Pricing and hedging under delay constraints

Huyên PHAM

Laboratoire de Probabilités et Modèles Aléatoires
Universities Paris 6-Paris 7
CREST
and Institut Universitaire de France

joint work with B. BRUDER, Société Générale and Paris 7

Séminaire FIME
Paris Dauphine, CREST, EDF
January 15, 2008

Basic motivations

- Pricing and hedging of an option on hedge funds :
  - Hedge funds : pooled investment vehicle administered by professional managers
  - Illiquid assets in hedge funds : debts, options ...
  - The hedge fund manager needs time to find a counterpart to trade these assets
  - To buy or sell shares of hedge funds, investors must declare their orders one to three months before they are effectively executed
  - Once the order is passed, its execution is mandatory

▶ Execution delay → liquidity risk

- Implementation delay in financial decision-making problems
  - regulatory reasons, heavy preparatory work
  - e.g. management of a power plant

→ operational risk

- Our goal : provide a general but **tractable** mathematical framework for studying quantitatively the impact of execution delay.

▶ Impulse control with execution delay

Controlled process

- In absence of control, state system in $\mathbb{R}^d$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$:

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s$$

- **Impulse control with time lag** : a double sequence $(\tau_i, \xi_i)_{i \geq 1}$,
  - decision times : τ_i stopping times s.t. $\tau_{i+1} - \tau_i \geq h$, $h > 0$ minimal time lag between two interventions
  - impulse values : ξ_i valued in E (compact subset) and $\mathcal{F}_{\tau_i}$-measurable (based on information available at τ_i)

Controlled process

- In absence of control, state system in $\mathbb{R}^d$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$:

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s$$

- **Impulse control with time lag** : a double sequence $(\tau_i, \xi_i)_{i \geq 1}$,
  - decision times : τ_i stopping times s.t. $\tau_{i+1} - \tau_i \geq h$, $h > 0$ minimal time lag between two interventions
  - impulse values : ξ_i valued in E (compact subset) and $\mathcal{F}_{\tau_i}$-measurable (based on information available at τ_i)

- **Execution delay on the system** : the intervention ξ_i decided at τ_i is executed at time $\tau_i + \delta$, moving the system from

$$X_{(\tau_i+\delta)^-} \quad \rightarrow \quad X_{\tau_i+\delta} = \Gamma(X_{(\tau_i+\delta)^-}, \xi_i),$$

In the sequel, we set : $\delta = mh$, with $m \in \mathbb{N}$ (for simplicity of notations).

Control objective

- Total profit over a finite horizon $T < \infty$, associated to an impulse control $\alpha = (\tau_i, \xi_i)_{i \geq 1} \in \mathcal{A}$:

$$\Pi(\alpha) = \int_0^T f(X_t)dt + g(X_T) + \sum_{\tau_i + mh \leq T} c(X_{(\tau_i+mh)^-}, \xi_i),$$

f running profit function on $\mathbb{R}^d$, g terminal profit function on $\mathbb{R}^d$, c executed cost function on $\mathbb{R}^d \times E$.

▶ **Control problem** :

$$V_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\Pi(\alpha)\Big].$$

Financial example

• S asset price (e.g. spot price of a hedge fund) :

$$dS_t = \beta(S_t)dt + \gamma(S_t)dW_t$$

• Y_t cumulated number of shares in asset, Z_t amount of cash held by investor at time t : in absence of trading

$$dY_t = 0, \quad dZ_t = rZ_t dt \quad (r \text{ interest rate}).$$

• Portfolio strategy : $(\tau_i, \xi_i)_i$, where ξ_i represents the number of shares purchased or selled at time τ_i, but executed at $\tau_i + mh$.

