Valorisation d’options hors-la-monnaie par échantillonnage d’importance

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Deuxième Printemps de la Chaire Finance & Développement Durable, 26 mars 2008

Chaire Finance & Développement Durable

Option pricing: a new adaptive Monte carlo method

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1. Motivation: Option pricing
2. Importance Sampling for variance reduction
3. Particle methods to approximate the optimal importance law
4. Simulation results

General framework

- **We consider a Markov chain** $(X_n)_{n\geq 0} \in E = \mathbb{R}^d$ **with**
  - **Initial distribution** $\mu_0 = \mathcal{L}(X_0)$
  - **Transition kernels** $Q_k = \mathcal{L}(X_k \mid X_{k-1})$
  - **Joint distributions** $\mu_{0:k} = \mathcal{L}(X_0, \cdots, X_k) = \mu_{0:k-1} \times Q_k$
- **The Goal is to Compute efficiently the expectation** $\mu_{0:n}(H)$

$$\mu_{0:n}(H) = \mathbb{E}[H(X_{0:n})] = \mathbb{E}[H(X_0, \cdots, X_n)]$$

for a given $H \,:\, \mathbb{E}^{n+1} \longrightarrow \mathbb{R}$

Example of application: Pricing a european call

▶ We consider a price process $(S_k)_{k\geq 0}$ modelized by

$$S_k = V_k(X_k) = S_0 \exp(M_k + \sigma X_k)$$

where $(X_k)_{k\geq 0}$ is a Markov chain and $(M_k)_{k\geq 0}$ is such that S is martingale

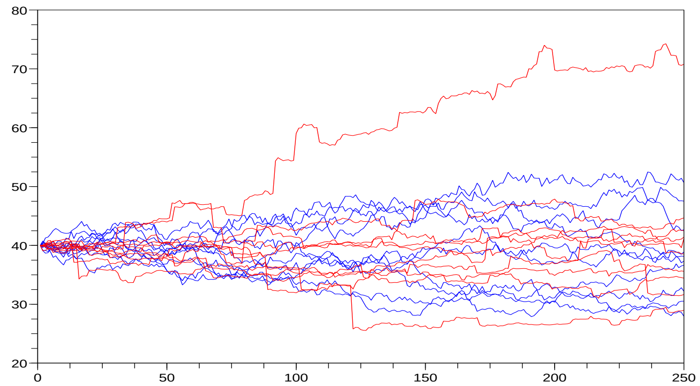
● Black-Scholes $=>$ No spikes

$X_k = W_{t_k}$ where W is the $BM(0,1)$

● NIG Lévy model $=>$ Spikes

$X_k = L_{t_k}$ where L is the Normal

Inverse Gaussian Lévy process



▶ The price of the European call with maturity t_n and strike K is given by

$$\mathbb{E}[H(X_{0:n})] = \mathbb{E}[(V_n(X_n) - K)^+]$$

▶ Crude Monte Carlo is inefficient when $K \gg S_0$ $=>$ variance reduction

Importance Sampling for variance reduction

- **Change of measure** $\mu \longrightarrow \nu$

$$\mu(H) = \mathbb{E}[H(X)] = \mathbb{E}[H(\tilde{X})\frac{d\mu}{d\nu}(\tilde{X})]\,, \quad \text{where} \quad X \sim \mu \quad \text{and} \quad \tilde{X} \sim \nu$$

- **Monte Carlo approximation**

$$\mathbb{E}[H(X)] \approx \frac{1}{M}\sum_{i=1}^{M} H(\tilde{X}_i)\frac{d\mu}{d\nu}(\tilde{X}_i)\,, \quad \text{where} \quad (\tilde{X}_1, \cdots, \tilde{X}_N) \quad \text{i.i.d.} \sim \nu$$

- **Optimal change of measure** $\mu \longrightarrow \nu^*$ **achieves zero variance if** $H \geq 0$

$$\nu^* = \frac{H\mu}{\mu(H)} = \frac{H\mu}{\mathbb{E}[H(X)]} \overset{\text{def}}{=} H \cdot \mu$$

- ν^* **depends on** $\mu(H) \Rightarrow$ **How to approximate** ν^* **?**

Progressive correction

- **We introduce some functions** $H_k \,:\, E^{k+1} \longrightarrow \mathbb{R}$ **for all** $0 \leq k \leq n$

$$H_0(x_0) = 1 \,, \quad \text{and} \quad H_n(x_{0:n}) = H(x_{0:n}) \,, \quad \text{for all} \quad x_{0:n} \in E^{n+1} \,.$$

