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# Valuation of a Power Plant Under Production Constraints and Market Incompleteness

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This paper presents a real option valuation model of a power plant, which accounts for physical constraints and market incompleteness. Switching costs, minimum on-off times, ramp rates, or non-constant heat rates are important characteristics that can lead, if neglected, to overestimated values. Existence of non-hedgeable uncertainties is also an important feature of energy markets whose impact on assets value is often neglected. We use the utility indifference approach to define the value of the physical asset. We derive the associated optimal control problems and provide a characterization of their solutions by means of a coupled system of reflected Backward Stochastic Differential Equations (BSDE). We relate this system to a system of variational inequalities, and we provide a numerical comparative study by implementing BSDE simulation algorithms, and PDE finite differences schemes.

*Key words*: Real Option; Backward Stochastic Differential Equation; Utility Indifference; Incomplete Market; Partial Differential Equation

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## 1. Introduction.

To assess the financial value of their production assets, power companies traditionally use the concept of Discounted Cash Flows (DCF), where future cash flows are estimated and discounted at a suitable rate. Subtracting investment costs to the DCF gives the expected Net Present Value (NPV), whose sign is an indicator of investment profitability. Nevertheless, this method usually underestimates the value of investments because it does not account for the flexibility or optionality embedded in the investments. To capture this optionality, valuation methods, inspired by the financial option pricing theory, have been developed. These methods are known as Real Option valuation, since the ownership of a production asset can be seen as a right to use this asset over its lifetime and then receive the generated cash flows. The fundamental concepts of this theory are presented in the books by Dixit and Pindyck (1994), Trigeorgis (1996) or Schwartz and Trigeorgis (2004), and are easily illustrated on the following example. Suppose a thermal power plant is built at time 0 and its estimated lifetime is $T$. Let $S_t^e$ and $S_t^f$ be respectively the electricity spot price and fuel spot price at time $t$. One would intuitively produce electricity at times $t$ when the spark spread $S_t^e - HS_t^f$ is positive, thus receiving a payoff $\int_0^T (S_t^e - HS_t^f)^+ dt$ at time $T$. Here $H$ is the heat rate of the power plant, representing the volume of fuel needed to produce 1 MWh. We can then identify the value of the plant to the price of an option on $(S^e, S^f)$ paying a stream of call options. Powerful option pricing methods, well developed in the financial market industry, can be used to price this option and estimate the power plant value. The investment decision would then be taken by comparing this value to the building cost of the plant.

The previous argument, valid on the above simple example, fails when the problem becomes more complex. Indeed, the above payoff supposes that the plant manager is able to start-up and shut-down the plant at any times, which is not the case in the real world. The payoff of the real option is in general much more exotic. In addition, electricity markets are usually incomplete and some uncertainties are not fully correlated to traded assets, which makes it more difficult to estimate the real option value. Recent works have tackled problems related to the optimal control of a power plant and its real option valuation. Deng and Oren (2003), Tseng and Barz (2002) or Gardner and Zhuang (2000) propose different methods to take production constraints into account, based on Stochastic Dynamic Programming. Hamadène and Jeanblanc (2005) study the starting and stopping problem of a power plant subject to start-up and shut-down costs in the Backward Stochastic Differential Equation (BSDE) framework. Carmona and Ludkovski (2006) studied a generalization to multiple mode switching. In all these papers, the real option price is defined as the maximum expected revenue of the power plant under risk-neutral or historic probability. In fact, it is argued in Dixit and Pindyck (1994) and Trigeorgis (1996), that the question of non-traded assets, and more generally market incompleteness, can be approached by solving a dynamic programming problem with a suitable discount rate accounting for the risk preferences of the agent. The discount rate is supposed to measure the agent's aversion towards the non-hedgeable risk.

In this paper, we propose an alternative method to account for both portfolio constraints (and market incompleteness, as a special case) and production constraints (mainly minimal on-off times, switching costs, ramp rates, non-constant heat rate). This method is based on the concept of utility indifference, and thus handles risk aversion consistently with classical economic theory. We verify that this pricing method reduces to the classical no-arbitrage pricing theory in the frictionless case.

The main tool of our analysis is the theory of Backward Stochastic Differential Equations (BSDE), which unifies the two methodologies of Dixit and Pindyck, dynamic programming, on the one hand, and contingent claims analysis, on the other hand, see e.g. El Karoui et al. (1997b).

Our main result provides a characterization of the utility indifference value of the production asset as the initial value of a coupled system of reflected BSDEs. Our results extend the methods of Rouge and El Karoui (2000) and Hu et al. (2005) who solved a similar valuation problem for European contingent claim. By the classical connection between BSDEs and semilinear PDEs, the above main result also implies a characterization of the utility indifference value as the solution of a coupled system of obstacle-semilinear PDEs. We next use these two representations to implement two alternative classes of numerical methods. The BSDE representation suggests to use a Monte Carlo based simulation algorithm (Bally et al. (2005), Bouchard and Touzi (2005), Gobet et al. (2004)), while the PDE representation suggests to use a finite differences scheme. We provide a comparative implementation of both methods in several examples with a maximum of three-dimensional state variable. We find that the finite differences scheme outperforms the BSDE simulation algorithm in all of our experiments. However, while the Monte Carlo algorithm is open for an implementation beyond the three-dimensional case, the finite differences method is limited by the curse of dimensionality.

The paper is organized as follows. Section 3 formulates the optimal control problem arising in the definition of the utility-based valuation. In Section 4, we state the verification result which relates this optimal control problem to a coupled system of reflected BSDEs. In Section 5, we show the existence of a solution to this system. The problem is specialized in Section 6 to the case where the market is complete. We also show that the no-arbitrage price in a complete market is obtained as a limiting case when switching costs and on-off times are sent to zero. In Section 7, we provide an equivalent formulation of the coupled system of BSDEs in terms of a coupled system of variational inequalities. We finally present in Section 8 a numerical comparative study between BSDE and PDE based methods for the valuation of a coal-fired power plant.

## 2. Notations.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ which satisfies the usual conditions. Let $T > 0$ be a given fixed maturity, and $\{W_t, 0 \leq t \leq T\}$ a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with values in $\mathbb{R}^n$. We denote by $\mathbb{E}[.]$ the expectation operator under $\mathbb{P}$ and $\mathbb{E}_t[.] := \mathbb{E}[.|\mathcal{F}_t]$ the conditional expectation operator with respect to $\mathcal{F}_t$. Expectation under another probability measure $\mathbb{Q}$ will be denoted by $\mathbb{E}^{\mathbb{Q}}[.]$.

We will make use of the following notation throughout the article. For a subset $K$ of $\mathbb{R}^n$, we denote by $L^\infty(K)$ the set of all bounded $\mathcal{F}_T$-measurable $K$-valued random variables, and by $\mathcal{H}^2(K)$ the set of all $\mathbb{F}$-adapted $K$-valued processes $C$ such that: $\mathbb{E}\left(\int_0^T C_t^2 dt\right) < \infty$. The subset of all continuous processes in $\mathcal{H}^2(K)$ is denoted $\mathcal{H}^2_0(K)$. The set of all $\mathbb{F}$-adapted, $K$-valued and bounded processes is denoted by $\mathcal{H}^\infty(K)$. Similarly, $\mathcal{H}^\infty_0(K)$ consists of all continuous processes of $\mathcal{H}^\infty(K)$. The set of all $\mathbb{F}$-adapted, $K$-valued, continuous, non-decreasing processes, starting from 0 is denoted $\mathcal{J}(K)$.

The set $\mathcal{M}_n(K)$ is the collection of all $n \times n$ matrices with entries in $K$. For a matrix $M \in \mathcal{M}_n(K)$, we denote by $M^*$ its transpose. Given two vectors $x, y \in \mathbb{R}^d$, we denote by $x \cdot y$ the Euclidean scalar product, by $|x| = \sqrt{x \cdot x}$ the Euclidean norm, and by $\text{diag}[x]$ the diagonal matrix with diagonal elements given by the components of $x$. Finally, for $x, y \in \mathbb{R}$, we shall use the notations $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

## 3. Problem Formulation.

Throughout the paper we consider an agent whose preferences are described by the exponential utility function with parameter $\eta > 0$:

$$U(x) := -e^{-\eta x}, \; x \in \mathbb{R} \; .$$

The parameter $\eta > 0$ corresponds to the constant absolute risk aversion level of the agent. This agent is allowed to manage a physical asset and to invest on a financial market.

### 3.1. Input and Output Commodity Market.

We consider a financial market on which are traded the input and output commodities, and containing a non-risky financial asset, whose price process is normalized to unity, by the usual change of numéraire. In order to allow for market incompleteness, we assume that the financial market is defined by a multidimensional stochastic price process $S$ with values in $\mathbb{R}^N$, solution of the multivariate stochastic differential equation:

$$dS_t = \hat{\mu}(t, S_t)dt + \hat{\Sigma}(t, S_t)dW_t\,,$$

where $\hat{\mu}(t, S_t) = \text{diag}[S_t]\mu_t$, $\hat{\Sigma}(t, S_t) = \text{diag}[S_t]\Sigma_t$, and the stochastic processes $(\mu, \Sigma)$, valued respectively in $\mathbb{R}^N$ and $\mathcal{M}_N(\mathbb{R})$, are bounded predictable processes. We also suppose that $\Sigma$ has full rank and $\Sigma^{-1}$ is bounded.

### 3.2. Management Strategies.

The physical asset can be in $M$ different modes. We denote by $\psi_t^i$, $1 \le i \le M$, the instantaneous rate of benefit in mode $i$. Throughout this paper, we assume that $\psi^i \in \mathcal{H}^\infty(\mathbb{R})$. Our analysis is based on the approach of quadratic BSDEs developed by Kobylanski (see Kobylanski (2000)), which requires the boundedness of the terminal condition. A possible extension to unbounded terminal conditions may be obtained by following the recent paper by Briand and Hu (2005).

Example 1 (linear production cost). The simplest example has two states off (0) and on (1), no maintenance costs, i.e. $\psi^0 \equiv 0$, and a linear production cost function of the type $\widetilde{\psi}_t^1 = q\,(S_t^1 - HS_t^2)$, where $S^1$ is the electricity spot price, $S^2$ the gas spot price, $q$ is a constant production capacity, and $H$ is a constant heat rate. As $\psi^1$ needs to be bounded, we can define $\psi_t^1 = h\left(\widetilde{\psi}_t^1\right)$ where the function $h$ is the threshold function: $h(x) := x\mathbf{1}_{[\underline{C},\overline{C}]} + \underline{C}\mathbf{1}_{(-\infty,\underline{C}]} + \overline{C}\ \mathbf{1}_{[\overline{C},\infty)}$.

In addition to the benefit rate functions $\psi^i$, the production asset is characterized by an horizon $T$, a terminal payoff $\chi \in L^\infty(\mathbb{R})$ at time $T$, switching costs $C_{i,j} \ge 0$ when switching from mode $i$ to $j \ne i$ and minimal times $\delta_i$ in each mode. In words, this means that switching the production asset from mode $i$ to mode $j \neq i$ at some time $t$ induces the cost $C_{i,j}$, and implies that the production regime can not be changed before time $t+\delta_j$. We suppose throughout the paper the conditions:

$$\forall n \geq 1,\ \forall (i_0,\ldots,i_n),\ C_{i_0,i_1} + \cdots + C_{i_{n-1},i_n} + C_{i_n,i_0} + \delta_{i_0} + \cdots + \delta_{i_n} > 0 \tag{1}$$

$$\forall i,j,k,\ C_{i,j} + C_{j,k} \geq C_{i,k}\ . \tag{2}$$

Condition (1) implies that a management strategy with infinitely many switches either impossible or non-optimal. Condition (2) is a natural condition on the structure of switching costs.

In order to define the set of admissible management strategies of the production asset, we need to introduce the functions:

$$\bar{\delta}_i(t) := (t+\delta_i) \wedge T,\ 1 \leq i \leq M\ .$$

Definition 1. A management strategy of the production asset is an $\mathbb{F}$-adapted càdlàg pure jump process $\{\xi_t,\ t \in [0,T]\}$ with values in $\{1,\ldots,M\}$, with jump times $(\theta_n, n \geq 0)$ and states $(\xi^n,\ n \geq 0)$, such that, for all $n \geq 0$, $\bar{\delta}_{\xi^n}(\theta_n) \leq \theta_{n+1}$. In this setting, we have:

$$\xi_t = \sum_{n \geq 0} \xi^n \mathbf{1}_{\{\theta_n \leq t < \theta_{n+1}\}}\ .$$

An admissible management strategy is such that $N(\xi) := \inf\{n \geq 0,\ \theta_n = T\} - 1 < \infty$ a.s., i.e. which is composed a.s. of a finite number of switches. We denote by $\mathcal{X}_0$ the set of such admissible strategies. Given a management strategy $\xi \in \mathcal{X}_0$, we denote by $\mathcal{X}_t(\xi)$ the set of all admissible strategies $\xi'$ such that $\xi' = \xi$ on $[0,t]$.