Financial example

• State process $X = (S, Y, Z)$

▶ when the order (τ_i, ξ_i) is executed at time $\tau_i + mh$, the system moves from $X_{(\tau_i+mh)^-}$ to $\Gamma(X_{(\tau_i+mh)^-}, \xi_i)$ given by :

$$\begin{aligned} S_{\tau_i+mh} &= S_{(\tau_i+mh)^-} \quad (\text{or} \quad P(S_{(\tau_i+mh)^-}, \xi_i) \quad \text{if large investor}) \\ Y_{\tau_i+mh} &= Y_{(\tau_i+mh)^-} + \xi_i \\ Z_{\tau_i+mh} &= Z_{(\tau_i+mh)^-} - \xi_i S_{\tau_i+mh}. \end{aligned}$$

• Optimal investment : maximize the expected utility of terminal wealth

$$\mathbb{E}\Big[U(Z_T + Y_T S_T)\Big].$$

Financial example

- State process $X = (S, Y, Z)$

▶ when the order (τ_i, ξ_i) is executed at time $\tau_i + mh$, the system moves from $X_{(\tau_i+mh)^-}$ to $\Gamma(X_{(\tau_i+mh)^-}, \xi_i)$ given by :

$$\begin{aligned} S_{\tau_i+mh} &= S_{(\tau_i+mh)^-} \quad (\text{or} \quad P(S_{(\tau_i+mh)^-}, \xi_i) \quad \text{if large investor}) \\ Y_{\tau_i+mh} &= Y_{(\tau_i+mh)^-} + \xi_i \\ Z_{\tau_i+mh} &= Z_{(\tau_i+mh)^-} - \xi_i S_{\tau_i+mh}. \end{aligned}$$

- Optimal investment with option delivery (indifference pricing) : maximize the expected utility

$$\mathbb{E}\Big[U(Z_T + Y_T S_T - g(S_T))\Big].$$

Some references : stochastic systems with memory

- Elsanosi, Larssen (01), Gozzi, Marinelli (04),

$\longrightarrow$ infinite dimensional system : HJB in Hilbert space

$\longrightarrow$ interesting analysis but abstract theoretical results

$\longrightarrow$ except in very special cases, it is usually not practical

Some references : impulse control

• PDE variational formulation of impulse control problems :
Bensoussan-Lions (82) : no delay $m = 0$

Bar-Ilan, Sulem (95), Oksendal, Sulem (06) : delay but with particular controlled process (Lévy process for X and additive intervention operator Γ) on infinite horizon

• Probabilistic calculation for particular threshold strategy :
Bayraktar, Egami (06) : $\delta = h$ i.e $m = 1$, infinite horizon and impulse value chosen at time of execution, i.e. ξ_i $\mathcal{F}_{\tau_i + h}$-measurable

• Financial applications : liquidity risk and execution delay ($m = 1$)
Subramanian, Jarrow (01), Alvarez, Keppo (02), Keppo, Peura (06)

New features and contributions in our model

• General diffusion framework on finite horizon

• New orders can be decided between the period of execution delay, i.e. $\delta = mh \geq h$ (delay larger than time lag intervention)

Main goal

• Obtain a unique PDE characterization of the original control problem

• Provide an implementable algorithm

• Measure impact of execution delay

Markovian setting

• Extend definition of control problem V_0 to general initial conditions :

▶ **Important issue** : the state process X is not Markovian in itself :

given an impulse control, the state of the system is not only defined by its current state value at time t but also by the pending orders : the orders not yet executed, i.e. decided in $(t - mh, t]$.

Remark : Due to the time decision lag h, the number of pending orders is $\leq m$.

▶ **How to make the state process Markovian!**

Some notations (I)

- Set of k ($k = 0, \ldots, m$) pending orders at time $t \in [0, T]$:

$$P_t(k) = \Big\{ p = (t_i, e_i)_{1 \leq i \leq k} \in ([0, T] \times E)^k \ : \ t_i - t_{i-1} \geq h, \ \text{and} \ \ t - mh < t_i \leq t \Big\},$$

- State domains for $k = 0, \ldots, m$:

$$\mathcal{D}_k = \big\{ (t, x, p) \ : (t, x) \in [0, T] \times \mathbb{R}^d, \ p \in P_t(k) \big\}.$$

Remark

For $k = 0$, $P_t(0) = \emptyset$, and $\mathcal{D}_0 = [0, T] \times \mathbb{R}^d$.

