HEDGING UNDER LIQUIDITY RISK

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INTRODUCTION : THE BLACK-SCHOLES MODEL

The financial market

- 1 non-risky asset $S^0 \equiv 1$ (change of numéraire)

- 1 risky asset S : $dS_t \;=\; S_t \left[\mu \, dt + \sigma \, dW_t\right]$

Option / contingent claim : $g(S_T)$, where

$$g \;:\; \mathbb{R}_+ \longrightarrow \mathbb{R}$$

Main problem Valuation of the option $g(S_T)$

• Portfolio strategy Y_t : number of shares of S in portfolio

$\Longrightarrow$ self-financing condition : $dX_t \;=\; Y_t\,dS_t + (X_t - Y_t S_t) \times 0$

Given an initial capital $X_0 = x$, denote : $X_t^{x,Y} := x + \int_0^t Y_u\,dS_u$

$\mathcal{A}$: set of admissible Strategies

$$\int_0^T |Y_u|^2 du \;<\; \infty \quad \text{and} \quad X^{x,Y} \text{ bounded from below}$$

contrast with discrete-time models...

Super-hedging problem

$$V_0 \;:=\; \inf\left\{x \;:\; X_T^{x,Y} \geq g(S_T) \text{ a.s. for some } Y \in \mathcal{A}\right\}$$

Explicit solution in complete market

$$V_t \;=\; v(t, S_t) \;:=\; \mathbb{E}^{\mathbb{P}^0}\left[g(S_T) | S_t\right]$$

<u>PDE characterization</u>

$$-\mathcal{L}v \;:=\; -\frac{\partial v}{\partial t} - \frac{1}{2}\,\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} \;=\; 0 \;\text{ and }\; v(T, s) \;=\; g(s)$$

<u>Greeks (Risk control variables)</u>

- $\Delta_t := \dfrac{\partial v}{\partial s}(t, S_t)$: optimal hedging portfolio $Y_t^* \equiv$ replicating portfolio

- $\Gamma_t := \dfrac{\partial^2 v}{\partial s^2}(t, S_t)$: variation of the hedging portfolio in a stress scenario

Itô's lemma

Optimal wealth process $X_t^* := v(t, S_t)$. Then :

$$X_t^* \;=\; X_0^* + \int_0^t \mathcal{L}v(u, S_u)du + \int_0^t Y_u^* dS_u \;=\; X_0^* + \int_0^t Y_u^* dS_u$$

where $Y_u^* = v_s\,(u, S_u)$.

Since v_s is smooth, it follows from another application of Itô's lemma

$$Y_t^* \;=\; Y_0^* + \int_0^t \mathcal{L}v_s(u, S_u)du + \int_0^t \Gamma_u dS_u$$

MODELING ILLIQUIDITY

There are two classes of models

– Large trader models : investor affects the dynamics of the stock price by means of its position, total wealth, trade. Permanent impact <Frey '98-'02, Platen-Schweizer '98, Schönbucher-Wilmott '05>

– Supply function models are more in the spirit production with increasing technology, or orders book : sellers place orders

Quantity	10	35	20	100
Price	110	112	117	125

so that the price by share is non-increasing. But there is no influence of a large trade on the next moment orders book... <Çetin-Jarrow-Protter '06, Rogers-Singh '05>

Liquidity cost à la Çetin, Jarrow and Protter (2004, 2006)

Risky asset price is defined by a supply curve :

$$\mathbf{S}(S_t, \nu) \quad : \quad \text{price per share of } \nu \text{ risky assets}$$

$\mathbf{S}(S_t, 0) = S_t$ is the zero volume price defined by

$$\frac{dS_t}{S_t} = \mu(S_t)dt + \sigma(S_t)dW_t$$

Examples

1. infinite liquidity : $\mathbf{S}(s, \nu) = s$ for any $\nu \in \mathbb{R}$
2. Prop. transaction costs : $\mathbf{S}(s, \nu) = (1+\lambda)s\mathbb{1}_{\{\nu \geq 0\}} + (1-\mu)s\mathbb{1}_{\{\nu < 0\}}$
3. Exponential supply function : $\mathbf{S}(s, \nu) = se^{\alpha s \nu}$