We will also make use of the following notation. The set of all $\mathbb{F}$-adapted stopping times with values in $[t,T]$ will be denoted by $\mathcal{T}_t$. Given a management strategy $\xi \in \mathcal{X}_0$, we define the sequence $(\theta_n^* := \bar{\delta}_{\xi^n}(\theta_n),\ n \geq 0)$ of the switching times increased by the minimal times. We also define the sequence $(C_n^* := C_{\xi^n,\xi^{n+1}})$ of the switching costs.

Conditions (1)-(2) ensure that a management strategy $\xi$ such that $\mathbb{P}(N(\xi)) > 0$ is either not possible (presence of minimal times) or not optimal (presence of switching costs). This justifies the choice of the admissible set $\mathcal{X}_0$. Without loss of generality, we suppose that the power plant has just been switched to mode 1 at time 0 ($\theta_0 = 0$ and $\xi_0 = 1$). Given a management strategy of the plant $\xi \in \mathcal{X}_0$, we define its cumulated benefit at time $t \in [0,T]$:

$$B_t^\xi := \int_0^t \psi_u^{\xi_u} du - \sum_{n\geq 1,\, \theta_n \leq t} C_{\xi^{n-1},\xi^n} \ .$$

REMARK 1. The analysis of this paper can be easily extended to include a smooth transition from one mode to the other, so as to account for the so-called ramp rates. It can also be extended to include minimal times $\delta_{i,j}$ depending on both previous and current states.

### 3.3. Investment Strategies.

In addition to the production activity, the producer is allowed to invest continuously in the financial market. We shall denote by $\pi_t$ the amount invested in the market at time $t$. By the usual self-financing condition, the wealth process $X$ is defined for any $t \in [0,T]$ by:

$$X_t^{x,\pi} := x + \int_0^t \sum_{i=1}^N \pi_u^i \frac{dS_u^i}{S_u^i} \ = \ x + \int_0^t \pi_u \cdot (\mu_u du + \Sigma_u dW_u) \ ,$$

where $x$ denotes the initial capital. In order to account for possible portfolio constraints, we assume that the process $\pi$ takes values in some given closed convex subset $K$ of $\mathbb{R}^N$. We follow the definition of Hu et al. (2005) of admissible investment strategies on the financial market.

DEFINITION 2. An investment strategy is an $\mathbb{F}-$predictable $K$-valued process $\pi = \{\pi_t, 0 \leq t \leq T\}$ with $\mathbb{E}\int_0^T |\Sigma_t^* \pi_t|^2 dt < \infty$ a.s. such that $\left\{e^{-\eta X_\tau^{0,\pi}} : \tau \in \mathcal{T}_0\right\}$ is a uniformly integrable family. We denote by $\mathcal{A}_0$ the collection of all such investment strategies. For $\tau \in \mathcal{T}_0$ and $\pi^0 \in \mathcal{A}_0$, we denote by $\mathcal{A}_\tau(\pi^0)$ the subset of $\mathcal{A}_0$ consisting of all investment strategies $\pi \in \mathcal{A}_0$ such that $\pi = \pi^0$ on $[0,\tau]$.

EXAMPLE 2. *Incomplete market.* Let $K = \{(u_1, \dots, u_N) \in \mathbb{R}^N,\ u_{k+1} = \dots = u_N = 0\}$, for some $k \in \{1, \dots, N\}$. Then only the first $k$ components of $S$ represent prices of financial assets which can be traded by the producer.

### 3.4. Utility Valuation of the Production Asset

The variable $\chi$ represents some terminal payoff associated with the presence of the power plant. For instance, it may represent the dismantling cost of the power plant. Similarly, we introduce the random variable $\chi' \in L^\infty$ as the terminal payoff in the absence of the power plant, that may be different from $\chi$. Let

$$V_0(x) := \sup_{(\xi,\pi)\in\mathcal{X}_0\times\mathcal{A}_0} \mathbb{E}\left[U\left(X_T^{x,\pi} + B_T^\xi + \chi\right)\right] \tag{3}$$

$$v_0(x) := \sup_{\pi\in\mathcal{A}_0} \mathbb{E}\left[U\left(X_T^{x,\pi} + \chi'\right)\right] \tag{4}$$

be the indirect utility function of the manager respectively in the presence and absence of the power plant. Then, the utility valuation of the power plant is defined by:

$$p_0(x) := \sup\{p \geq 0 \ : \ V_0(x-p) \geq v_0(x)\} \ . \tag{5}$$

It is the highest price the agent is ready to pay to buy the power plant. In the context of the exponential utility, we can write:

$$v_0(x) = -e^{-\eta(x+y_0)} \text{ and } V_0(x) = -e^{-\eta(x+\overline{Y}_0^1)} \ ,$$

where $y_0$ and $\overline{Y}_0^1$ are independent of the initial capital $x$. Then the value of the plant is given by:

$$p_0 = \overline{Y}_0^1 - y_0 \ . \tag{6}$$

The main result of this paper provides a characterization of $(y_0, \overline{Y}_0^1)$ by means of a coupled system of reflected Backward Stochastic Differential Equations.

## 4. A Verification Result.

In this section we relate $v_0$ and $V_0$ to the solution of a coupled system of reflected BSDEs.

### 4.1. Non-Linear $g-$Expectation

The analysis of this paper appeals to the notion of non-linear $g$-expectation introduced by Peng (2003). Appendix A1 provides a quick review and a straightforward extension of this notion for a quadratic generator $g \ : \ \Omega\times[0,T]\times\mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying for all $t\in[0,T]$:

$$g(t,0)=0, \ |g(t,z)| \leq a_0 + b_0|z|^2 \text{ and } \left|\frac{\partial g}{\partial z}(t,z)\right| \leq a_1 + b_1|z| \text{ a.s. } , \tag{7}$$

for some constants $a_0,\ a_1,\ b_0,\ b_1$. For all bounded $\mathcal{F}_T$-measurable random variable $\zeta$, consider the following BSDE:

$$Y_t = \zeta - \int_t^T g(u, Z_u)du - \int_t^T Z_u dW_u \ . \tag{8}$$

The existence of a unique solution $(Y,Z) \in \mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N)$ to this BSDE was proved by Kobylanski (2000). The nonlinear $g$-expectation operator is defined by:

$$\mathcal{E}^g_{\tau,T}[\zeta] := Y_\tau \text{ for every } \tau \in \mathcal{T}_0 \text{ and } \zeta \in L^\infty(\mathbb{R}) \ .$$

### 4.2. Optimal Investment Decision.

The main result of this section requires the following additional notations. For $1 \leq i \leq M$, we introduce the random functions:

$$g_t(z) := \frac{\eta}{2} \left|\Sigma_t^* z - \Pi_t(\Sigma_t^* z))\right|^2 + \Pi_t(\Sigma_t^* z) \cdot \Pi_t(\Sigma_t^{-1}\mu_t) \tag{9}$$

$$f_t^i(z) := g_t(z) - \frac{1}{2\eta} \left|\Pi_t(\Sigma_t^{-1}\mu_t)\right|^2 - \psi_t^i \ , \tag{10}$$

where $\Pi_t(x)$ represents the orthogonal projection of $x$ on the closed convex set $\Sigma_t^* K$, the image of $K$ by $\Sigma_t^*$. We then consider the BSDEs:

$$Y_t^i = \zeta - \int_t^T f_u^i(Z_u^i)du - \int_t^T Z_u^i \cdot \Sigma_u dW_u \ , \tag{11}$$

for some $\zeta \in L^\infty(\mathbb{R})$. Since the random function $\tilde{g}_t(z) := g_t\left(\left(\Sigma_t^*\right)^{-1} z\right)$ is a quadratic generator satisfying the conditions (7), and $\psi^i$, $\Sigma$, $\mu$ are bounded, existence and uniqueness of a solution $(Y^i, Z^i) \in \mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N)$, with:

$$Y_t^i = \mathcal{E}^g_{t,T}\left[\zeta + \int_t^T \left(\frac{1}{2\eta}\left|\Pi_u(\Sigma_u^{-1}\mu_u)\right|^2 + \psi_u^i\right) du\right] \ ,$$

follows from the general results of Appendix A. We are then able to relate the value function $v_0$ to the initial value of this BSDE:

Proposition 1 **(Hu et al. (2005))**. *The indirect utility function of the manager in the absence of the production asset is given by:*

$$v_0(x) = -\exp\left(-\eta x - \eta \mathcal{E}^g_{0,T}\left[\chi' + \frac{1}{2\eta}\int_0^T \left|\Pi_t(\Sigma_t^{-1}\mu_t)\right|^2 dt\right]\right) \ .$$

### 4.3. Optimal Management-Investment Decision.

This section relates the value function $V_0$ to the initial value of a system of reflected BSDEs. We consider the coupled system of Reflected BSDEs (RBSDE), for $t\in[0,T]$ and $1\le i\le M$:

$$Y_t^i = \chi - \int_t^T f_u^i\left(Z_u^i\right)du - \int_t^T Z_u^i\cdot\Sigma_u dW_u + \left(K_T^i - K_t^i\right) \tag{12}$$

$$Y_t^i \ge \max_{j\neq i}\left\{\overline{Y}_t^j - C_{i,j}\right\} \tag{13}$$

$$\overline{Y}_t^i = \mathcal{E}^g_{t,\overline{\delta}_i(t)}\left[Y^i_{\overline{\delta}_i(t)} + \int_t^{\overline{\delta}_i(t)}\left(\frac{1}{2\eta}\left|\Pi_u(\Sigma_u^{-1}\mu_u)\right|^2 + \psi_u^i\right)du\right] \tag{14}$$

$$K^i \in \mathcal{J}(\mathbb{R})\ ,\int_0^T\left(Y_t^i - \max_{j\neq i}\left\{\overline{Y}_t^j - C_{i,j}\right\}\right)dK_t^i = 0\ . \tag{15}$$

Given $\overline{Y}^{1-i}$, (12)-(13)-(15) define the process $Y^i$ as the value function of the optimal investment-management problem when there is no constraint on the first switching time. Given $Y^i$, the process $\overline{Y}^i$ defines the value of the optimal investment problem with time duration $\delta_i$, corresponding to the switching delay constraint. The existence and uniqueness of a solution $(Y^i,Z^i,K^i)\in\mathcal{H}_0^\infty(\mathbb{R})\times\mathcal{H}_0^2(\mathbb{R}^N)\times\mathcal{J}(\mathbb{R})$, $1\le i\le M$ to the system of coupled RBSDEs (12)-(13)-(14)-(15) will be discussed in the subsequent section.

The main result of this section provides a characterization of the value function $V_0$, defined in (3), in terms of the component $\overline{Y}^1$ of the solution of the RBSDEs (12)-(13)-(14)-(15). We recall the assumption made previously that the plant has just been switched to mode 1 at time 0, which explains why the following proposition involves the component $\overline{Y}^1$.