Some notations (II)

• Set of admissible controls from a given pending order $p = (t_i, e_i)_{1 \leq i \leq k} \in P_t(k)$:

$$\mathcal{A}_{t,p} = \Big\{\alpha = (\tau_i, \xi_i)_{i \geq 1} \in \mathcal{A} \ : (\tau_i, \xi_i) = (t_i, e_i), \ i = 1, \ldots, k \text{ and } \tau_{k+1} \geq t\Big\},$$

▶ **Finite dimensional controlled Markov process** :
Given $(t, x, p) \in \mathcal{D}_k$, $k \leq m$, $\alpha \in \mathcal{A}_{t,p}$, we denote $\{X_s^{t,x,p,\alpha}, t \leq s \leq T\}$ the controlled process starting from $X_t = x$, with pending order p, and controlled by α.

Control objective : dynamic version

- Criterion : for $(t,x,p) \in \mathcal{D}_k$, $k \leq m$, $\alpha = (\tau_i, \xi_i)_i \in \mathcal{A}_{t,p}$,

$$
\begin{aligned}
J_k(t,x,p,\alpha) &= \mathbb{E}\Big[\int_t^T f(X_s^{t,x,p,\alpha}) ds + g(X_T^{t,x,p,\alpha}) \\
&\quad + \sum_{t < \tau_i + mh \leq T} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-}, \xi_i) \Big],
\end{aligned}
$$

- Corresponding **value functions** :

$$
v_k(t,x,p) = \sup_{\alpha \in \mathcal{A}_{t,p}} J_k(t,x,p,\alpha), \quad k \leq m, \ (t,x,p) \in \mathcal{D}_k.
$$

Remark

$V_0 = v_0(0, X_0, \emptyset)$.

Assumptions

- **(H1)** f, g, c and Γ are continuous and satisfy a linear growth condition on x
- **(H2)** $g(x) \geq g(\Gamma(x,e)) + c(x,e)$, for all $(x,e)\ \mathbb{R}^d \times E$.

Remarks

- Economic interpretation of **(H2)** satisfied in financial examples

- If **(H2)** is not satisfied, the value functions may be discontinuous

Example : $b = \sigma = f = g = 0$, $c(x,e) = 1$. Then,

$$v_0(t,x) = \begin{cases} 0, & T - mh < t \leq T \\ i, & T-(m+i)h < t \leq T-(m+i-1)h,\ i \geq 1. \end{cases}$$

$\rightarrow$ Discontinuities of v_0 at $t = T-(m+i-1)h$, $i \geq 1$.

State domain partition

- Partition the set of pending orders into $P_t(k) = P_t^1(k) \cup P_t^2(k)$:

$$
\begin{aligned}
P_t^1(k) &= \Big\{ p = (t_i, e_i)_{1 \leq i \leq k} \in P_t(k) \ : t_k > t - h \Big\} \\
P_t^2(k) &= \Big\{ p = (t_i, e_i)_{1 \leq i \leq k} \in P_t(k) \ : t_k \leq t - h \Big\}.
\end{aligned}
$$

and define the corresponding state domains $\mathcal{D}_k = \mathcal{D}_k^1 \cup \mathcal{D}_k^2$:

$$
\begin{aligned}
\mathcal{D}_k^1 &= \big\{ (t, x, p) \ : (t, x) \in [0, T] \times \mathbb{R}^d, \ p \in P_t^1(k) \big\} \\
\mathcal{D}_k^2 &= \big\{ (t, x, p) \ : (t, x) \in [0, T] \times \mathbb{R}^d, \ p \in P_t^2(k) \big\}.
\end{aligned}
$$

State domain of no possible order decision

- If $(t, x, p) \in \mathcal{D}_k^1$, the controller cannot take action in $[t, t + dt]$. Only the diffusion X operates

State domain of no possible order decision

• If $(t, x, p) \in \mathcal{D}_k^1$, the controller cannot take action in $[t, t + dt]$. Only the diffusion X operates

▶ **Linear PDE's on $\mathcal{D}_k^1$, $k = 1, \ldots, m$:**

$$-\frac{\partial v_k}{\partial t} - \mathcal{L} v_k - f \quad = \quad 0 \qquad \text{on } \mathcal{D}_k^1$$

where

$$\mathcal{L}\varphi \quad = \quad b(x).D_x\varphi + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x)D_x^2\varphi)$$

is the generator of the diffusion X.