LIQUIDITY COST : Portfolio dynamics

X_t : holdings in cash, Y_t : holdings in risky asset (number of shares)

$$X_{t+dt} - X_t + \left(Y_{t+dt} - Y_t\right) \mathbf{S}\left(S_t, Y_{t+dt} - Y_t\right) = 0$$

$$\begin{aligned} \Longrightarrow \quad X_T &= X_0 - \sum \left(Y_{t+dt} - Y_t\right) \mathbf{S}\left(S_t, Y_{t+dt} - Y_t\right) \\ &= X_0 + \sum Y_t \left(S_t - S_{t+dt}\right) + \ldots. \end{aligned}$$

Direct computation leads to

$$\begin{aligned} Z_T := X_T + Y_T S_T &= Z_0 + \sum Y_t \left(S_t - S_{t+dt}\right) \\ & \quad - \underbrace{\sum \left(Y_{t+dt} - Y_t\right) \left[\mathbf{S}\left(S_t, Y_{t+dt} - Y_t\right) - \mathbf{S}\left(S_t, 0\right)\right]}_{\text{Liquidity cost}} \end{aligned}$$

LIQUIDITY COST : Continuous-time limit

FROM NOW ON, ASSUME $\nu \longmapsto \mathbf{S}(s,\nu)$ is smooth at $\nu = 0$, and define the Liquidity indicator :

$$\ell(s) \;:=\; \left[4\,\frac{\partial \mathbf{S}}{\partial \nu}(s,0)\right]^{-1}$$

THEN :

finite continuous-time liquidity cost <u>iff</u> $[Y,Y]_T < \infty$

Under this condition, the continuous-time limit of process Z is

$$\begin{aligned} Z_T \;&=\; Z_0 + \int_0^T Y_t dS_t - \int_0^T \frac{\partial \mathbf{S}}{\partial \nu}(S_t,0)\, d[Y,Y]_t^c - \sum_{t\le T} \Delta Y_t \left[\mathbf{S}\left(S_t, \Delta Y_t\right) - S_t\right] \\ &=\; Z_0 + \int_0^T Y_t dS_t - \frac{1}{4}\int_0^T \ell(S_t)^{-1} d[Y,Y]_t^c - \sum_{t\le T} \Delta Y_t \left[\mathbf{S}\left(S_t, \Delta Y_t\right) - S_t\right] \end{aligned}$$

LIQUIDITY COST : the super-hedging problem

ASSUME NO LIQUIDITY COST AT MATURITY T

Super-hedging problem :

$$V_0 \ := \ \text{inf}\left\{z \ : \ Z_T^{z,Y} \geq g(S_T) \text{ a.s. for some } Y \in \mathcal{A}\right\}$$

Remarks

1. Jumps in the Y process are allowed, so the problem "selects" the optimal initial position in the stock
2. Liquidity costs at maturity : static problem !

LIQUIDITY COST : The Çetin-Jarrow-Protter paradox

Without further restrictions on trading strategies, **the problem reduces to Black-Scholes** ! (Çetin, Jarrow and Protter). **BUT no existence of optimal strategy**. Reason for this result is the following <Bank-Baum 04>

Lemma *For all predictable $W-$integrable càdlàg process ϕ, and $\varepsilon > 0$*

$$\sup_{0\le t\le 1} \left| \int_0^t \phi_r dW_r - \int_0^t \phi_r^\varepsilon dW_r \right| \;\le\; \varepsilon$$

for some a.c. predictable process $\phi_t^\varepsilon = \phi_0^\varepsilon + \int_0^t \alpha_r dr$, $\int_0^1 |\alpha_r| dr < \infty$ *a.s.*

