Simulation of delta hedging of an option with volume uncertainty

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Agenda

1. Introduction : volume uncertainty
2. Test description: a simple option
3. Results when the market is complete: price is the only uncertainty
4. Results when the market is incomplete: volume is random
5. Conclusions

1 – Introduction : volume uncertainty

EDF's activity subject to several risks

- EDF's economic result in France may vary because of :
  - Uncertainty of the demand, depending mainly on temperature
    - ±1℃ in winter ≈ ±1,5 GW
  - Uncertainty of the hydro inflows
    - Hydro ≈ 9 % of EDF production
  - Uncertainty of the availability of power plants
    - One nuclear power plant ≈ 1 GW
  - Uncertainty of market prices
    - Power, coal, fuel, CO_2

- No counterparty exists for the major part of uncertainties which impact EDF's results
  - A big part of the risk is not "hedgeable"

Uncertainty hedging

- The (almost) only available counterparty is the forward market (to handle price uncertainty)
  - Markets of options or climatic derivatives are not mature in France
  - A part of market activity deals with spot market (linked mainly to week ahead forwards and futures)
- What it the best solution to hedge price uncertainty in this situation?
  - Hedging purpose: reduce the influence of price uncertainty on the dispersion of results
  - One possibility : to use the classical delta hedging strategy

Delta hedging : classical theory

- Perfect (no arbitrage) and complete market hypothesis : a hedging portfolio is set to replicate the value of the considered contract
  - Considering an option whose payoff is $H(S_T)$ depending only on the commodity price S_T at time T, the hedging portfolio is then composed at time t by the volume Δ_t of the commodity itself :

$$\Delta_t = \frac{\partial V_t}{dS_t} \quad \text{with} \quad V_t = E_t^{\mathbf{Q}}\left[H\left(S_T\right)\right]$$

  - The balancing of the hedging portfolio is performed continuously
  - Under those conditions: whatever price evolution, the value of the hedging portfolio is always equal to the difference between the payoff and the initial value of the option V_0

$$V_0 + \int_0^T \Delta_t dS_t = H\left(S_T\right) \quad \text{with} \quad V_0 = E_0^{\mathbf{Q}}\left[H\left(S_T\right)\right]$$

Delta hedging in our context

- Theoretical hypothesis are not verified
  - The market is not complete: the hedging strategy will not replicate every uncertainties
  - Continuous hedging is not realistic
    - The cotation of products is not continuous
    - Calculation duration of the value of the portfolio do not allow frequent rebalancing of the hedging portfolio

- What is the efficiency of a delta hedging in incomplete market?
  - When the balancing of the portfolio is done periodically?
  - When « volume » uncertainties are not hedgeable?

⇒ Simulations of a simple portfolio (toy example)

2 – Test description : a simple option

Option and price

- We <u>own</u> a European-type option
  - Strike *K*
  - Underlying spot market, maturity *T*
  - Volume P sold at *T* : deterministic ($P=P_{max}$) or random ($P\leq P_{max}$)

- Forward price model : 2 gaussian factors model

$$\frac{dF(t,T)}{F(t,T)} = \sigma_S e^{-a_F(T-t)} dz_S(t) + \sigma_L dz_L(t)$$

Short term volatility

Mean reversion

Long term volatility

  - *F(0,T)* = K
  - Spot price *S* at *T* : *S* = *F(T,T)*
  - *Martingale probability : F(t,T)* = E_t[S]

Volume

- Volume uncertainty

  - We model a random energy P_F which may limit the energy sold at maturity (≈ "availability" of the option)

$$dP_F(t,T) = \sigma_F e^{-a_F(T-t)} dz_F(t)$$

  - $P_F(0,T)=P_{max}$
  - $P_S = P_F(T,T)$
  - At maturity *T*, if *S* > *K*, the sold energy is P = min(P_{max},P_S)

Option value and initial delta

- The delta-hedging strategy is first defined as the sensitivity of the expectation of the payoff, under a martingale probability.

$$V_t = E_t\left[P\left(S-K\right)^+\right] \qquad \Delta_t = \frac{\partial V_t}{\partial F\left(t,T\right)}$$

- Option without volume uncertainty : $P = P_{max} = 12\ 000\ MWh$
  - Expectation of the option payoff at initial time : V_0 = 95 k€
  - Delta value at initial time : Δ_0 = 6 768 MWh
- Option with volume uncertainty : $P = min(P_{max}, P_S)$
  - Expectation of the option payoff at initial time : V_0 = 85 k€
  - Delta value at initial time : Δ_0 = 5 962 MWh

Hedging process

- At initial date
  - We sell the volume Δ_0 of forward
- At time t< T
  - We calculate the delta Δ_t
  - We update the hedging portfolio by selling (if $d\Delta_t>0$) or by buying (if $d\Delta_t <0$) the volume $d\Delta(t) = \Delta(t) - \Delta(t-1)$ at forward price *F(t,T)*
- At maturity T
  - The hedging portfolio is composed of a sold volume of Δ_{T-1} and has generated cash-flows corresponding to:

$$\sum_{t=0}^{T-1} d\Delta(t) F(t)$$

  - If $S=F(T,T) > K$, the volume Δ_{T-1} is furnished by the exercise of the option for a cost *K*; remaining power $(P- \Delta_{T-1})^+$ is sold on the spot market at price *S*.
  - If $S < K$, the volume Δ_{T-1} must be bought on the market at price *S*.

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Cash-flows at maturity

- Cash-flows

$$\Phi = \sum_{t=0}^{T-1} d\Delta_t F(t) \quad \leftarrow \text{Cash-flows linked to the balancing of the hedging portfolio}$$

$$+1_{S>K}\left\{\left(P-\Delta_{T-1}\right)\left(S-K\right)-\Delta_{T-1}K\right\} \quad \leftarrow \text{if } S > K$$

$$-1_{S<K}\Delta_{T-1}S \quad \leftarrow \text{if } S < K$$

- This expression can be rewritten

$$\Phi = \sum_{t=0}^{T-1} d\Delta_t F(t) - \Delta_{T-1}S + P\left(S-K\right)^+$$

- We compare the distribution of cash-flows Φ to the expectation of payoff at t=0

$$\Phi \overset{?}{=} V_0 = E_0\left[P\left(S-K\right)^+\right]$$

  - If the equality is verified, we have a discrete formulation of the previous equation:

