\sum_{i\in I}\sum_{j\in J^{i,k}} \bar{\xi}_t^{i,j,k} 1_{\{\bar{S}_t^k = C_t^{i,j,k} + e^{i,j,k}\bar{A}_t\}}(\omega) > 0 \text{ for all } \omega \in \Omega$$

which follows from the definition of $\bar{S}$. Moreover (4.27) implies that

$$\begin{aligned}\sum_{i\in I}\sum_{j\in J^{i,k}} \kappa^{i,j,k} 1_{\{S_t^{*k} \geq C_t^{i,j,k} + e^{i,j,k}\bar{A}_t\}}(\omega) \\ \leq \sum_{i\in I}\sum_{j\in J^{i,k}} \kappa^{i,j,k} 1_{\{\bar{S}_t^k > C_t^{i,j,k} + e^{i,j,k}\bar{A}_t\}}(\omega) \text{ for all } \omega \in B.\end{aligned} \tag{4.30}$$

From (4.28), (4.29) and (4.30) we conclude that there exists $C \subseteq B$ with

$$\sum_{i\in I}\sum_{j\in J^{i,k}} \xi_t^{*i,j,k}(\omega) < \sum_{i\in I}\sum_{j\in J^{i,k}} \bar{\xi}_t^{i,j,k}(\omega) = D_t(\omega)$$

for all $\omega \in C$, which implies that S^* cannot be a set of equilibrium prices for the goods. □

REMARK 5. *The above proof shows that because we are using a linear utility function (due to the fact that our firms are risk neutral), waiting* until the last minute *and purchasing the required amount of allowances on the last time trading is possible, constitutes an optimal trading strategy. However, trading is possible at all times and since there is no reason to believe that the optimal trading strategy is unique, one can imagine that there are less extreme strategies leading to the optimum.*

REMARK 6. *On the basis of what is known for merit-order equilibria with discontinuous cost functions, we do not expect uniqueness of the price process S^{*k}.*

REMARK 7. *In the introduction, we referred to* **reduction costs** *or* **abatement costs** *as the costs of regulation, i.e. the pollution reduction costs. We now give a formal definition of what we mean by reduction costs. For each regulatory allocation $((\Lambda_t^i)_{t=0}^{T-1})_{i\in I}$, and for any choice of an equilibrium production schedule $\xi^* \in \mathcal{U}$, we define the reduction costs RC as the random variable given by the difference between the production costs $C(\xi^*)$ under this production schedule and the production costs incurred in the same random scenarios had we used the BAU equilibrium production schedule. In other words, the reduction costs are given by the random variable:*

$$RC = C(\xi^*) - C(\xi^*_{BAU}) = \sum_{t=0}^{T-1} \sum_{(i,j,k)\in M} (\xi_t^{*i,j,k} - \xi_t^{BAU,i,j,k}) C_t^{i,j,k}. \tag{4.31}$$

Notice also that as defined, the reduction costs do not depend upon the trading strategies of the individual firms in the emissions market.

REMARK 8. *The results of this section were derived under the assumption that the emission coefficients $e^{i,j,k}$ were constant. However, by mere inspection of the proofs, the reader will easily be convinced that all the results remain true if these emission coefficients are instead adapted stochastic processes in $\mathcal{L}^1_{T-1}(\mathbb{R})$.*

5. Prices and Windfall Profits in the Standard Scheme. The previous sections were devoted to the introduction and the mathematical analysis of what we called the standard emission trading scheme. This cap-and-trade scheme was chosen because it is representative of the implementation of the first phase of the EU-ETS. In this section, we define rigorously the concept of windfall profit. This is done in Subsection 5.1. In Subsection 5.2, we elucidate the fact that free allocation of allowances can be used to create strong incentives for the use of the cleaner technologies.

But because our analysis of cap-and-trade schemes would not be complete without numerical evidence from realistic examples based on market data, we spend half of this section introducing a case study which we use for illustration purposes throughout the paper. For the purpose of numerical computations, we restrict our attention to an economy where one single good is produced. We choose the example of *electricity* because the power sector is worldwide one of the most important sources of green house gas emissions. We study the impact of regulation on spot prices and producer profits. In order to provide specific insight on the effects of cap-and-trade legislations, we performed numerical simulations of equilibrium prices and optimal production schedules by solving the global optimization problem (4.4) using data from the Texas electricity market. We introduce some of the intricacies of this market in Subsection 5.3 where we also describe the elements of the economy. Details on the stochastic models used for the exogenous processes of demand and production costs, together with details on the calibration of their parameters are given in the appendix at the end of the paper. Subsection 5.4 gives a first set of numerical results illustrating some of the major criticisms voiced against cap-and-trade schemes. We report numerical findings from this case study throughout the remainder of the paper.

5.1. Windfall Profits and Penalty Under the Standard Scheme. As explained above, the pricing mechanism of the standard emissions trading scheme induces a significant wealth transfer from consumers to producers.

Another way of understanding the extra profits made by the producers is to consider the windfall profits defined as follows. In the general framework of a standard cap-and-trade system with multiple goods, if ξ^* is an optimal production strategy associated with the equilibrium (A^*, S^*), we define the target price $\hat{S}_t^k$ of good k as:

$$\hat{S}_t^k := \max_{i \in I, j \in J^{i,k}} C_t^{i,j,k} \mathbf{1}_{\{\xi_t^{*i,j,k} > 0\}}. \tag{5.1}$$

This price is the marginal cost under the optimal production schedule without taking into account the cost of pollution. We then define the windfall profits of firm i as:

$$WP^i = \sum_{t=0}^{T-1} \sum_{(j,k) \in M_i} (S_t^{*k} - \hat{S}_t^k) \xi_t^{*i,j,k},$$

and the overall windfall profits as

$$WP = \sum_{i \in I} WP^i = \sum_{t=0}^{T-1} \sum_{k \in K} (S_t^{*k} - \hat{S}_t^k) D_t^k. \tag{5.2}$$

These windfall profits measure the profits for the production of goods in excess over what the profits would have been, had the same dispatching schedule been used, and the target prices (e.g. the marginal fuel costs) be charged to the end consumers without the cost of pollution.

REMARK 9. *A reasonable definition of the windfall profits of firm i could have been*

$$\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S_t^{*k} - \hat{S}_t^k)\xi_t^{*i,j,k} - \pi \left(\Gamma^i - \Pi^i(\xi^{*i})\right)^+ \tag{5.3}$$

in which the penalty payments due to the scheme are withdrawn from the extra profits. Since producers decide upon their production strategies and therewith the risk to pay the penalty, we take the point of view that they should pay the penalty and not the endconsumer. However as can be seen in Figure 5.1 the penalty payments vanish in comparison to the windfall profits as defined in (5.2). Hence in practical applications, both definitions give similar results.

Figure 5.1 shows the distribution of windfall profits as computed in the example of the Texas electricity market which we describe in Subsection 5.3. We observe that the windfall profits are in average almost 10 times higher than actual abatement costs. Furthermore it also shows that the costs of expected future penalty passed to the customers are much higher (4637 times) than the penalty actually paid. This is consistent with the deterministic example presented in the introduction.

5.2. Incentives for the Use of Cleaner Technologies. Using (4.20) we see that the expected profits and losses of firm $i \in I$ in an equilibrium (A^*, S^*) with associated production schedules ξ^* are given by

$$\begin{aligned} \mathbb{E}[L^{A^*,S^*,i}(\theta^{*i}, \xi^{*i})] = \mathbb{E}\left[(-\Delta^i + \sum_{t=0}^{T-1} \Lambda_t^i) A_T^*\right] \\ + \mathbb{E}\left[\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S_t^{*k} - C_t^{i,j,k} - e^{i,j,k} A_t^*)\xi_t^{*i,j,k}\right] \end{aligned} \tag{5.4}$$

As shown in Theorem 4.6, both the equilibrium price processes (A^*, S^*) and the production strategies ξ^* are preserved under a change of the regulatory allocation from $((\Lambda_t^i)_{t=0}^{T-1})_{i\in I}$ to $((\tilde{\Lambda}_t^i)_{t=0}^{T-1})_{i\in I}$ as long as

$$\sum_{i\in I} \sum_{t=0}^{T-1} \Lambda_t^i = \sum_{i\in I} \sum_{t=0}^{T-1} \tilde{\Lambda}_t^i$$

holds almost surely. However, such an adjustment of the allocation changes the expected profits and losses of producer $i \in I$ by the amount:

$$\mathbb{E}\left[(\tilde{\Lambda}^i - \Lambda^i) A_T^*\right]. \tag{5.5}$$

Obviously this creates a money transfer from producers with $\tilde{\Lambda}^i - \Lambda^i < 0$ to producers with $\tilde{\Lambda}^i - \Lambda^i > 0$. If the initial allocation can vary with the type of production plant, it is possible to set it up in order to increase or decrease the income of clean or dirty plants respectively. In other words, the initial allocation can be used to adjust incentives to build and run cleaner plants.

Even though our model concerns only the management of existing technologies, and does not address the important issue of investment in the development of cleaner technologies, the mechanism highlighted above is one of the most powerful tools put in the hands of policy makers by a cap-and-trade scheme. Clearly, this leverage will disappear if auctioning of the allowances is used, for example to control windfall profits.

