Measures of Multivariate Risks

Alfred Galichon

Département d'Economie, Ecole polytechnique

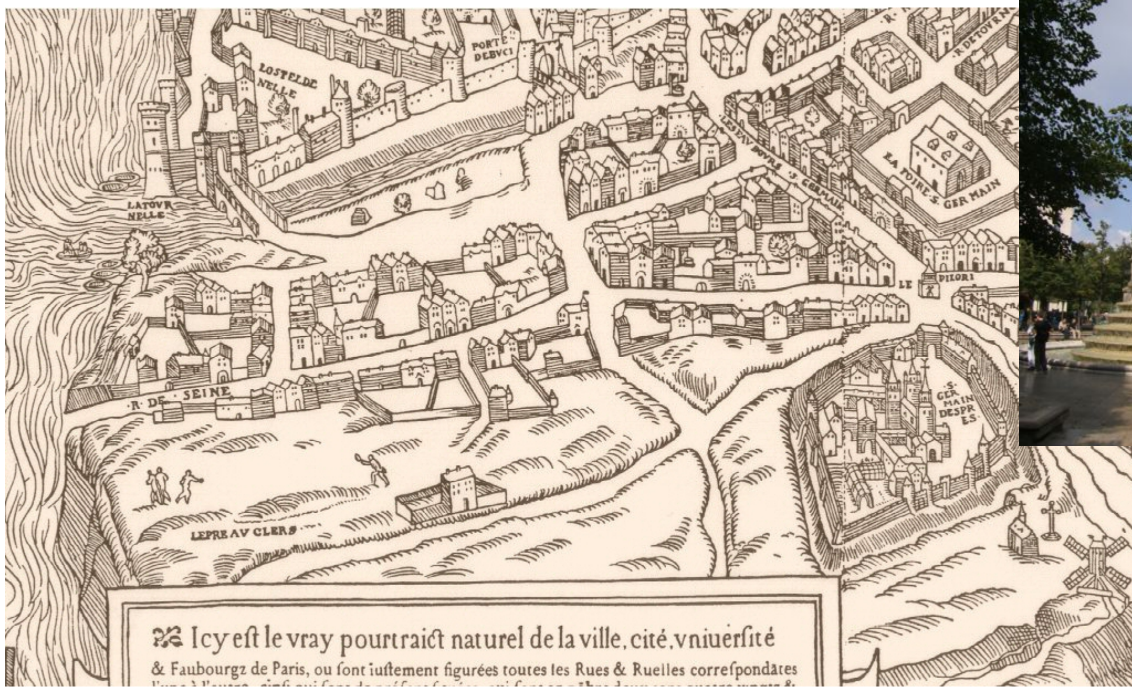
Journée « Chaire Finance & Développement Durable »

March 26 2008

This talk

1. (Very) brief reminder on optimal transportation
2. Generalizing coherent and regular risk measures to the multivariate case
3. Generalizing Kusuoka's theorem on coherent regular risk measures

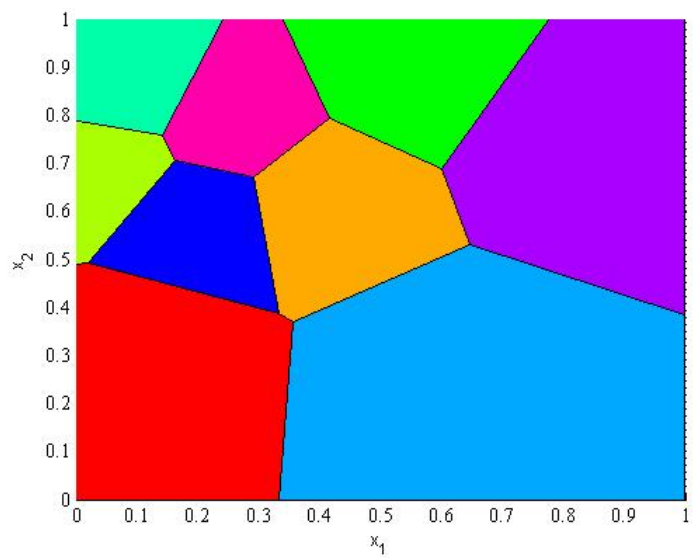
What is optimal transport?



