Risk neutral dynamics of spot and forward electricity prices

II Printemps de la Chaire Finance & Développement Durable

Luciano Campi, Jean-Michel Marin, Nizar Touzi

Laboratoire FiME

Paris, 26 Mars 2008

Contents

Introduction I : Motivations

- In standard finacial markets : $F_t(T) = S_t e^{r(T-t)}$. This equality relies heavily on costless storability of financial assets, it breaks down when S_t is spot price of electricity
- A priori, no relations between spot and forward at least in a market composed of electricity and bank account (see, e.g., Geman-Vasicek (2001))
- Geman-Vasicek (2001) and Bessembinder-Lemon (2002) show that *short-term* forward contract are (upward- or downward-) biased estimator of spot prices, so ...
- ... in mathematical terms, when $t \uparrow T$, $F_t(T)$ does not necessarily tend to S_T

Introduction I : Motivations

- In standard finacial markets : $F_t(T) = S_t e^{r(T-t)}$. This equality relies heavily on costless storability of financial assets, it breaks down when S_t is spot price of electricity
- A priori, no relations between spot and forward at least in a market composed of electricity and bank account (see, e.g., Geman-Vasicek (2001))
- Geman-Vasicek (2001) and Bessembinder-Lemon (2002) show that *short-term* forward contract are (upward- or downward-) biased estimator of spot prices, so ...
- ... in mathematical terms, when $t \uparrow T$, $F_t(T)$ does not necessarily tend to S_T

Introduction I : Motivations

- In standard finacial markets : $F_t(T) = S_t e^{r(T-t)}$. This equality relies heavily on costless storability of financial assets, it breaks down when S_t is spot price of electricity
- A priori, no relations between spot and forward at least in a market composed of electricity and bank account (see, e.g., Geman-Vasicek (2001))
- Geman-Vasicek (2001) and Bessembinder-Lemon (2002) show that *short-term* forward contract are (upward- or downward-) biased estimator of spot prices, so ...
- ... in mathematical terms, when $t \uparrow T$, $F_t(T)$ does not necessarily tend to S_T

Introduction I : Motivations

- In standard finacial markets : $F_t(T) = S_t e^{r(T-t)}$. This equality relies heavily on costless storability of financial assets, it breaks down when S_t is spot price of electricity
- A priori, no relations between spot and forward at least in a market composed of electricity and bank account (see, e.g., Geman-Vasicek (2001))
- Geman-Vasicek (2001) and Bessembinder-Lemon (2002) show that *short-term* forward contract are (upward- or downward-) biased estimator of spot prices, so ...
- ... in mathematical terms, when $t \uparrow T$, $F_t(T)$ does not necessarily tend to S_T

Introduction II : Main idea

- Nevertheless, imagine an fictitious economy where electricity is produced only out of coal, so that electrcity spot price $P_t = q_c S_t^c$, and agents can invest in coal, electricity and bank account
- Assume no-arbitrage in the market of coal and bank account, s.t. there exists a risk-neutral measure $\mathbb{Q}$ for $\tilde{S}_t^c = e^{-rt} S_t^c$
- A forward contract on spot electricity P_T can be viewed as a contract on coal necessary to produce 1 MWh of electricity, with price $c_c S_t^c$, so that

$$F_0^e(T) = \mathbb{E}_{\mathbb{Q}}[P_T] = \mathbb{E}_{\mathbb{Q}}[c_c S_T^c] = c_c F_0^c(T)$$

Introduction II : Main idea

- Nevertheless, imagine an fictitious economy where electricity is produced only out of coal, so that electrcity spot price $P_t = q_c S_t^c$, and agents can invest in coal, electricity and bank account
- Assume no-arbitrage in the market of coal and bank account, s.t. there exists a risk-neutral measure $\mathbb{Q}$ for $\tilde{S}_t^c = e^{-rt} S_t^c$
- A forward contract on spot electricity P_T can be viewed as a contract on coal necessary to produce 1 MWh of electricity, with price $c_c S_t^c$, so that

$$F_0^e(T) = \mathbb{E}_{\mathbb{Q}}[P_T] = \mathbb{E}_{\mathbb{Q}}[c_c S_T^c] = c_c F_0^c(T)$$