We now elaborate on this last point. First we notice that even an auction of 100% of the allowances cannot reduce windfall profits to zero with certainty. This is obvious in a market with many nuclear power plants and where coal is always on the margin. Indeed, in such a market, nuclear power producers make huge windfall profits, and since their emissions are essentially zero, they do not need any allowances. Hence auctions can only affect the windfall profits of the coal fired plant owners. In summary, if the regulator chooses an auction to control windfall profits, a huge part of the initial allocation, if not all the allowances, should be auctioned. However in this case, the regulator loses the control of the incentives afforded by the free allocation of allowances. The analysis presented in the next section was motivated by this dilemma: we propose alternative cap-and-trade schemes that not only reduce windfall profits essentially to zero, but also provide a considerable amount of allowances that can be allocated for free. But before, we describe the case study we use for illustration purposes.

5.3. A Model for Electricity and Carbon Trading in Texas. To perform numerical simulations, we chose to focus on the electricity sector in Texas. Texas has an installed capacity of 81,855 MW, mainly split into gas-fired (51,489 MW), coal-fired (23,321 MW), and nuclear (9,019 MW) power plants. These figures are based on the installed capacity in 2007, including also additional nuclear and coal fired power plants that are planned to come online over the next 7 years. Including upcoming capacity slightly changes the *production stack* and leads to more interesting results than using the actual 2007 installed capacity. Nuclear technology has close to zero emissions, and it is always running in base-load. The source of emission reduction thus essentially comes from switching fuel between gas and coal.

So for all practical purposes, our model for Texas can be assumed to involve one good, electricity, produced from two different technologies, gas and coal. Stochastic costs of production are equal to $C_t^{i,j,k} = H^j P_t^j$, where $j \in \{g, c\}$, H^j is the heat rate of technology j and P_t^j is the corresponding fuel price. D_t stands for the electricity demand from which nuclear capacity has already been subtracted. We set the emission rates to 0.42 ton/MWh for gas technology (typical CCGT-like plant) and 0.95 ton/MWh for coal technology respectively. These average emission rates have been chosen to give a faithful representation of Texas' park of power plants.

The global optimization problem reads:

$$\inf_{\xi \in \mathcal{U}} \mathbb{E}\left[\sum_{t=0}^{T-1}(C_t^g \xi_t^g + C_t^c \xi_t^c) + \pi\left(\Gamma + \sum_{t=0}^{T-1}(e^g \xi_t^g + e^c \xi_t^c)\right)^+\right]$$

where Γ defined in (4.2) as the adjusted uncontrolled emissions. In the particular case of two technologies, we can proceed to the change of variable $(\xi_t^g, \xi_t^c) \longmapsto (\mathcal{E}_t, \mathcal{C}_t)$, where

$$\mathcal{E}_t = e^c \xi_t^c + e^g \xi_t^g \quad \text{and} \quad \mathcal{C}_t = C_t^c \xi_t^c + C_t^g \xi_t^g$$

are respectively the total emission and the cost of production for the period $[t, t+1)$. Using the constraint that the demand has to be met, we obtain an equivalent formulation in terms of an emission abatement problem:

$$\min_{\underline{\mathcal{E}} \leq \mathcal{E} \leq \overline{\mathcal{E}}} \mathbb{E}\left[\sum_{t=0}^{T-1}(D_t(e^c F_t + C_t^c) - F_t\mathcal{E}_t) + \pi\left(\Gamma + \sum_{t=0}^{T-1}\mathcal{E}_t\right)^+\right] \tag{5.6}$$

where:

$$\underline{\mathcal{E}}_t = e^g \min(D_t, \kappa^g) + e^c(D_t - \kappa^g)^+$$
$$\overline{\mathcal{E}}_t = e^c \min(D_t, \kappa^c) + e^g(D_t - \kappa^c)^+$$

are respectively the minimal and maximal emissions at time t, and

$$F_t := \frac{C_t^g - C_t^c}{e^c - e^g} \tag{5.7}$$

is the fuel spread per ton of CO_2 (or abatement cost). The fuel spread F represents the marginal switching cost necessary to decrease emissions by 1 unit. We observe that the above formulation (5.6) only involves 2 exogenous stochastic processes: D and F. Finally, we set the aggregated uncontrolled emissions $\sum_{i \in I} \Delta^i$ infinitesimally small to stay in the realm of the assumptions of Theorem 4.6, and solve the global optimization problem by stochastic dynamic programming on a 2-dimensional trinomial tree. Further details are given in the appendix at the end of the paper.

5.4. Electricity Price Impact Under the Standard Scheme. In this subsection, we discuss the impact of the regulation on electricity prices. We already emphasized that uniqueness of equilibrium electricity prices was not guaranteed. However, we identified the minimal price among all the possible equilibrium prices in Equation (4.18). In what follows, we focus on this price.

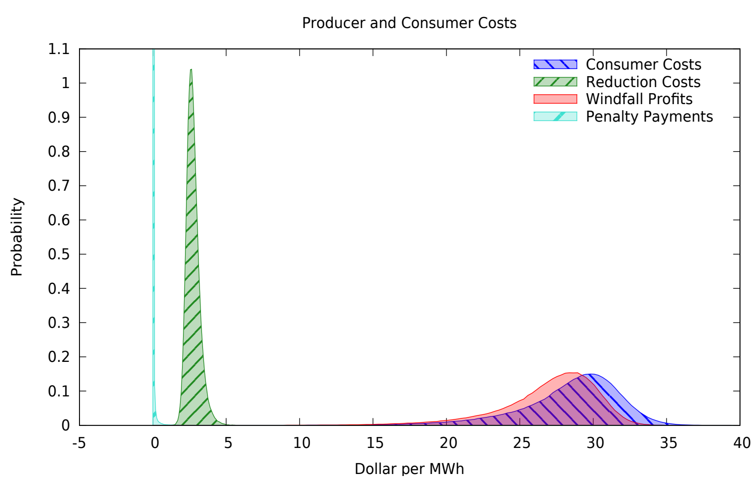

FIGURE 5.1. *Histograms of the consumer costs, reduction costs, windfall profits and penalty payments under a standard trading scheme scenario.*

Equation (4.18) shows two sources of change in the spot price compared to business as usual. First, the marginal technology may be different: this induces a variation in marginal cost. This variation is likely to be positive but a negative variation is possible. Suppose for example that in a BAU scenario, coal is started first but that demand is high enough so that gas is the marginal technology. Suppose that in the presence of the trading scheme, allowance price is high enough to induce a fuel switch, so that gas is started first. Assume also that demand is high so that coal is the marginal technology. In this case, the variation in marginal cost can be negative. The second source of variation is the price of pollution $e^{i,j,k}A_t^*$ for the marginal technology. The producers pass through the cost of expected penalties to end-consumers. This second contribution is always positive and is such that the spot price under the trading scheme is always greater than the spot price in BAU.

A possible interpretation of formula (4.18) is that the allowance price enters the electricity price as the price of an additional commodity that is used for power generation besides fuels. Producing the last infinitesimal unit of electricity at time t induces not only costs due to extra fuel consumption, but also increases the emissions by $e^{i,j,k}$ and hence also the expected penalty at time T by $e^{i,j,k}A_t^*$. Consequently these costs have to be covered by the end-consumers, for the marginal production of product k to be profitable. Since this amount is passed on to the endconsumer in each timestep the consumer cost $\sum_{t=0}^{T-1}(S_t^* - S_t^{BAU})D_t$ is much bigger than the penalty that is actually paid. As we will see in the following the consumer costs exceed also by far the reduction cost of the scheme.

Figure 5.1 quantifies both the penalty payments and the consumer cost, and compares them to reduction costs and windfall profits (as defined in the next section) under a standard trading scheme for the Texas electricity sector. The penalty and initial allocation for this example are $\pi = 100\$$ and $\theta_0 = 1.826 \times 10^8$ allowances respectively. This allocation corresponds to a reduction target of 10%, i.e. a cap of 1.827×10^8t Carbon, to be reached with 95% probability.

The results depicted in Figure 5.1 illustrate the major criticism articulated by some of the opponents of the cap-and-trade systems: end-consumer costs are approximately more than 10 times higher than social costs due to the trading scheme. Hence *the consumers' burden exceeds by far the overall reduction costs*, which gives rise for significant extra profits for the producers.

6. Alternative Designs for Emission Trading Schemes. The main objective of emission trading schemes is to use market mechanisms to force producers to reach a certain reduction target, and at the same time, to give incentives to develop and build cleaner production facilities. As explained in the introduction, this last objective cannot be fully investigated in the framework of our equilibrium models since they are limited to the optimization of the production from the already existing technologies. Despite this limitation, and in view of the shortcomings of the standard cap-and-trade scheme demonstrated in the previous section, we propose alternative designs which fulfil both objectives at low reduction costs, low windfall profits and hence low costs transferred to the consumer.

This is possible because the mathematical theory developed in the previous sections allows us to study emissions reduction policies that are different from the standard EU-ETS scheme.

In the first Subsection 6.1 below, we introduce a general (and fairly complex) cap-and-trade scheme including taxes and subsidies. We argue that the theoretical results derived earlier in the paper for standard schemes, can be transferred to this more general framework. The remaining part of the section is devoted to the identification and the calibration of two of the simplest particular cases: a relative scheme introduced in Subsection 6.3, and a *carbon tax* introduced in Subsection 6.4 and which can equivalently be viewed as a particular case of the Business As Usual scenario discussed in Subsection 3.2. The final Subsection 6.5 provides comparative statics highlighting the differences between these schemes on the case study of the Texas electricity market.