At the start of the XVth century, Paris had 17 public fountains, and 250.000 inhabitant… that is a fountain for every 15.000 inhabitants

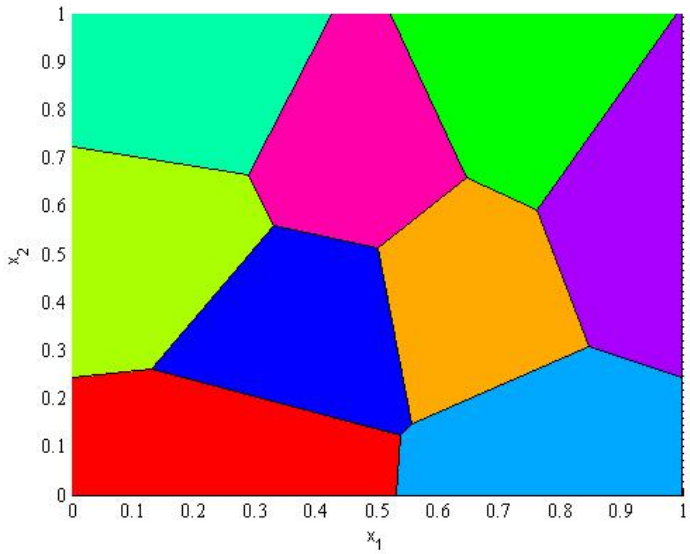
A problem of supply and demand…

- Suppose each fountain has a capacity of 15.000 users
- Inhabitants are uniformly spread on the surface of the city
- Fountains are not uniformly spread…
- Without a regulating mechanism, people will choose the closest fountain. Some will be overused, others will be underused.



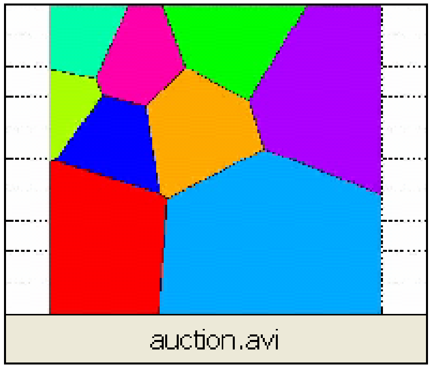
… regulated by prices

- One can use a system of differentiated prices for each fountains: raise the prices of the fontains in excess demand, and decrease the prices of the fountains in excess supply
- there is a system of **equilibrium prices** which adjusts supply to demand:



… attained by a Walrasian auction

- On this animation one can see an example of **Walrasian auction** process which leads to determination of equilibrium prices.



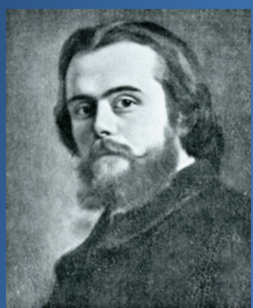
Some remarks

- The mechanism adjusting prices to regulate demand is nothing else than a **Walrasian “tâtonnement”** algorithm, which leads to numerical determination of prices, as put forward by Paul Samuelson in 1947.

- The need for differentiated prices to regulate demand is related to the fact that fountain distribution is not uniform. There is actually a strong connection between this problem and the **Gini index**.

- The fact that the distribution of facilities (here, fountains) is discrete is the present case but can be taken **continuous** without conceptual modifications.

Formalisation: Walrasian equilibrium



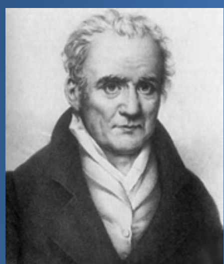
Let μ be the inhabitants distribution on $[0,1]^2$
(normalized by $\iint_{[0,1]^2} d\mu = 1$);

Let P_n be the distribution of the fountains
(located in $\{Y_1, ..., Y_n\}$, where Y_k has capacity p_k, $\sum_{k=1}^{n} p_k = 1$);

Then there exists a **price system** $w_1, ..., w_n$ such that each inhabitant in $u \in [0,1]^2$ choses the fountain $\varphi(u) \in \{1, ..., n\}$ which maximizes his/her utility
$V^w(u) = \max_k \{\langle u, Y_k \rangle - w_k\}$, and one has $Y_{\varphi(u)} = \nabla V(u)$.