Introduction II : Main idea

- Nevertheless, imagine an fictitious economy where electricity is produced only out of coal, so that electrcity spot price $P_t = q_c S_t^c$, and agents can invest in coal, electricity and bank account
- Assume no-arbitrage in the market of coal and bank account, s.t. there exists a risk-neutral measure $\mathbb{Q}$ for $\tilde{S}_t^c = e^{-rt} S_t^c$
- A forward contract on spot electricity P_T can be viewed as a contract on coal necessary to produce 1 MWh of electricity, with price $c_c S_t^c$, so that

$$F_0^e(T) = \mathbb{E}_{\mathbb{Q}}[P_T] = \mathbb{E}_{\mathbb{Q}}[c_c S_T^c] = c_c F_0^c(T)$$

The Model

- **Randomness :** $(W^0, W) = (W^0, W^1, \ldots, W^n)$ Wiener process defined on a given $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{F}_t^W$ models the market information flow.
- Riskless asset $S_t^0 = \exp \int_0^t r_u du$, $t \geq 0$, r is $\mathcal{F}_t^0$-adapted and ≥ 0.
- **Commodities market:** $n \geq 1$ commodities (coal, gas, ...) whose prices S^i to produce 1 MWh of electricity follows

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_j \sigma_t^{ij} dW_t^j), \quad t \geq 0.$$

  For simplicity, assume that convenience yields $y^i = 0$ for all $i = 1, \ldots, n$.
- **Electricity demand:** $D = (D_t)_{t \geq 0}$ $\mathcal{F}_t^0$-adapted, positive process; notice that D is independent of each S^i.

The Model

- **Randomness :** $(W^0, W) = (W^0, W^1, \ldots, W^n)$ Wiener process defined on a given $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{F}_t^W$ models the market information flow.
- Riskless asset $S_t^0 = \exp \int_0^t r_u du$, $t \geq 0$, r is $\mathcal{F}_t^0$-adapted and ≥ 0.
- **Commodities market:** $n \geq 1$ commodities (coal, gas, ...) whose prices S^i to produce 1 MWh of electricity follows

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_j \sigma_t^{ij} dW_t^j), \quad t \geq 0.$$

For simplicity, assume that convenience yields $y^i = 0$ for all $i = 1, \ldots, n$.

- **Electricity demand:** $D = (D_t)_{t \geq 0}$ $\mathcal{F}_t^0$-adapted, positive process; notice that D is independent of each S^i.

The Model

- **Randomness :** $(W^0, W) = (W^0, W^1, \ldots, W^n)$ Wiener process defined on a given $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{F}_t^W$ models the market information flow.
- Riskless asset $S_t^0 = \exp \int_0^t r_u du$, $t \geq 0$, r is $\mathcal{F}_t^0$-adapted and ≥ 0.
- **Commodities market:** $n \geq 1$ commodities (coal, gas, ...) whose prices S^i to produce 1 MWh of electricity follows

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_j \sigma_t^{ij} dW_t^j), \quad t \geq 0.$$

  For simplicity, assume that convenience yields $y^i = 0$ for all $i = 1, \ldots, n$.
- **Electricity demand:** $D = (D_t)_{t \geq 0}$ $\mathcal{F}_t^0$-adapted, positive process; notice that D is independent of each S^i.

Risk neutral dynamics of spot and forward electricity prices
Luciano Campi, Jean-Michel Marin, Nizar Touzi
Introduction
The Model
Electricity forward prices
Constant coefficients
What's next

The Model

- **Randomness :** $(W^0, W) = (W^0, W^1, \ldots, W^n)$ Wiener process defined on a given $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{F}_t^W$ models the market information flow.
- Riskless asset $S_t^0 = \exp \int_0^t r_u du$, $t \geq 0$, r is $\mathcal{F}_t^0$-adapted and ≥ 0.
- **Commodities market:** $n \geq 1$ commodities (coal, gas, ...) whose prices S^i to produce 1 MWh of electricity follows

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_j \sigma_t^{ij} dW_t^j), \quad t \geq 0.$$

  For simplicity, assume that convenience yields $y^i = 0$ for all $i = 1, \ldots, n$.
- **Electricity demand:** $D = (D_t)_{t \geq 0}$ $\mathcal{F}_t^0$-adapted, positive process; notice that D is independent of each S^i.