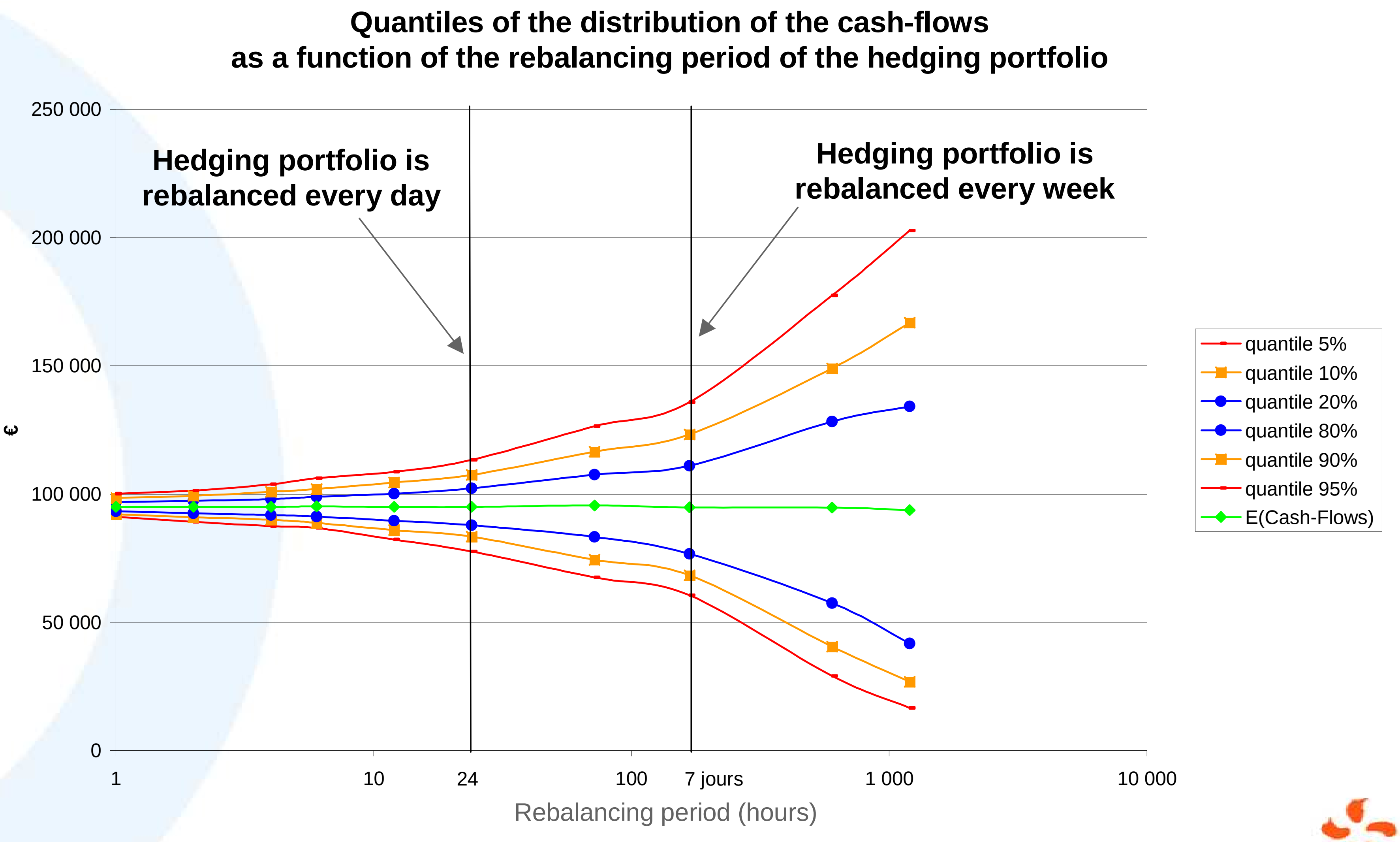
$$V_0 + \int_0^T \Delta_t dS_t = H\left(S_T\right)$$

Simulations

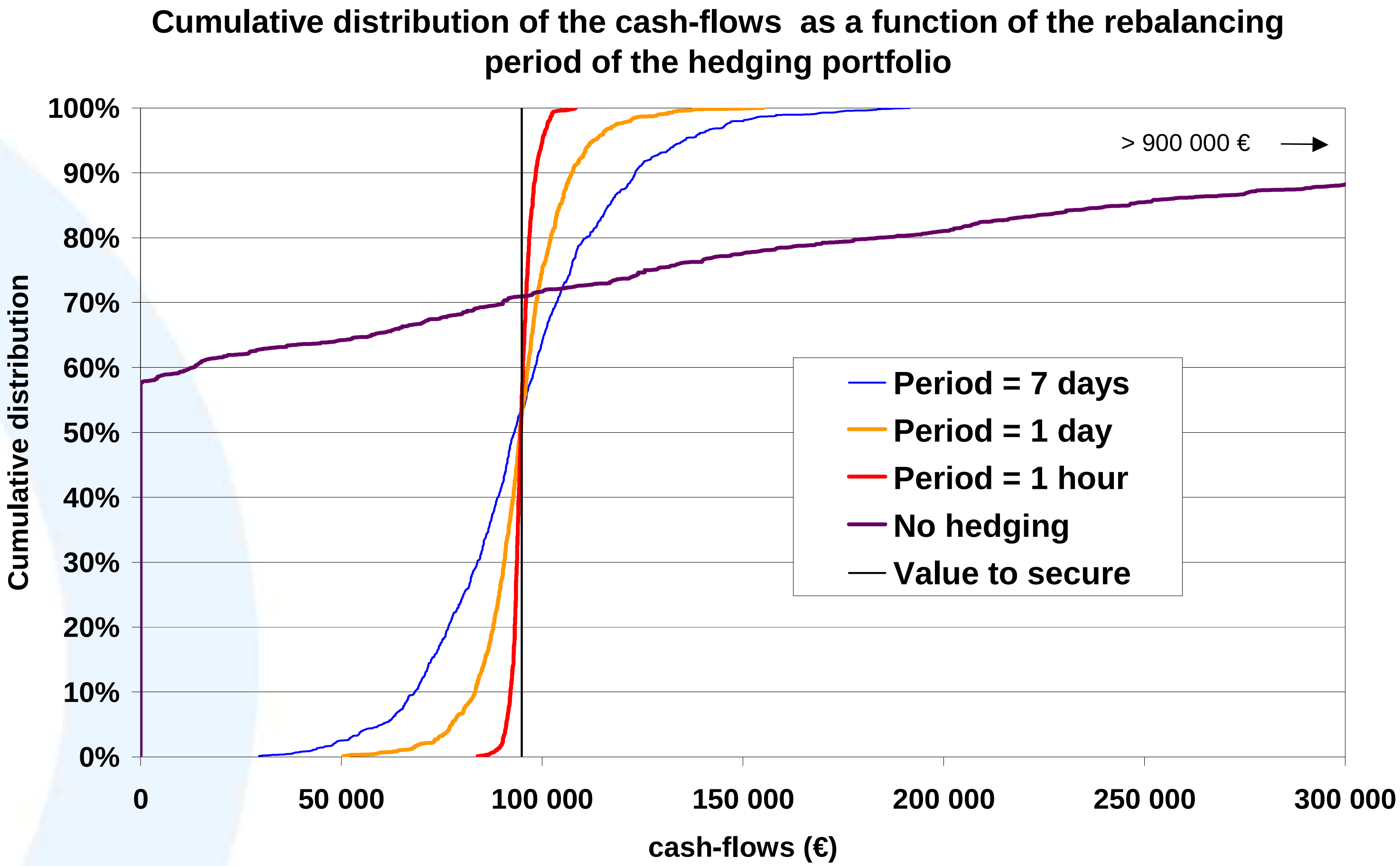
- We simulate 1000 paths of forward prices at hourly granularity
- The deltas are estimated for the corresponding forward prices over 5000 simulations of spot price.
- Result comparisons are performed with similar random variables
- Transaction costs are considered to be null
- We are only interested by the value of the hedging portfolio at the maturity T (we are not considering its value along the existence of the option)

