A class of GARCH models with exogenously-driven volatility for electricity spot prices

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26/03/2008

Chaire Finance & Développement Durable

Outline

Introduction : stationarity in time series

$(X_t)_{t\in\mathbb{Z}}$ a real process.

Définition

(X_t) is strictly stationary if, for all h and all $k \geq 1$,

$(X_1, X_2, \ldots, X_k)$ and $(X_{1+h}, X_{2+h}, \ldots, X_{k+h})$ have the same distribution.

(X_t) is second-order stationary if $EX_t^2 < \infty$, and
i)EX_t is independent of t,
ii)Cov(X_t, X_{t+h}) is independent of t, for all h.

Standard models for stationary processes: ARMA (autoregressive moving-average)
Extensions ARI(ntegrated)MA, S(easonal)ARIMA

Volatility models

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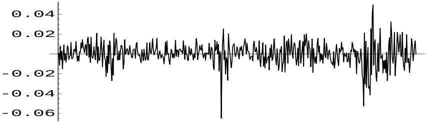

with empirical autocorrelations close to those of a white noise.

But empirical autocorrelations of the squares are often statistically significant.

+ Volatility clustering, Leptokurticity of the marginal distribution

Classes of volatility Models

$$\epsilon_t = \sigma_t \eta_t$$

where

- (η_t) is an iid (0,1) process
- (σ_t) is a process (volatility), $\sigma_t > 0$
- the variables σ_t and η_t are independent

Two main classes of models:

- GARCH-type (Generalized Autoregressive Conditional Heteroskedasticity): $\sigma_t \in \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots)$
- Stochastic volatility

Standard GARCH(1,1) Model

$$\begin{cases} \epsilon_t &= \sigma_t \eta_t, \qquad (\eta_t)\ iid\ (0,1) \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2, \qquad \omega > 0, \alpha, \beta \geq 0 \end{cases}$$

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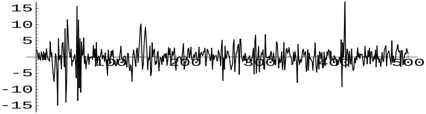

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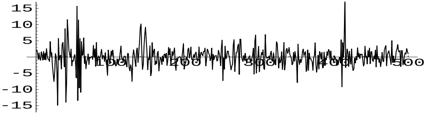
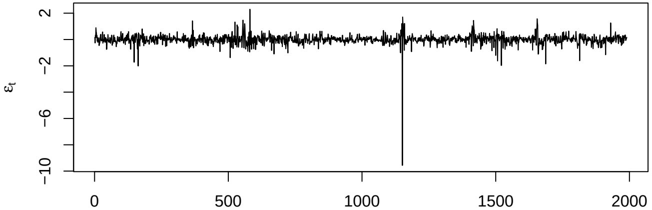


The coefficients can be constrained to produce stationary solutions:
- either in both senses,
- or only in the strict sense.

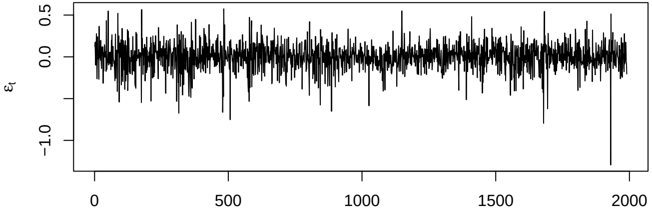
Without these conditions, the model is explosive.

Electricity spot prices

- Trends
- Volatility clustering, leptokurticity
- Seasonalities (weekly, monthly..)
- Dependency with respect to exogenous variables : temperature..