6.1. General Market Designs for Emission Trading Schemes. We first generalize the allocation procedure. Beyond the allocation Λ_t^i for firm i at time t, the regulator is now allowed to distribute credits dynamically and proportionally to production. To be more specific, at each time $0 \leq t < T$, firm i is provided with an allocation

$$\Lambda_t'^i(\xi^i) = X_t^i + \sum_{(j,k)\in M_i} Y_t^{i,j,k}\xi_t^{i,j,k}, \tag{6.1}$$

where X^i and $Y^{i,j,k}$ are adapted processes in $\mathcal{L}^1_{T-1}(\mathbb{R})$.

In addition, the regulator can also *tax* or *subsidize* the various firms by means of financial incentives or disincentives similar to the credit endowments described above. In this case, the profits of the firms are lowered at time t by an amount

$$V_t^i + \sum_{(j,k)\in M_i} Z_t^{i,j,k}\xi_t^{i,j,k}, \tag{6.2}$$

where V^i and $Z^{i,j,k}$ are also adapted processes in $\mathcal{L}^1_{T-1}(\mathbb{R})$. Notice that V^i and $Z^{i,j,k}$ represent a tax when positive and a subsidy when negative. Examples of positive $Z^{i,j,k}$ include fuel and carbon taxes. The combination of V^i and $Z^{i,j,k}$ allows for the introduction of alternative regulation such as a system of reward/penalty with respect to a given production (or equivalently emission) target. Under such a generalized cap-and-trade scheme, the terminal wealth (or profits and losses) of firm $i \in I$ reads:

$$\begin{aligned} H^{A,S,i}(\theta^i,\xi^i) := &-\sum_{t=0}^{T-1} V_t^i + \sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t^k - C_t^{i,j,k} - Z_t^{i,j,k})\xi_t^{i,j,k} \\ &+ \sum_{t=0}^{T-1}\theta_t^i(A_{t+1}-A_t) - \theta_T^i A_T \\ &- \pi\left(\Delta^i + \Pi^i(\xi^i) - \sum_{t=0}^{T-1}\left(X_t^i + \sum_{(j,k)\in M_i} Y_t^{i,j,k}\xi_t^{i,j,k}\right) - \theta_T^i\right)^+ . \end{aligned} \tag{6.3}$$

We can still use Definition 1 of an equilibrium for such a generalized scheme. Indeed Definition 1 carries over by merely eplacing $L^{A,S,i}(\theta^i,\xi^i)$ by $H^{A,S,i}(\theta^i,\xi^i)$, and Λ^i by $\sum_{t=0}^{T-1}\Lambda_t'^i(\xi^i)$. The present formulation gives a general framework for the analysis of a broader class of cap-and-trade schemes. We mostly focus on two important particular cases: 1) the case where $Z_t^{i,j,k} \geq 0$ varies with i and j which leads to a scheme based on a fuel or emission tax scheme, and 2) the case where $Z_t^{i,j,k} \leq 0$ only depends on k, which corresponds to a subsidy for the production of good k.

Given that the dependence of the generalized allocation (6.1) and the taxes and subsidies (6.2) upon the production schedule ξ is affine, and given the repeated remarks we made to this effect, most of the results of the first part of the paper (including the existence and the properties of the equilibrium prices) do apply to the present situation. However, because the statements of these generalizations would have been more involved and the proof more technical, we chose to present them first in the seemingly restrictive framework of the standard cap-and-trade schemes not to distract from the gist of the analysis. In the next section, we give equivalence results providing rigorous and enlightening proofs of these generalizations.

For an equilibrium (A^*, S^*) of the generalized scheme with associated strategies (θ^*, ξ^*), it is straightforward to extend the definition of windfall profits of firm i as:

$$GWP^i = \sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S_t^{*k} - \hat{S}_t^k)\xi_t^{*i,j,k} - \sum_{t=0}^{T-1} \left(V_t^i + \sum_{(j,k)\in M_i} Z_t^{i,j,k} \xi_t^{*i,j,k} \right),$$

the overall windfall profits being defined as

$$GWP = \sum_{i\in I} GWP^i = \sum_{t=0}^{T-1} \sum_{k\in K} (S_t^{*k} - \hat{S}_t^k) D_t^k - \sum_{t=0}^{T-1} \left(V_t^i + D_t^k \sum_{i\in I, j\in J^{i,k}} Z_t^{i,j,k} \right), \quad (6.4)$$

and used in this form in the case study of the Texas electricity market.

6.2. Equivalence between Equilibria. In this subsection, we restrict ourselves to allocation coefficients of proportionality $Y_t^{i,j,k}$ which depend upon the good, but which are the same for all the firms and all the technologies, and we often use the notation $Y_t^{i,j,k} = Y_t^k$. In this case, the proportional part of the new allocation (6.1) can be rewritten as

$$\sum_{(i,j,k)\in M} Y_t^{i,j,k} \xi_t^{i,j,k} = \sum_{k\in K} Y_t^k \left(\sum_{i\in I} \sum_{j\in J^{i,k}} \xi_t^{i,j,k} \right) = \sum_{k\in K} Y_t^k D_t^k.$$

The main thrust of this subsection is to identify one-to-one correspondences between equilibria of generalized schemes with allocations given by (6.1) and equilibria of standard schemes with allocations of the form

$$\Lambda_t^i = X_t^i + \Xi_t^i \quad \text{for all } i \in I, t = 0, \dots, T-1, \quad (6.5)$$

where the stochastic process Ξ in $\mathcal{L}_{T-1}^1(\mathbb{R}^{|I|})$ is chosen so that:

$$\sum_{i\in I} \Xi_t^i = \sum_{k\in K} Y_t^k D_t^k, \qquad t = 0, 1, \cdots, T-1. \quad (6.6)$$

The following proposition is the main theoretical results of this section.

PROPOSITION 6.1. *The following statements hold when $\Lambda = \sum_{i\in I} \sum_{t=0}^{T-1} \Lambda_t^i$ fulfills Assumption 0 and the processes $Y^{i,j,k}$ depend only upon the good $k \in K$ and not on the firm $i \in I$ or the technology $j \in J^{i,k}$.*
(i) If the processes $Z^{i,j,k} = Z^k$ depend only upon the good $k \in K$, and (A^*, S^*) is an equilibrium with production strategy ξ^* for a standard cap-and-trade scheme with adjusted uncontrolled emissions given by

$$\Gamma^i = \Delta^i - \sum_{t=0}^{T-1} (X_t^i + \Xi_t^i) \quad \textit{for all } i \in I, \quad (6.7)$$

then the couple of price processes $(A^*, S^\dagger)$ where

$$S_t^{\dagger k} = S_t^{*k} + Z_t^k - Y_t^k A_t^* \quad \textit{for all } k \in K, t = 0, \dots, T-1 \quad (6.8)$$

is an equilibrium of the generalized cap-and-trade scheme with the same production strategies ξ^*. *The converse statement also holds.*
(ii) If the processes $Y^{i,j,k}$ *and* $Z^{i,j,k}$ *depend only upon the good* $k \in K$ *as in (i) above, if Assumptions 1 and 2 hold and if* $\overline{\xi} \in \mathcal{U}$ *is a solution of the global optimization problem (4.4) with adjusted uncontrolled emission (6.7), then the couple of processes* $(\overline{A}, \overline{S})$ *defined by*

$$\overline{A}_t = \pi \mathbb{P}_t[\Gamma + \Pi(\overline{\xi}) \geq 0], \qquad t = 0, \dots, T \tag{6.9}$$

and

$$\overline{S}_t^k = \max_{i \in I,\, j \in J^{i,k}} (C_t^{i,j,k} + e^{i,j,k} \overline{A}_t) \mathbf{1}_{\{\overline{\xi}_t^{i,j,k} > 0\}} + Z_t^k - Y_t^k \overline{A}_t, \qquad t = 0, \dots, T-1 \;\; k \in K, \tag{6.10}$$

form a market equilibrium. Moreover, the equilibrium allowance price process is almost surely unique, while the process $\overline{S}$ *is the smallest equilibrium price in the sense of Theorem 4.6.*
(iii) The conclusions of (ii) still hold when the processes $Z^{i,j,k}$ *can also depend upon the firm* $i \in I$ *and the technology* j *if we use the* adjusted parameters *defined by*

$$\hat{Z}_t^k := 0, \quad \textit{and} \quad \hat{C}_t^{i,j,k} := C_t^{i,j,k} + Z_t^{i,j,k} \tag{6.11}$$

instead of the original parameters of the model.

In particular if we choose $(X^i)_{i \in I}$ and $(Y^k)_{k \in K}$ so that

$$\sum_{i \in I} \Lambda^i = \sum_{t=0}^{T-1} \left(\sum_{i \in I} X_t^i + \sum_{(k) \in K} Y_t^k D_t^k \right)$$

we deduce that there is a one-to-one correspondence between generalized schemes and standard schemes with initial allocation Λ^i. We will elaborate on this in Section 7.
In order to prove Proposition 6.1, we shall need the following lemma.