Proposition 2. *Suppose that there exists a solution to* (12)-(13)-(14)-(15), *then the value of the optimal problem* (3) *is given by:*

$$V_0(x) = U\left(x + \overline{Y}_0^1\right)\ .$$

*Moreover, define the management strategy $\hat{\xi}$ by $\hat{\theta}_0 = 0$, $\hat{\xi}_0 = 1$, and:*

$$\hat{\theta}_{n+1} = \inf\left\{t\ge\overline{\delta}_{\hat{\xi}^n}(\hat{\theta}_n),\ Y_t^{\hat{\xi}^n} = \max_{j\neq\hat{\xi}^n}\left\{\overline{Y}_t^j - C_{\hat{\xi}^n,j}\right\}\right\}\wedge T$$
$$\hat{\xi}^{n+1} = \inf\left\{j\neq\hat{\xi}^n,\ Y^{\hat{\xi}^n}_{\hat{\theta}_{n+1}} = \overline{Y}^j_{\hat{\theta}_{n+1}} - C_{\hat{\xi}^n,j}\right\}\ ,$$

*for $n \geq 0$, $1 \leq i \leq M$, and the investment strategy $\hat{\pi}$ by:*

$$\hat{\pi}_t = (\Sigma_t^*)^{-1}\Pi_t\left((\eta\Sigma_t)^{-1}\mu_t - \Sigma_t^*\overline{Z}_t^{\hat{\xi}^n}\right) \text{ for } \hat{\theta}_n \leq t < \overline{\delta}_{\hat{\xi}^n}(\hat{\theta}_n)$$
$$\hat{\pi}_t = (\Sigma_t^*)^{-1}\Pi_t\left((\eta\Sigma_t)^{-1}\mu_t - \Sigma_t^* Z_t^{\hat{\xi}^n}\right) \text{ for } \overline{\delta}_{\hat{\xi}^n}(\hat{\theta}_n) \leq t < \hat{\theta}_{n+1} \ ,$$

*for $1 \leq i \leq M$ and $n \geq 0$. Then $(\hat{\xi}, \hat{\pi})$ defines an optimal management-investment strategy.*

*Proof.* See Appendix B. □

As a straightforward corollary, we obtain a characterization of the power plant value $p_0$.

Corollary 1. *The utility indifference price of the production asset is given by:*

$$p_0 = \overline{Y}_0^1 - \mathcal{E}_{0,T}^g\left[\zeta + \frac{1}{2\eta}\int_0^T \left|\Pi_t(\Sigma_t^{-1}\mu_t)\right|^2 dt\right].$$

## 5. Existence of a Solution of the RBSDE System

To prove the existence of a solution of the system of RBSDEs, we adapt the method developed in Hamadène and Jeanblanc (2005). We define the sequences of processes $Y^{i,n}$, $Z^{i,n}$, $K^{i,n}$, $n \geq 0$, for $1 \leq i \leq M$ as follows. We start from:

$$Y_t^{i,0} := \mathcal{E}_{t,T}^g\left[\chi + \int_t^T \left(\frac{1}{2\eta}\left|\Pi_u(\Sigma_u^{-1}\mu_u)\right|^2 + \psi_u^i\right) du\right] \ . \tag{16}$$

Given $Y^{i,n-1}$, we compute $\overline{Y}^{i,n-1}$ as:

$$\overline{Y}_t^{i,n-1} = \mathcal{E}_{t,\overline{\delta}_i(t)}^g\left[Y_{\overline{\delta}_i(t)}^{i,n-1} + \int_t^{\overline{\delta}_i(t)} \left(\frac{1}{2\eta}\left|\Pi_u(\Sigma_u^{-1}\mu_u)\right|^2 + \psi_u^i\right) du\right] \ , \tag{17}$$

and $Y^{i,n}, Z^{i,n}, K^{i,n}$ as the solution of a reflected BSDE:

$$Y_t^{i,n} = \chi - \int_t^T f_u^i\left(Z_u^{i,n}\right) du - \int_t^T Z_u^{i,n}\cdot\Sigma_u dW_u + \left(K_T^{i,n} - K_t^{i,n}\right) \tag{18}$$
$$Y_t^{i,n} \geq \max_{j\neq i}\left\{\overline{Y}_t^{j,n-1} - C_{i,j}\right\} \tag{19}$$
$$K^{i,n} \in \mathcal{J}(\mathbb{R}) \text{ and } \int_0^T \left(Y_t^{i,n} - \max_{j\neq i}\left\{\overline{Y}_t^{j,n-1} - C_{i,j}\right\}\right) dK_t^{i,n} = 0 \ . \tag{20}$$

We thus compute sequentially the processes $Y^{i,n}$ for all $n$. The existence of the triple $(Y^{i,n}, Z^{i,n}, K^{i,n}) \in \mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N) \times \mathcal{J}(\mathbb{R})$ for each $n$ follows from Kobylanski et al. (2002).

We shall give an interpretation of the processes $Y^{i,n}$ in terms of the value function of an optimal control problem with $n$ possible switches. Indeed $Y^{i,0}$ defined by (16) corresponds to the maximal utility when no switch is allowed (cf Hu et al. (2005)). An optimal switch shall be decided when a reflexion of the BSDE system occurs. After a switch from mode $i$ to $j$, the plant is subject to the minimal time constraint and only $n-1$ possible switches remain. This provides an intuitive explaination on how the sequence $(Y^{i,n})_{n\geq 0}$ is built. To prove this result, let us introduce the following notation. Let $\xi \in \mathcal{X}_0$, $n, m \in \mathbb{N}$. We denote by $\mathcal{X}^{n,m}(\xi)$ the set of management strategies $\xi' \in \mathcal{X}_{\theta_n}(\xi)$ such that: $N(\xi') \leq n+m$. In words, $\mathcal{X}_t^{n,m}(\xi)$ is the set of all admissible management strategies in $\mathcal{X}_{\theta_n}(\xi)$ which have less than $m$ mode switches between $\theta_n$ and $T$.

Proposition 3. *Let $(\xi, \pi)$ be a pair of management-investment strategies in $\mathcal{X}_0 \times \mathcal{A}_0$. Then*

$$U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n,m}\right)_{\theta_n} = \underset{(\xi',\pi') \in \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n}\left[U\left(X_T^{0,\pi'} + B_T^{\xi'} + \chi\right)\right] .$$

*Proof.* See Appendix C. □

The process $Y^{i,n}$ can thus be seen as an approximation of $Y^i$ when the number of possible switches is restricted to $n$. We then deduce the following corollary:

Corollary 2. *For $i = 1, \ldots, M$, the sequences $(Y^{i,n})_{n\geq 0}$ and $(\overline{Y}^{i,n})_{n\geq 0}$ are non-decreasing.*

*Proof.* See Appendix C. □

From this monotonicity property we obtain the convergence of the processes $Y^{i,n}, Z^{i,n}$ and $K^{i,n}$.

Proposition 4. *The sequences of processes $(Y^{i,n}, Z^{i,n}, K^{i,n})$, $n \geq 0$, converge uniformly to processes $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{K}^i)$ in $\mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N) \times \mathcal{J}(\mathbb{R})$. Moreover $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{K}^i)$ is a solution of the coupled system of BSDEs* (12)-(13)-(14)-(15).

*Proof.* See Appendix C. □

## 6. The Complete Market Case

In this section, we verify that the utility indifference pricing rule is indeed an extension of the no-arbitrage price in a frictionless market framework. To this end, we show that in a complete market the power plant value does not depend on the risk aversion coefficient $\eta$ and that the no-arbitrage pricing formula is obtained as a limiting case when production constraints vanish. The proof of the following result, similar to that of Proposition 2 with no investment, is omitted.

PROPOSITION 5. *If $K = \mathbb{R}^N$, the value of the power plant is given by:*

$$p_0 = \sup_{\xi \in \mathcal{X}_0} \mathbb{E}^{\mathbb{Q}^*} \left[ B_T^\xi + \chi - \chi' \right] ,$$

*where the equivalent martingale measure $\mathbb{Q}$ is given by its density:*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left( - \int_0^T \Sigma_t^{-1} \mu_t \cdot dW_t - \frac{1}{2} \int_0^T \left| \Sigma_t^{-1} \mu_t \right|^2 dt \right) , \tag{21}$$

REMARK 2. When there are only 2 modes 1,2, the producer only faces switching costs and the market is complete, we find the results of Hamadène and Jeanblanc (2005). In this case, if we define $\widetilde{Y}_t := Y_t^1 - Y_t^2$ and $\widetilde{Z}_t := Z_t^1 - Z_t^2$, then the quadruple $(\widetilde{Y}, \widetilde{Z}, K^1, K^2)$ satisfies the following doubly-reflected BSDE with constant barriers:

$$\widetilde{Y}_t = - \int_t^T \left( g_u \left( \widetilde{Z}_u \right) - \psi_u^1 + \psi_u^2 \right) du - \int_t^T \widetilde{Z}_u \cdot \Sigma_u dW_u + \left( K_T^1 - K_t^1 \right) - \left( K_T^2 - K_t^2 \right).$$

Finally, we study the limiting case when the switching costs and delays are zero, i.e. $\delta_i = C_{i,j} = 0$ for all $i, j$, and the market is complete. In this case, we shall prove that the value of the power plant is given by the classical no-arbitrage pricing formula in complete markets, which can be easily explained as follows. In the absence of production constraints and terminal payoffs ($\chi = \chi' = 0$), the producer would choose at each time $t$ the production mode that gives the best benefit rate and would gain $\max_j \psi_t^j$ at each time $t$. The optimal management strategy is to have the plant switched in mode $i$ when $\psi_t^i \geq \psi_t^j$, $j \neq i$. Then, the plant is equivalent to a financial option with payoff $\int_0^T \max_j \psi_t^j dt$ at time $T$. We are thus expecting the value of the power plant to be the expectation under the risk neutral measure of the above payoff.

Observe however that the above optimal management strategy is not in $\mathcal{X}_0$, as it exhibits an infinite number of switching times. We will then use the following strategy for the proof of our limiting result. Set $\delta_i = 0$ for $1 \leq i \leq M$ and consider identical positive switching costs $C_{i,j} = C > 0$, $i \neq j$, in order to ensure the existence of an optimal management strategy in $\mathcal{X}_0$. We next prove that the utility indifference value converges to the no-arbitrage price by sending $C$ to zero.

In order to emphasize the dependence of the variables on the switching costs, we will denote by $(Y^{i,C}, Z^{i,C}, K^{i,C})$ the solution of the backward system (12)-(13)-(14)-(15) with switching costs $C_{i,j} = C$, and by $p_0^C$ the utility indifference value of the plant.

PROPOSITION 6. *Let $K = \mathbb{R}^N$, $\delta_i = 0$, $C_{i,j} = C > 0$ for all $i \neq j$. Then*

$$p_0^C \longrightarrow p^0 := \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T \max_j \psi_t^j dt + \chi - \chi' \right] \ , \ as \ C \to 0 \ ,$$

*where $\mathbb{Q}$ is the equivalent martingale measure defined in* (21).

*Proof.* The proof of this proposition is omitted and available under request. □

## 7. Relation with a PDE Obstacle Problem.

In this section, we consider a Markovian setting and suppose that $\chi = \phi(S_T)$, $\mu_t = \mu(t, S_t)$, $\Sigma_t = \Sigma(t, S_t)$, $\psi_t^i = \psi^i(t, S_t)$ and $f_t^i(z) = f^i(t, S_t, z)$. For $t \in [0, T]$, $s \in \mathbb{R}^N$, we denote by $\mathbb{F}^t := \{\mathcal{F}_u^t, t \leq u \leq T\}$ the natural filtration of the Brownian motion $W_u^t := W_u - W_t$, $t \leq u \leq T$. We denote by $(Y^{i,t,s}, Z^{i,t,s}, K^{i,t,s})$ the solution of the BSDE system, adapted to $\mathbb{F}^t$, when $S_t = s$. We define $u^i(t,s) := Y_t^{i,t,s}$ and $\overline{u}^i(t,s) := \overline{Y}_t^{i,t,s}$. Then, it follows from Kobylanski (2000) and Kobylanski et al. (2002) that $u^i$ and $\overline{u}^i$ are viscosity solutions of the following PDE system:

$$0 = \min \left\{ u^i - \overline{u}^{1-i} + C_{1-i}, -\mathcal{L}u^i + f^i \left( .,., \frac{\partial u^i}{\partial s} \right) \right\}, \text{ on } [0,T] \times \mathbb{R}^N \text{ and } u^i(T,s) = \phi(s) \ ,$$

for $1 \leq i \leq M$, and for all $t_0 \in [0, T]$, $\overline{u}^i(t_0, s) = w^i(t_0, t_0, s)$, where $w^i(t_0, t, s)$ solves:

$$0 = -\mathcal{L}w^i + f^i \left( .,., \frac{\partial w^i}{\partial s} \right), \text{ on } [t_0, \overline{\delta}_i(t_0)] \times \mathbb{R}^N \text{ and } w^i(t_0, \overline{\delta}_i(t_0), s) = u^i(\overline{\delta}_i(t_0), s) \ , \tag{22}$$

for every fixed $t_0 \in [0, T]$ and $\mathcal{L}$ is the Dynkin operator associated to the diffusion process $S$:

$$\mathcal{L}u(t,s) := \frac{\partial u}{\partial t}(t,s) + \mu(t,s) \frac{\partial u}{\partial s}(t,s) + \frac{1}{2} \mathrm{Tr} \left( \Sigma \Sigma^*(t,s) \frac{\partial^2 u}{\partial s^2}(t,s) \right) \ .$$

## 8. Numerical Implementation in a Complete Market

In this section, we discuss and implement several numerical schemes to solve the RBSDE or PDE system and compute the value of a coal-fired power plant with two modes 0,1 in a complete market.