Electricity prices residuals (peak): 05/01/02−26/06/07



Electricity prices residuals (off peak): 05/01/02−26/06/07

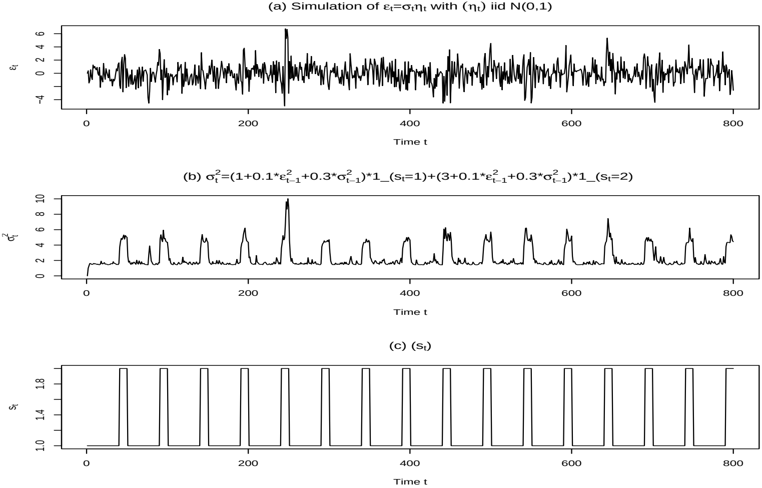
A GARCH(1,1) Model driven by an exogenous process

$$\begin{cases} \epsilon_t &= \sigma_t \eta_t, \qquad (\eta_t) \; iid \; (0,1) \\ \sigma_t^2 &= \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2, \end{cases}$$

where
- $\omega(\cdot) > 0, \alpha(\cdot), \beta(\cdot) \geq 0$
- (s_t) is a sequence of real numbers $s_t \in E = \{1, \ldots, d\}$.

For electricity prices, s_t could be an integer giving information about : the day in the week (e.g. week-end or not), the level of temperature or the excess temperature over a curve of average temperature.

Example : (s_t) periodic



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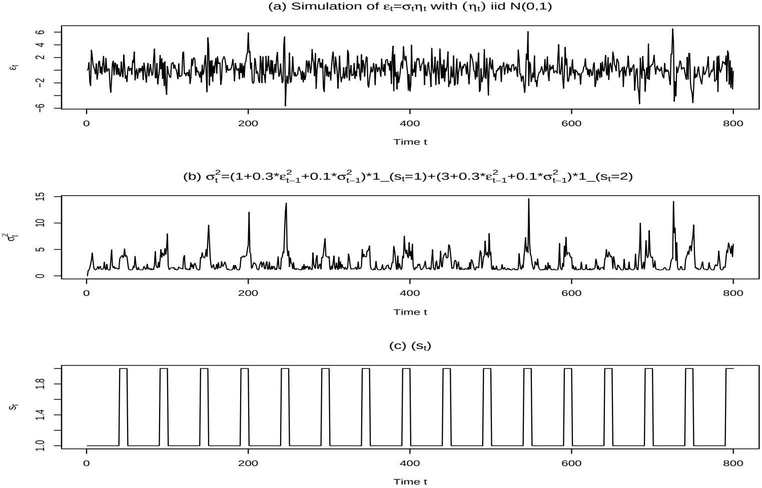
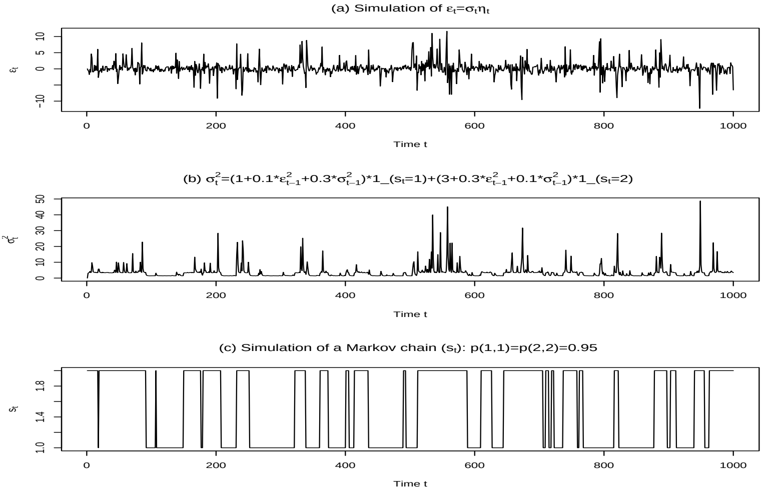