State domain of possible order decision

- If $(t, x, p) \in \mathcal{D}_k^2$, the controller has the choice of :
  - doing nothing, i.e. let the diffusion X operate on $[t, t+dt] \rightarrow$ linear PDE's
  - passing immediately an order (t, e), so that the pending orders switch from p (with cardinal k) to $p \cup (t, e)$ (with cardinal $k+1$) $\rightarrow$

$$v_k(t, x, p) \quad \geq \quad \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e))$$

State domain of possible order decision

- If $(t, x, p) \in \mathcal{D}_k^2$, the controller has the choice of :
  - doing nothing, i.e. let the diffusion X operate on $[t, t+dt] \rightarrow$ linear PDE's
  - passing immediately an order (t, e), so that the pending orders switch from p (with cardinal k) to $p \cup (t, e)$ (with cardinal $k+1$) $\rightarrow$

$$v_k(t, x, p) \quad \geq \quad \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e))$$

▶ **Variational inequalities on $\mathcal{D}_k^2$, $k = 0, \ldots, m-1$:**

$$\min \Big[-\frac{\partial v_k}{\partial t} - \mathcal{L} v_k - f \ ,$$

$$v_k(t, x, p) - \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e)) \Big] \quad = \quad 0 \qquad \text{on} \quad \mathcal{D}_k^2$$

Dynamic programming system

- PDE system for the value functions v_k, $k = 0, \ldots, m$:

$$
\begin{aligned}
-\frac{\partial v_k}{\partial t} - \mathcal{L} v_k - f &= 0 \quad \text{on } \mathcal{D}_k^1, \quad k \geq 1, \\
\min\Big[-\frac{\partial v_k}{\partial t} - \mathcal{L} v_k - f \, , & \\
v_k(t,x,p) - \sup_{e \in E} v_{k+1}(t,x,p \cup (t,e)) \Big] &= 0 \quad \text{on } \mathcal{D}_k^2, \quad k \leq m-1.
\end{aligned}
$$

Time-boundary conditions

- (Standard) terminal condition at T :

$$v_k(T^-, x, p) = g(x), \quad x \in \mathbb{R}^d, \ p \in P_T(k), \ k = 1, \ldots, m.$$

Time-boundary conditions

- (Standard) terminal condition at T :

$$v_k(T^-, x, p) \;=\; g(x), \quad x \in \mathbb{R}^d,\ p \in P_T(k),\ k = 1, \ldots, m.$$

- Non standard condition on the time-boundary of $\mathcal{D}_k \leftrightarrow$ execution of the first pending order (t_1, e_1) of $p = (t_i, e_i)_{1 \leq i \leq k}$:

$$v_k((t_1 + mh)^-, x, p) \;=\; c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-),$$

where $p_- = p \setminus (t_1, e_1) = (t_i, e_i)_{2 \leq i \leq k}$.

(Technical difficulty due to continuity issue for v_{k-1}).

Non standard features

- Form of the domain $\mathcal{D}_k = \mathcal{D}_k^1 \cup \mathcal{D}_k^2 = \{(t,x,p) \; : (t,x) \in [0,T] \times \mathbb{R}^d, \; p \in P_t(k)\}$
- Coupled system both on the PDE and on the boundary conditions :
  - v_k depends on v_{k+1} on the variational inequality
  - v_{k+1} depends on v_k via a time-boundary condition
- Discontinuity of the differential operator for v_k
  - linear PDE on $\mathcal{D}_k^1$
  - free-boundary problem on $\mathcal{D}_k^2$

Main theoretical result

Theorem

The family of value functions v_k, $k = 0, \ldots, m$, is the unique viscosity solution to the PDE system, satisfying the time-boundary conditions, a linear growth condition on x, and

$$v_k(t, x, p) \geq \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e)), \ (t, x, p) \in \mathcal{D}_k, \ t = t_k + h.$$

Moreover, v_k is continuous on $\mathcal{D}_k$.