Electricity spot price P_t when all technologies are available :

- $\Delta_i > 0$ denotes the capacity of i-th commodity for electricity at every instant, a constant known to the producer
- order commodities prices $S_t^{(1)}(\omega) \leq \ldots \leq S_t^{(n)}(\omega)$ from the cheapest to the most expensive, giving an $\mathcal{F}_t^W$-adapted *random* permutation $\pi_t(\omega)$ of $\{1, \ldots, n\}$
- look at the demand D_t:

$$D_t \in I_k^{\pi_t} := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)} \right) \Rightarrow P_t = S_t^{(k)}$$

- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}}$ for $t \geq 0$.

Electricity spot price P_t when all technologies are available :

- $\Delta_i > 0$ denotes the capacity of i-th commodity for electricity at every instant, a constant known to the producer
- order commodities prices $S_t^{(1)}(\omega) \leq \ldots \leq S_t^{(n)}(\omega)$ from the cheapest to the most expensive, giving an $\mathcal{F}_t^W$-adapted *random* permutation $\pi_t(\omega)$ of $\{1, \ldots, n\}$
- look at the demand D_t:

$$D_t \in I_k^{\pi_t} := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)}\right) \Rightarrow P_t = S_t^{(k)}$$

- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}}$ for $t \geq 0$.

Electricity spot price P_t when all technologies are available :

- $\Delta_i > 0$ denotes the capacity of i-th commodity for electricity at every instant, a constant known to the producer
- order commodities prices $S_t^{(1)}(\omega) \leq \ldots \leq S_t^{(n)}(\omega)$ from the cheapest to the most expensive, giving an $\mathcal{F}_t^W$-adapted *random* permutation $\pi_t(\omega)$ of $\{1, \ldots, n\}$
- look at the demand D_t:

$$D_t \in I_k^{\pi_t} := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)}\right) \Rightarrow P_t = S_t^{(k)}$$

- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}}$ for $t \geq 0$.

Electricity spot price P_t when all technologies are available :

- $\Delta_i > 0$ denotes the capacity of i-th commodity for electricity at every instant, a constant known to the producer
- order commodities prices $S_t^{(1)}(\omega) \leq \ldots \leq S_t^{(n)}(\omega)$ from the cheapest to the most expensive, giving an $\mathcal{F}_t^W$-adapted *random* permutation $\pi_t(\omega)$ of $\{1, \ldots, n\}$
- look at the demand D_t:

$$D_t \in I_k^{\pi_t} := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)}\right) \Rightarrow P_t = S_t^{(k)}$$

- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}}$ for $t \geq 0$.

Technologies failures I : The case of two commodities

- If $n = 2$ we have $S_t^1 \leq S_t^2$ or $S_t^2 \leq S_t^1$, let's consider the first case $\pi_t = \{1, 2\}$
- Introduce two r.v.'s ϵ_t^i, $i = 1, 2$ such that
  - $\epsilon_t^i = 1$ when technology i is available, otherwise $\epsilon_t^i = 0$
  - $\epsilon_t^i = 0$ implies that $\epsilon_t^j = 1$ for $i \neq j$
- Only three cases may happen at each time t
  1. $\epsilon_t^1 = \epsilon_t^2 = 1$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1,\Delta_1+\Delta_2)}(D_t)$
  2. $\epsilon_t^1 = 1, \epsilon_t^2 = 0$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t)$
  3. $\epsilon_t^1 = 0, \epsilon_t^2 = 1$ then $P_t = S_t^2 \mathbf{1}_{[0,\Delta_1+\Delta_2)}(D_t)$
- To sum up:
  $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1 \epsilon_t^1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1 \epsilon_t^1, \Delta_1 \epsilon_t^1 + \Delta_2 \epsilon_t^2)}(D_t)$

Technologies failures I : The case of two commodities

- If $n = 2$ we have $S_t^1 \leq S_t^2$ or $S_t^2 \leq S_t^1$, let's consider the first case $\pi_t = \{1, 2\}$
- Introduce two r.v.'s ϵ_t^i, $i = 1, 2$ such that
  - $\epsilon_t^i = 1$ when technology i is available, otherwise $\epsilon_t^i = 0$
  - $\epsilon_t^i = 0$ implies that $\epsilon_t^j = 1$ for $i \neq j$
- Only three cases may happen at each time t
  1. $\epsilon_t^1 = \epsilon_t^2 = 1$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1,\Delta_1+\Delta_2)}(D_t)$
  2. $\epsilon_t^1 = 1, \epsilon_t^2 = 0$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t)$
  3. $\epsilon_t^1 = 0, \epsilon_t^2 = 1$ then $P_t = S_t^2 \mathbf{1}_{[0,\Delta_1+\Delta_2)}(D_t)$
- To sum up:
  $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1 \epsilon_t^1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1 \epsilon_t^1, \Delta_1 \epsilon_t^1 + \Delta_2 \epsilon_t^2)}(D_t)$