Example : (s_t) realization of a Markov chain



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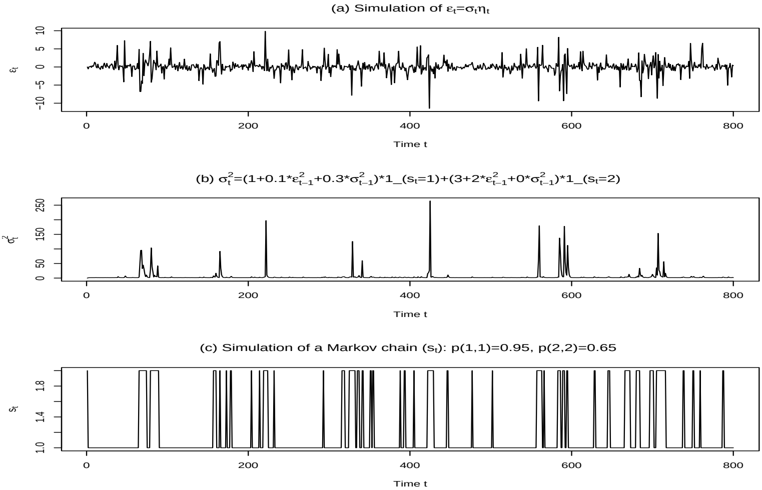


TS models with time-dependent coefficients

- **Nonstationary processes:** Priestley (1965), Whittle (1965), Hallin (1986)
- **Locally stationary processes:** Dalhaus (1997)
- **Periodic models:** Periodic ARMA (Anderson and Vecchia (1983), Lund and Basawa (2000)); Periodic GARCH (Bollerslev and Ghysels (1996))
- **Time-varying ARMA models:** Kwoun and Yajima (1986), Bibi and Francq (2003), Francq and Gautier (2004), Azrak and Mélard (2006)
- **Non-stationary volatility models:** Engle and Rangel (2005), Dalhaus and Subba Rao (2006), Amado and Teräsvirta (2008)

Existence of non-explosive solutions

$$\begin{cases} \epsilon_t &= \sigma_t \eta_t, \qquad (\eta_t)\ iid\ (0,1) \\ \sigma_t^2 &= \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2 \end{cases}$$

Let $a(x,y) = \alpha(x)y^2 + \beta(x)$. $[\sigma_t^2 = \omega(s_t) + a(s_t, \eta_{t-1})\sigma_{t-1}^2]$

Theorem

For $j = 1, \dots d$ *let* $\mathcal{T}(t,j,n) = \{\tau \in \{0,\dots,n\} \mid s_{t-\tau} = j\}$.
Assume that $\forall t, \quad |\mathcal{T}(t,j,n)|/n \to \pi_j,$ *as* $n \to \infty$, *for some* $\pi_j \geq 0$ *with* $\sum_{j=1}^d \pi_j = 1$. *Then, if*

$$\gamma_0 := \sum_{j=1}^d \pi_j E\{\log a(j,\eta_0)\} < 0,$$

the model admits a nonanticipative solution (ϵ_t).

Existence of non-explosive solutions

The nonanticipative solution is given by

$$\epsilon_t = \left\{ \omega(s_t) + \sum_{n=1}^{+\infty} a(s_t, \eta_{t-1}) \dots a(s_{t-n+1}, \eta_{t-n}) \omega(s_{t-n}) \right\}^{1/2} \eta_t.$$

If $\gamma_0 > 0$, for any starting value h_0 we have

$$\sigma_t^2 \to +\infty, \ a.s. \quad t \to \infty.$$

If, in addition, $E|\log \eta_0^2| < \infty$ then

$$\epsilon_t^2 \to +\infty, \ a.s. \quad t \to \infty.$$

Remarks

- The *local* stationarity condition,

$$E\{\log a(j, \eta_0)\} < 0, \qquad j = 1, \ldots, d$$

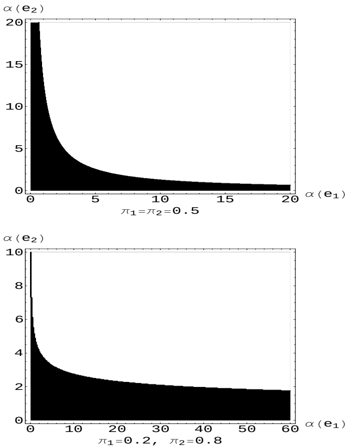
  implies the existence of a solution.
- Simple necessary condition for the existence of a solution:

$$\prod_{j=1}^{d} \beta^{\pi_j}(j) < 1.$$

- In the ARCH(1) case, a more explicit condition is:

$$\prod_{j=1}^{d} \alpha^{\pi_j}(j) < e^{-E \log \eta_0^2}.$$

Example of stability region : ARCH(1), 2 regimes



Remarks

- If for one regime $\alpha(j) = \beta(j) = 0$ and $\pi_j > 0$, a non explosive solution always exists.
- The condition coincides with the strict stationarity condition of a Markov Switching GARCH.
- Both the conditional and unconditional variances are time-varying: under appropriate conditions

$$\text{var}(\epsilon_t) = \omega(s_t) + \sum_{n=1}^{\infty} \left(\prod_{i=0}^{n-1} (\alpha + \beta)(s_{t-i}) \right) \omega(s_{t-n}).$$

Existence of moments

Theorem

Let m a strictly positive integer such that $E\eta_t^{2m} < \infty$. If

$$\gamma_m := \prod_{j=1}^{d} \{Ea(j, \eta_0)^m\}^{\pi_j} < 1,$$

the model has a non anticipative solution (ϵ_t) with $E\epsilon_t^{2m} < \infty$. If $\gamma_m > 1$ there is no nonanticipative solution (ϵ_t) such that $E\epsilon_t^{2m} < \infty$.

Remark:

- The condition does not coincide with the moment condition of a Markov Switching GARCH.

Comparison with Markov-Switching models

$$\begin{cases} \epsilon_t & = \sigma_t \eta_t, \qquad (\eta_t) \ iid \ (0,1) \\ \sigma_t^2 & = \omega(S_t) + \alpha(S_t)\epsilon_{t-1}^2 + \beta(S_t)\sigma_{t-1}^2 \end{cases}$$

where (S_t) is an irreducible, aperiodic and stationary Markov chain on $\{1, \dots, d\}$.

- Existence of a strictly stationary solution under the same condition $\gamma_0 < 0$
- But the moment conditions are different (depend on the transition probabilities)

From a statistical point of view, (S_t) is non observed (hidden) which makes the likelihood intractable (path dependence).

Optimal prediction of squares

Time-varying ARMA(1,1) representation for ϵ_t^2:

$$\epsilon_t^2 = \omega(s_t) + (\alpha + \beta)(s_t)\epsilon_{t-1}^2 + u_t - \beta(s_t)u_{t-1}.$$

The optimal prediction of ϵ_t^2 at horizon 1 is

$$\hat{\epsilon}_t^2 \quad = \quad \omega(s_t) + (\alpha + \beta)(s_t)\epsilon_{t-1}^2 - \sum_{k \geq 0} \beta(s_t) \dots \beta(s_{t-k}) v_{t-k-1}$$

where $v_t = \epsilon_t^2 - \omega(s_t) - (\alpha + \beta)(s_t)\epsilon_{t-1}^2$.

Estimation

Data generating mechanism:

$$\epsilon_t = \sqrt{h_t}\eta_t, \quad h_t = \omega_0(s_t) + \alpha_0(s_t)\epsilon_{t-1}^2 + \beta_0(s_t)h_{t-1}.$$

Vector of parameters:

$$\theta = (\omega(1), \ldots, \omega(d), \alpha(1), \ldots, \alpha(d), \beta(1), \ldots, \beta(d))'$$

assumed to belong to a parameter space $\Theta \subset]0, +\infty[^d \times [0, \infty[^{2d}$.

The sequence (s_t) is known.