$u \rightarrow \nabla V(u)$ is the **gradient of a convex function**, pushing forward the distribution of the inhabitants μ towards the fountain distribution P_n, which is denoted $\varphi \# \nu = P_n$.

Monge-Kantorovich problem and Brenier theorem



Let μ and P be two probability measures on $\mathbb{R}^d$ with second moments, such that μ is absolutely continuous. Then

$$\sup_{U \sim \mu, X \sim P} E\left[\langle U, X \rangle\right]$$

where the supremum is over all the couplings of μ and P if attained for a coupling such that one has $X = \nabla V(U)$ almost surely, where V is a convex function $\mathbb{R}^d \to \mathbb{R}$ which happens to be the solution of the dual Kantorovich problem

$$\inf_V \int V(u)\, d\mu(u) + \int V^*(x)\, dP(x)\,.$$

Applications of the Monge-Kantorovich problem to Economics

Many existing works:

- Hedonic models (Chiappori, Ekeland, Heckman)
- Mechanism design (Carlier)
- Urban economics (Ekeland, Carlier)
- General equilibrium (Levine)

Many perspectives…

1. Partial identification in econometrics (with I. Ekeland et M. Henry)
2. Specification tests (with V. Chernozhukov)
3. Risk measures (with M. Henry)
4. New matching algorithms (with G. Carlier et F. Santambrogio)

Today

COMONOTONIC MEASURES OF MULTIVARIATE RISKS

(joint work with Marc Henry, Université de Montréal)

paper available at
http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1115729

Motivation

why risk measures?

- current events...
- incomplete markets

2 uses:

- measurement: provide management with indicators of the level of risk
- management: determine capital budgeting rules (Basle II etc.)

problem with current literature:

- divergence practice/theory
- axioms are sometimes difficult to justify (and teach)
- does not extend well to natural situations (eg. multivariate risk)

This paper

- recall recent literature on risk measures and reinterpret the axioms

- propose multivariate extension

- propose a re-interpretation in terms of collective surplus on an Arrow-Debreu market

- provide a computational algorithm

The Value-at-Risk (VaR)

Aim: measure & manage risk of portfolio's contingent loss Y.

- $VaR_\alpha(Y)$ = smallest capital amount to cover losses in $\alpha\%$ cases...
- is robust to tail behaviour (eg. more than variance)
- has become a market standard for market risk measurement (Basle II 1st pillar)
- is however criticized among both practitioners and academics

Problem. VaR can fail to be subadditive: $VaR_\alpha(Y_1 + Y_2)$ can be greater than $VaR_\alpha(Y_1) + VaR_\alpha(Y_2)$...

Why is this a problem?... "creative accounting", "financial shennanigans" etc.

Desirable axioms for a risk measure

Definition. A functional $\varrho : L_d^\infty \to \mathbb{R}$ is called a *coherent risk measure* if it satisfies the following properties:

- Monotonicity (MON): $X \leq Y \Rightarrow \varrho(X) \leq \varrho(Y)$
- Translation invariance (TI): $\varrho(X+m) = \varrho(X) + m\varrho(1)$
- Convexity (CO): $\varrho(\lambda X + (1-\lambda)Y) \leq \lambda\varrho(X) + (1-\lambda)\varrho(Y)$ for all $\lambda \in (0,1)$.
- Positive homogeneity (PH): $\varrho(\lambda X) = \lambda\varrho(X)$ for all $\lambda \geq 0$.

Definition. $\varrho : L^\infty \to \mathbb{R}$ is called a *regular risk measure* if it satisfies:

- Law invariance (LI): $\varrho(X) = \varrho(\tilde{X})$ when $X \sim \tilde{X}$.
- Comonotonic additivity (CA): $\varrho(X+Y) = \varrho(X) + \varrho(Y)$ when X, Y are comonotonic, i.e. weakly increasing transformation of each other.