### 8.1. The BSDE-Based Numerical Scheme

We fix a discretization step $\Delta$, such that $T = N_0\Delta$ for some $N_0 \in \mathbb{N}$, and $\delta_i = \kappa_i \Delta$. We denote by $(y_n^i, z_n^i)$ (resp. $(\overline{y}_n^i, \overline{z}_n^i)$) the approximation at time $n\Delta$ of the processes $(Y^i, Z^i)$ (resp. $(\overline{Y}^i, \overline{Z}^i)$). We adapt the Euler scheme for RBSDEs proposed in Bouchard and Touzi (2005) and consider the following scheme, for $0 \le n < N_0$:

$$z_n^i = \frac{1}{\Delta}\left(\Sigma_{n\Delta}^*\right)^{-1}\mathbb{E}_{n\Delta}\left[y_{n+1}^i\left(W_{(n+1)\Delta} - W_{n\Delta}\right)\right],\ y_n^i = \max\left\{\mathbb{E}_{n\Delta}[y_{n+1}^i] - \Delta f_{n\Delta}^i(z_n^i),\ \overline{y}_n^{1-i,\kappa_{1-i}} - C_{1-i}\right\}\ ,$$

with terminal condition $y_{N_0}^i = \xi$ and, for $0 \le k \le \kappa_i$:

$$\begin{aligned}
\overline{y}_{N_0}^{i,k} &= \xi + \sum_{l=1}^{\kappa_i - k}\Delta\left(\frac{1}{2\eta}\left|\Pi(\Sigma^{-1}\mu)\right|^2 + \psi^i\right)_{(N_0-l)\Delta}, \quad \overline{y}_n^{i,0} = y_n^i + \sum_{l=1}^{\kappa_i}\Delta\left(\frac{1}{2\eta}\left|\Pi(\Sigma^{-1}\mu)\right|^2 + \psi^i\right)_{(n-l)\Delta} \\
\overline{z}_n^{i,k} &= \frac{1}{\Delta}\left(\Sigma_{n\Delta}^*\right)^{-1}\mathbb{E}_{n\Delta}\left[\overline{y}_{n+1}^{i,k-1}\left(W_{(n+1)\Delta} - W_{n\Delta}\right)\right], \quad \overline{y}_n^{i,k} = \mathbb{E}_{n\Delta}[\overline{y}_{n+1}^{i,k-1}] - \Delta g_{n\Delta}^i(\overline{z}_n^{i,k})\ .
\end{aligned}$$

The main difference with Bouchard and Touzi (2005) is that, at each time $n\Delta$, we need to look forward until time $n\Delta + \delta_i$ in order to decide whether a mode switch is profitable. We therefore need to compute (an approximation of) the solution of BSDE (14). This is done in $\kappa_i + 1$ steps and justifies the use of the vector $(\overline{y}_n^{i,k})_{0 \le k \le \kappa_i}$: at each time $n$, $\overline{y}_n^{i,k}$ is the approximation of

$$\mathcal{E}_{n\Delta,(n+k)\Delta}^g\left[Y_{(n+k)\Delta}^i + \sum_{l=1}^{\kappa_i}\Delta\left(\frac{1}{2\eta}\left|\Pi(\Sigma^{-1}\mu)\right|^2 + \psi^i\right)_{(n+k-l)\Delta}\right]\ .$$

The numerical approximation of the conditional expectation operator can be tackled by different methods: kernel regression methods Carrière (1996), projection methods Longstaff and Schwartz (2001), quantization Bally et al. (2005), Malliavin calculus Bouchard and Touzi (2005). In this paper we implement the projection-based method of Gobet et al. (2004).

### 8.2. The PDE-Based Numerical Scheme.

Regarding the PDE system, we denote by $\mathcal{D}_{n\Delta}^i\phi$ the solution of the PDE (22) at time $n\Delta$ when the terminal condition at time $(n+1)\Delta$ is $\phi$. We also denote by $u_n^i$ (resp. $\overline{u}_n^i$) the approximation at time $n\Delta$ of the function $u^i$ (resp. $\overline{u}^i$). A natural numerical scheme for solving the PDE is: $u_{N_0}^i = \phi$, $u_n^i = \max\left\{\mathcal{D}_{n\Delta}^i u_{n+1}^i\ ,\ \overline{u}_n^{1-i,\kappa_{1-i}} - C_{1-i}\right\}$, and for $0 \le k \le \kappa_i$, $\overline{u}_{N_0}^{i,k} = \phi$, $\overline{u}_n^{i,0} = u_n^i$, $\overline{u}_n^{i,k} = \overline{\mathcal{D}}_{n\Delta}u_{n+1}^{i,k-1}$. The differential operator can be approximated by classical methods. In our numerical implementation, we use the finite differences approximation.

| | | |
|---|---|---|
| $\mu_F = 0.03$ | $\mu_G = 0.0003$ | $\sigma_F = 0.1$ |
| $\sigma_G = 0.01$ | $a = 0.13$ | $b = 0$ |

| | | |
|---|---|---|
| $\delta_0 = 24$ h | $\delta_1 = 8$ h | $T = 8760$ h |
| $C_0 = 0$ € | $C_1 = 35530$ € | $H = 0.5$ ton/MWh |

**Table 1** **Example 1 - Price process parameters and power plant characteristics.**

### 8.3. Valuation of a Coal-Fired Power Plant.

We consider the example of a coal-fired power plant. The process $S_t = (F_t(T), G_t(T))$ is defined by:

$$\frac{dF_t(T)}{F_t(T)} = \mu_F e^{-a(T-t)} dt + \sigma_F e^{-a(T-t)} dW_t^1 \ , \ \frac{dG_t(T)}{G_t(T)} = \mu_G e^{-b(T-t)} dt + \sigma_G e^{-b(T-t)} dW_t^2 \ .$$

Here, $F_t(T)$ is the forward price of electricity at time $t$ with delivery at time $T$ and $G_t(T)$ is the forward price of coal at time $t$ with delivery at time $T$. This model is the well known one-factor model used in energy markets (see Clewlow and Strickland (2000)). We suppose that there is no correlation between the two assets, which is approximately the case for coal. We also suppose that the spot prices of electricity and gas at time $t$ are defined by $F_t(t)$ and $G_t(t)$. In this context, the spot prices can be expressed in terms of the forward prices: $F_t(t) = \phi_F(t, F_t(T))$ and $G_t(t) = \phi_G(t, G_t(T))$ , where $\phi_F$ and $\phi_G$ are deterministic functions defined by:

$$\begin{aligned} \phi_F(t,\alpha) &= F_0(t) \exp\left(\frac{1}{2}\sigma_F^2 \int_0^t e^{-a(t-u)} \left(e^{-a(T-u)} - e^{-a(t-u)}\right) du\right) \left[\frac{\alpha}{F_0(T)}\right]^{e^{a(T-t)}} \\ \phi_G(t,\beta) &= G_0(t) \exp\left(\frac{1}{2}\sigma_G^2 \int_0^t e^{-b(t-u)} \left(e^{-b(T-u)} - e^{-b(t-u)}\right) du\right) \left[\frac{\beta}{G_0(T)}\right]^{e^{b(T-t)}} \end{aligned} ,$$

and $F_0(.)$, $G_0(.)$ are initial deterministic forward curves. The power plant can be in two modes: on (denoted 1) or off (denoted 0). The instantaneous rates of benefit of the power plant at time $t$ are given by: $\psi_t^0 := 0$, $\psi_t^1 := q\,(F_t(t) - HG_t(t))$. In what follows, we make use of the notation $C_0 := C_{0,1}$ and $C_1 := C_{1,0}$. The terminal payoffs $\chi$ and $\chi'$ are set to 0. We compute the power plant value on two examples. The first example is a toy example described in Table 1. The price process parameters roughly correspond to data from 2004, with a constant electricity future curve of 20 €/MWh, a constant coal future curve of 30 \$/ton, and a constant Euro/Dollar exchange rate of 1. Parameters

| | | |
|---|---|---|
| $\mu_F = 0.03$ | $\mu_G = 0.0003$ | $\sigma_F = 0.07$ |
| $\sigma_G = 0.011$ | $a = 0.013$ | $b = 0.0005$ |

| | | |
|---|---|---|
| $\delta_0 = 24$ h | $\delta_1 = 8$ h | $T = 8760$ h |
| $C_0 = 0$ € | $C_1 = 35530$ € | $H = 0.3627$ ton/MWh |

**Table 2 Example 2 - Price process parameters and power plant characteristics.**

of the power plant are close to those of a real power plant. This toy example has been designed to isolate the impact of the frictions on the power plant value.

The second example corresponds to a similar power plant with a lower and more realistic heat rate, and a constant maintenance cost of 10000 €/day. The price process parameters correspond to data from 2006 in France, where higher electricity prices have been observed. The Euro/Dollar rate for 2006 was around 1.2. With these values of the parameters, we observed almost no impact of the constraints on the power plant value. Indeed, if electricity price is much higher than coal price, it is never optimal to shut down the power plant. Recall that the plant is assumed to be turned off at time 0. Then, the optimal managing strategy is to wait until time $\delta_0$ when a mode switch is allowed, and keep on producing until $T$. The conclusion of this example is that constraints have little impact if the plant is highly profitable. In order to have significant results, we choose to set the Euro/Dollar rate at 0.8 for this example . The reference risk aversion coefficient is taken equal to $\eta = 1$ for both examples. In agreement with the results of Section 6, we observe that the power plant value does not depend on $\eta$. Results for both examples are shown in Subsection 8.5.

### 8.4. The Corresponding BSDE and PDE.

The drivers $f^i$ associated to the above problem are given by:

$$f_t^i(z) = \mu_F e^{-a(T-t)} z_1 + \mu_G e^{-b(T-t)} z_2 - \frac{1}{2\eta}\left(\left(\frac{\mu_F}{\sigma_F}\right)^2 + \left(\frac{\mu_G}{\sigma_G}\right)^2\right) - \psi_t^i \ .$$

and the Dynkin operator by:

$$\begin{aligned}\mathcal{L}u(t,\alpha,\beta) = u_t(t,\alpha,\beta) + \frac{1}{2}\alpha^2\sigma_F^2 e^{-2a(T-t)} + \mu_F e^{-a(T-t)} u_\alpha(t,\alpha,\beta) + \mu_G e^{-b(T-t)} u_\beta(t,\alpha,\beta) \\ + u_{\alpha\alpha}(t,\alpha,\beta) + \frac{1}{2}\beta^2\sigma_G^2 e^{-2b(T-t)} u_{\beta\beta}(t,\alpha,\beta) \ .\end{aligned}$$

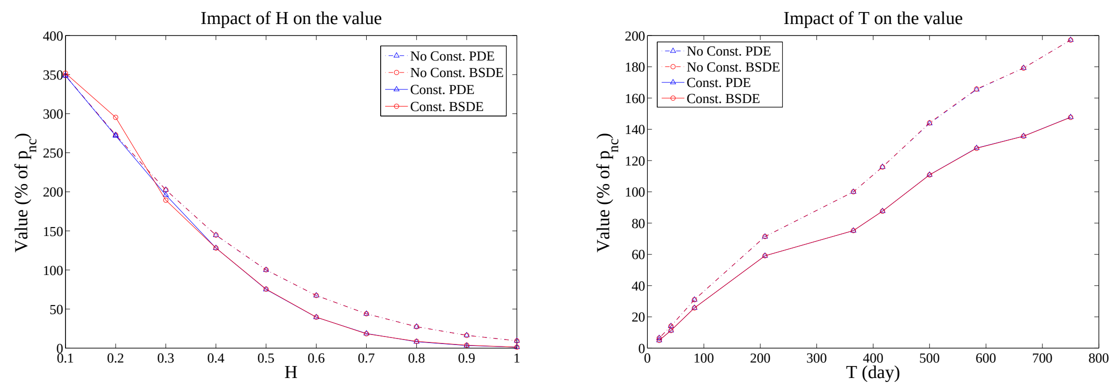


**Figure 1** **Example 1 - Impact of $H$ (left) and $T$ (right).**

For the purpose of our numerical implementation, we first proceed to the change of variable:

$$\xi_t = \int_0^t \sigma_F e^{-a(t-u)}(dW_t^1 + \frac{\mu_F}{\sigma_F}dt) \ , \ \zeta_t = \int_0^t \sigma_G e^{-b(t-u)}(dW_t^2 + \frac{\mu_G}{\sigma_G}dt) \ ,$$

which avoids dealing with exponential coefficients of the form $e^{-a(T-t)}$, and allows the use of Brownian bridge techniques. Regarding the BSDE approximation, we choose to follow the methodology developed by Gobet et al. (2004) with an $8 \times 8$ grid, linear approximation inside each domain, and 25600 simulations. The time step is set to 1 hour. For the PDE, we approximate the operator $\mathcal{D}^i$ by a Crank-Nicholson scheme, within a domain $[-5, 5] \times [-5, 5]$ in $(\xi, \zeta)$. In each direction, we mesh the interval with 100 steps. Time step is 1 hour.