Gaussian Quasi-likelihood

Observations: $(\epsilon_1, \dots, \epsilon_n)$ [and also $(s_1, \dots, s_n)$].

$$L_n(\theta) = L_n(\theta; \epsilon_1, \dots, \epsilon_n) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi\tilde{\sigma}_t^2}} \exp\left(-\frac{\epsilon_t^2}{2\tilde{\sigma}_t^2}\right),$$

where for $t \geq 2$,

$$\tilde{\sigma}_t^2 = \tilde{\sigma}_t^2(\theta) = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\tilde{\sigma}_{t-1}^2.$$

with $\tilde{\sigma}_1^2 = \omega(s_1) + \alpha(s_1)\tilde{\epsilon}_0^2 + \beta(s_1)\tilde{\sigma}_0^2$. A QMLE of θ is defined as any measurable solution $\hat{\theta}_n$ of

$$\hat{\theta}_n = \arg\max_{\theta \in \Theta} L_n(\theta) = \arg\min_{\theta \in \Theta} \tilde{\mathrm{I}}_n(\theta),$$

where

$$\tilde{\mathrm{I}}_n(\theta) = n^{-1} \sum_{t=1}^{n} \tilde{\ell}_t, \qquad \text{and} \quad \tilde{\ell}_t = \tilde{\ell}_t(\theta) = \frac{\epsilon_t^2}{\tilde{\sigma}_t^2} + \log \tilde{\sigma}_t^2.$$

Use of the unconditional model

A0: (s_t) is a realization of a process (S_t) which is stationary, ergodic, and independent of (η_t).

Thus $\pi_j = P[S_t = j]$.
If

$$\gamma_0 = \sum_{j=1}^{d} \pi_j E\{\log a(j, \eta_0)\} = E\{\log a(S_t, \eta_0)\} < 0,$$

there exists a unique nonanticipative and strictly stationary solution $(\epsilon_{S,t})$ to the model

$$\epsilon_{S,t} = \sigma_{S,t}\eta_t, \quad \sigma^2_{S,t} = \omega_0(S_t) + \alpha_0(S_t)\epsilon^2_{S,t-1} + \beta_0(S_t)\sigma^2_{S,t-1}.$$

Assumptions

A1: $\theta_0 \in \Theta$ and Θ is compact

A2: $\sum_{j=1}^{d} \pi_j E\{\log a_0(j, \eta_0)\} < 0 \quad (a_0(j, \eta_0) = \alpha_0(j)\eta_0^2 + \beta_0(j))$
$\forall \theta \in \Theta,\ \prod_{j=1}^{d} \beta^{\pi_j}(j) < 1.$

A3: $\exists r, \rho \in (0, 1),\ C > 0,$

$$\forall i > 0, \quad E\left\{a_0^r(S_t, \eta_{t-1}) \dots a_0^r(S_{t-i}, \eta_{t-i-1})\right\} < C\rho^{i+1}.$$

A4: η_t^2 has a nondegenerate distribution with $E\eta_t^2 = 1$.

A5: For all j, $\alpha_0(j) + \beta_0(j) \neq 0$ and $\pi_j > 0$.
$\exists \ell \in \{1, \dots, d\}, \quad \alpha_0(\ell) > 0.$

Remark: A3 holds automatically if (S_t) is iid or if the local stationarity conditions hold.

Consistency and Asymptotic Normality

Theorem

Under **A0-A5**, *for* $\mathbb{P}_S$*-almost all sequence* (s_t), $\quad \hat{\theta}_n \to \theta_0, \quad a.s.$ *as* $n \to \infty$.

A6: θ_0 belongs to the interior of Θ $(\alpha_0(\cdot) > 0, \beta_0(\cdot) > 0)$

A7: $\kappa_\eta = E\eta_t^4 < \infty$

Theorem

Under **A0-A7**, *for* $\mathbb{P}_S$*-almost all sequence* (s_t),
$\sqrt{n}(\hat{\theta}_n - \theta) \overset{d}{\rightsquigarrow} \mathcal{N}(0, (\kappa_\eta - 1)J^{-1})$, *where*

$$J = E_{S,\eta}\left(\frac{1}{\sigma^4_{S,t}(\theta_0)} \frac{\partial \sigma^2_{S,t}(\theta_0)}{\partial \theta} \frac{\partial \sigma^2_{S,t}(\theta_0)}{\partial \theta'}\right)$$

$\sigma^2_{S,t} = \omega(S_t) + \alpha(S_t)\epsilon^2_{t-1} + \beta(S_t)\sigma^2_{S,t-1}$.