Technologies failures I : The case of two commodities

- If $n = 2$ we have $S_t^1 \leq S_t^2$ or $S_t^2 \leq S_t^1$, let's consider the first case $\pi_t = \{1, 2\}$
- Introduce two r.v.'s ϵ_t^i, $i = 1, 2$ such that
  - $\epsilon_t^i = 1$ when technology i is available, otherwise $\epsilon_t^i = 0$
  - $\epsilon_t^i = 0$ implies that $\epsilon_t^j = 1$ for $i \neq j$
- Only three cases may happen at each time t
  1. $\epsilon_t^1 = \epsilon_t^2 = 1$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1, \Delta_1+\Delta_2)}(D_t)$
  2. $\epsilon_t^1 = 1, \epsilon_t^2 = 0$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t)$
  3. $\epsilon_t^1 = 0, \epsilon_t^2 = 1$ then $P_t = S_t^2 \mathbf{1}_{[0,\Delta_1+\Delta_2)}(D_t)$
- To sum up:
  $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1 \epsilon_t^1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1 \epsilon_t^1, \Delta_1 \epsilon_t^1 + \Delta_2 \epsilon_t^2)}(D_t)$

Technologies failures I : The case of two commodities

- If $n = 2$ we have $S_t^1 \leq S_t^2$ or $S_t^2 \leq S_t^1$, let's consider the first case $\pi_t = \{1, 2\}$
- Introduce two r.v.'s ϵ_t^i, $i = 1, 2$ such that
  - $\epsilon_t^i = 1$ when technology i is available, otherwise $\epsilon_t^i = 0$
  - $\epsilon_t^i = 0$ implies that $\epsilon_t^j = 1$ for $i \neq j$
- Only three cases may happen at each time t
  1. $\epsilon_t^1 = \epsilon_t^2 = 1$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1,\Delta_1+\Delta_2)}(D_t)$
  2. $\epsilon_t^1 = 1, \epsilon_t^2 = 0$ then $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1)}(D_t)$
  3. $\epsilon_t^1 = 0, \epsilon_t^2 = 1$ then $P_t = S_t^2 \mathbf{1}_{[0,\Delta_1+\Delta_2)}(D_t)$
- To sum up:
  $P_t = S_t^1 \mathbf{1}_{[0,\Delta_1 \epsilon_t^1)}(D_t) + S_t^2 \mathbf{1}_{[\Delta_1 \epsilon_t^1, \Delta_1 \epsilon_t^1 + \Delta_2 \epsilon_t^2)}(D_t)$

Technologies failures II : The general case

- Let η be a new process, values in $\{0, 1, \ldots, n\}$, with interpretation
  - event $\{\eta_t = i\}$ means "i-th technology not available", for $1 \leq i \leq n$
  - event $\{\eta_t = 0\}$ means that all technologies are available

Hidden assumption: only one failure at the time is allowed.

- Define $\epsilon_t^i := \mathbf{1}_{\{\eta_t \neq i\}}$, $1 \leq i \leq n$, so that
  - $\epsilon_t^i = 1$ means "i-th technology available at time t
  - $\epsilon_t^i = 0$ means "i-th technology **not** available at time t
- Set $I_k^{\pi_t}(t) := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}\right)$
- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}(t)}$ for $t \geq 0$.

Technologies failures II : The general case

- Let η be a new process, values in $\{0, 1, \ldots, n\}$, with interpretation
  - event $\{\eta_t = i\}$ means "i-th technology not available", for $1 \leq i \leq n$
  - event $\{\eta_t = 0\}$ means that all technologies are available

  Hidden assumption: only one failure at the time is allowed.
- Define $\epsilon_t^i := \mathbf{1}_{\{\eta_t \neq i\}}$, $1 \leq i \leq n$, so that
  - $\epsilon_t^i = 1$ means "i-th technology available at time t
  - $\epsilon_t^i = 0$ means "i-th technology **not** available at time t
- Set $I_k^{\pi_t}(t) := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}\right)$
- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}(t)}$ for $t \geq 0$.