3 – Results when the market is complete: price is random, volume is deterministic

Cash-flows quantiles

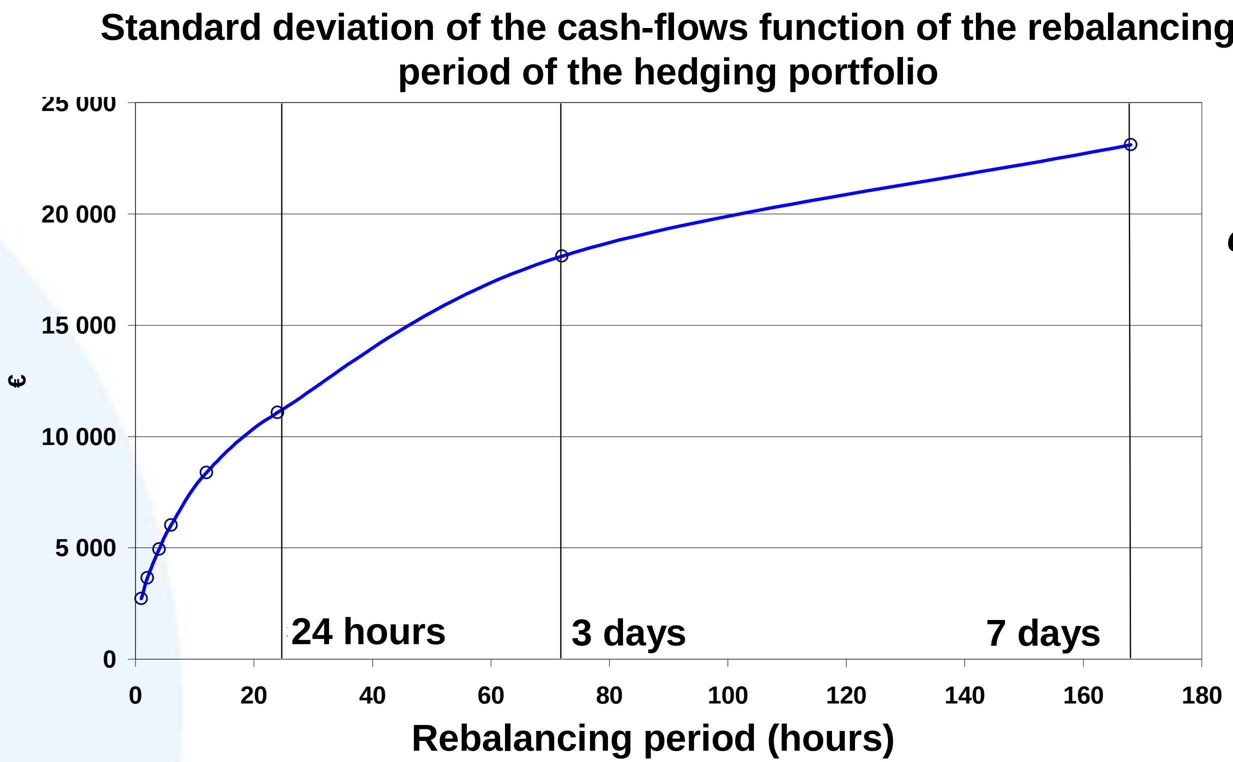


Cumulative distribution of cash-flows



- The efficiency of the hedging is verified if the hedging is continuously rebalanced (theoretical result in complete market)