### 8.5. Numerical Results in a Complete Market

Example 1. The coal-fired power plant price in the presence of production constraints is $p_c = 15.91 \ 10^6$ €, while the price in the absence of production constraints (i.e. $\delta_0 = \delta_1 = C_0 = C_1 = 0$) is $p_{nc} = 21.17 \ 10^6$ €. In the context of this example, we observe that the presence of production constraints reduces the power plant value by 25% over one year, which is very significant.

To highlight the contribution of the various frictions, we vary each parameter separately, starting from two reference configurations, corresponding respectively to $p_c$ and $p_{nc}$. Figure 1-left shows the variations of the value (in % of $p_{nc}$) with respect to the heat rate $H$ (left) and the horizon $T$ (right). Each time we show the value without constraints (dotted lines) and with constraints (solid lines), computed both by PDE (diamonds) and BSDE (circles). This figure confirms the decrease of the value with the heat rate $H$, as expected. PDE and BSDE computations lead to very close results in both cases. We also observe that the value with constraints is always smaller than the no-constraint value. On the other hand, the value is increasing with the horizon $T$ (Figure 1-right), as expected, almost linearly in both cases, with a larger slope in the absence of constraints.

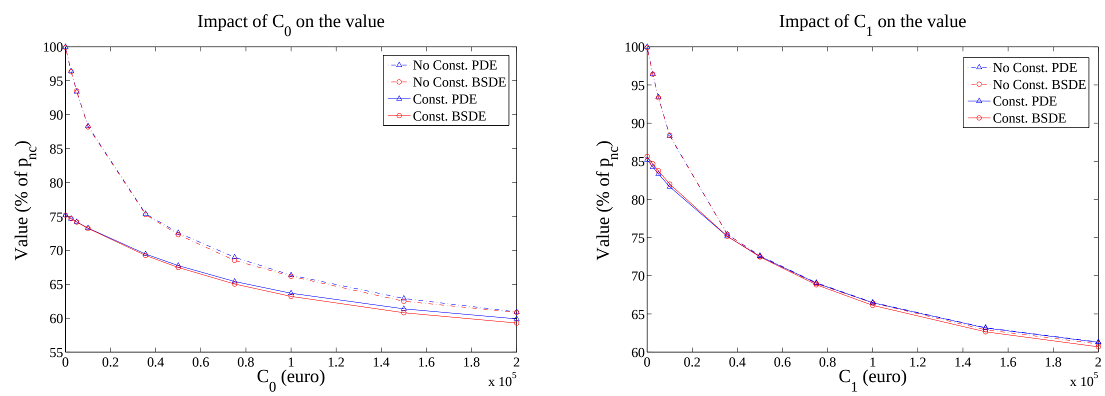


**Figure 2** **Example 1 - Impact of $C_0$ (left) and $C_1$ (right).**

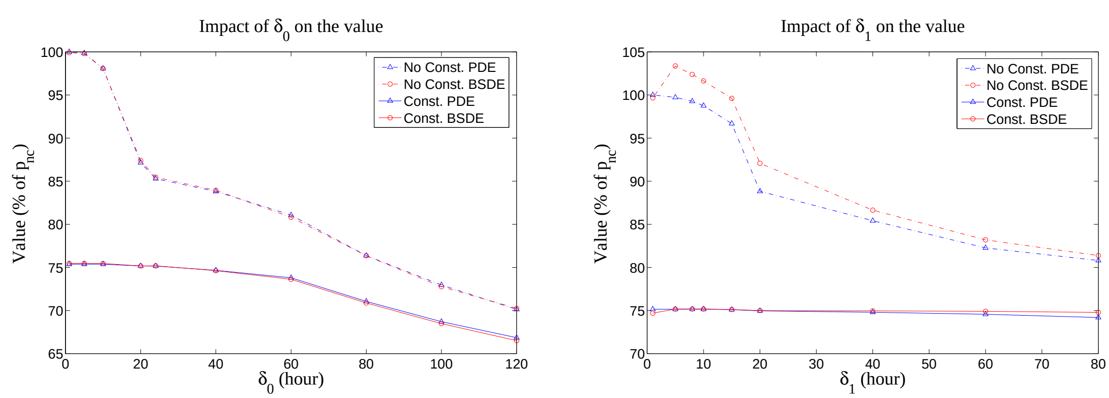


**Figure 3** **Example 1 - Impact of $\delta_0$ (left) and $\delta_1$ (right).**

The power plant value is also decreasing with the shut-down and start-up costs $C_0$ and $C_1$, as expected (see figures 2-left and 2-right). We notice that variations of the switching costs have a more limited impact on the value in the presence of other constraints.

We also observe a decrease of the value with the start-up and shut-down minimal times $\delta_1$ and $\delta_0$ (see respectively figures 3-left and 3-right). Nonetheless, the switching delays have less impact on the value, especially for small values where the curves present an horizontal tangent. We also observe that the impact of $\delta_0$ is larger than that of $\delta_1$ in the presence of other constraints. This can be explained by the large difference between $C_0$ and $C_1$ and the fact that $\psi^0 = 0$. Thus there is a strong asymmetry between modes on and off. High start-up costs limit the number of switches and thus increase the time between two switches, which limits the impact of the switching delays.

Example 2. The coal-fired power plant value in the presence of production constraints is $p_c = 119.3\ 10^6$ €, while the price in the absence of production constraints (i.e. $\delta_0 = \delta_1 = C_0 = C_1 = 0$ and no maintenance cost) is $p_{nc} = 125.9\ 10^6$ €. In this case, the presence of production constraints only reduces the power plant value by 5% over one year. We also observe that the value of the power plant is more than 10 times higher than in Example 1. As we highlighted above, the higher the profitability of the power plant, the lower the impact of the constraints on the value.

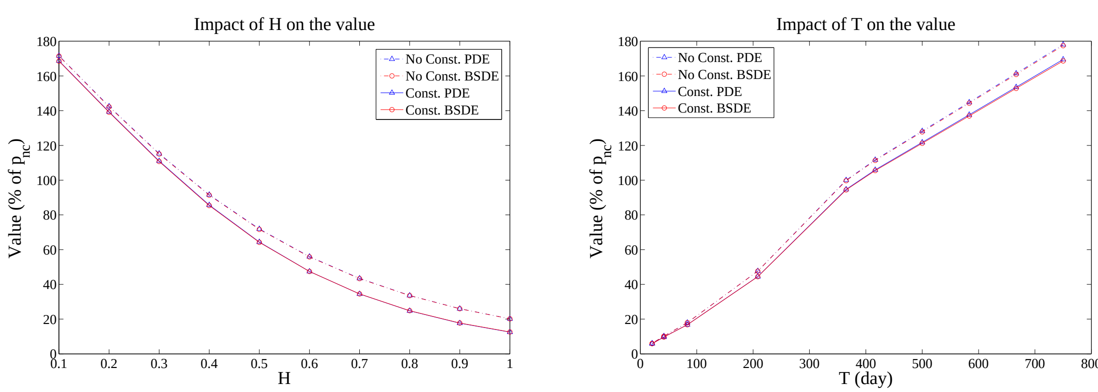


**Figure 4** **Example 2 - Impact of $H$ (left) and $T$ (right).**

Figures 4 to 6 show the variations of the value with the parameters. These variations are similar to those of Example 1, with a lower magnitude. We observe here that the BSDE method converges more slowly than in Example 1, but still provides very close results to those of the PDE method.

In terms of computational time for this 2-dimensional example, the PDE method is much faster than the BSDE. Table 3 shows some time comparisons (in minutes) referring to the computation

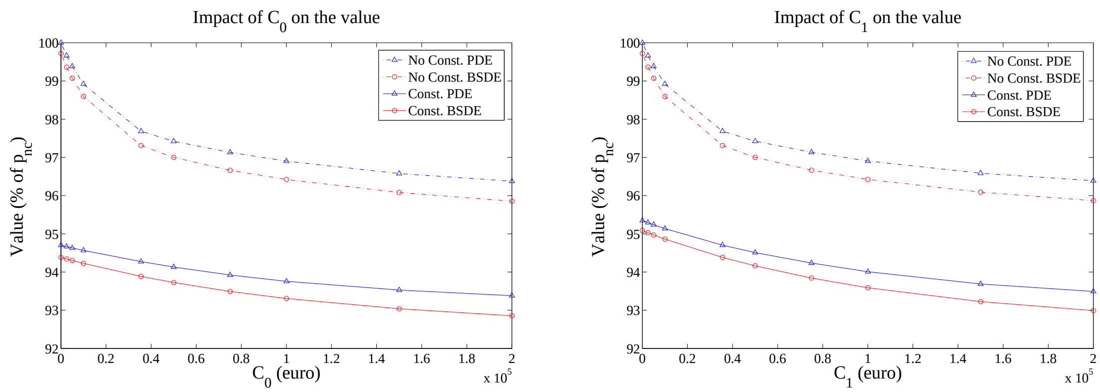


**Figure 5** **Example 2 - Impact of $C_0$ (left) and $C_1$ (right).**

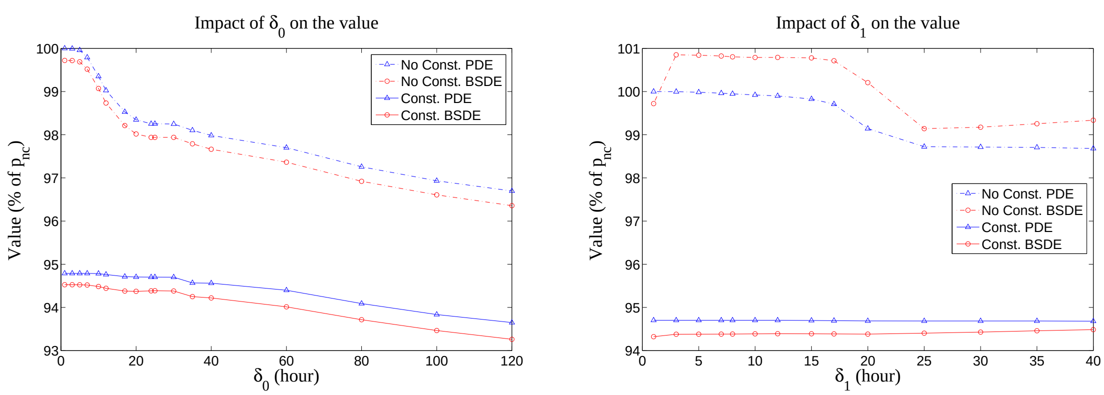


**Figure 6** **Example 2 - Impact of $\delta_0$ (left) and $\delta_1$ (right).**

of the value with constraints for different time horizons. As expected, computation times are linear with the horizon. In higher dimension, the finite differences method for PDEs becomes untractable.

| Horizon $T$ (day) | 83 | 208 | 365 | 500 | 750 |
|---|---|---|---|---|---|
| PDE (CPU mn) | 14 | 20 | 35 | 62 | 79 |
| BSDE (CPU mn) | 99 | 230 | 302 | 477 | 718 |

**Table 3** **Comparison of time performances between the PDE and BSDE algorithms.**

On the other hand, BSDE methods do not depend so heavily on dimension, and we expect them to become more efficient when dimension increases.