Maximal correlation risk measure

Result (Kusuoka, 2001). A coherent risk measure ϱ is regular if and only if for some increasing and nonnegative function ϕ on $[0,1]$, we have

$$\varrho(X) := \int_0^1 \phi(t) F_X^{-1}(t) dt,$$

where F_X denotes the cumulative distribution functions of the random variable X (thus $Q_X(t) = F_X^{-1}(t)$ is the associated quantile).

ϱ is called a *Maximal correlation risk measure*. Examples include:
- **Expected shortfall**: $\phi(t) = 1_{\{t \geq \alpha\}}$
- **Exponential risk measure**: $\phi(t) = 1 - e^{-\alpha t}$.

Other classes of risk measures exist, without comonotonic additivity.

The problem

- **Problem:** what can be said for risks which are multidimensional?
- **Interest?** risk usually has several dimension (price/liquidity ; multicurrency portfolio ; environmental/financial risk, etc.)
- Literature on multivariate risk measure: Jouini, Meddeb, & Touzi (2004); Rüschendorf (2006) focus on **coherent measures**. We look to generalize **regular measures** as well.

Higher dimension extension

What are the difficulties in extending risk measures to the multivariate case?

- **COHERENCE**
  - **Monotonicity** ⇐ **Not obvious**
  - **Translation invariance** ⇐ **OK**
  - **Convexity** ⇐ **OK**
  - **Positive homogeneity** ⇐ **OK**

- **REGULARITY**
  - **Law invariance** ⇐ **OK**
  - **Comonotonic additivity** ⇐ **Not obvious**

How to extend comonotonicity?

By the rearrangement inequality of Hardy and Littlewood, we have:

Two random vectors X and Y in L^∞ are comonotonic if for some random vector $U \sim \mu$, we have

$$U \in argmax_{\tilde{U}} \left\{ \mathbb{E}[X\tilde{U}],\ \tilde{U} \sim \mu \right\}, \text{ and}$$
$$U \in argmax_{\tilde{U}} \left\{ \mathbb{E}[Y\tilde{U}],\ \tilde{U} \sim \mu \right\}$$

which is equivalent to the existence of φ_1 and φ_2 nondecreasing and a random variable U such that $X = \varphi_1(U)$ and $Y = \varphi_2(U)$ almost surely.

- **Financial interpretation: X and Y are comonotonic if they share a common maximal risk exposure.**
- **Geometric interpretation: X and Y are comonotonic if they have the same L2 projection on the equidistribution class of U.**

Extending comonotonicity

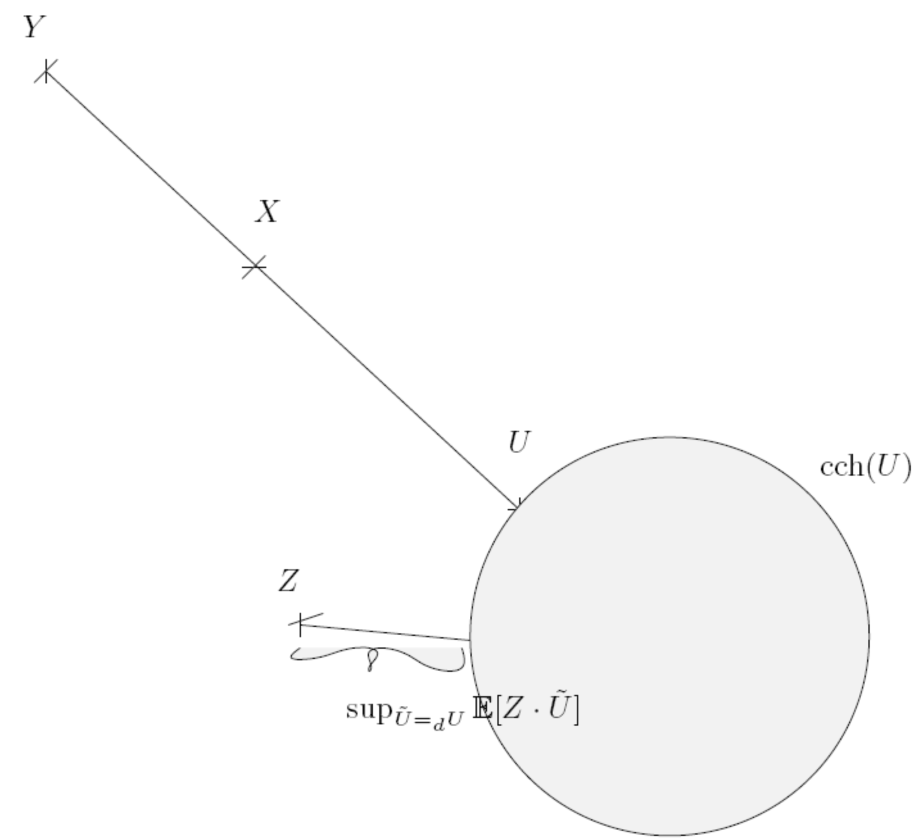
A variational characterization will be the basis for our generalized notion of comonotonicity.