Cash-flows standard deviation

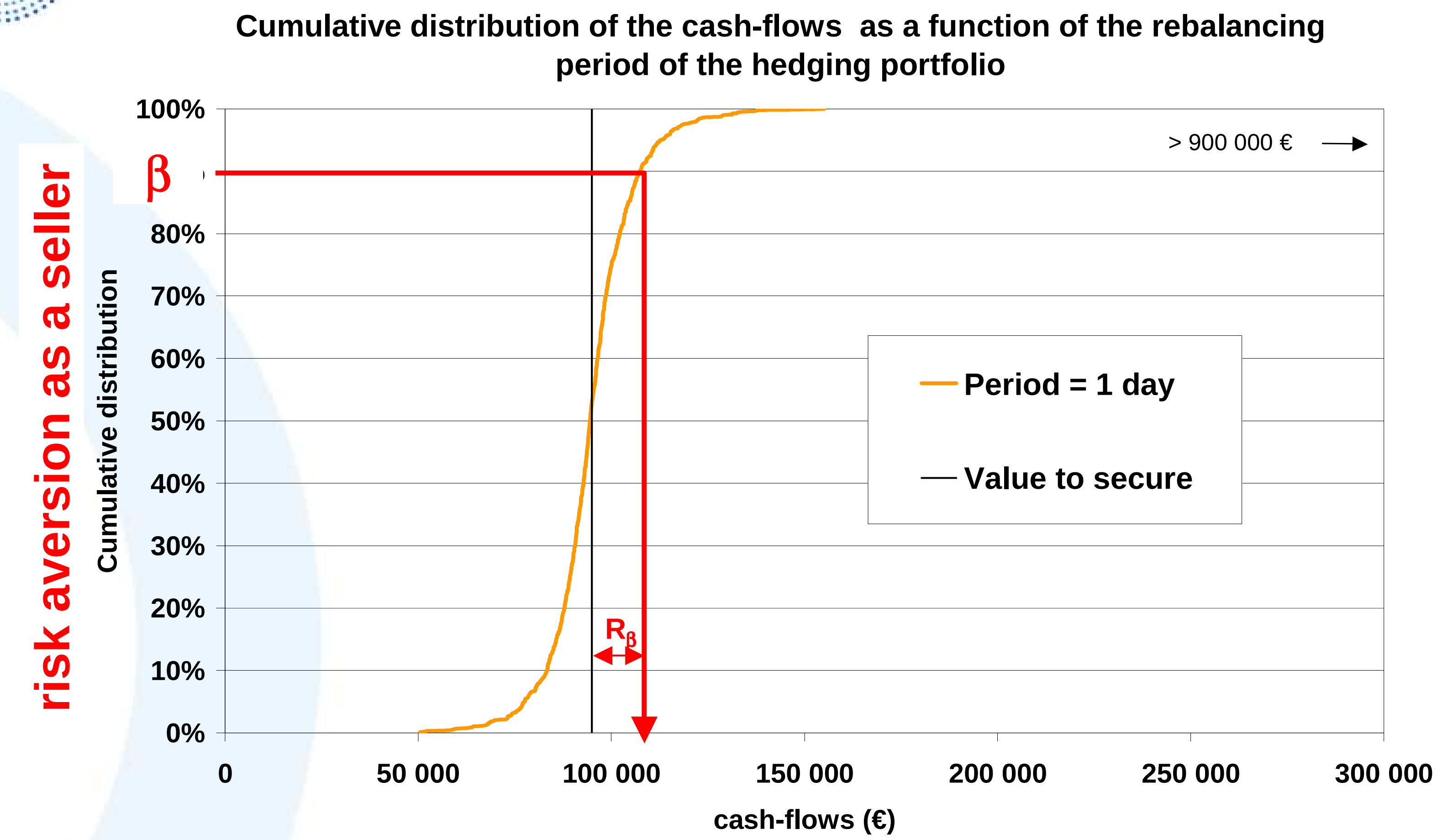


$$\sigma_{\phi}^{2} \approx \frac{\pi}{4n}\left(\sigma \frac{\partial V_0}{\partial \sigma}\right)^2$$

n the number of hedging operations

- Theoretical result : standard deviation is proportional to the square of hedging period
  - For an hourly balancing: coefficient of variation is around 3%
  - For a daily balancing : coefficient of variation is around 9%
  - For a weekly balancing : coefficient of variation is around 24%