Remarks

- No moment assumption on the observed process
- A consistent estimator of J is

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\tilde{\sigma}_t^4(\hat{\theta}_n)}\frac{\partial\tilde{\sigma}_t^2(\hat{\theta}_n)}{\partial\theta}\frac{\partial\tilde{\sigma}_t^2(\hat{\theta}_n)}{\partial\theta'},$$

where

$$\tilde{\sigma}_t^2(\hat{\theta}_n) = \hat{\omega}_n(s_t) + \hat{\alpha}_n(s_t)\epsilon_{t-1}^2 + \hat{\beta}_n(s_t)\tilde{\sigma}_{t-1}^2(\hat{\theta}_n).$$

(no need to specify a process (S_t))

Examples of asymptotic variance matrices

$$\epsilon_t = \begin{cases} (1 + 0.1\epsilon_{t-1}^2)^{1/2}\eta_t & \text{if } s_t = 1 \\ \\ (3 + 0.1\epsilon_{t-1}^2)^{1/2}\eta_t & \text{if } s_t = 2 \end{cases} \qquad (S_t) \text{ Markov chain}$$

(η_t) iid $\mathcal{N}(0,1)$

- $p(1,1) = p(2,2) = 0.5$

$$\text{Var}_{as}(\sqrt{n}(\hat{\theta}_n - \theta)) = \begin{pmatrix} 7.41 & 0 & -1.62 & 0 \\ 0 & 56.78 & 0 & -8.96 \\ -1.62 & 0 & 1.30 & 0 \\ 0 & -8.96 & 0 & 5.28 \end{pmatrix}$$

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(η_t) iid $\mathcal{N}(0,1)$

- $p(1,1) = p(2,2) = 0.95$

$$\text{Var}_{as}(\sqrt{n}(\hat{\theta}_n - \theta)) = \begin{pmatrix} 3.83 & 0 & -1.33 & 0 \\ 0 & 300.51 & 0 & -53.24 \\ -1.33 & 0 & 1.58 & 0 \\ 0 & -53.24 & 0 & 32.39 \end{pmatrix}$$

Examples of asymptotic variance matrices

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(η_t) iid, mixture of normal distributions ($\kappa_\eta \approx 9$)

- $p(1,1) = p(2,2) = 0.95$

$$\text{Var}_{as}(\sqrt{n}(\hat{\theta}_n - \theta)) = \begin{pmatrix} 11.39 & 0 & -1.92 & 0 \\ 0 & 918.26 & 0 & -77.02 \\ -1.92 & 0 & 4.21 & 0 \\ 0 & -77.02 & 0 & 87.99 \end{pmatrix}$$

Useful Lemma

Main technical difficulty (for the proofs): absence of standard ergodic and CLT theorems.

Lemma (Francq and Gautier, 2004)

Let f be a measurable function, $f : \{1, \ldots, d\}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$, such that $Ef(S_t, S_{t-1}, \ldots, \eta_t, \eta_{t-1}, \ldots)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$. Then, for $\mathbb{P}_S$-almost all sequence (s_t),

$$\frac{1}{n}\sum_{t=1}^{n} f(s_t, s_{t-1}, \ldots, \eta_t, \eta_{t-1}, \ldots) \to Ef(S_t, S_{t-1}, \ldots, \eta_t, \eta_{t-1}, \ldots),$$

$\mathbb{P}_\eta - a.s.$

Summary and conclusions

- Standard GARCH and volatility models are inappropriate for series displaying nonstationarities
- The proposed model is conditional on the realizations of an exogenous discrete process
- It follows that, when existing, its solutions are non stationary. The conditions of existence, and of existence of moments, depend on the asymptotic frequencies of the states of the exogenous process, and on the model coefficients.
- QML estimation requires additional assumptions on the underlying sequence. The asympotic distribution of the QMLE depends on the whole distribution of the underlying process. Numerical implementation is not more difficult than in standard GARCH.