LEMMA 6.2. *Let* A *be a martingale,* $\xi^i \in \mathcal{U}^i, \theta^i \in \mathcal{V}^i(A, \Lambda^i), \theta'^i \in \mathcal{V}^i(A, \Lambda'^i(\xi^i))$, *and* S, S' *be two integrable price processes, such that:*

$$\theta'^i_t = \theta^i_t \quad \textit{for all } t \leq T-1 \tag{6.12}$$

$$\theta'^i_T = \theta^i_T - \sum_{t=0}^{T-1} \left(\Xi_t^i - \sum_{(j,k) \in M_i} Y_t^k \xi_t^{i,j,k} \right) \tag{6.13}$$

$$S'^k_t = S_t^k + Z_t^k - Y_t^k A_t \quad \textit{for all } k \in K, t = 0, \dots, T-1$$

where $Z^k \leq 0$ *is a subsidy, then we have:*

$$\mathbb{E}[L^{A,S,i}(\theta^i, \xi^i)] + \mathbb{E}\left[\sum_{t=0}^{T-1} \Xi_t^i A_t \right] = \mathbb{E}[H^{A,S',i}(\theta'^i, \xi^i)] + \mathbb{E}\left[\sum_{t=0}^{T-1} V_t^i \right]. \tag{6.14}$$

Proof. The right hand side of (6.14) is equal to

$$\mathbb{E}\Bigg[\sum_{t=0}^{T-1} \sum_{(j,k) \in M_i} (S'_t - C_t^{i,j,k} - Z_t^k) \xi_t^{i,j,k} - \theta'^i_T A_T$$
$$- \pi \left(\Delta^i + \Pi^i(\xi^i) - \sum_{t=0}^{T-1} (X_t^i + \sum_{(j,k) \in M_i} Y_t^k \xi_t^{i,j,k}) - \theta'^i_T \right)^+ \Bigg]$$

and using the martingale property of A, it can be expressed as

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t - C_t^{i,j,k} - Y_t^k A_t)\xi_t^{i,j,k} \\
&\qquad -\theta_T^i A_T - \sum_{t=0}^{T-1}\Bigg(\Xi_t^i - \sum_{(j,k)\in M_i}\xi_t^{i,j,k}Y_t^k\Bigg)\mathbb{E}(A_T|\mathcal{F}_t) \\
&\qquad\qquad -\pi\Bigg(\Delta^i + \Pi^i(\xi^i) - \sum_{t=0}^{T-1}(X_t^i + \Xi_t^i) - \theta_T^i\Bigg)^+\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t - C_t^{i,j,k})\xi_t^{i,j,k} - \theta_T^i A_T - \pi\Bigg(\Delta^i + \Pi^i(\xi^i) - \sum_{t=0}^{T-1}\Lambda_t^i - \theta_T^i\Bigg)^+\Bigg] \\
&\qquad + \mathbb{E}\Bigg[\sum_{t=0}^{T-1}\Xi_t^i A_t\Bigg]
\end{aligned}$$

which is the left hand side of (6.14). □

We can now turn to the proof of Proposition 6.1.

Proof. (i) Let (A^*, S^*) be equilibrium price processes of a standard scheme with strategies (ξ^*, θ^*). Let ${\theta'}^*$ be the adjusted optimal strategy as in Lemma 6.2. The assertion follows by checking that conditions (i) to (iii) of Definition 1 are fulfilled by the pair of price processes $(A^*, S^\dagger)$ and strategies $(\xi^*, {\theta'}^*)$. Since θ^* satisfies the market clearing condition (3.5), so does ${\theta'}^*$. This proves (i) while condition (ii) follows directly from (6.6). Moreover, given $(\theta', \xi^i) \in \mathcal{V}^i(A^*, {\Lambda'}^i(\xi^i)) \times \mathcal{U}^i$, we define strategies θ^i such that (6.12) and (6.13) hold which guaranties that $\theta^i \in \mathcal{V}^i(A^*, \Lambda^i)$ is satisfied as well. Furthermore Proposition 3.1 implies that A^* is a martingale and the result of Lemma 6.2 yields

$$\begin{aligned}
\mathbb{E}[H^{A^*,S^\dagger,i}({\theta'}^i, \xi^i)] &= \mathbb{E}[L^{A^*,S^*,i}(\theta^i, \xi^i)] - \mathbb{E}\left[\sum_{t=0}^{T-1}(V_t^i - \Xi_t^i A_t^*)\right] \\
&\leq \mathbb{E}[L^{A^*,S^*,i}(\theta^{*i}, \xi^{*i})] - \mathbb{E}\left[\sum_{t=0}^{T-1}(V_t^i - \Xi_t^i A_t^*)\right] \qquad (6.15) \\
&= \mathbb{E}[H^{A^*,S^\dagger,i}({\theta'}^{*i}, \xi^{*i})]
\end{aligned}$$

where we used the optimality of the equilibrium strategies (ξ^*, θ^*) of the standard scheme in (6.15). This holds for all $(\theta', \xi^i) \in \mathcal{V}^i(A^*, {\Lambda'}^i(\xi^i)) \times \mathcal{U}^i$ which proves condition (iii). The converse can be proved in exactly the same way.

Assertion (ii) follows directly from (i), Theorem 4.6 while Remark 3 gives the martingale property of A^*. Finally, the proof of part (iii) is straightforward. □

Note that the above proof also shows that, not only do allocation prices coincide, but also equilibrium production strategies. Thus the switching costs of the generalized cap-and-trade schemes are the same as for the standard cap-and-trade schemes with adjusted uncontrolled emissions $(\Gamma^i)_{i\in I}$.

The above discussion suggests that windfall profits could be reduced with a relative allocation rule constant over time. This motivates the following analysis.

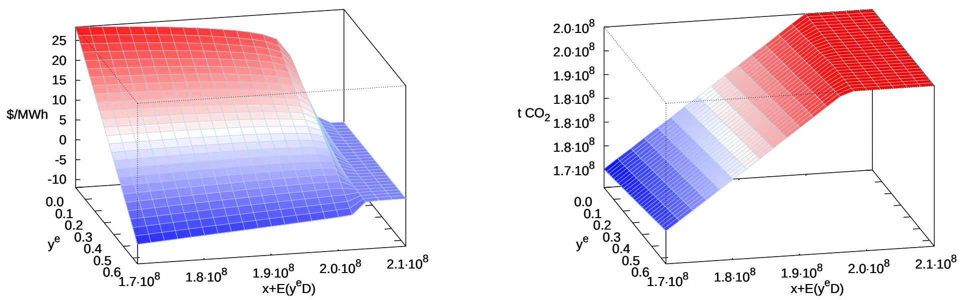

FIGURE 6.1. *Windfall profits (left) and* 95% *percentile of total emissions (right) as functions of the relative allocation parameter and the expected allocation. Here* $D = \sum_{t=0}^{T-1} D_t^e$ *denotes the total electricity demand over one compliance period.*

6.3. Cap-and-Trade Schemes with Relative Allowance Allocation. A positive relative allocation for good $k \in K$, i.e. the assumption that $Y_t^{i,j,k} = y^k > 0$ for every $i \in I$ and those $j \in J^{i,k}$, can be seen as a subsidy for good k that is given in the form of allowances rather than in cash. Under such an assumption, when producing one unit of good k, the marginal expected penalty increases only by $(e^{i,j,k} - y^k)A_t^*$ rather than by $e^{i,j,k}A_t^*$ as in a standard scheme. Thus the net marginal overall production costs of the firms are lower when compared to the standard scheme. This should result in a decrease of the price of good k. In the present subsection, we study the simplest generalized cap-and-trade scheme taking advantage of this mechanism by setting:

$$
\begin{aligned}
Y_t^{i,j,k} &= y^k \in \mathbb{R} \qquad \text{for all } t = 0, \dots, T-1 \\
X_0^i &= x^i \in \mathbb{R} \quad , \quad X_t^i = 0 \qquad \text{for all } t = 1, \dots, T-1 \\
V_t^i &= 0, \quad Z_t^{i,j,k} = 0 \qquad \text{for all } t = 0, \dots, T-1
\end{aligned}
$$

for all $(i, j, k) \in M$. In what follows, not only do we discuss this relative cap-and-trade scheme, but we also gain new insight into the standard cap-and-trade scheme by treating it as a relative cap-and-trade scheme with $y^k = 0$ for all $k \in K$.

In the case study of the Texas electricity market, we do not give this relative allocation to nuclear production. This does not influence the equilibrium prices nor the equilibrium production strategies since marginal production costs for nuclear power plants are far below marginal cost for gas and coal power plants, even if emissions and relative allocation are priced into gas and coal. Under these considerations it is straightforward to extend Proposition 6.1 to apply to such an allocation rule as well.

But for any comparison of the different cap-and-trade schemes to be meaningful, we need to calibrate their respective parameters to common characteristics. We now proceed to the discussion of such a calibration procedure.

6.3.1. Calibration of the Parameters. The relative scheme introduced in this section has three regulatory parameters. Using the notation of this section, they are: 1) the penalty π, 2) the relative allocation coefficients $(y^k)_{k \in K}$, 3) the total initial allocations $x = \sum_{i \in I} x^i$ given to the firms $i \in I$. In this subsection we show, using again the example of the Texas electricity market, how one should choose these parameters in order to guarantee an emissions reduction target with given probability while keeping the expected windfall profits near zero and controlling the reduction costs to keep them as low as possible. In the particular simulation used to illustrate the strategy, we choose an emissions reduction target of 1.827×10^8 to be reached with probability 95%.