- **We introduce some potential functions** $G_k \,:\, E^{k+1} \longrightarrow \mathbb{R}$

$$G_0(x_0) = 1 \,, \quad \text{and} \quad G_k(x_{0:k}) = \frac{H_k(x_{0:k})}{H_{k-1}(x_{0:k-1})} \,, \quad \text{for all} \quad x_{0:k} \in E^{k+1} \,.$$

- **We introduce the sequence of measures** $(\nu_{0:k})_{0 \leq k \leq n}$ **on** $(E^{k+1})_{0 \leq k \leq n}$

$$\nu_{0:k} = G_{0:k} \cdot \mu_{0:k} = \frac{G_{0:k}\mu_{0:k}}{\mu_{0:k}(G_{0:k})} \,, \quad \text{where} \quad G_{0:k} = \prod_{p=0}^{k} G_p \,.$$

- $G_{0:n} = H \Rightarrow$ $\nu_{0:n}$ **is the optimal importance distribution for** $\mu_{0:n}(H)$

$$\nu_{0:n} = H \cdot \mu_{0:n} = \nu^*_{0:n}$$

Evolution of $(\nu_{0:k})_{0\leq k\leq n}$

- **Evolution of** $(\nu_{0:k})_{0\leq k\leq n}$

$$\nu_{0:k-1} \xrightarrow[\text{Mutation}]{(1)} \eta_{0:k} = \nu_{k-1} \times Q_k \xrightarrow[\text{Correction}]{(2)} \nu_{0:k} = G_k \cdot \eta_{0:k}$$

- **[Del Moral & Garnier 2005] consider the sequence of measures** $(\gamma_{0:k})_{0\leq k\leq n}$ **on** $(E^{k+1})_{0\leq k\leq n}$ **such that for all test function** ϕ **on** E^{k+1}**,**

$$\gamma_{0:k}(\phi) = \mathbb{E}[\prod_{p=0}^{k} G_p(X_{0:p})\phi(X_{0:k})] \; ,$$

=> $\gamma_{0:n}(1) = \mu_{0:n}(H) = \mathbb{E}[H(X_{0:n})]$

- **Link between** $\gamma_{0:n}(1)$ **and** $(\eta_{0:k})_{0\leq k\leq n}$

$$\gamma_{0:n}(1) = \prod_{k=0}^{n} \eta_{0:k}(G_k)$$

Approximation of $\nu_{0:n}$ by particle methods

▶ The idea is to replace $\eta_{0:k} = \nu_{0:k-1} \times Q_k$ by its empirical measure

$$\eta_{0:k}^N = S^N(\nu_{0:k-1} \times Q_k) = \frac{1}{N}\sum_{i=1}^N \delta_{X_{0:k}^i}$$

where $(X_{0:k}^1, \cdots, X_{0:k}^N)$ **are i.i.d.** $\sim \nu_{0:k-1} \times Q_{k-1}$

▶ Particle approximation $(\nu_{0:k}^N)_{0\leq k\leq n}$

$$\nu_{0:k-1}^N \xrightarrow[\text{Selection and mutation}]{(1)} \eta_{0:k}^N = S^N(\nu_{0:k-1}^N \times Q_k) \xrightarrow[\text{Correction}]{(2)} \nu_{0:k}^N = G_k \cdot \eta_{0:k}^N$$

▶ Particle approximation $(\gamma_{0:k}^N)_{0\leq k\leq n}$

$$\gamma_{0:k}^N = G_k \eta_{0:k}^N \gamma_{0:k-1}^N(1) \quad \textbf{hence} \quad \gamma_{0:n}^N(1) = \prod_{k=0}^n \eta_{0:k}^N(G_k)$$

Algorithm

- **Initialization:**

Set $\nu_0^N = \nu_0 = \mu_0$

- **Selection: Generate independantly**

$$(\tilde{X}_{0:k}^1, \cdots, \tilde{X}_{0:k}^N) \quad \text{i.i.d.} \quad \sim \quad \nu_{0:k}^N = \sum_{i=1}^N \omega_k^i \, \delta_{X_{0:k}^i}$$

- **Mutation: Generate independantly for each** $i \in \{1, \cdots, N\}$**,**

$$X_{k+1}^i \quad \sim \quad Q_{k+1}(\tilde{X}_k^i, \cdot) \quad \text{then set} \quad \eta_{0:k+1}^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_{0:k+1}^i}$$

- **Weighting: For each particle** $i \in \{1, \cdots, N\}$**, compute**

$$\omega_{k+1}^i = \frac{G_{k+1}(X_{0:k+1}^i)}{\sum_{j=1}^N G_{k+1}(X_{0:k+1}^j)} \quad \text{then set} \quad \nu_{0:k+1}^N = \sum_{i=1}^N \omega_{k+1}^i \, \delta_{X_{0:k+1}^i}$$

Density estimation

- At the end of the algorithm, we get $\nu_{0:n}^N \approx \nu_{0:n}^*$.