## 9. Numerical Implementation in an Incomplete Market.

### 9.1. Source of Incompleteness.

In this section, we present a simple example of incomplete market where the spot price of electricity at time $t$ is no longer given by $F_t(t)$. Due to non-storability, standardization of forward contracts, illiquidity and a restricted number of actors in electricity markets, large discrepancies can be observed between the last quoted forward price for a given maturity and the spot price at that maturity. This aspect is discussed for example in Skantze and Ilic (2000), where a general model is proposed for the relationship between the forward price $F_t(T)$ and the spot price $P_T$: $F_t(T) = \Psi(\mathbb{E}_t[P_T], \mathrm{Var}_t[P_T], \varepsilon_t)$, where $\mathrm{Var}_t[S_T]$ is the conditional variance of $P_T$ and $\varepsilon$ is a random disturbance. In particular, $F_t(t) = \Psi(P_t, 0, \varepsilon_t)$ is not necessarily equal to $P_t$.

To take into account this specificity, we choose the simplest model: $P_t := F_t(t) + \varepsilon_t$, where $\varepsilon_t$ is some exogenous, non-tradable, stochastic shock with dynamics:

$$d\varepsilon_t = -\kappa\varepsilon_t dt + \gamma dW_t^3 ,$$

and $W^3$ is independent of $(W^1, W^2)$. The instantaneous rate of benefit in production mode is now given by: $\psi_t^1 := q\left(F_t(t) + \varepsilon_t - HG_t(t)\right)$. Since the shock $\varepsilon$ does not correspond to any tradable asset, the market is incomplete and the RBSDE system is non-linear. A quadratic term in $Z^{i,3}$ appears:

$$\begin{aligned} dY_t^i = &\left(\frac{\eta\gamma^2}{2}(Z_t^{i,3})^2 + \mu_F e^{-a(T-t)} Z_t^{i,1} + \mu_G e^{-b(T-t)} Z_t^{i,2} - \frac{1}{2\eta}\left|\Sigma_t^{-1}\mu_t\right|^2 - \psi_t^i\right) dt \\ &+ \sigma_F e^{-a(T-t)} Z_t^{i,1} dW_t^1 + \sigma_G e^{-b(T-t)} Z_t^{i,2} dW_t^2 + \gamma Z_t^{i,3} dW_t^3 - dK_t^i \ . \end{aligned}$$

We restrict to the case where there are no delays: $\delta_0 = \delta_1 = 0$, and $T$ is equal to 6 months. The characteristics of the power plant and the price process are those of Example 2, except that we set the drifts to $\mu_F = \mu_G = 0$. This allows us to use a smaller domain for the PDE mesh. Parameter $\kappa$ is set to 0.02. We focus our analysis on the impact of $\eta$ and $\gamma$ on the value.

### 9.2. Numerical Scheme and Results.

No theoretical analysis of the approximation of BSDEs with quadratic generator is available in the literature. A consistency result can be obtained, following the lines of Bouchard and Touzi (2005), by approximating the quadratic generator by a sequence of Lipschitz functions. However, the rate of convergence of such an approximation method is difficult to obtain. We tested the numerical convergence of a direct Monte Carlo computation via the Gobet-Lemor-Warin method, as in the complete market case, but we were unsuccessful in obtaining satisfactory results.

In the particular case where the generator is a second-order polynomial in $Z$, which is the case in this article, it is possible to transform the quadratic BSDE into a linear Forward-Backward SDE by means of the Girsanov theorem. The methodology developed by Delarue and Menozzi (2006) can then be followed and provides both a numerical scheme and a convergence result for this scheme.

Another approach, developed by Chaumont et al. (2005), uses the connection between BSDEs and PDEs. They introduced a finite differences scheme for the quadratic PDE and proved its convergence.

**FBSDE algorithm.** We first applied Delarue-Menozzi's algorithm for Forward Backward SDEs. Since the generator of the BSDE is a second order polynomial in $Z$, an application of Girsanov Theorem allows us to rewrite the quadratic BSDE into a linear coupled Forward Backward SDE:

$$\begin{aligned} d\varepsilon_t &= \left(-\kappa\varepsilon_t - \frac{\eta\gamma^2}{2}Z_t^{i,3}\right)dt + \gamma d\overline{W}_t^3 \ , \\ dY_t^i &= \left(\mu_F e^{-a(T-t)}Z_t^{i,1} + \mu_G e^{-b(T-t)}Z_t^{i,2} - \frac{1}{2\eta}\left|\Sigma_t^{-1}\mu_t\right|^2 - \psi_t^i\right)dt \\ &\quad + \sigma_F e^{-a(T-t)}Z_t^{i,1}dW_t^1 + \sigma_G e^{-b(T-t)}Z_t^{i,2}dW_t^2 + \gamma Z_t^{i,3}d\overline{W}_t^3 - dK_t^i \ , \end{aligned}$$

where $d\overline{W}_t^3 := dW_t^3 + \frac{\eta\gamma}{2}Z_t^{i,3}dt$, and the other components of the forward process are unchanged. Numerical methods are available for these equations in Delarue and Menozzi (2006). We observed that the computational time is very high in dimension 3. As an illustration, we computed the BSDE associated to the "on" mode (i.e. same BSDE as $(Y^1, Z^1)$ with no reflexion) over an horizon of 4 days. We obtain an initial value of 530000 (domain = (0.5, 0.5, 0.5), space step = (3.125E-3,3.125 E-3,3.125E-3), time step = 1 hour), compared to the PDE result 527300 (domain =(1,1,2), space step = (0.05,0.05,0.05), time step = 1/100 hour). The FBSDE algorithm thus converges to the PDE value. Nevertheless, the computational time for this example is 16 hours for the BSDE against 26 mn for the PDE.

**PDE algorithm.** We also solved the non-linear PDE using the same scheme as Chaumont et al. (2005). This scheme is totally explicit and imposes very strong Courant-Friedrichs-Lewy conditions. We solved the PDE on a domain $[-1,1]\times[-1,1]$ in $(\xi,\zeta)$ and $[-2,2]$ in $\varepsilon$, meshed by $40\times 40\times 80$ steps. The time step is taken to 1/100 hour (=36 sec!). This is why we only computed the power plant value over an horizon of 6 months. Computational time for this example was in the range of 1 week. An implicit version of this scheme can be implemented but we did not implement it.

We are then able to study the impact of the parameters $\gamma$ and $\eta$ on the value. The results are shown in Figure 7. We observe a decrease of the value with $\eta$ (Figure 7-left, the value is expressed in % of

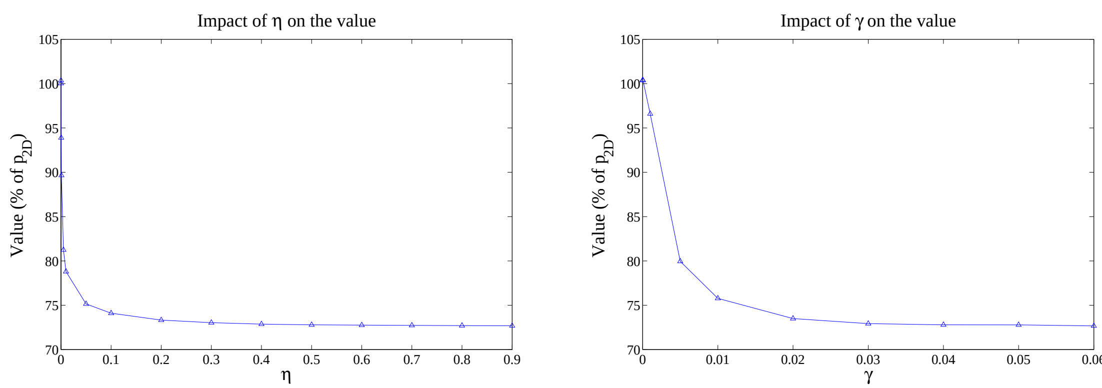


**Figure 7 Impact of $\eta$ (left) and $\gamma$ (right) on the value in % of the value in complete market.**

the value in complete market, i.e. when $\gamma=0$), which is natural since we deal with a buying price. On the other hand, as $\gamma$ increases, the power plant value decreases and converges to the complete market price when $\gamma$ tends to 0. We finally conclude that the real option value can be impacted by a 25 % decrease in the presence of a non-traded uncertainty.

## 10. Conclusion.

Utility indifference provides a nice framework for the valuation of physical assets in incomplete markets. In a complete market, this methodology leads to the usual no-arbitrage value, and is thus an extension of the arbitrage free valuation method. This method takes into account the agent's preference via a utility function and, from this point of view, is more satisfactory than the discount rate $\rho$ introduced by Dixit and Pindyck (1994), which has a poor economic interpretation. Indeed, a discount rate reveals the agent's preference over time but not really over risk. In our setting, the exponential utility is parameterized by a single coefficient $\eta$ and may thus not be more difficult to calibrate than the discount rate $\rho$ in the methodology of Dixit and Pindyck (1994).

Nevertheless, this methodology has two main drawbacks. The first one is the computational complexity of the utility maximization problems in incomplete markets, as we saw in the above paragraph. The second is the non-linearity of the pricing rule with respect to the payoff. If the agent owns a portfolio of physical assets, the value of the portfolio is not the sum of the assets values. The value of a power plant is impacted by the presence of other assets (physical or financial) in the agent's portfolio. Rigourously, the agent should compute the utility indifference value of its whole portfolio of assets, implying the resolution of a hard multi-asset commitment problem.

## Appendix A: Properties of the non-linear $g$-expectation.

PROPOSITION 7. *Let $\tau$ be an arbitrary stopping time in $\mathcal{T}_0$. Then:*

(i) $\mathcal{E}^g_{\tau,T}[\zeta] \le \mathcal{E}^g_{\tau,T}[\zeta']$ *a.s., whenever $\zeta \le \zeta'$ a.s. (Monotonicity)*

(ii) $\mathcal{E}^g_{T,T}[\zeta] = \zeta$

(iii) $\mathcal{E}^g_{\tau,T}[\mathbf{1}_A\zeta] = \mathbf{1}_A\mathcal{E}^g_{\tau,T}[\zeta]$ *a.s. for every $A \in \mathcal{F}_t$ (0-1 law).*

*Proof.* (i) follows from the comparison theorem for quadratic BSDEs (cf. Kobylanski (2000)). (ii) is trivial. (iii) Multiplying both sides of (8) by $\mathbf{1}_A$, we see that:

$$\mathbf{1}_A Y_t = \mathbf{1}_A\zeta - \int_t^T g(u, \mathbf{1}_A Z_u)du - \int_t^T \mathbf{1}_A Z_u dW_u \ ,$$

where we used the fact that $g(t,0) = 0$ a.s. Hence $(Y\mathbf{1}_A, Z\mathbf{1}_A)$ is a solution of the BSDE with terminal condition $\zeta\mathbf{1}_A$, and therefore $\mathcal{E}^g_{\tau,T}[\mathbf{1}_A\zeta] = \mathbf{1}_A\mathcal{E}^g_{\tau,T}[\zeta]$. □

By following the lines of the arguments of Peng, it can be shown that the family $\{\mathcal{E}^g_{\tau,T}[\zeta],\ \tau \in \mathcal{T}_0\}$ can be aggregated by a càdlàg adapted process $\{\mathcal{E}^g_{t,T}[\zeta],\ 0 \le t \le T\}$ in the sense that $\mathcal{E}^g_{\tau,T}[\zeta] = \mathcal{E}^g_{t,T}[\zeta]$ a.s. on $\{\tau = t\}$. For later use, we isolate the following general result on quadratic BSDEs.