(Short) Elements of Proof

• Viscosity properties : as usual, consequences of a suitable version of **dynamic programming principle**

• Uniqueness and comparison principles : more delicate!
In addition to usual dedoubling variables techniques and Ishii's lemma, arguments in the proofs involve **backward and forward iterations on the domains and value functions** due to the coupling.

Initialization phase

First step of the algorithm based on the following remark :

• An order decided after $T - mh$ is executed after T, and so does not influence the state process X_t for $t \leq T$.

Initialization phase

First step of the algorithm based on the following remark :

• An order decided after $T - mh$ is executed after T, and so does not influence the state process X_t for $t \leq T$.

▶ Therefore, if $(t, x, p) \in \mathcal{D}_k$ is s.t. the pending order $p = (t_i, e_i)_{1 \leq i \leq k} \in \Theta_k \times E^k$ satisfies : $t_1 > T - mh$, i.e. all the pending orders are executed after T, then

$$v_k(t, x, p) = \mathbb{E}\Big[\int_t^T f(X_s^{t,x,0}) ds + g(X_T^{t,x,0}) \Big],$$

which is easily computable.

Step n

- For $k = 1, \ldots, m$, we introduce the increasing sequence of sets :

$$\begin{aligned} \mathcal{D}_k(n) &= \Big\{(t, x, p) \in \mathcal{D}_k \ : t_1 > T - nh\Big\}, \\ N &= \inf\{n \geq 1 \ : T - nh < 0\}. \end{aligned}$$

Step n

- For $k = 1, \ldots, m$, we introduce the increasing sequence of sets :

$$\mathcal{D}_k(n) = \Big\{(t, x, p) \in \mathcal{D}_k \ : t_1 > T - nh\Big\},$$
$$N = \inf\{n \geq 1 \ : T - nh < 0\}.$$

▶ From the initialization phase, we know the value of v_k on $\mathcal{D}_k(m)$

Step n

- For $k = 1, \ldots, m$, we introduce the increasing sequence of sets :

$$\mathcal{D}_k(n) = \Big\{(t, x, p) \in \mathcal{D}_k \ : t_1 > T - nh\Big\},$$
$$N = \inf\{n \geq 1 \ : T - nh < 0\}.$$

- From the initialization phase, we know the value of v_k on $\mathcal{D}_k(m)$
- $\mathcal{D}_k(N) = \mathcal{D}_k$

Step n

- For $k = 1, \ldots, m$, we introduce the increasing sequence of sets :

$$\mathcal{D}_k(n) = \Big\{(t, x, p) \in \mathcal{D}_k \ : t_1 > T - nh\Big\},$$
$$N = \inf\{n \geq 1 \ : T - nh < 0\}.$$

- From the initialization phase, we know the value of v_k on $\mathcal{D}_k(m)$
- $\mathcal{D}_k(N) = \mathcal{D}_k$
- We shall compute v_k on $\mathcal{D}_k(n)$ by **forward induction on** $n = m, \ldots, N$.

From step n to $n+1$

- *Induction hypothesis at step n* : we know the values of v_k, $k = 0, \ldots, m$, on $\mathcal{D}_k(n)$

▶ *Step $n \rightarrow n+1$* : Computation of v_k, $k = 0, \ldots, m$, on $\mathcal{D}_k(n+1)$

- by **backward recursion on** k!