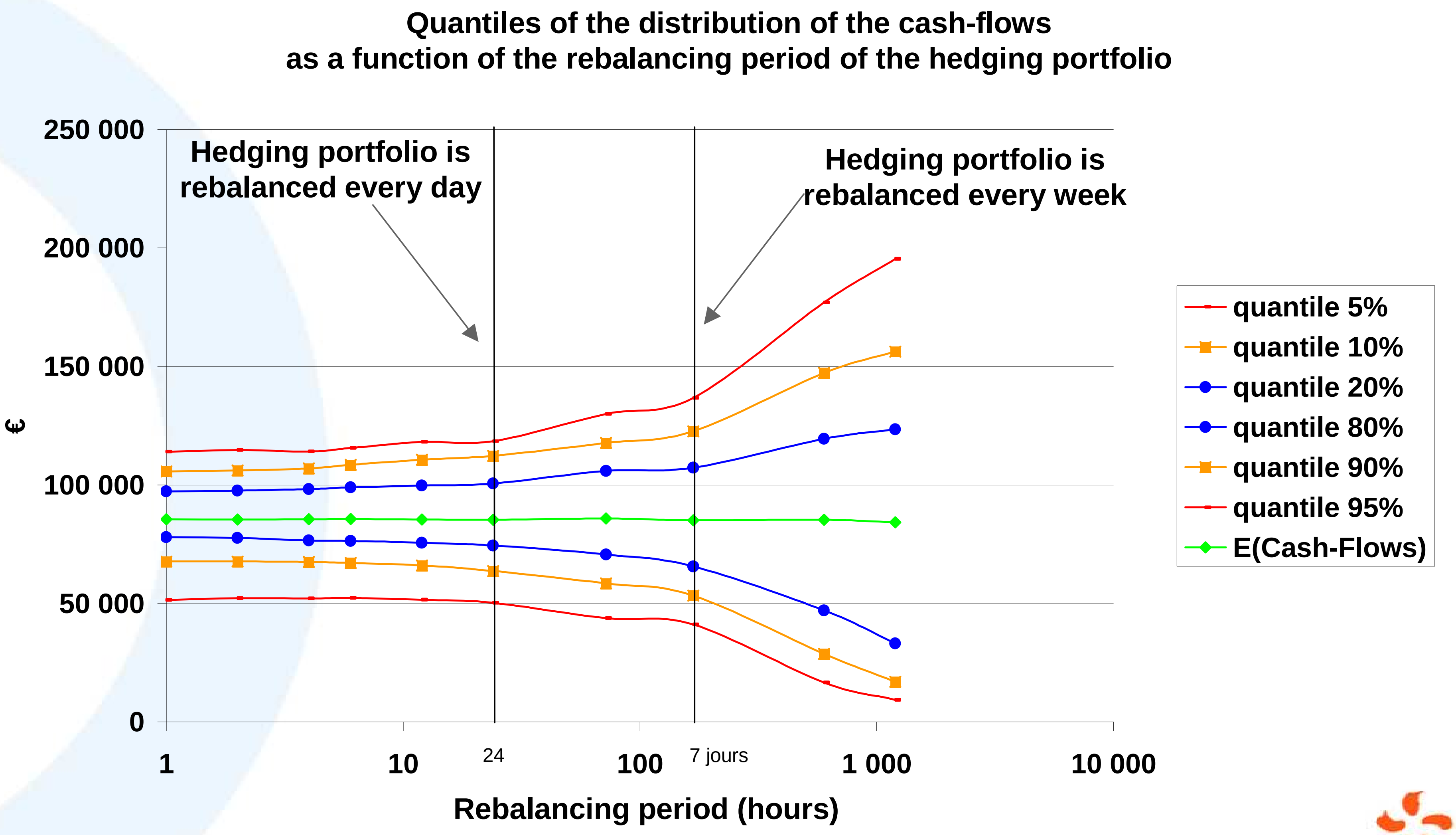
Risk aversion



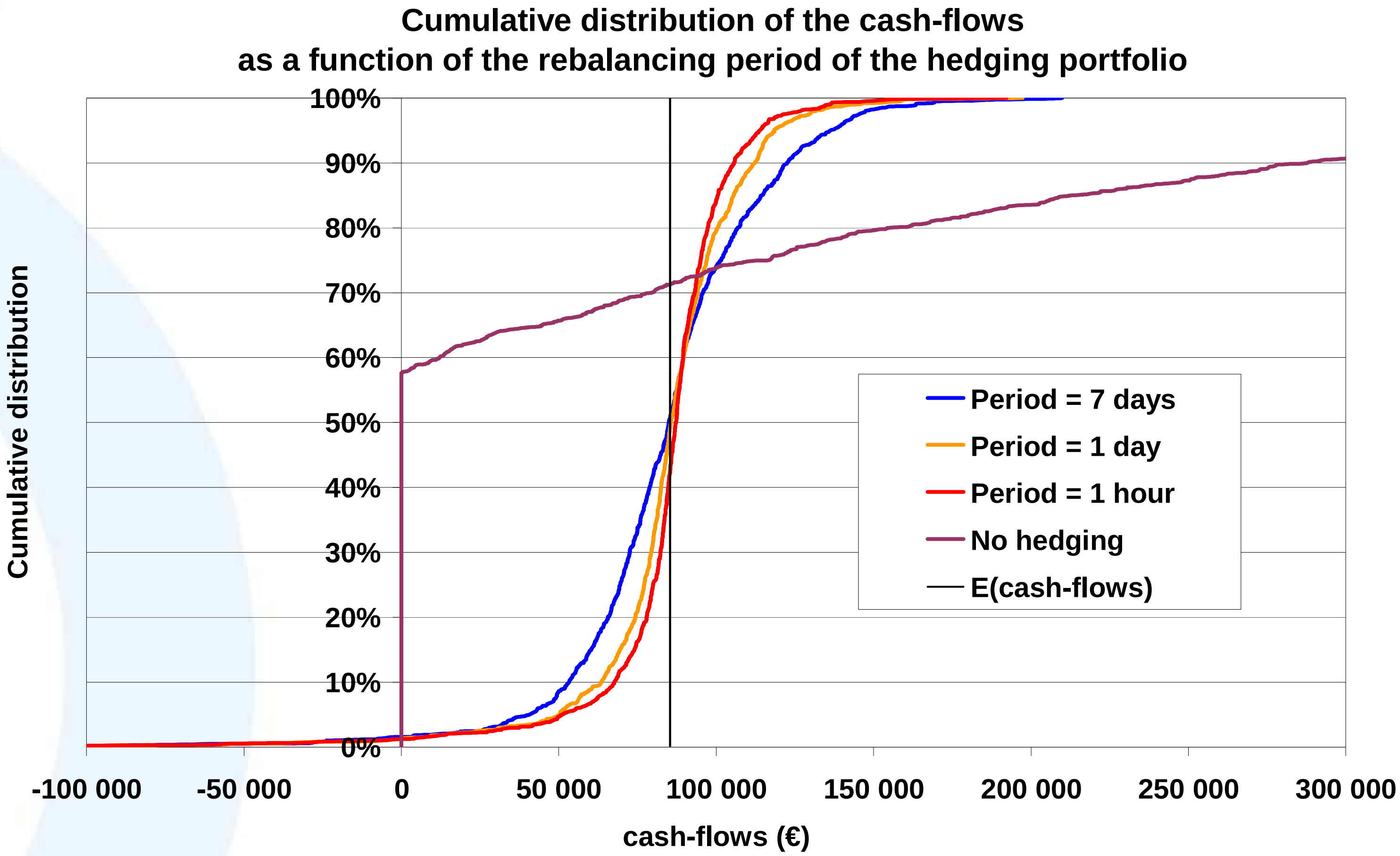
- As a seller of the option, if we are not able to hedge more than once a day, we would ask a price depending of our risk aversion β

4 – Results when the market is incomplete : prices and volume are random

Cash-flows quantiles



Cumulative distribution of cash-flows



- Frequent balancing of the hedging portfolio is less efficient (influence of volume uncertainty)

- Negative cash-flows are possible (tail of distribution)

Why negative cash-flows?

- Example of a particular scenario
  - At the beginning of the period: moderate prices, average available power → we sell the delta to hedge the cash-flows of our option
  - At the end of the period
    - Prices increase → we should sell more…
    - …but the forecast available power is decreasing → we buy, at possible higher prices than the prices we sold
  - Due to volume uncertainty, cash-flows linked to the exercise of the option may not compensate the cost of the hedging
  - In other words, this strategy lead us to sell on the forward market more energy than the amount we really have at maturity
    - The volume seen in the delta is the <u>expectation</u> of the volume at maturity

Introducing a volumetric risk aversion in the delta

- Assuming a big aversion to negative cash-flows, we may use a heuristic rule to limit the risks of such scenarios :