But Importance sampling requires a smooth approximation of ν^*

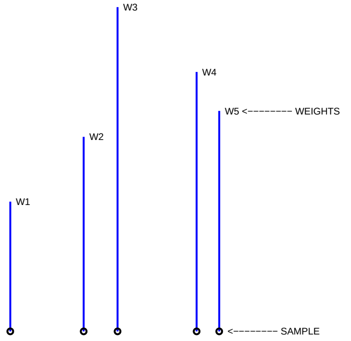
- **Kernel of order 2** K

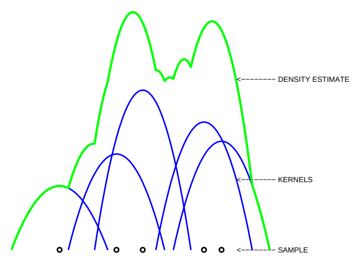
$$K \geq 0 \qquad \int K = 1 \qquad \int x_i \, K = 0 \qquad \int |x_i \, x_j| \, K < \infty$$

- **Rescaled kernel** K_h $\qquad K_h(x) = \frac{1}{h^d} \, K(\frac{x}{h})$

- $$\nu^N = \sum \omega^i \, \delta_{\xi^i} \xrightarrow[K_h * \cdot]{\text{Density estimation}} \nu^{N,h} = \sum \omega^i \, K_h(\cdot - \xi^i)$$

- **Optimal choice of** h $=>$ $\mathbb{E}\|\nu_{0:n}^{N,h} - \nu_{0:n}^*\|_1 \leq \frac{C}{N^{\frac{4}{2(d+4)}}}$





Adaptive choice of the sequence $(H_k)_{0 \leq k \leq n}$

[Cérou & al. 2006] [Hommem-de-Mello & Rubinstein 2002] [Musso & al. 2001]

► **In the case of european call pricing** $H(x_{0:n}) = (V_n(x_n) - K)^+$

$$\begin{cases} H_n(x_{0:n}) = (V_n(x_n) - K)^+ \ , \\ \\ H_k(x_{0:k}) = \max\left((V_k(x_k) - K_k), \varepsilon\right) \ , \quad \text{for all } 1 \leq k \leq n-1 \ , \quad \text{where} \end{cases}$$

- $\varepsilon > 0$ **ensures the positivity of** H_k **for** $1 \leq k \leq n-1$
- K_k **is a r.v. depending on** $(V_k^1 = V_k(X_{0:k}^1), \cdots, V_k^N = V_k(X_{0:k}^N))$ **and on parameter** $\rho \in (0,1)$**:**

$$K_k = V_k^{([\rho N])} \quad \text{where} \quad V_k^{(1)} \leq \cdots \leq V_k^{(N)} \ ;$$

Variance of the estimator

► **Importance sampling**

$$\mu(H) = \mathbb{E}[H(X)] \approx IS^{M,N} = \frac{1}{M}\sum_{i=1}^{M} H(\tilde{X}_i)\frac{d\mu_{0:n}}{d\nu_{0:n}^{N,h}}(\tilde{X}^i)\,,$$

where $(\tilde{X}^1, \cdots, \tilde{X}^M)$ **i.i.d.** $\sim \nu_{0:n}^{N,h}$

$$Var(IS^{M,N}) \leq \frac{\mu(H)^2}{M}\,\mathbb{E}\left[\left\|\frac{d\nu^*}{d\nu^{N,h}}\right\|_\infty \|\nu^* - \nu^{N,h}\|_{L^1}\right]$$

$$\leq C'\frac{\mu(H)^2}{MN^{\frac{4}{2(d+4)}}}\,.$$

Some simulation results

- **Pricing of a European call** **Maturiry : 1 year ; Volatility 20%/year**
- **Pricing of a European call** $N = 200$, $M = 10N$, $\rho = 20\%$

K	**Variance ratio BS**
60	45
65	194
70	690
75	4862
80	16190

References

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