Technologies failures II : The general case

- Let η be a new process, values in $\{0, 1, \ldots, n\}$, with interpretation
  - event $\{\eta_t = i\}$ means "i-th technology not available", for $1 \leq i \leq n$
  - event $\{\eta_t = 0\}$ means that all technologies are available

  Hidden assumption: only one failure at the time is allowed.
- Define $\epsilon_t^i := \mathbf{1}_{\{\eta_t \neq i\}}$, $1 \leq i \leq n$, so that
  - $\epsilon_t^i = 1$ means "i-th technology available at time t
  - $\epsilon_t^i = 0$ means "i-th technology **not** available at time t
- Set $I_k^{\pi_t}(t) := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}\right)$
- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}(t)}$ for $t \geq 0$.

Technologies failures II : The general case

- Let η be a new process, values in $\{0, 1, \dots, n\}$, with interpretation
  - event $\{\eta_t = i\}$ means "i-th technology not available", for $1 \leq i \leq n$
  - event $\{\eta_t = 0\}$ means that all technologies are available

Hidden assumption: only one failure at the time is allowed.

- Define $\epsilon_t^i := \mathbf{1}_{\{\eta_t \neq i\}}$, $1 \leq i \leq n$, so that
  - $\epsilon_t^i = 1$ means "i-th technology available at time t
  - $\epsilon_t^i = 0$ means "i-th technology **not** available at time t
- Set $I_k^{\pi_t}(t) := \left[\sum_{i=1}^{k-1} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)} \epsilon_t^{\pi_t(i)}\right)$
- ... so that $P_t = \sum_k S_t^{(k)} \mathbf{1}_{I_k^{\pi_t}(t)}$ for $t \geq 0$.

Risk neutral dynamics of spot and forward electricity prices

Luciano Campi, Jean-Michel Marin, Nizar Touzi

No-arbitrage assumption on commodities market.

Let $T > 0$. There exists $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T^W$ such that :

1. each $\tilde{S}^i/S^0$ is a $\mathbb{Q}$-martingale w.r.t. $\mathcal{F}^W$
2. the laws of W^0 and η do not change
3. $\mathcal{F}_T^0, \mathcal{F}_T^W, \mathcal{F}_T^\eta$ are independent conditionally to $\mathcal{F}_t$, $t < T$

Remarks

1. Property 3 above is satisfied if W^0, W and η are constructed on the canonical product space and the change of measure affects only the factor where W is defined.
2. Being D not tradable, this market is not complete. We choose $\mathbb{Q}$ as the pricing measure.

Electricity forward prices I

- The pay-off of a forward contract on spot electricity is $P_T = \sum_k S_T^{(k)} \mathbf{1}_{I_k^{\pi^T}(T)}$ so it can be viewed as an option on commodities
- Use no-arbitrage assumption on commodities to get

$$F_t(T) = \mathbb{E}^{\mathbb{Q}_T}[P_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}_T}\left[\sum_{k=1}^{n} S_T^{(k)} \mathbf{1}_{I_k^{\pi^T}(T)}|\mathcal{F}_t\right]$$

where $\mathbb{Q}_T$ is the forward risk-neutral measure on $\mathcal{F}_T$:

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{\exp \int_t^T r_u du}{\mathbb{E}^{\mathbb{Q}}[\exp \int_t^T r_u du|\mathcal{F}_t]}$$

(Notice that $\mathbb{Q}_T = \mathbb{Q}$ if r is non-random)

Electricity forward prices I

- The pay-off of a forward contract on spot electricity is $P_T = \sum_k S_T^{(k)} \mathbf{1}_{I_k^{\pi_T}(T)}$ so it can be viewed as an option on commodities
- Use no-arbitrage assumption on commodities to get

$$F_t(T) = \mathbb{E}^{\mathbb{Q}_T}[P_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}_T}\left[\sum_{k=1}^{n} S_T^{(k)} \mathbf{1}_{I_k^{\pi_T}(T)}|\mathcal{F}_t\right]$$

where $\mathbb{Q}_T$ is the forward risk-neutral measure on $\mathcal{F}_T$:

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{\exp\int_t^T r_u du}{\mathbb{E}^{\mathbb{Q}}[\exp\int_t^T r_u du|\mathcal{F}_t]}$$

(Notice that $\mathbb{Q}_T = \mathbb{Q}$ if r is non-random)