To gain a first insight into the numerics, we fix the penalty π at 100\$. The left pane of Figure 6.1 gives the expected windfall profits while the right pane gives the 95% percentile of the total emissions for different values of the relative allocation coefficient (y^e) and the expected total allocation. It appears that the expected allocation controls the amount by which carbon emissions are reduced, while the relative allocation coefficient y^e controls the windfall profits. Designing a cap-and-trade scheme with zero windfall profits and pre-assigned emissions target levels can be done by choosing the parameters of our relative scheme at the intersection of the zero windfall profit level set with the 1.827×10^8 emission percentile level set. This procedure is depicted in Figure 6.2. We find $y^e = 0.54$ and $\sum_{i \in I} x^i = 5.4 \times 10^7$.

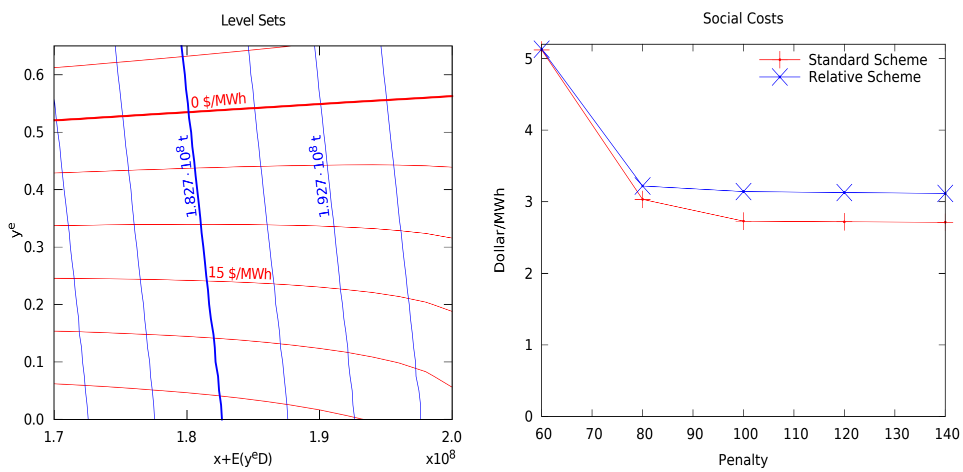

FIGURE 6.2. *The left pane shows the level sets of the two plots of Figure 6.1. The blue and the red lines indicate level sets of the windfall profits and the 95% quantile of emissions respectively. The right pane gives the plots of the overall production costs for electricity for one year as function of the penalty level for both the absolute and relative schemes. The free regulatory parameters are chosen to guarantee the desired emissions percentile, and in the case of the relative scheme, such that the windfall profits are zero.*

Since for the standard cap-and-trade scheme the parameter is zero (i.e. $y^e = 0$), we have one less regulatory parameter to calibrate. And since controlling the emissions level is of the utmost importance, it takes precedence over the control of the windfall profits. So in a standard cap-and-trade scheme, the initial allocation is chosen to reach the emission target and the windfall profits follow without being controlled. Hence the desired parameter values are obtained at the intersection of the 1.827×10^8 emission percentile level set with $y^e = 0$. Giving the initial allocation $\sum_{i \in I} x^i = 1.826 \times 10^8$.

Repeating the above procedure for different penalties levels gives regulatory settings with different production costs for the relative and the standard scheme in Figure 6.2. Obviously the reduction costs are lowered by increasing penalty for both schemes. As shown in the right pane of Figure 6.2, this decrease in reduction costs is significant until the penalty reaches the level $\pi = \mathbf{1}00\$$. After that, the reduction costs stay nearly the same, becoming independent of π for larger values of π. Hence, we conclude that in this setting a penalty of 100$ is a reasonable choice for both the relative and the standard scheme.

6.4. Emission Taxes. A static tax scheme (known in the popular press as a *carbon tax*) is a regulation that penalizes the emission of each ton of carbon by a fixed amount, say $z > 0$. Since there is no penalty, no allocation and no trading, it is natural to view a carbon tax as a Business As Usual scenario with adjusted production costs

$$\widehat{C}_t^{i,j,k} = C_t^{i,j,k} + e^{i,j,k} z$$

for all $(i, j, k) \in M$ and $t = 0, \ldots, T-1$. Using the results of Section 3.2, we see that in such a scheme, the prices of goods follow a merit order pricing rule with effective production costs given by $C_t^{i,j,k} + e^{i,j,k} z$ for all $(i, j, k) \in M$ and $t = 0, \ldots, T-1$. Earnings under a tax scheme are based on the spread of these effective production costs. Since this spread does not depend exclusively on the differences between the production costs associated with the various technologies available, it is in general not clear how to predict what the windfall profits will be. It is not even clear if they can be negative or positive. To gain some insight on this issue, consider a tax of $z = 60\$$ (which is realistic for a 10% reduction target as will be seen below), and assume that at some point in time, the marginal production costs of coal and gas are the same while all plants have to run to satisfy the demand. In this case the spread in effective production cost is $(e_c - e_g)z = 31.8\$$ and will be earned for each MWh that is produced with gas. Hence the windfall profits are 31.8$ per MWh of electricity produced with gas. However, in the case of BAU, the earnings are zero.

Like a standard cap-and-trade scheme, a tax scheme has only one regulatory control parameter: the tax level z. As explained above, this single parameter should be calibrated in order to guarantee a specified reduction target. Thus the windfall profits are automatically given by the reduction target and cannot be controlled.

6.5. Comparison of the Various Abatement Schemes. We now compare the characteristics of the standard and the relative cap-and-trade schemes with the regulatory parameters chosen in the previous subsection.

We first consider the windfall profits and the consumer costs. The results are given in Figure 6.3.

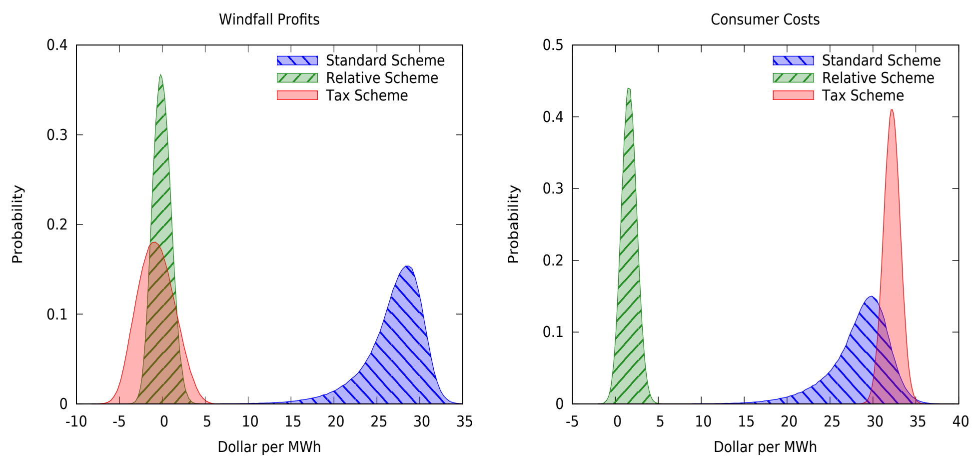

FIGURE 6.3. *Histograms (computed from* 500000 *simulation scenarios) of the yearly distribution of windfall profits (left) and consumer costs (right) for the Standard Scheme, a Relative Scheme and a Tax Scheme.*

As expected the relative scheme gives much lower consumer costs than the standard scheme. This is related to the fact that the windfall profits have a narrow distribution around zero in the case of the relative scheme, while the windfall profits of the standard scheme are 10 times higher than the reduction costs. When compared to the standard scheme, the only drawback of the relative scheme seems to be the slightly higher level of reduction costs which can be observed on the right pane of Figure 6.4. However since this cost increase corresponds to approximately 0.4$ per MWh, it is small in comparison to production costs, and thus can be neglected in practice. Moreover those higher production costs are not just wasted money, they are paid for higher emission reduction in many scenarios as can be seen on the left pane of Figure 6.4. In particular the relative scheme takes advantage of cheap fuel switches when the standard scheme cannot reduce emissions anymore. Moreover the relative scheme is less sensitive to weather, since in warm winters less allowances are allocated pushing the price up. This in turn is responsible for higher emission abatements and consequently higher abatement costs.

In this example of a relative scheme, approximately 30% of the allowances are given as initial allocation, by allocating these to clean plants, further incentives can be set to build cleaner plants. This seems to be an important advantage of the relative scheme over other mechanisms such as auctioning and tax.