Definition (μ-comonotonicity). Let μ be an atomless probability measure on $\mathbb{R}^d$. Two random vectors X and Y in L^∞_d are called μ-comonotonic if for some random vector $U \sim \mu$, we have

$$\begin{aligned} U &\in argmax_{\tilde{U}} \left\{ \mathbb{E}[X \cdot \tilde{U}],\ \tilde{U} \sim \mu \right\}, \text{ and} \\ U &\in argmax_{\tilde{U}} \left\{ \mathbb{E}[Y \cdot \tilde{U}],\ \tilde{U} \sim \mu \right\} \end{aligned}$$

equivalentely, X and Y are μ-comonotonic if there exists two convex functions V_1 and V_2 and a random variable U such that $X = \varphi_1(U)$ and $Y = \varphi_2(U)$, where $\varphi_1 = \nabla V_1$ and $\varphi_2 = \nabla V_2$.

Illustrating comonotonicity



The subtleties of higher-dimensional comonotonicity

In dimension one, one recovers the classical notion of comotonicity regardless of the choice of μ. However, in dimension greater than one, the comonotonicity relation crucially depends on the baseline distribution μ, unlike in dimension one. The following lemma makes this precise:

Lemma 1 *Let μ and ν be atomless probability measures on $\mathbb{R}^d$. Then:*
- In dimension $d = 1$, μ-comonotonicity always implies ν-comonotonicity.
- In dimension $d \geq 2$, μ-comonotonicity implies ν-comonotonicity if and only if $\nu = T\#\mu$ for some location-scale transform $T(u) = \lambda u + u_0$ where $\lambda > 0$ and $u_0 \in \mathbb{R}^d$. In other words, comonotonicity is an invariant of the location-scale family classes.

Extending maximal correlation risk measures

By the rearrangement inequality of Hardy and Littlewood, we can write:

$$\int_0^1 \phi(t) F_X^{-1}(t)dt = \max\left\{\mathbb{E}[X\tilde{U}] : \tilde{U} \sim \mu\right\}.$$

where μ if the probability distribution of ϕ, and the maximum is taken over all the random variables with distribution μ.

- interest of this variational formulation? admits a natural generalization in higher dimension.

A representation result

The following result is a multivariate extension of Kusuoka's theorem.

Theorem. Let ϱ be a measure with the subadditivity, law invariance, μ-comonotonic additivity and positive homogeneity properties. Then there exists a measure $\hat{\mu}$

$$\varrho(X) = \max\{\mathbb{E}[X.U] : U \sim \hat{\mu}\}$$

thus ϱ is a maximal correlation risk measure. Further the distribution of $U \sim \hat{\mu}$ is a obtained from μ by a location scale transform, that is there exist $\lambda > 0$ and $u_0 \in \mathbb{R}^d$ such that

$$\lambda(U - u_0) \sim \mu.$$

Conclusion

- Examples of application: measures of risks which have several components which are not perfect substitutes for each other
  - environmental/financial risk
  - price/liquidity risk
  - multi-currency portfolio
  - etc.

- Link with Non-Expected-Utility theory (Schmeidler, Yaari…): risk measures can be interpreted as (the opposite of) utility functionals over lotteries.

Thank you !

alfred.galichon@polytechnique.edu