Electricity forward prices II : The main formula

Proposition

Under previous assumptions and if $S^i_T \in L^1(\mathbb{Q}_T)$, $1 \le i \le n$ *: for all* $t \in [0, T]$

$$F_t(T) = \sum_{i=1}^{n} \sum_{\pi \in \Pi_n} c_{\pi(i)} F^{\pi(i)}_t(T) \mathbb{Q}_T[D_T \in I^\pi_i(T) | \mathcal{F}^0_t] \times \mathbb{Q}^{\pi(i)}_T[\pi_T = \pi | \mathcal{F}^W_t]$$

where :

- Π_n *is the set of all permutations of* $\{1, \ldots, n\}$
- $F^i_t(T)$ *is forward price of i-th commodity, delivery date* T
- $d\mathbb{Q}^{\pi(i)}_T / d\mathbb{Q}_T = S^{\pi(i)}_T / \mathbb{E}^{\mathbb{Q}_T}[S^{\pi(i)}_T]$ *on* $\mathcal{F}^W_T$

Electricity forward prices II : The main formula

Proposition

Under previous assumptions and if $S_T^i \in L^1(\mathbb{Q}_T)$, $1 \leq i \leq n$ *: for all* $t \in [0, T]$

$$F_t(T) = \sum_{i=1}^{n} \sum_{\pi \in \Pi_n} c_{\pi(i)} F_t^{\pi(i)}(T) \mathbb{Q}_T[D_T \in I_i^{\pi}(T) | \mathcal{F}_t^0] \times \mathbb{Q}_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t^W]$$

where :

- Π_n *is the set of all permutations of* $\{1, \ldots, n\}$
- $F_t^i(T)$ *is forward price of i-th commodity, delivery date T*
- $d\mathbb{Q}_T^{\pi(i)}/d\mathbb{Q}_T = S_T^{\pi(i)}/\mathbb{E}^{\mathbb{Q}_T}[S_T^{\pi(i)}]$ *on* $\mathcal{F}_T^W$

Electricity forward prices II : The main formula

Proposition

Under previous assumptions and if $S^i_T \in L^1(\mathbb{Q}_T)$, $1 \le i \le n$ *: for all* $t \in [0, T]$

$$F_t(T) = \sum_{i=1}^{n} \sum_{\pi \in \Pi_n} c_{\pi(i)} F_t^{\pi(i)}(T) \mathbb{Q}_T[D_T \in I_i^{\pi}(T) | \mathcal{F}_t^0] \times \mathbb{Q}_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t^W]$$

where :

- Π_n *is the set of all permutations of* $\{1, \ldots, n\}$
- $F^i_t(T)$ *is forward price of i-th commodity, delivery date* T
- $d\mathbb{Q}_T^{\pi(i)}/d\mathbb{Q}_T = S_T^{\pi(i)}/\mathbb{E}^{\mathbb{Q}_T}[S_T^{\pi(i)}]$ *on* $\mathcal{F}_T^W$

Electricity forward prices III : some remarks

This model is able to explain three basic features of electricity market as :

- *Observed spikes in electricity spot prices dynamics*
- *Non-convergence of electricity forward prices towards spot (day-ahead) prices as* $t \uparrow T$*. Indeed,*

$$F_t(T) \to F_T(T) = \sum_{i=1}^{n} S_T^{(i)} \mathbb{Q}_T[x \in I_i^{\pi}(T)]|_{x=D_T,\pi=\pi_T}.$$

  $F_T(T) \neq P_T$ *whenever* η *is non-degenerate.*
- *The paths of electricity forward prices are much smoother than the corresponding spot prices.*

The constant coefficients model : more explicit formulae

- Commodities prices S^i follow n-dim Black-Scholes model : volatilities $\sigma^{ij} > 0$ and interest rate $r > 0$ constant so that, in particular, $\mathbb{Q}_T = \mathbb{Q}$
- $F^i_t(T) = e^{r(T-t)} S^i_t$ for all commodities $1 \le i \le n$
- Demand of electricity : D follows a CIR process

$$dD_t = a(b - D_t)dt + \delta\sqrt{D_t}dW^0_t, \quad D_0 > 0$$

with $a, b > 0$ and δ such that $2ab \ge \delta^2 \Rightarrow D_t > 0$ all $t \ge 0$.