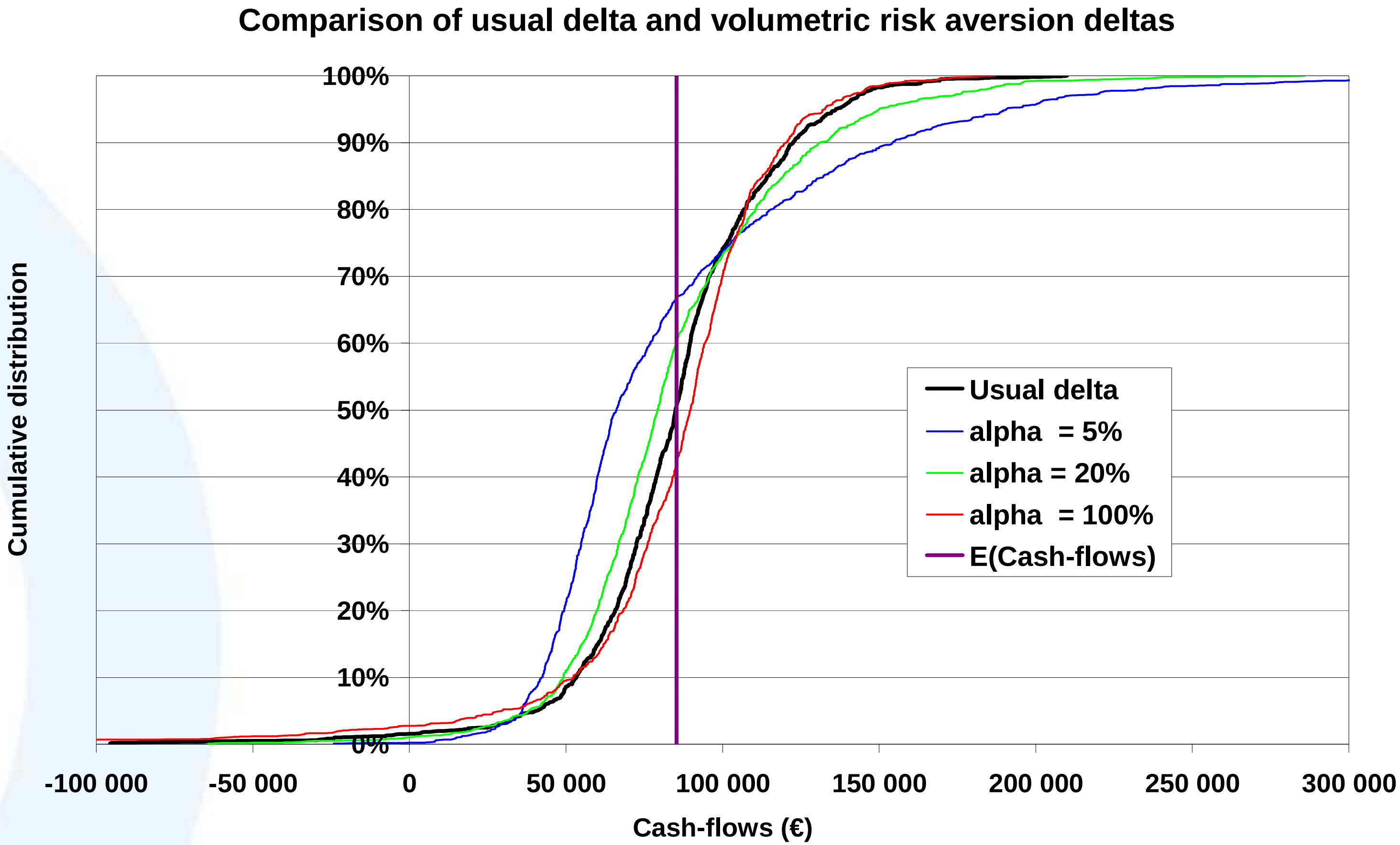
  - Instead of defining the delta as the sensitivity of the expected cash-flows for any available energy P at maturity, we define it as the sensitivity of the expected cash-flows for a given quantile α of P : P_α.

$$\Delta_t = \frac{\partial E_t\left[P_a \left(S-K\right)^+ \right]}{\partial F\left(t,T\right)}$$

  - If α is small enough, we limit the risk of “selling more than we have”

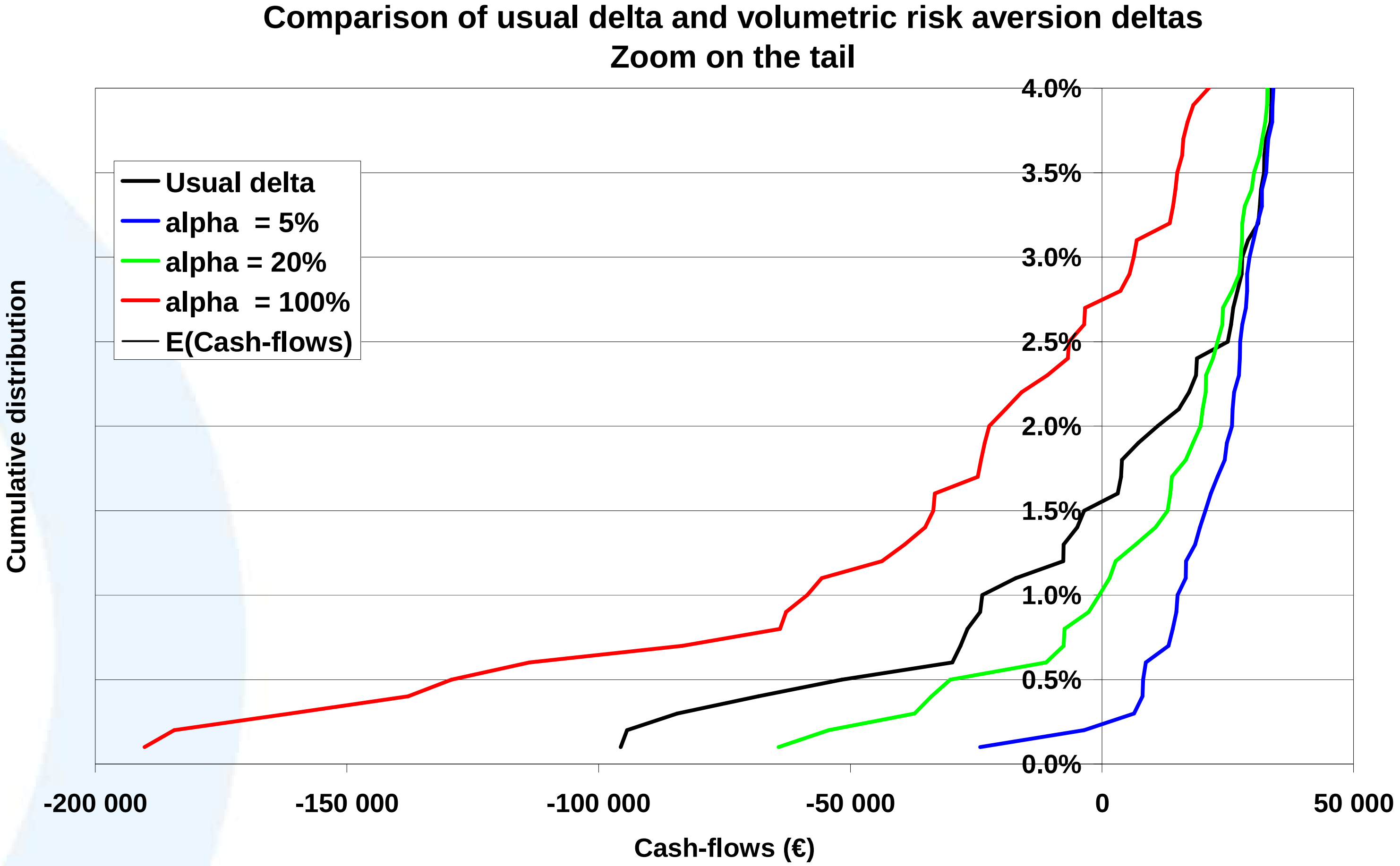
  - Same kind of approach developed in “pricing volumetric risk”, Kolos & Mardanov, Energy risk, october 2008, pp 54-60