$\Longrightarrow$ Allow for arbitrary a.c. $Y_t = Y_0 + \int_0^t \alpha_u du \Longrightarrow V = \mathrm{BS}$ (with $\Gamma = 0$)

LIQUIDITY COST : importance of admissible strategies

We show that **liquidity cost does affect V_0, perfect replication is possible, and hedging strategy can be described (formally)**

Definition $Y \in \mathcal{A}$ if it is of the form

$$Y_t \;=\; \sum_{n=0}^{N-1} y_n \mathbb{1}_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_u du + \int_0^t \Gamma_u dS_u$$

• (τ_n) is an $\nearrow$ seq. of stop. times, y_n are $\mathcal{F}_{\tau_n}-$measurable, $\|N\|_\infty < \infty$

• Y and Γ are $\mathbb{L}^\infty-$bounded up to some polynomial of S

• $\Gamma_t = \Gamma_0 + \int_0^t a_u du + \int_0^t \xi_u dW_u$, $0 \le t \le T$, and

$$\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty \quad \text{where} \quad \|\phi\|_{B,b} := \left\| \sup_{0\le t\le T} \frac{|\phi_r|}{1+S_t^B} \right\|_{\mathbb{L}^b}$$

MAIN RESULT 1 : Optimality of continuous portfolios

Let $\mathcal{A}^{\text{cont}} := \{Y \in \mathcal{A} : \; Y \text{ is continuous}\}$ and

$$V_0^{\text{cont}} \; := \; \inf\left\{z \; : \; Z_T^{z,Y} \geq g(S_T) \text{ a.s. for some } Y \in \mathcal{A}^{\text{cont}}\right\}$$

Theorem $\quad V = V_0^{\text{cont}}$

Under liquidity costs, it is better to perform consecutive small trades instead of a large one

Process Z can be interpreted as the *short-time* liquidation value of the portfolio

MAIN RESULT 2 : PDE characterization

Theorem Let $-C \leq g(.) \leq C(1+\ .)$ for some $C > 0$. Then $V(t,s)$ is the unique continuous viscosity solution of the dynamic programming equation

$$-V_t(t,s) + \frac{1}{4}s^2\sigma(t,s)^2\ell(s)\left[1 - \left(\frac{V_{ss}(t,s)}{\ell(s)} + 1\right)^{+^2}\right] = 0$$

with $V(T,s) = g(s)$ and $-C \leq V(t,s) \leq C(1+s)$ for every (t,s).

- Main technical tool : small time behavior of double stochastic integrals

- Notice that there is no boundary layer i.e. face-lifting

MAIN RESULT 3 : When do liquidity costs matter ?

Recall that Çetin, Jarrow and Protter showed that, under their larger set of strategies, the minimal super-hedging cost coincides witth the Black-Scholes price. With our set of strategies $\mathcal{A}$, we have :

Corollary $V_0 = \mathbb{E}^{\mathbb{P}^0}[g(S_T)]$ if and only if g is an affine function

FORMAL DESCRIPTION OF HEDGING STRATEGY :

concavity versus convexity

• **For a convex payoff :** only possibility to super-hedge is the Black-Scholes perfect replication strategy

• **For a concave payoff :** two possibilities to super-hedge

1. Black-Scholes perfect replication $\Longrightarrow \Gamma \neq 0$ so pay liquidity cost

2. Buy-and-hold $\Longrightarrow \Gamma = 0$ no liquidity cost, but hedge might be too expensive $\Longrightarrow$

$v_{ss} < -\ell(s)$: buy-and-hold strategy is more interesting because liquidity cost is too expensive

$v_{ss} \geq -\ell(s)$: perfect replication

OPTIMAL HEDGING STRATEGY : example of an exponential liquidity function

Let $\mathbf{S}(s,\nu) := s\, e^{(\alpha s \nu/4)}$. Then $\ell(s) = \dfrac{1}{\alpha s^2}$

Define $\phi(t,s) := \dfrac{1}{4\alpha}(\sigma^2 t + 4 \ln s)$ so that $\phi_{ss} = -\ell(s)$ and $\phi_t = \dfrac{1}{4} s^2 \sigma^2 \ell(s)$