- Under these assumptions probabilities $\mathbb{Q}[D_T \in I^\pi_k(T)|\mathcal{F}^0_t]$ and $\mathbb{Q}^{\pi(i)}_T[\pi_T = \pi|\mathcal{F}^W_t]$ can be *computed explicitly as functions of the parameters.*

The constant coefficients model : more explicit formulae

- Commodities prices S^i follow n-dim Black-Scholes model : volatilities $\sigma^{ij} > 0$ and interest rate $r > 0$ constant so that, in particular, $\mathbb{Q}_T = \mathbb{Q}$
- $F_t^i(T) = e^{r(T-t)} S_t^i$ for all commodities $1 \leq i \leq n$
- Demand of electricity : D follows a CIR process

$$dD_t = a(b - D_t)dt + \delta\sqrt{D_t}dW_t^0, \quad D_0 > 0$$

with $a, b > 0$ and δ such that $2ab \geq \delta^2 \Rightarrow D_t > 0$ all $t \geq 0$.
- Under these assumptions probabilities $\mathbb{Q}[D_T \in I_k^\pi(T)|\mathcal{F}_t^0]$ and $\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi|\mathcal{F}_t^W]$ can be *computed explicitly as functions of the parameters.*

The constant coefficients model : more explicit formulae

- Commodities prices S^i follow n-dim Black-Scholes model : volatilities $\sigma^{ij} > 0$ and interest rate $r > 0$ constant so that, in particular, $\mathbb{Q}_T = \mathbb{Q}$
- $F_t^i(T) = e^{r(T-t)} S_t^i$ for all commodities $1 \leq i \leq n$
- Demand of electricity : D follows a CIR process

$$dD_t = a(b - D_t)dt + \delta\sqrt{D_t}dW_t^0, \quad D_0 > 0$$

with $a, b > 0$ and δ such that $2ab \geq \delta^2 \Rightarrow D_t > 0$ all $t \geq 0$.

- Under these assumptions probabilities $\mathbb{Q}[D_T \in I_k^\pi(T)|\mathcal{F}_t^0]$ and $\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi|\mathcal{F}_t^W]$ can be *computed explicitly as functions of the parameters.*

The constant coefficients model : more explicit formulae

- Commodities prices S^i follow n-dim Black-Scholes model : volatilities $\sigma^{ij} > 0$ and interest rate $r > 0$ constant so that, in particular, $\mathbb{Q}_T = \mathbb{Q}$
- $F_t^i(T) = e^{r(T-t)} S_t^i$ for all commodities $1 \leq i \leq n$
- Demand of electricity : D follows a CIR process

$$dD_t = a(b - D_t)dt + \delta\sqrt{D_t}dW_t^0, \quad D_0 > 0$$

with $a, b > 0$ and δ such that $2ab \geq \delta^2 \Rightarrow D_t > 0$ all $t \geq 0$.

- Under these assumptions probabilities $\mathbb{Q}[D_T \in I_k^\pi(T)|\mathcal{F}_t^0]$ and $\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi|\mathcal{F}_t^W]$ can be *computed explicitly as functions of the parameters.*

What's next ?

- Pricing of options on forward electricity : it can be reduced to pricing of basket options of commodities
- Simulations and estimation of parameters in progress ...
- Make the model more complex, e.g. add stochastic convenience yields and interest rate, more than one failure at the time ...
- Study the risk premium $\pi(t, T) = F_t(T) - P_t$ in our model, compare with other models

What's next ?

- Pricing of options on forward electricity : it can be reduced to pricing of basket options of commodities
- Simulations and estimation of parameters in progress ...
- Make the model more complex, e.g. add stochastic convenience yields and interest rate, more than one failure at the time ...
- Study the risk premium $\pi(t,T) = F_t(T) - P_t$ in our model, compare with other models

What's next ?

- Pricing of options on forward electricity : it can be reduced to pricing of basket options of commodities
- Simulations and estimation of parameters in progress ...
- Make the model more complex, e.g. add stochastic convenience yields and interest rate, more than one failure at the time ...
- Study the risk premium $\pi(t,T) = F_t(T) - P_t$ in our model, compare with other models

What's next ?

- Pricing of options on forward electricity : it can be reduced to pricing of basket options of commodities
- Simulations and estimation of parameters in progress ...
- Make the model more complex, e.g. add stochastic convenience yields and interest rate, more than one failure at the time ...
- Study the risk premium $\pi(t, T) = F_t(T) - P_t$ in our model, compare with other models