Comparison of strategies for weekly hedging



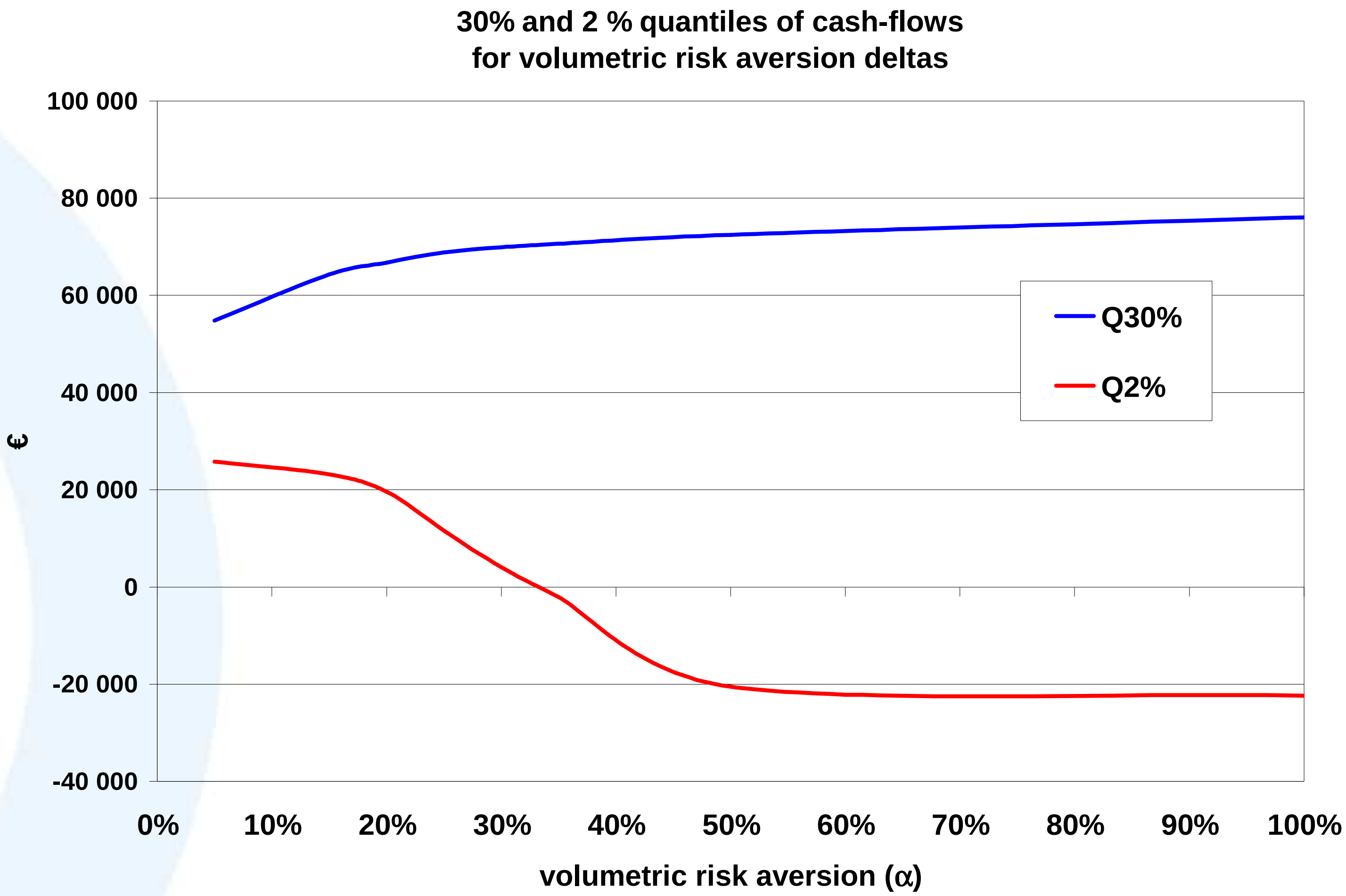
- As expected, the delta with volumetric risk aversion can limit the negative cash-flows (see following zoom on the tail)
- As a consequence, all the distribution of final cash-flows is changed