- Let (t,s) be such that $V_{ss}(t,s) < -\ell(s)$, set

$$\theta \;:=\; \inf\left\{u > t \;:\; V_{ss}(u,S_u) \geq -\ell(S_u)\right\}$$

and observe that

- $-\frac{1}{4} s^2 \sigma^2 \phi_{ss}^2(t,s) \ell(s) = \mathcal{L}\phi(t,s)$
- $(V - \phi)$ concave V
- and $(V - \phi)_t(u, S_u) = 0$ for $t \leq u \leq \theta$

OPTIMAL HEDGING STRATEGY : "Hedge perfectly ϕ and Buy-and-hold the difference $(V-\phi)$"

Set $Z_0 := V(t,s)$, $Y_0 := V_s(t,s)$, $\Gamma_t := \phi_{ss}(u,S_u)$, and $\alpha_t := \mathcal{L}\phi_s(u,S_u)$

$$
\begin{aligned}
\Longrightarrow \quad Z_\theta &= V(t,s) + \int_t^\theta Y_u dS_u - \frac{1}{4}\int_t^\theta \ell(s)^{-1}\Gamma_t^2 \sigma^2 S_t^2 dt \\
&= V(t,s) + + \int_t^\theta \left(V_s(t,s) + \int_t^u \mathcal{L}\phi_s(r,S_r)dr + \phi_{ss}(r,S_r)dS_r\right) dS_u \\
&\quad + \int_t^\theta \mathcal{L}\phi(u,S_u)du \\
&= (V-\phi)(t,s) + (V-\phi)_s(t,s)\ [S_\theta - s] + \phi(t,s) + \int_t^\theta \mathcal{L}\phi(u,S_u)du \\
&\quad + \int_t^\theta \left(\phi_s(t,s) + \int_t^u \mathcal{L}\phi_s(r,S_r)dr + \phi_{ss}(r,S_r)dS_r\right) dS_u \\
&= (V-\phi)(t,s) + (V-\phi)_s(t,s)\ [S_\theta - s] + \phi(t,s) + \int_t^\theta \mathcal{L}\phi(u,S_u)du \\
&\quad + \int_t^\theta \phi_s(u,S_u)dS_u
\end{aligned}
$$

Hence

$$
\begin{aligned}
Z_\theta &= (V-\phi)(t,s) + (V-\phi)_s(t,s)\ [S_\theta - s] + \phi(\theta, S_\theta) \\
&\geq (V-\phi)(t,S_\theta) + \phi(\theta, S_\theta) \quad \text{by concavity of } (V-\phi)(t,.) \\
&= V(\theta, S_\theta) \quad \text{by the fact that } (V-\phi)_t(u,S_u) = 0 \text{ for } t \leq u \leq \theta
\end{aligned}
$$

LARGE LIQUIDITY EXPANSION

Let $\mathbf{S}^\varepsilon(s,\nu) := \mathbf{S}(s,\varepsilon\nu)$, $\varepsilon > 0$

Then $\ell^\varepsilon(s) = \varepsilon^{-1}\ell(s)$ and V^ε is the unique vis. sol. of

$$-V_t^\varepsilon(t,s) + \frac{1}{4\varepsilon}s^2\sigma(t,s)^2\ell(s)\left[1 - \left(\frac{\varepsilon V_{ss}^\varepsilon(t,s)}{\ell(s)} + 1\right)^{+^2}\right] = 0$$

with $V^\varepsilon(T,s) = g(s)$

Proposition With $V^0(t,s) = \mathbb{E}_{t,s}[g(S_T)]$, we have

$$V^\varepsilon(t,s) = V^0(t,s) + \mathbb{E}^{\mathbb{P}^0}\left[\int_t^T \frac{V_{ss}^{0^2}}{4\ell}(u,S_u)S_u^2\sigma_u^2 du\right] + \circ(\varepsilon)$$