From step n to $n+1$: $k = m$

- Computation of v_m on $\mathcal{D}_m(n+1)$:

From step n to $n+1$: $k = m$

- Computation of v_m on $\mathcal{D}_m(n+1)$:
  - v_m satisfies the linear PDE

$$-\frac{\partial v_m}{\partial t} - \mathcal{L} v_m - f = 0, \quad \text{on } \mathcal{D}_m(n+1)$$

From step n to $n+1$: $k = m$

- Computation of v_m on $\mathcal{D}_m(n+1)$:
  - v_m satisfies the linear PDE

$$-\frac{\partial v_m}{\partial t} - \mathcal{L}v_m - f \quad = \quad 0, \quad \text{on } \mathcal{D}_m(n+1)$$

  - together with the boundary data of $\mathcal{D}_m(n+1)$

$$v_m((t_1+mh)^-, x, p) \quad = \quad c(x, e_1) + v_{m-1}(t_1+mh, \Gamma(x, e_1), p_-).$$

▶ Notice that since $t_1 > T-(n+1)h$, then $t_2 > T-nh$, and so $p_- = (t_i, e_i)_{2 \leq i \leq m}$ is s.t. $(t_1+mh, \Gamma(x, e_1), p_-) \in \mathcal{D}_{m-1}(n)$
$\longrightarrow$ $v_{m-1}(t_1+mh, \Gamma(x, e_1), p_-)$ is known from step n

From step n to $n+1$: $k = m$

- Computation of v_m on $\mathcal{D}_m(n+1)$:

▶ Linear Feynman-Kac (F-K) representation

$$v_m(t,x,p) = \mathbb{E}\Big[\int_t^{t_1+mh} f(X_s^{t,x,0})ds + c(X_{t_1+mh}^{t,x,0}, e_1) + v_{m-1}(t_1+mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\Big].$$

From step n to $n+1$: $k+1 \longrightarrow k$

- *Recursion hypothesis at order* $k+1$: we know the values of v_{k+1} on $\mathcal{D}_{k+1}(n+1)$.

▶ Computation of v_k on $\mathcal{D}_k(n+1)$:

From step n to $n+1$: $k+1 \longrightarrow k$

- *Recursion hypothesis at order* $k+1$: we know the values of v_{k+1} on $\mathcal{D}_{k+1}(n+1)$.

▶ Computation of v_k on $\mathcal{D}_k(n+1)$:

- Known boundary data of $\mathcal{D}_k(n+1)$ from step n

$$v_k((t_1+mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1+mh, \Gamma(x, e_1), p_-).$$

From step n to $n+1$: $k+1 \rightarrow k$

- *Recursion hypothesis at order* $k+1$: we know the values of v_{k+1} on $\mathcal{D}_{k+1}(n+1)$.

▶ Computation of v_k on $\mathcal{D}_k(n+1)$:

- Known boundary data of $\mathcal{D}_k(n+1)$ from step n

$$v_k((t_1+mh)^-, x, p) \;=\; c(x, e_1) + v_{k-1}(t_1+mh, \Gamma(x, e_1), p_-).$$

- Depending on whether $(t, x, p) \in \mathcal{D}_k^1$ or $\mathcal{D}_k^2$, the PDE for v_k is either linear or a variational inequality with obstacle

$$\sup_{e \in E} v_{k+1}(t, x, p \cup (t, e)),$$

which is known from recursion hypothesis at order $k+1$.

Summary

- Computation of the family $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k$:

Summary

- Computation of the family $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k$:

▶ **Initialization :** Linear F-K computation of $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k(m)$

Summary

- Computation of the family $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k$:

▶ **Initialization :** Linear F-K computation of $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k(m)$

▶ **Step** $n \rightarrow n + 1$ (from $n = m$ to $n = N$) :

Computation of $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k(n + 1)$ by backward recursion from $k = m$ to 0 :

Summary

- Computation of the family $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k$:

▶ **Initialization :** Linear F-K computation of $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k(m)$

▶ **Step** $n \to n+1$ (from $n = m$ to $n = N$) :

Computation of $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k(n+1)$ by backward recursion from $k = m$ to 0 :

- Initialization : Linear F-K computation of v_m on $\mathcal{D}_m(n+1)$ from step n

Summary

- Computation of the family $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k$:

▶ **Initialization :** Linear F-K computation of $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k(m)$

▶ **Step** $n \to n+1$ (from $n = m$ to $n = N$) :

Computation of $\{v_k, k = 0, \ldots, m\}$ on $\mathcal{D}_k(n+1)$ by backward recursion from $k = m$ to 0 :