Zoom on the tail of the distributions



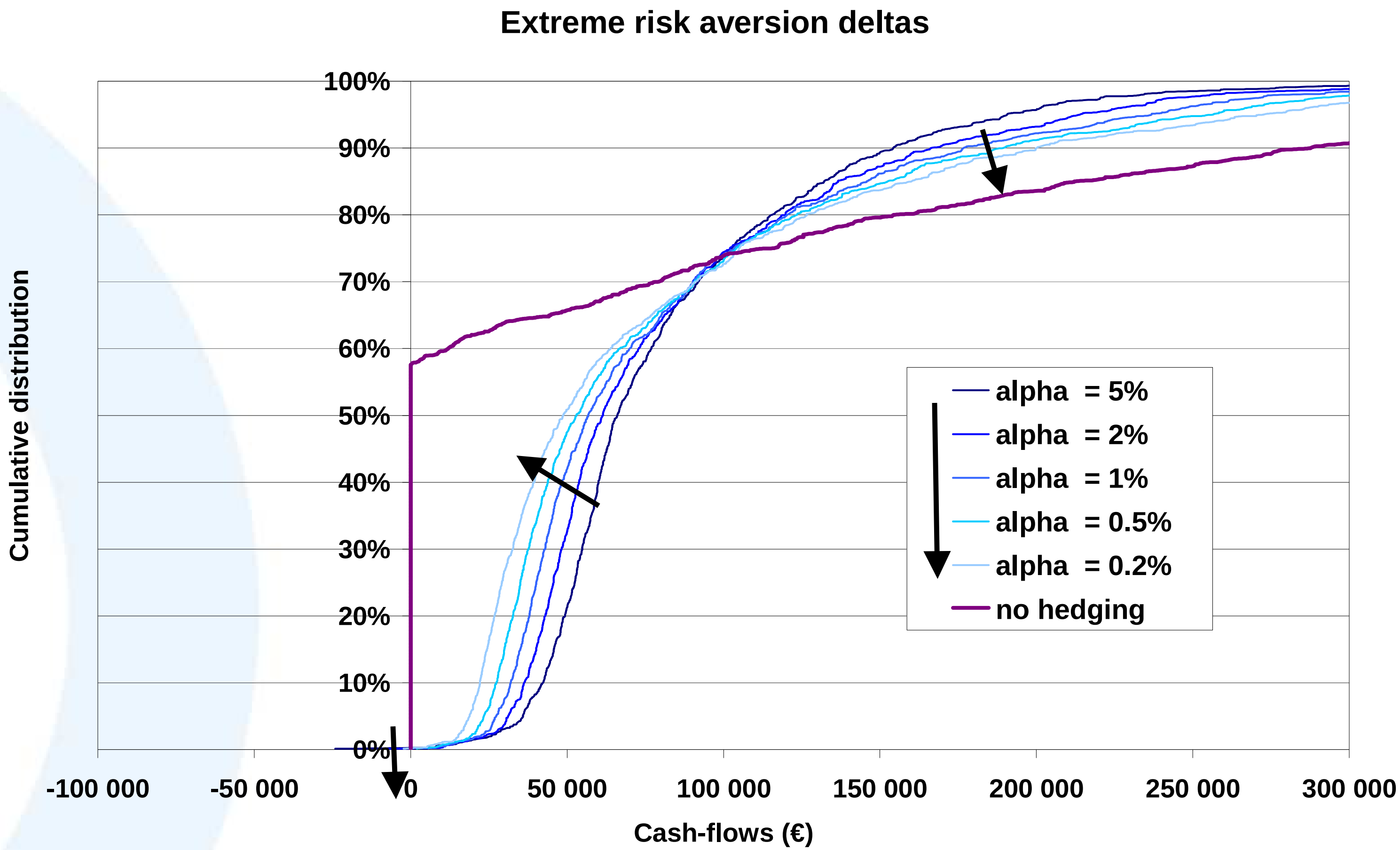
- The lower α, the lower probability of negative cash-flows

Compromise between « extreme » risk and « normal » risk (30% quantile)



- As the expected cash-flows remains the same, the cost for decreasing the extreme risks (negative cash-flows) is a reduction of gain in more likely scenarios

Pushing the extreme risk aversion to the limit



- With such an option, the only way of avoiding negative cash-flows (P=0) is not to hedge

5 – Conclusions

Main conclusions

- Even in complete market hypothesis, a realistic (non continuous) delta hedging strategy leads to residual risks that must be taken into account in pricing options
- With volume uncertainties, to shorten the rebalancing period of a delta hedging strategy reduces the variation of the cash-flows until a non compressible value due to the non-hedgeable volume uncertainty
- The hedging can be counter-productive (cash-flows can be negative because of conjunction of adverse prices/volume scenarios)
- These extreme risks can be limited (but not suppressed) while introducing a simple volumetric risk aversion heuristic rule in the delta calculation
  - It shows that a compromise between the reduction of extreme and more likely risks is needed
  - There is a big issue in the expression of risk aversion

For future studies (1/3): 2 categories of optimisation methods

- Optimisation under explicit risk constraints
  - Hedging strategy π such that :

$$\max_{\pi} E\left[CashFlows\right] \quad \text{under constraints} \quad \varphi\left[CashFlows\right] \leq \beta$$

  - where φ gives the risk constraints
  - Methods exist to take into account global constraints like EEaR (Extreme Earnings at Risk) or CVaR (Conditional Value at Risk), but
    - Local constraints or probability constraints are difficult to include in the problem
    - Solving this type of problems is generally time consuming (iterative methods)
- Maximisation of a utility function
  - Hedging strategy π such that :

$$\max_{\pi} E\left[g\left(CashFlows\right)\right]$$

  - Where g is a utility function which gives the risk aversion (typically : exponential functions which give penalties to adverse cash-flows)
    - The utility function is often complex is to define

For future studies (2/3)

- Simulation of hedging strategies
  - Simulation is a way to understand underlying mechanisms
  - Different hedging strategies which may take into account
    - Transaction costs
    - Liquidity issue market depth issue
    - Market Operational constraints which reduce the balancing frequency…
  - Back-testing over real data

For future studies (3/3)

- Use the link between risk factors: example in 1 dimension, correlation between forward price F and volume Q uncertainty
  - One portfolio with value V(F,Q), hedge C(F)

$$dV(F,Q) + dC(F) = \underbrace{\frac{\partial V}{\partial F}}_{\Delta_F} dF + \underbrace{\frac{\partial C}{\partial F}}_{\Delta_C} dF + \underbrace{\frac{\partial V}{\partial Q}}_{\Delta_Q} dQ$$

  - Gaussian log ratio for F and Q with volatility σ_F and σ_Q, correlation ρ
  - dV + dC variance

$$\sigma^2_{dV+dC} = \left(\underbrace{\Delta_{F+C}}_{\Delta_F + \Delta_C} \sigma_F F \right)^2 + \left(\Delta_Q \sigma_Q Q \right)^2 + 2\rho \, \Delta_{F+C} \Delta_Q \sigma_F \sigma_Q FQ$$

  - Position which minimises the variance of the evolution of the value of the hedged portfolio

$$\Delta^*_{F+C} = \underset{\Delta_{F+C}}{\arg\min} \left(\sigma^2_{dV+dC} \right) = -\rho \, \Delta_Q \frac{\sigma_Q Q}{\sigma_F F}$$