Next, we study the effect of an emission tax on the Texas electricity market. Figure 6.4 shows that a pure tax scheme that fulfills the above reduction target of $1.827 \times 10^8 t_{CO_2}$ with 95% probability, is on average, more than twice (2.4×) as expensive as the standard cap-and-trade-scheme. These extra costs are paid for extra emission reductions. However in contrast with the results in the case of the relative scheme, the average cost increase per reduced ton of carbon is considerable when we compare it to the case of the standard scheme. The reason is that a tax is not flexible enough to control emissions when abatement costs are stochastic. This results in an emission uncertainty that exceeds even the BAU uncertainty with several orders of magnitude. Notice moreover that it carries a significant risk to reduce nearly no

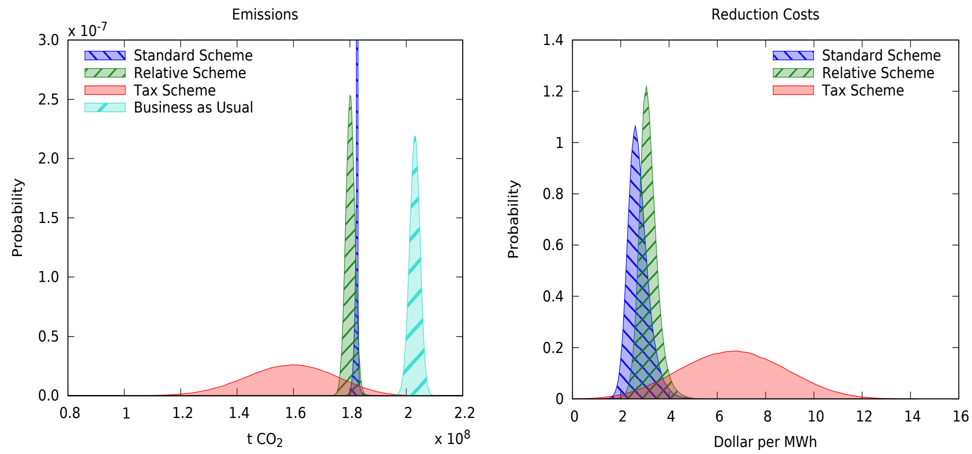

FIGURE 6.4. *Yearly emissions from electricity production (left) for the Standard Scheme, the Relative Scheme, a Tax Scheme and BAU, and yearly abatement costs (right).*

emissions. In such a scenario the tax corrections for upcoming years will be extremely expensive. Needless to say a tax scheme induces a huge money transfer from consumers to the regulator, which as can be seen in Figure 6.4 is even bigger than the costs transferred to the consumer in a standard cap-and-trade scheme.

7. More Financial Incentives. One of the main arguments in favor of the relative schemes studied in the previous section is the fact that they reduce windfall profits. However, this reduction comes with slightly higher reduction costs than in the case of the absolute scheme. While this cost increase is negligible in practice, it is of great theoretical interest to understand how and why one can design schemes that give exactly zero windfall profits at exactly the same reduction costs as the standard cap-and-trade scheme. In order to do so, we identified in Proposition 6.1 the generalized schemes which are in a one-to-one correspondence with the production policies of the standard scheme. The latter are given by a subclass of generalized schemes for which $Z_t^{i,j,k}$ and $Y_t^{i,j,k}$ depend only on k. The terminal wealth of firm $i \in I$ under such a scheme reads:

$$\begin{aligned} H^{A,S,i}(\theta^i, \xi^i) := &- \sum_{t=0}^{T-1} V_t^i + \sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S_t^k - C_t^{i,j,k} - Z_t^k)\xi_t^{i,j,k} \\ &+ \sum_{t=0}^{T-1} \theta_t^i (A_{t+1} - A_t) - \theta_T^i A_T \\ &- \pi\left(\Delta^i + \Pi^i(\xi^i) - \sum_{t=0}^{T-1}\left(X_t^i + \sum_{(j,k)\in M_i} Y_t^{i,j,k}\xi_t^{i,j,k}\right) - \theta_T^i\right)^+ . \end{aligned} \tag{7.1}$$

The results of this section will demonstrate the versatility and the flexibility of the generalized framework introduced in this paper. However, because of the level of complexity of their implementations, and despite the high degree of control they provide the regulator with, it is unlikely that the schemes identified here will be used by policy makers or regulators.

7.1. Design of Financial Incentives. In this subsection, we discuss the design of financial incentives. We propose to adjust the financial positions of each firm $i \in I$ by

$$-\sum_{t=0}^{T-1}(V_t^i + \sum_{(j,k)\in M_i} \xi_t^{i,j,k} Z_t^k) \tag{7.2}$$

depending on its production strategy ξ^i. Obviously, the results of Proposition 6.1 still hold in this case. In equilibrium, allowance prices, production strategies, and penalty are identical to those of a standard scheme. However, the price of goods increases by the amount Z_t^k at $t = 0, \ldots, T-1$. Hence, as depicted in Figure 7.1 the scheme induces a money transfer

$$\sum_{t=0}^{T-1}(V_t^i + \sum_{(j,k)\in M_i} \xi_t^{i,j,k} Z_t^k)$$

from producers to the regulator. In the meantime, the quantity

$$\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} \xi_t^{i,j,k} Z_t^k$$

is entirely passed on to the end consumer, so that the price increase $(Z_t^k)_{t=0}^{T-1} \geq 0$ results in a money transfer from consumers to the regulator.

Remark 10. *This discussion is a good illustration of the relevance of the inclusion of incentives in the model, and the difficulties in reducing windfall profits. It seems natural to think that one could do so by keeping track of the number of MWh traded, the marginal technology used, and the price of carbon with the intention to charge the windfall profits to the producers and reimburse consumers later. In this case, given a production strategy ξ^i and a price process S, each agent $i \in I$ would be charged the amount*

$$\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (S_t^k - \hat{S}_t^k)\xi_t^{i,j,k}. \tag{7.3}$$

This would correspond to a standard scheme where (7.3) would be withdrawn from (2.16), or alternatively S^k would be replaced by $\hat{S}^k$. But since $\bar{S}^k$ is the lowest price for good $k \in K$, the demand cannot be satisfied if $\hat{S}^k$ drops below $\bar{S}^k$. Due to (5.1) this happens whenever $e^{i,j,k} > 0$ for some marginal production plant (i,j). Hence, it is very unlikely that an equilibrium exists in such a scheme. On the other hand, if we change the amount to be charged to

$$\sum_{t=0}^{T-1} \sum_{(j,k)\in M_i} (\overline{S}_t^k - \hat{S}_t^k)\xi_t^{i,j,k}, \tag{7.4}$$

it follows from the above discussion that the entire amount is passed on to the end-consumer and windfall profits are not reduced.

Proposition 6.1 happens to be a very versatile tool when it comes to designing new schemes with required properties. As corollaries to this proposition, two appropriate adjustments with zero windfall profits are given in Sections 7.2 and 7.3.

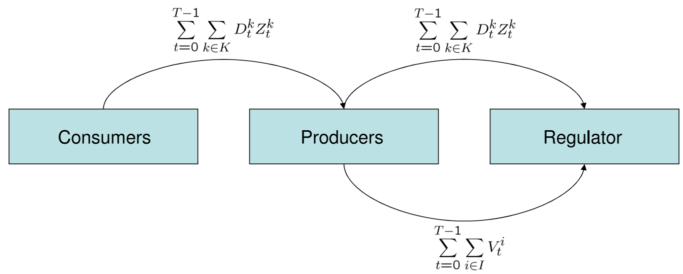

FIGURE 7.1. *$V > 0$ gives a money transfer from producers to the regulator, while $Z > 0$ gives a money transfer from consumer to the regulator. By choosing V and Z in an appropriate way it is possible to avoid a money transfer to/from the regulator.*

7.2. Zero Windfall Profit Scheme with Tax and Subsidy. In this section we consider the generalized allocation scheme given by

$$X_t^i = \Lambda_0^i \mathbf{1}_{\{t=0\}}$$
$$Y_t^{i,j,k} = 0$$

for all $i \in I$, $k \in K$, $j \in J^{i,k}$ and $t = 0, \dots, T-1$. We show here how the tax/subsidy system comprised into the generalized scheme can theoretically lead to zero windfall profits at equilibrium. This result is a direct corollary of Proposition 6.1.

COROLLARY 7.1. *Consider a generalized cap-and-trade scheme such that for all $i \in I$ and $k \in K$*

$$V_t^i = \sum_{(j,k)\in M_i} \overline{\xi}_t^{i,j,k} (\overline{S}_t - \hat{S}_t^k) \text{ for all } t = 0, \dots, T-1$$
$$Z_t^k = 0 \text{ for all } k \in K, t = 0, \dots, T-1. \tag{7.5}$$

Then each equilibrium (A^*, S^*) of the standard cap-and-trade scheme is also an equilibrium of this generalized scheme. In particular the equilibrium with lowest product prices is given by $(\overline{A}, \overline{S})$ from Theorem 4.6. For this equilibrium the windfall profits for the aggregated producing sector are zero.

REMARK 11. *The processes $\overline{S}$, $\hat{S}$ and $\overline{\xi}$ appearing in (7.5) are defined in Theorem 4.6 and formula (5.1). They are independent of the actually realized equilibrium $(A^\dagger, S^\dagger)$ and and its corresponding strategies $(\theta^\dagger, \xi^\dagger)$. Consequently, their computation involves solving the global optimal control problem.*

In order to maintain the end-consumer costs at a reasonable level, the amount $\sum_{i\in I} \sum_{t=0}^{T-1} V_t^i$ has to be redistributed from the regulator to the end consumers in an appropriate way. As shown in Figure 7.1 this is not needed if the financial incentives fulfill

$$\sum_{t=0}^{T-1} \sum_{i\in I} V_t^i = - \sum_{t=0}^{T-1} \sum_{k\in K} D_t^k Z_t^k, \tag{7.6}$$

and in this case, there is no money transfer from producers to regulators. This is the object of the following corrolary.