- Initialization : Linear F-K computation of v_m on $\mathcal{D}_m(n+1)$ from step n
- $k+1 \to k$: Computation of v_k on $\mathcal{D}_k(n+1)$ by linear F-K or optimal stopping problems involving data of v_{k-1} on $\mathcal{D}_{k-1}(n)$ and v_{k+1} on $\mathcal{D}_{k+1}(n+1)$

Impact of execution delay on option pricing

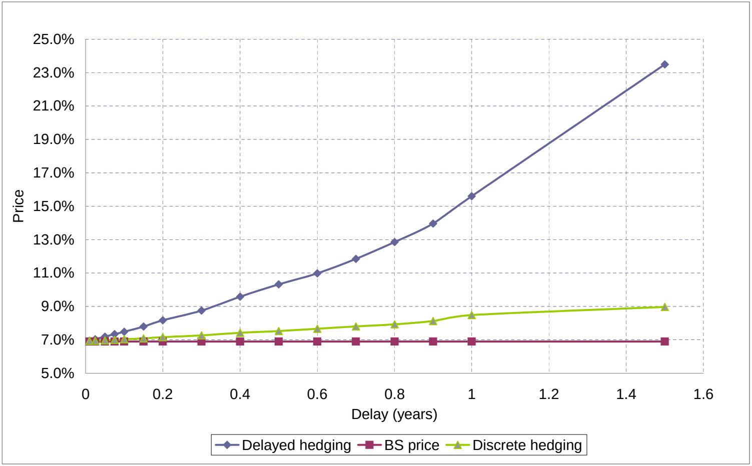
- Indifference price π of a call option $g(S_T) = (S_T - K)_+$:
  - $v_0(S_0, Y_0, Z_0)$: value function of the optimal investment problem without option
  - $v_g(S_0, Y_0, Z_0)$: value function of the optimal investment problem with option delivery
  - $\pi = \pi(S_0, Y_0, Z_0)$ s.t. $v_g(S_0, Y_0, Z_0 + \pi) = v_0(S_0, Y_0, Z_0)$
- Numerical illustrations with :
  - BS model : $r = 0$, $\sigma = 10\%$, $K = S_0$ (At The Money)
  - CARA utility : $U(x) = 1 - e^{-\eta x}$ with $\eta = 20$. $\rightarrow \pi = \pi(S_0, Y_0)$

▶ Dependence of π on delay mh and maturity T

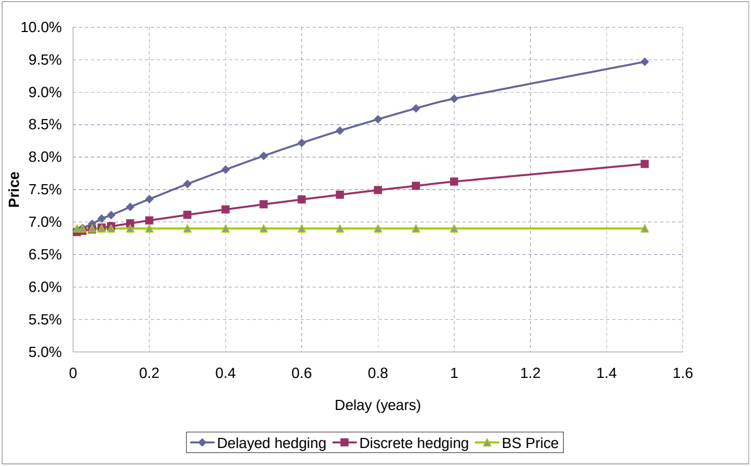
Indifference price for a $T = 3$ years ATM call option for different values of h, in percentage of the initial spot price