LEMMA 1. *The process $\left\{\int_0^t Z_s dW_s,\ t \in [0,T]\right\}$ is a BMO martingale (see Hu et al. (2005) for the definition of a BMO martingale).*

*Proof.* Since $Y \in \mathcal{H}_0^\infty(\mathbb{R})$, we can find $y$ such that $|Y_t| < y$, $\forall t \le T$. Following Kobylanski (2000), we define the function $\Phi(x) := \frac{e^{2b_0 y}}{2b_0}\left(e^{2b_0 y} - e^{-2b_0 x}\right) - x - y$ for $|x| \le y$ and we observe that:

$$\Phi \ge 0\ ,\ \ \Phi' \ge 0\ ,\ \ \Phi \in C^2([-y,y],\mathbb{R}) \text{ and } \frac{1}{2}\Phi'' + b_0\Phi' + b_0 = 0\ .$$

Let $\tau$ be an arbitrary stopping time valued in $[0,T]$. Since $(Y,Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^n)$, it follows that:

$$\mathbb{E}_\tau\left[\Phi\left(Y_T\right) - \Phi\left(Y_\tau\right)\right] = \mathbb{E}_\tau\left[\int_\tau^T \left(\frac{1}{2}\Phi''(Y_t)|Z_t|^2 + \Phi'(Y_t)g(t,Z_t)\right)dt\right]\ ,$$

by application of Itô's lemma. Since $g\left(t,z\right) \le a_0 + b_0|z|^2$, for some constants $a_0$, $b_0$, this provides:

$$\begin{aligned}\mathbb{E}_\tau\left[\Phi\left(Y_T\right) - \Phi\left(Y_\tau\right)\right] &\le \mathbb{E}_\tau\left[\int_\tau^T \left(\frac{1}{2}\Phi'' + b_0\Phi'\right)(Y_t)|Z_t|^2 dt\right] + a_0\mathbb{E}_\tau\left[\int_\tau^T \Phi'(Y_t)dt\right] \\ &\le -b_0\mathbb{E}_\tau\left[\int_\tau^T |Z_t|^2 dt\right] + a_0\mathbb{E}_\tau\left[\int_\tau^T \Phi'(Y_t)dt\right]\ .\end{aligned}$$

By the boundedness of $Y$ and the continuity of $\Phi$ and $\Phi'$, we deduce the existence of a constant $c_0$ such that:

$$\mathbb{E}_\tau\left[\int_\tau^T |Z_t|^2 dt\right] \le c_0\ .$$

Since $c_0$ does not depend on the arbitrary stopping time $\tau$, this provides the required result. □

## Appendix B: Proof of Proposition 2

Let $n \in \mathbb{N}$ and $(\xi,\pi) \in \mathcal{X}_0 \times \mathcal{A}_0$. We now use a verification argument as in El Karoui et al. (1997a) and Hu et al. (2005). Define the family of processes:

$$R_t^{n,\xi,\pi}(\xi',\pi')\ :=\ U\left(X_t^{0,\pi'} + B_t^{\xi'} + Y_t^{\xi^n}\right)\ ,\ t \in [0,T] \text{ for } (\xi',\pi') \in \mathcal{X}_{\theta_n}(\xi) \times \mathcal{A}_{\theta_n}(\pi)\ . \tag{23}$$

For the sake of simplicity, we will simply write $R_t^{n,\xi,\pi} := R_t^{n,\xi,\pi}(\xi',\pi')$ whenever $t \le \theta_n$ or whenever the latter quantity does not depend on $(\xi',\pi')$. Observe that, since $\psi^i$ and $Y^i$ are bounded, the process $R^{n,\xi,\pi}(\xi',\pi')$ is of class $D$ and is thus well defined and integrable. We start with the following lemma:

Lemma 2. *Assume that the coupled system of RBSDEs* (12)-(13)-(14)-(15) *has a solution with bounded processes $Y^i$. Let $n \in \mathbb{N}$, $(\xi,\pi) \in \mathcal{X}_0 \times \mathcal{A}_0$ be fixed.*

(i) *For every $(\xi',\pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)$, the process $\left\{R_t^{n,\xi,\pi}(\xi',\pi'),\ \theta_n^* \le t \le \theta'_{n+1}\right\}$ is a super-martingale and:*

$$\mathbb{E}_{\theta_n^*}\left[R_{\theta'_{n+1}}^{n,\xi,\pi}(\xi',\pi')\right] \le e^{\eta C_n'^*} R_{\theta_n^*}^{n,\xi,\pi}\ . \tag{24}$$

(ii) *Let $(\hat{\xi},\hat{\pi}) \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)$ such that:*

$$\hat{\theta}_{n+1} := \inf\left\{t \ge \theta_n^*,\ dK_t^{\xi^n} > 0\right\} \wedge T \tag{25}$$

$$\hat{\pi}_t := \pi_t^0\,\mathbf{1}_{[0,\theta_n^*)}(t) + (\Sigma_t^*)^{-1}\Pi_t\left[(\eta\Sigma_t)^{-1}\mu_t - \Sigma_t^* Z_t^{\xi^n}\right]\,\mathbf{1}_{\left[\theta_n^*,\hat{\theta}_{n+1}\right]}(t)\ . \tag{26}$$

*Then, the process $\left\{R^{n,\xi,\pi}(\hat{\xi},\hat{\pi}),\ \theta_n^* \le t < \hat{\theta}_{n+1}\right\}$ is a martingale and:*

$$\mathbb{E}_{\theta_n^*}\left[R_{\hat{\theta}_{n+1}}^{n,\xi,\pi}(\hat{\xi},\hat{\pi})\right] = e^{\eta C_n'^*} R_{\theta_n^*}^{n,\xi,\pi}\ .$$

*Proof.* (i) Since the processes $Y^i$, $\psi^i$ are bounded, and $\pi' \in \mathcal{A}_0$, we only need to check that the process $R^{n,\xi,\pi}(\xi',\pi')$ is a local super-martingale on $[\theta_n^*, \theta'_{n+1}]$. On this interval, we can decompose this process into:

$$R_t^{n,\xi,\pi}(\xi',\pi') = R_{\theta_n^*}^{n,\xi,\pi} M_t^{n,\pi}(\pi') A_t^{n,\xi,\pi}(\xi',\pi') ,$$

where $M_t^{n,\pi}(\pi')$ is a local martingale defined by $M_{\theta_n^*}^{n,\pi}(\pi') = 1$,

$$\frac{dM_t^{n,\pi}(\pi')}{M_t^{n,\pi}(\pi')} = -\eta(\pi'_t + Z_t^{\xi^n}) \cdot \Sigma_t dW_t ,$$

and $A^{n,\xi,\pi}(\xi',\pi')$ is a bounded variation process defined by $A_{\theta_n^*}^{n,\xi,\pi}(\xi',\pi') = 1$ and

$$\frac{A_t^{n,\xi,\pi}(\xi',\pi')}{A_t^{n,\xi,\pi}(\xi',\pi')} = \left( -\eta f_t^{\xi^n}(Z_t^{\xi^n}) - \eta\pi'_t \cdot \mu_t + \frac{\eta^2}{2} \left| \Sigma_t^* \left( \pi'_t + Z_t^{\xi^n} \right) \right|^2 \right) dt - \eta dB_t^{\xi'} + \eta dK_t^{\xi^n} .$$

Observing that $dB_t^{\xi'} = \psi_t^{\xi^n} dt - C_n'^* \mathbf{1}_{\{t=\theta'_{n+1}\}}$ for $\theta_n^* \le t \le \theta'_{n+1}$, that $K^{\xi^n}$ is non-decreasing, and that:

$$f_t^i(Z_t^i) = \inf_{\pi \in K} -\psi_t^i - \pi \cdot \mu_t + \frac{\eta}{2} \left| \Sigma_t^* \left( \pi + Z_t^i \right) \right|^2 , \tag{27}$$

for all $1 \le i \le M$, we deduce that $A^{n,\xi,\pi}(\xi',\pi')$ is a non-decreasing bounded variation process on $[\theta_n^*, \theta'_{n+1}]$. Therefore $R^{n,\xi,\pi}(\xi',\pi')$ is a local super-martingale, and $dA_{\theta'_{n+1}}^{n,\xi,\pi}(\xi',\pi') \ge \eta C_n'^* A_{\theta'_{n+1}}^{n,\xi,\pi}(\xi',\pi')$ implies (24).

(ii) a) In this step, we show that $R^{n,\xi,\pi}(\hat{\xi},\hat{\pi})$ is a martingale on $[\theta_n^*, \hat{\theta}_{n+1})$. Observe that the process $\hat{\pi}_t$ defined by (25) is the (unique) minimizer of the problem (27). From this and the definition of $\hat{\theta}_{n+1}$, $A_t^{n,\xi,\pi}(\hat{\xi},\hat{\pi}) = A_{\theta_n^*}^{n,\xi,\pi}$ for $t \in [\theta_n^*, \hat{\theta}_{n+1})$. Then $R_t^{n,\xi,\pi}(\xi',\pi') = R_{\theta_n^*}^{n,\xi,\pi} M_t^{n,\pi}(\pi') A_{\theta_n^*}^{n,\xi,\pi}$ is a local martingale. By Lemma 1, it follows that the process $\int_0^\cdot Z_t^i \cdot \Sigma_t dW_t$ is a BMO martingale, for all $1 \le i \le M$. In order to show that $R^{n,\xi,\pi}(\hat{\tau},\hat{\pi})$ is a martingale on $[\theta_n^*, \hat{\theta}_{n+1})$, it is sufficient to prove that the process $\int_0^\cdot \hat{\pi}_t \cdot \Sigma_t dW_t^1$ is a also a BMO martingale, as it is proved in Hu et al. (2005). Observe that, for $t \in [\theta_n^*, \hat{\theta}_{n+1}]$,

$$|\Sigma_t^* \hat{\pi}_t|^2 = \left| \Pi_t \left( \eta^{-1} \Sigma_t^{-1} \mu_t - \Sigma_t^* Z_t^{\xi^n} \right) \right|^2 \le 2\eta^{-2} \left| \Pi_t \left( \Sigma_t^{-1} \mu_t \right) \right|^2 + 2 \left| \Pi_t \left( \Sigma_t^* Z_t^{\xi^n} \right) \right|^2 .$$

We then deduce that, for all stopping time $\tau$ with values in $[\theta_n^*, \hat{\theta}_{n+1})$,

$$\mathbb{E}\left[ \int_\tau^T |\Sigma_t^* \hat{\pi}_t|^2 dt \middle| \mathcal{F}_\tau \right] \le c_1 + 2\mathbb{E}\left[ \int_\tau^T \left| \Pi_t \left( \Sigma_t^* Z_t^{\xi^n} \right) \right|^2 dt \middle| \mathcal{F}_\tau \right] \le c_1 + 2c_0 ,$$

for some constant $c_1$. Since the latter bound does not depend on the arbitrary stopping time $\tau$, this shows that the process $\int_0^\cdot \hat{\pi}_t \cdot \Sigma_t dW_t$ is a BMO martingale on $[\theta_n^*, \hat{\theta}_{n+1}]$.

(b) We now prove that $\hat{\pi}$ is in $\mathcal{A}_0$. On $[0, \theta_n^*]$, $\hat{\pi}$ is equal to $\pi^0 \in \mathcal{A}_0$. The BMO martingale property of $\int_0^\cdot \hat{\pi}_t \cdot \Sigma_t dW_t$ on $[\theta_n^*, \hat{\theta}_{n+1}]$ implies that $\mathbb{E}\left[ \int_{\theta_n^*}^{\hat{\theta}_{n+1}} |\Sigma_t^* \hat{\pi}_t|^2 dt \right] < \infty$, and therefore $\mathbb{E}\left[ \int_0^T |\Sigma_t^* \hat{\pi}_t|^2 dt \right] < \infty$. Using now the BMO martingale property of $\int_0^\cdot \overline{Z}_t^{\xi^n} \cdot \Sigma_t dW_t$, we prove that $M^{n,\pi}(\hat{\pi})$ is a uniformly integrable martingale on $[\theta_n^*, \hat{\theta}_{n+1}]$. As $A^{n,\xi,\pi}(\hat{\xi},\hat{\pi})$ is bounded on $[\theta_n^*, \hat{\theta}_{n+1}]$, $R^{n,\xi,\pi}(\hat{\xi},\hat{\pi})$ is a uniformly integrable family on $[\theta_n^*, \hat{\theta}_{n+1}]$, and so is $e^{-\eta X_t^{0,\hat{\pi}}}$. Hence $\hat{\pi}$ is an admissible portfolio.