Corollary 7.2. *Consider a generalized cap-and-trade scheme such that for all* $i \in I$ *and* $k \in K$

$$V_t^i = \sum_{(j,k)\in M_i} \overline{\xi}_t^{i,j,k}(\overline{S}_t^k - \hat{S}_t^k) \qquad \textit{for all } t = 0,\dots,T-1$$

$$Z_t^k = -(\overline{S}_t^k - \hat{S}_t^k) \qquad \textit{for all } t = 0,\dots,T-1$$

where $\overline{S}$ *denotes the equilibrium prices of goods in the standard scheme (recall Theorem 4.6), and* $\hat{S}$ *denotes the pure merit order prices as defined in* (5.1). *In this setting each equilibrium* (A^*, S^*) *of the standard cap-and-trade scheme corresponds to an equilibrium of the generalized cap-and-trade scheme given by* $(A^*, S^* - (\overline{S} - \hat{S}))$. *In particular the equilibrium with lowest product prices is given by* $(\overline{A}, \hat{S})$ *with* $\overline{A} = A^*$ *because of the uniqueness of the equilibrium allowance prices. In this equilibrium, producers have zero windfall profits.*

If a firm i follows the production strategy ξ^i, then the net money transfer to the regulator it is responsible of is given by

$$\sum_{(j,k)\in M_i} (\overline{\xi}_t^{i,j,k} - \xi_t^{i,j,k})(\overline{S}_t^k - \hat{S}_t^k). \tag{7.7}$$

Notice that this is zero if the firm follows the strategy $\overline{\xi}$. If it decides to produce more or less than in equilibrium, it is rewarded or respectively penalized by (7.7). This increases the incentives to produce, and hence results in lower product prices. As shown by (7.6), the only money transfer in this scheme is the transfer of penalty payments from penalized firms to rewarded firms.

7.3. Zero Windfall Profit Scheme with Dynamic Allocation. Here we set $V \equiv Z \equiv 0$, and show how dynamic stochastic allocation of allowances can also lead to null windfall profits for the producers.

Corollary 7.3. *Consider a cap-and-trade scheme with dynamic allowance allocation where*

$$X_t^i = -\sum_{(j,k)\in M_i} \overline{\xi}_t^{i,j,k} \frac{\overline{S}_t^k - \hat{S}_t^k}{\overline{A}_t} + \Lambda_t^i 1_{\{t=0\}}$$

$$Y_t^{i,j,k} = Y_t^k = \frac{\overline{S}_t^k - \hat{S}_t^k}{\overline{A}_t}$$

at each time point $t = 0,\dots,T-1$ *and where we use the convention* $0/0 = 0$. *Here,* $\overline{A}$ *and* $\overline{S}$ *denote the equilibrium allowance and lowest prices for goods respectively in the standard scheme from Theorem 4.6. Further let* $\hat{S}$ *be the pure merit order price defined in* (5.1). *In this setting it holds that* $(X_t^i)_{t=0}^{T-1} \in \mathcal{L}^\infty_{T-1}(\mathbb{R})$ *and* $(Y_t^k)_{t=0}^{T-1} \in \mathcal{L}^\infty_{T-1}(\mathbb{R})$ *for all* $i \in I$ *and* $k \in K$. *Each equilibrium* (A^*, S^*) *of the standard cap-and-trade scheme corresponds to an equilibrium of the generalized cap-and-trade scheme given by* $(A^*, S^* - (\overline{S} - \hat{S}))$. *In particular the equilibrium with lowest prices for goods is given by* $(A^*, \hat{S})$. *In this equilibrium, the windfall profits of each firm are zero.*

Proof. Notice that both X_t^i and Y_t^k are well defined despite the presence in the denominator of $\overline{A}_t$. Indeed by definition, $\overline{S}_t^k = \hat{S}_t^k$ whenever $\overline{A}_t = 0$. The equilibrium result is a direct consequence of Proposition 6.1. Hence it remains only to prove that

$(X_t^i)_{t=0}^{T-1} \in \mathcal{L}^\infty_{T-1}(\mathbb{R})$ and $(Y_t^k)_{t=0}^{T-1} \in \mathcal{L}^\infty_{T-1}(\mathbb{R})$ for all $i \in I$. Let us first prove that $\frac{\overline{S}_t^k - \hat{S}_t^k}{\overline{A}_t}$ is bounded from above. Since

$$\begin{aligned}
\overline{S}_t^k &= \max_{i\in I, j\in J^{i,k}} (C_t^{i,j,k} + e^{i,j,k}\overline{A}_t)\mathbf{1}_{\{\overline{\xi}_t^{i,j,k}>0\}} \\
&\leq \max_{i\in I, j\in J^{i,k}} C_t^{i,j,k}\mathbf{1}_{\{\overline{\xi}_t^{i,j,k}>0\}} + \max_{i\in I, j\in J^{i,k}} e^{i,j,k}\overline{A}_t\mathbf{1}_{\{\overline{\xi}_t^{i,j,k}>0\}} \\
&\leq \hat{S}_t^k + \overline{A}_t \max_{i\in I, j\in J^{i,k}} e^{i,j,k}
\end{aligned}$$

it follows that

$$\frac{\overline{S}_t^k - \hat{S}_t^k}{\overline{A}_t} \leq \max_{i\in I, j\in J^{i,k}} e^{i,j,k} \text{ for all } k \in K, t = 0, \dots, T-1 \tag{7.8}$$

which gives the upper bound. To prove the lower bound, we notice that for all $k \in K$ and $t = 0, \dots, T-1$ it holds that $\overline{S}_t^k - \hat{S}_t^k \geq 0$ and $\overline{A}_t \geq 0$. Thus

$$\frac{\overline{S}_t^k - \hat{S}_t^k}{\overline{A}_t} \geq 0 \text{ for all } k \in K, t = 0, \dots, T-1 \tag{7.9}$$

which concludes the proof. □

This scheme ensures the existence of an equilibrium with zero windfall profits and exactly the same reduction costs as for the standard cap-and-trade scheme, without any extra money transfer from consumer/producer to regulator, as e.g. would be the case for auctioning. A further advantage compared to auctioning is that it allows to distribute the amount $\sum_{i\in I} X_t^i$ differently to control the incentives to build cleaner plants.

The drawback of this scheme is that it requires a random allocation (following a prespecified rule) of allowances dynamically through time. Moreover, the daily allocation can be negative at times for some producers. But this should not be a problem as seen in the Green Certificate Schemes.

Notice further that it suffices to give a relative allocation only to those plants which are frequently on the margin. Therefore, in the electricity markets, the relative allocation does not need to be given to nuclear plants.

The theoretical results of this section suppose market rules that may be difficult to enforce in a real market. However they show how the different levers brought by the generalized schemes can be used to keep the prices of goods at a low level.

8. Conclusions. In this paper we introduced a new mathematical framework for competitive equilibrium, in which emissions trading schemes can be analyzed. This framework is general enough to accommodate tax based abatement policies, existing cap-and-trade schemes such as those implemented in the first phase of the European Union ETS, as well as new market designs.

The main thrust of the paper is to provide policy makers and regulators with the tools necessary to design and implement cap-and-trade schemes capable of reaching reasonable pollution targets at low reduction costs while controlling windfall profits and incentives for cleaner production technologies.

On one hand, we develop a rigorous mathematical theory for competitive economic models in which stochastic demand and production costs are given exogenously. We prove existence of an equilibrium in which price processes for goods and pollution appear endogenously and pollution prices are unique.

On the other hand, we provide analytic and computational tools to analyze and compare the various emissions trading schemes. Regulators and policy makers need to understand the structure and the role of these new markets vis-a-vis pollution control, and we view these tools as crucial in the design and the implementation of sound environmental economic policy.

The computational tools developed in this study provide, for each market design

- Monte Carlo scenarios generators for equilibrium prices of goods and pollution allowance certificates
- Computations of, for each scenario of the demands for goods and fuel costs,
  - pollution levels
  - end-consumers costs
  - producers windfall profits
  - reduction costs

Finally, as illustration of the versatility of the tools developed for the purpose of the qualitative analysis of cap-and-trade schemes considered in the paper, we implemented them in a case study of the Texas electricity market.

As observed in the SOx and NOx California RECLAIM program and at the end of the first implementation phase of the EU ETS, cap-and-trade systems can fail as too generous an allocation of pollution permits will serve as a disincentive for emissions reductions and deflate pollution prices. However, our numerical experiments prove that **cap-and-trade schemes can work** in the sense that emissions targets properly chosen can be reached at low costs.

Moreover we use our computational tools to provide a thorough comparison of a sample of alternative schemes: the standard scheme inspired by the first EU implementation phase, an emission tax scheme, and a relative scheme in which allowances distributions are driven by instantaneous (as opposed to historical) production levels. Among other things, we show that the relative allocation scheme which we propose can reduce average windfall profits essentially to zero while keeping reduction cost nearly at the same level as the standards cap-and-trade scheme. Moreover, this relative scheme allows to control the incentives to build new plants.

The following table summarizes the results of our comparative analysis of standard cap-and-trade schemes (whether or not they include auctioning of allowances) with tax schemes and the relative scheme which we introduced in this paper.