h (years)	BS price	discrete hedging, $m = 0$ $Y_0 = 0$	delayed hedging, $m = 1$ $Y_0 = 0$	discrete hedging optimal Y_0	delayed hedging optimal Y_0
0.01	6.90	6.94	6.94	6.85	6.86
0.025	6.90	6.94	7.03	6.87	6.91
0.05	6.90	6.97	7.19	6.89	6.97
0.075	6.90	6.99	7.34	6.92	7.05
0.1	6.90	7.03	7.48	6.94	7.11
0.15	6.90	7.08	7.79	6.98	7.23
0.2	6.90	7.16	8.16	7.03	7.35
0.3	6.90	7.26	8.75	7.11	7.59
0.4	6.90	7.42	9.58	7.19	7.81
0.5	6.90	7.53	10.32	7.27	8.02
0.6	6.90	7.66	10.98	7.35	8.22
0.7	6.90	7.80	11.84	7.42	8.41
0.8	6.90	7.93	12.86	7.49	8.58
0.9	6.90	8.12	13.97	7.56	8.75
1	6.90	8.48	15.60	7.62	8.90
1.5	6.90	8.97	23.49	7.89	9.47

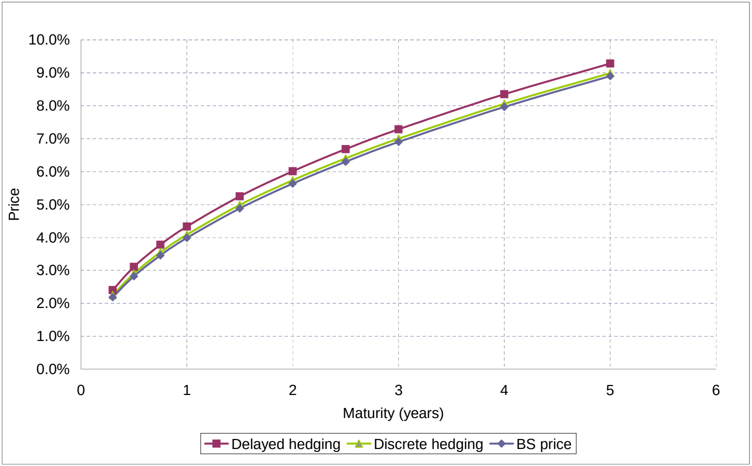
Indifference price for a $T = 3$ years ATM call option, with no initial endowment $Y_0 = 0$ in stock, for discrete and delayed hedging in function of h ($m = 1$).



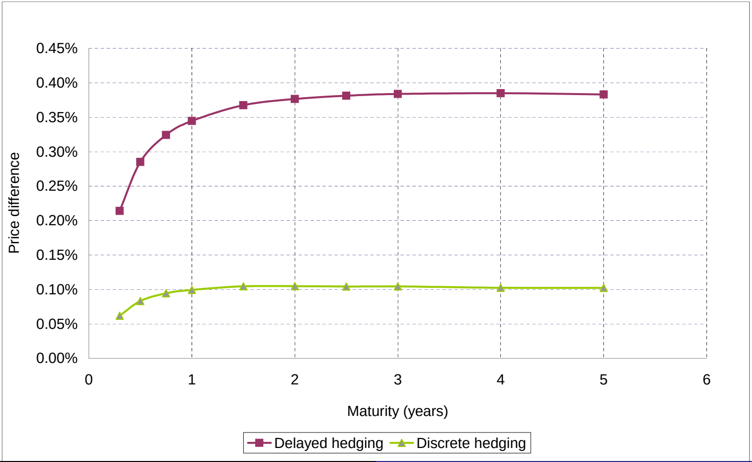
Indifference price for a $T = 3$ years ATM call option, with optimal initial endowment in stock, for discrete and delayed hedging in function of h $(m = 1)$.



Indifference price for discrete and delayed hedging with $h = 2$ months ($m = 1$), with optimal initial endowment Y_0 in stock, in function of the maturity.



Difference of the Indifference price w.r.t. BS price for discrete and delayed hedging with $h = 2$ months ($m = 1$), with optimal initial endowment Y_0 in stock, in function of the maturity.



Conclusion

- General and tractable mathematical formulation of control problem with execution delay
- Other applications in corporate finance
- Probabilistic numerical methods :

$\longrightarrow$ Work in progress with I. Kharoubbi, J. Ma and J. Zhang.