(c) We complete the proof by noticing that at time $\hat{\theta}_{n+1}$: $A_{\hat{\theta}_{n+1}}^{n,\xi,\pi}(\hat{\xi},\hat{\pi}) = e^{\eta C_n'^*} A_{\theta_n^*}^{n,\xi,\pi}$. □

We then deduce the proposition:

PROPOSITION 8. *Let $n \in \mathbb{N}$, $(\xi,\pi) \in \mathcal{X}_0 \times \mathcal{A}_0$ be fixed. Then we have:*

$$\underset{(\xi',\pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^{n+1}}\right)_{\theta'_{n+1}}\right] = R^{n,\xi,\pi}_{\theta_n^*} \ .$$

*Proof.* Let $(\xi',\pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)$. Then, $\overline{Y}^j_{\theta'_{n+1}} \le C_{\xi^n,j} + Y^{\xi^n}_{\theta'_{n+1}}$ for all $j \neq \xi^n$, together with the supermartingale property of $R^{n,\xi,\pi}(\xi',\pi')$, yield:

$$\begin{aligned} \mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^{n+1}}\right)_{\theta'_{n+1}}\right] &\le \mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + C'^*_n + Y^{\xi^n}\right)_{\theta'_{n+1}}\right] \\ &\le \mathbb{E}_{\theta_n^*}\left[e^{-\eta C'^*_n} R^{n,\xi,\pi}_{\theta'_{n+1}}(\xi',\pi')\right] \\ &\le R^{n,\xi,\pi}_{\theta_n^*} \ . \end{aligned} \tag{28}$$

Thus,

$$\underset{(\xi',\pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^{n+1}}\right)_{\theta'_{n+1}}\right] \le R^{n,\xi,\pi}_{\theta_n^*} \ .$$

The converse inequality is obtained by observing that (28) is in fact an equality for the choice of a pair $(\hat{\xi},\hat{\pi})$ characterized in the previous lemma. □

We can then turn to the proof of Proposition 2:

*Proof of Proposition 2.* Since $V_0(x) = e^{-\eta x} V_0(0)$, we only deal with the case of a zero initial capital. Let $(\xi,\pi)$ be a pair of management-investment strategies in $\mathcal{X}_0 \times \mathcal{A}_0$. Results from Hu et al. (2005) to the processes $\overline{Y}^i$, $1 \le i \le M$, on intervals of the form $[\theta_n, \theta_n^*]$ allow us to derive the following properties:

$$\underset{(\xi',\pi') \in \mathcal{X}_{\theta_n}(\xi) \times \mathcal{A}_{\theta_n}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n}\left[R^{n,\xi,\pi}_{\theta_n^*}(\xi',\pi')\right] = R^{n,\xi,\pi}_{\theta_n} \ ,$$

the argument of the supremum depending only on $\pi'$ and where the supremum is attained for $\pi' = \hat{\pi}$. Using this result together with Lemma 2 we get:

$$\mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^{n+1}}\right)_{\theta_{n+1}}\right] \le U\left(X^{0,\pi} + B^{\xi} + Y^{\xi^n}\right)_{\theta_n^*} \ ,$$

thus

$$\mathbb{E}_{\theta_n}\left[U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^{n+1}}\right)_{\theta_{n+1}}\right] \le U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n}\right)_{\theta_n} \ . \tag{29}$$

Using the fact that $\xi$ has a finite number of switches almost surely ($N(\xi) < \infty$ a.s.), a direct iteration of these inequalities implies:

$$\mathbb{E}\left[U\left(X_T^{0,\pi} + B_T^{\xi} + \chi\right)\right] \le U\left(X_{\theta_0}^{0,\pi} + B_{\theta_0}^{\xi} + \overline{Y}^1_{\theta_0}\right) = U(\overline{Y}^1_0) \ , \tag{30}$$

We therefore get $V_0(0) \leq -e^{-\eta \overline{Y}^1_0}$. The converse inequality is obtained by observing that, first, (29) is in fact an equality for the choice of the management-investment strategy $(\hat{\xi}, \hat{\pi})$. Second, $(\hat{\xi}, \hat{\pi})$ is indeed an admissible strategy since:

$$\mathbb{E}\left[U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^{n+1}}\right)_{\theta_{n+1}}\right] = U\left(\overline{Y}^1_0\right) \ ,$$

showing that $\mathbb{P}(N(\hat{\xi}) = \infty) > 0$ is not possible. □

## Appendix C: Proofs of Section 5

*Proof of Proposition 3.* Consider the sequence of management strategy $\hat{\xi} \in \mathcal{X}_{\theta_n}(\xi)$ defined by:

$$\begin{aligned}
\hat{\theta}_{k+1} &= \inf\left\{t \geq \overline{\delta}_{\hat{\xi}_k}(\hat{\theta}_k),\ Y_t^{\hat{\xi}^k, n+m-k} = \max_{j \neq \hat{\xi}^k}\left\{\overline{Y}_t^{j,n+m-k-1} - C_{\hat{\xi}^k, j}\right\}\right\} \\
\hat{\xi}^{k+1} &= \min\left\{j \neq \hat{\xi}^k,\ Y_{\hat{\theta}_{k+1}}^{\hat{\xi}^k, n+m-k} = \overline{Y}_{\hat{\theta}_{k+1}}^{j,n+m-k-1} - C_{\hat{\xi}^k, j}\right\}
\end{aligned}$$

for $n \leq k \leq n+m-1$ and $\hat{\theta}_{n+m+1} = T$. Consider also the investment strategy $\hat{\pi} \in \mathcal{A}_{\theta_n}(\xi)$ defined as:

$$\begin{aligned}
\hat{\pi}_t &= (\Sigma_t^*)^{-1}\Pi_t\left((\eta\Sigma_t)^{-1}\mu_t - \Sigma_t^* Z_t^{\hat{\xi}^k, n+m-k}\right) \text{ for } \ \hat{\theta}_k^* \leq t \leq \hat{\theta}_{k+1} \\
\hat{\pi}_t &= (\Sigma_t^*)^{-1}\Pi_t\left((\eta\Sigma_t)^{-1}\mu_t - \Sigma_t^* \overline{Z}_t^{\hat{\xi}^k, n+m-k}\right) \text{ for } \ \hat{\theta}_k \leq t \leq \hat{\theta}_k^* \ .
\end{aligned}$$

Following the same argument as in Section B, we prove that the processes $U(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^k, n+m-k})$ and $U(X^{0,\pi'} + B^{\xi'} + Y^{\xi'^k, n+m-k})$ defined respectively on $[\theta'_k, \theta'^*_k]$ and $[\theta'^*_k, \theta'_{k+1}]$ are super-martingales for every $(\xi', \pi') \in \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n}(\pi)$, and martingales with $(\pi', \xi') = (\hat{\pi}, \hat{\xi})$. The only difference with Section B lies in the fact that the number of switches is bounded by $m$. This implies:

$$\begin{aligned}
U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n, m}\right)_{\theta_n} &= \mathbb{E}_{\theta_n}\left[U\left(X^{0,\hat{\pi}} + B^{\hat{\xi}} + Y^{\hat{\xi}^n, m}\right)_{\hat{\theta}_n^*}\right] \\
&= \mathbb{E}_{\theta_n}\left[U\left(X^{0,\hat{\pi}} + B^{\hat{\xi}} + \overline{Y}^{\hat{\xi}^{n+1}, m-1}\right)_{\hat{\theta}_{n+1}}\right] \ .
\end{aligned}$$

Direct iteration of this argument provides:

$$U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n, m}\right)_{\theta_n} = \mathbb{E}_{\theta_n}\left[U\left(X_T^{0,\hat{\pi}} + B_T^{\hat{\xi}} + \chi\right)\right] \ .$$

On the other hand, for any management-investment strategies $(\xi', \pi') \in \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n}(\pi)$, the same super-martingale argument yields:

$$U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n, m}\right)_{\theta_n} \geq \mathbb{E}_{\theta_n}\left[U\left(X_T^{0,\pi'} + B_T^{\xi'} + \chi\right)\right],$$

which completes the proof. □

*Proof of Corollary 2.* Notice that $\mathcal{X}^{n,m}(\xi) \subset \mathcal{X}^{n,m+1}(\xi)$, so $\overline{Y}_t^{i,n} \leq \overline{Y}_t^{i,n+1}$, a.s. Since $\overline{Y}^{i,n}$ and $\overline{Y}^{i,n+1}$ are continuous processes, this implies that $\overline{Y}^{i,n} \leq \overline{Y}^{i,n+1}$, a.s. The comparison principle for quadratic reflected BSDE (Theorem 3.2 in Kobylanski et al. (2002)) shows that $Y_t^{i,n} \leq Y_t^{i,n+1}$ a.s. for all $t$ and, by continuity $Y^{i,n} \leq Y^{i,n+1}$ a.s. □

*Proof of Proposition 4.* By Corollary 2, the sequence $(Y^{i,n})$ is non-decreasing. Then it converges pointwise to a process $\widetilde{Y}^i$. We now provide uniform bounds for this sequence. Let $(\xi,\pi)$ be a pair of management-investment strategies in $\mathcal{X}_0 \times \mathcal{A}_0$. From the proof of Proposition 3, we also deduce:

$$\begin{aligned} U\left(X^{0,\pi} + B^{\xi} + Y^{\xi^n,m}\right)_{\theta_n^*} &= \underset{(\xi',\pi')\in\mathcal{X}^{n,m}(\xi)\times\mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} + B_T^{\xi'} + \chi\right)\right] \\ &\leq \underset{(\xi',\pi')\in\mathcal{X}^{n,m}(\xi)\times\mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} + \int_0^T \max_j \psi_t^j dt + \chi\right)\right] \\ &\leq \underset{\pi'\in\mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} + \overline{\kappa}T + \overline{\chi}\right)\right] \end{aligned}$$

where $\overline{\kappa}$ is a bound for $\max_j |\psi^j|$, and $\overline{\chi}$ is an upper bound for $|\chi|$. Following Hu et al. (2005), we get:

$$U\left(X_t^{0,\pi} + \int_0^t \psi_u^i du + \overline{\Upsilon}_t^i\right) = \underset{\pi'\in\mathcal{A}_t(\pi)}{\text{ess. sup}}\mathbb{E}_t\left[U\left(X_T^{0,\pi'} + \int_0^T \psi_u^i du + \chi\right)\right] \ ,$$

where $\overline{\Upsilon}_t^i = \mathcal{E}_{t,T}^g\left[\chi + \int_t^T \left(\frac{1}{2\eta}\left|\Pi_{\Sigma_u^* K}(\Sigma_u^{-1}\mu_u)\right|^2 + \psi_u^i\right) du\right]$, thus:

$$U\left(X^{0,\pi} + \int_0^{\cdot} \psi_u^i du + \overline{\Upsilon}^i\right)_{\theta_n^*} \geq \underset{\pi'\in\mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} - \overline{\kappa}T - \overline{\chi}\right)\right] \ ,$$

and we end up with:

$$U\left(X^{0,\pi} + B^{\xi} + Y^{\xi^n,m}\right)_{\theta_n^*} \leq U\left(2(\overline{\kappa}T + \overline{\chi}) + X^{0,\pi} + \int_0^{\cdot} \psi_u^{\xi^n} du + \overline{\Upsilon}^{\xi^n}\right)_{\theta_n^*} \ .$$

This being true for all management strategy $\xi$, we obtain:

$$Y_t^{i,n} \leq 2\overline{\kappa}T + 2\overline{\xi} + \overline{\Upsilon}_t^i \ .$$

On the other hand, $Y^{0,n} \geq Y^{i,0} = \overline{\Upsilon}_t^i$ because the sequence $Y^{i,n}$ is non-decreasing. Since $\overline{\Upsilon}^i \in \mathcal{H}^\infty(\mathbb{R})$, as a solution of a quadratic BSDE with bounded terminal condition, the sequence $(Y^{i,n})_{n\geq 0}$ is uniformly bounded by some constant. In particular, this implies that $\widetilde{Y}^i \in \mathcal{H}^\infty(\mathbb{R})$. Using relation (17), we deduce that the sequences $(\overline{Y}^{i,n})_{n\geq 0}$, $1 \leq i \leq M$, are uniformly bounded.

We are thus in the conditions of proposition 2.4 in Kobylanski (2000), and we conclude that the sequences $(\overline{Y}^{i,n})_{n\geq 0}$, $1 \leq i \leq M$, converge to processes $\check{Y}^i \in \mathcal{H}_0^\infty(\mathbb{R})$. We are also in the conditions of theorem 4 in Kobylanski et al. (2002) and we conclude that $(Y^{i,n})_{n\geq 0}$ converges uniformly on $[0,T]$ to $\widetilde{Y}^i \in \mathcal{H}_0^\infty(\mathbb{R})$, $(Z^{i,n})_{n\geq 0}$ converges to $\widetilde{Z}^i \in \mathcal{H}_0^2(\mathbb{R}^N)$ and $(K^{i,n})_{n\geq 0}$ converges uniformly on $[0,T]$ to $\widetilde{K}^i \in \mathcal{J}(\mathbb{R})$. Moreover $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{K}^i)$ satisfies the backward system (12)-(13)-(14)-(15). □

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