Comparison of Schemes					
	Red. Target	Incentives	Windfall	Social Cost	Cons. Cost
Standard	+	+	−	+	−
Std&Auct	+	−	−	+	−
Relative	+	+	+	+	+
Tax	−	−	+	−	−

9. Appendix. This final section contains the technical details of the implementations used to produce the numerical results presented in the paper.

9.1. Model Calibration. We chose to run the numerical experiments with data from the Texas electricity market, because it relies on an independent grid, with few interconnections to the rest of the country. For this reason, it is possible to analyze the impact of an emission reduction policy without having to take into account emission leakage. In subsections 9.1.1 and 9.1.2, the demand for electricity and the fuel switch price process are specified using continuous-time processes

$$(D(t))_{t\in[0,T]}, \quad (F(t))_{t\in[0,T]}. \tag{9.1}$$

Note that we write the time parameter in parenthesis instead using subscript, to indicate continuous-time processes. Moreover, the horizon for continuous time is $[0,T]$, where we suppose that T equals to one year. By sampling (9.1) at discrete times, we obtain discrete-time versions of the processes with a daily time step resolution, and we use these versions for numerical purposes.

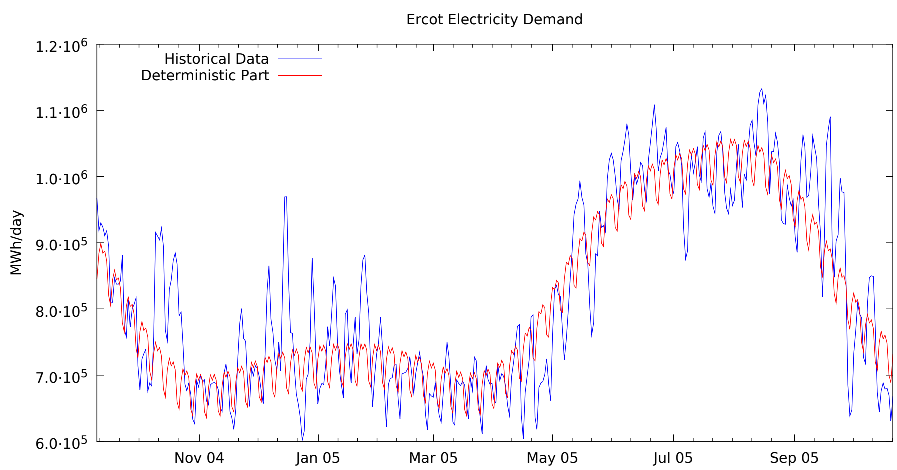

FIGURE 9.1. *Depicted are the historical daily electricity demand for the ERCOT supply area from 18/9/04 to 17/9/05 and the corresponding deterministic part* $(P(t))_{t=[0,T]}$.

9.1.1. Electricity Demand Process. The continuous–time demand process is modeled by

$$D(t) = \min\{(P_D(t) + X_D(t))^+, \kappa^n + \kappa^c + \kappa^g\} \qquad t \in [0,T]$$

where κ^n, κ^c and κ^g represent Texas nuclear, coal and natural gas capacities respectively, and where $P_D(t)$ represents the mean/deterministic component of the demand, and is modeled as

$$P_D(t) = a_D + b_D t + \sum_{j=0}^{6} c_j \cos(2\pi\varphi_j t + l_j) \qquad t \in [0,T] \tag{9.2}$$

the superposition of a linear demand superimposed to a seasonal component including several frequency, among them, a weekly demand contribution. The stochastic part $(X(t))_{t\in[0,T]}$ is modeled by an Ornstein-Uhlenbeck process whose evolution follows the stochastic differential equation

$$dX_D(t) = \gamma_D(\alpha_D - X_D(t))dt + \sigma_D dW(t) \tag{9.3}$$

driven by a process of Brownian motion $(W(t))_{t\in[0,T]}$. The parameters γ_D, α_D and $\sigma_D \in \mathbb{R}$ are estimated from historical load data for the time period 12/23/03-12/23/06 available on ERCOT's website, and reproduced in Figure 9.1.1. These parameters were identified in two steps. First the deterministic harmonics appearing in (9.2) were identified from peaks in the Fourier transform. Secondly, after removing the deterministic part $\{P_D(t)\}_{t\in[0,T]}$ (red line in the figure), the residual component $\{X_D(t)\}_{t\in[0,T]}$ was estimated by standard linear regression as explained in Subsection 9.1.3. The resulting estimates are given in the following tables:

Stochastic Part $(X_D(t))_{t\in[0,T]}$		
γ_D	α_D	σ_D
102	0	819340

Deterministic Part $(P_D(t))_{t\in[0,T]}$									
a_D	b_D		$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
766381	38027	c_i	157526	71355	19047	15056	30608	16601	6364
		φ_i	1	2	3	4	52.14	104.29	208.57
		l_i	−3.56	−0.82	−0.83	−4.36	0.77	−1.55	−2.15

Notice that, for the computation of the estimates of $(\varphi_i)_{i=0}^6$, long periodicities where computed with a yearly periodic Fourier transform while short periodicities were computed with a weekly periodic fourier transform.

9.1.2. Fuel Switch Price Process. The continuous–time fuel switch price process is modeled as

$$F(t) = a_F + X_F(t) \qquad t \in [0,T]$$

where the stochastic part $\{X_F(t)\}_{t\in[0,T]}$ is again modeled as an Ornstein-Uhlenbeck process whose time evolution is given by a stochastic differential equation of the form

$$dX_F(t) = \gamma_F(\alpha_F - X_F(t))dt + \sigma_F dW(t) \tag{9.4}$$

with parameters γ_F, α_F and σ_F. For estimation purposes we used Katy gas spot prices (ICE Katy Exxon Plant Tailgate East Texas) and Platts coal prices (PRB 8400B .35S Dly 1-Mo) from January 2004 to February 2007. Using expected long term means for gas and coal prices we chose $a = 60\$$ ignoring the recent fuel switch price increase. As for the electricity demand process the parameters of the stochastic component $\{X_F(t)\}_{t\in[0,T]}$ were calibrated using the procedure described in Subsection 9.1.3. We obtained the following estimates for the parameters of the fuel switch price process are:

Fuel Switch Price Process $(F(t))_{t\in[0,T]}$			
a_F	γ_F	α_F	σ_F
60	15.26	0	77.29

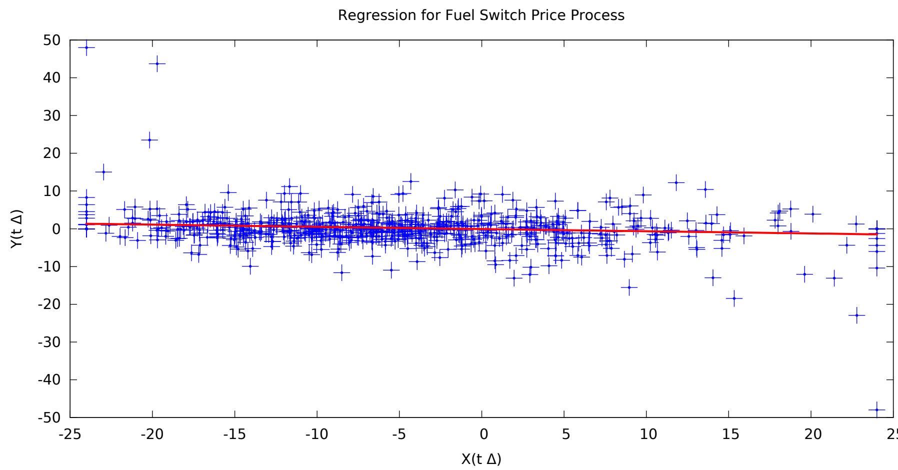

FIGURE 9.2. *Scatter plot of* $(X(t\Delta), Y(t\Delta))$ *calculated by (9.5) based on historical fuel switch prices for the ERCOT region. The straight lines depicts the respective estimated linear regressions.*

9.1.3. Linear Regression. The parameters γ_i, α_i and σ_i for $i \in \{D, F\}$ of the Ornstein-Uhlenbeck processes (9.3) and (9.4) are estimated by a standard linear regression method applied as follows. From the formulas for conditional mean and variance

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)e^{-\gamma(t-s)} + \alpha(1 - e^{-\gamma(t-s)}) \quad s \le t$$
$$\mathrm{Var}[X(t)|\mathcal{F}_s] = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma(t-s)}) \quad s \le t$$

we obtain the regression

$$Y(t\Delta) := X((t+1)\Delta) - X(t\Delta) = \beta_0 + \beta_1 X(t\Delta) + \beta_2 \epsilon_t \qquad t = 1, \ldots, n-1 \quad (9.5)$$

where $\{\epsilon_t\}_{t=1}^{n-1}$ are independent, standard Gaussian random variables and $\beta_0, \beta_1, \beta_2$ are connected to α, γ, σ by

$$\alpha = -\frac{\beta_0}{\beta_1}$$
$$\gamma = -\frac{1}{\Delta}\ln(1 + \beta_1)$$
$$\sigma = \sqrt{\frac{2\gamma\beta_2}{1 - e^{-2\gamma\Delta}}}.$$

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Laboratoire de Finance des Marchés de l'Énergie

Institut de Finance de Dauphine, Université Paris-Dauphine
1 place du Maréchal de Lattre de Tassigny
75775 PARIS Cedex 16
www.fime-lab.org