Estimations d'erreur des schémas de différences finies pour la résolution de l'équation HJB du contrôle stochastique

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Stochastic optimal control problem

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(P_x) \quad \begin{cases} \text{Min } \boldsymbol{E} \displaystyle\int_0^\infty \ell(y(t), u(t)) e^{-\lambda t} \mathrm{d}t; \\[2ex] \begin{cases} \mathrm{d}y(t) = f(y(t), u(t))\mathrm{d}t + \sigma(y(t), u(t))\mathrm{d}w(t), \\ y(0) = x, \end{cases} \\[2ex] u(t) \in U, \quad t \in [0, \infty[. \end{cases}
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- $\lambda \geq 0$: discounting factor,
- $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$: distributed cost,
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$: trend,
- $\sigma(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to$ space of $n \times r$ matrices
- w :standard r dimensional Brownian motion

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Smoothness of the value function

Theorem

Assume that U is compact and that ℓ, f and σ are Lipschitz w.r.t. y, uniformly in $u \in U$.
Then the state equation is well-posed, and the value function $V(x)$ is Lipschitz if λ is large enough, and Hölder otherwise with constant say μ_V.

HJB equation

$$\lambda v(x) = \inf_{u \in U} \{ \ell(x,u) + f(x,u) \cdot v_x(x) + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x,u)\, v_{x_i x_j}(x) \}, \quad \text{for all} \quad x \in \mathbb{R}^n.$$

- Covariance matrix: $a(x,u) := \sigma(x,u)\sigma(x,u)^T, \ \forall \ (x,u) \in \mathbb{R}^n \times \mathbb{R}^m.$
- All functions Lipschitz continuous and bounded: V unique bounded *viscosity solution* of HJB.

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A simple case

$$\lambda v(x) = \ell(x) + \Delta v(x)$$

Finite differences scheme, case $n = 1$, stepsize $h > 0$:

$$\lambda v(x) = \ell(x) + \frac{v(x-h) - 2v(x) + v(x+h)}{h^2}$$

Equivalently, multiplying by $h_0 > 0$, and adding $v(x)$ on each side:

$$v(x) = \frac{1}{1 + h_0\lambda}\left(\frac{h_0}{h^2}(v(x-h) + v(x+h)) + \left(1 - \frac{2h_0}{h^2}\right) v(x)\right)$$

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(Generalized) finite differences

Upwind Finite difference operator with step size h:

$$\delta_u[f]v(x) := f(x,u)_+ \cdot \frac{v(x+h\mathbf{1}) - v(x)}{h} + f(x,u)_- \cdot \frac{v(x) - v(x-h\mathbf{1})}{h}$$

Decomposition into rank 1 diffusions (α scalar, $\eta_i(x,u) \in \mathbb{R}^n$):

$$a(x,u) = \sum_{i \in I} \alpha_i(x,u)\eta_i(x,u)\eta_i(x,u)^\top$$

Second order Finite difference operator along direction $\eta_i(x,u)$

$$\Delta_{i,u} v(x) := \frac{v(x - h\eta_i(x,u)) - 2v(x) + v(x + h\eta_i(x,u))}{h^2}$$

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Grids

If the decomposition of a is such that $\eta_i(x,u)$ belongs to $h\mathbb{Z}^n$:

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Then independant numerical schemes over each grid $x_0 + h\mathbb{Z}^n$

Integer feasibility problem for each (x,u): difficult

FB, Zidani: Characterization of feasibility given the stencil size p, i.e.

$$\eta_i(x,u) \in h[-p,p]^n.$$

FB, Ottenwaelter, Zidani: fast computation ($O(p)$) when $n = 2$, based on arithmetic properties (walk in the Stern-Broco tree).

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Contractant fixed-point form

h_0: fictitious time step
$\beta := (1+\lambda h_0)^{-1} < 1$.

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\begin{aligned}
v(x) = \quad & \beta \inf_{u\in U} \Big\{ h_0 \ell(x,u) \\
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& + (1 - |f(x,u)|_1 \frac{h_0}{h} - 2 \sum_{i\in I} \alpha_i(x,u) \frac{h_0}{h^2}) v(x)) \Big\} .
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Monotonicity condition for the time step

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h_0 \left(\frac{|f(x,u)|_1}{h} + \frac{2}{h^2} \sum_{i\in I} \alpha_i(x,u) \right) \le 1.
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"Markov chain" maximum principle form

Stochastic matrix $M(\chi, x, u) \geq 0$, $\sum_\chi M(\chi, x, u) = 1$:

$$v(x) = \beta \inf_u \left(h_0 \ell(x, u) + \sum_\chi M(\chi, x, u) v(\chi) \right).$$

If v' associated with the perturbed cost ℓ':

$$v'(x) - v(x) \leq \beta \sup_u \Big(h_0 \ell'(x, u) - h_0 \ell(x, u) + \sum_\chi M(\chi, x, u)(v'(\chi) - v(\chi)) \Big).$$

and hence using $v'(\chi) - v(\chi) \leq \sup(v' - v)$

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Uniqueness and maximum principle

Theorem

The scheme has a unique condition v_h such that, if v' is the solution associated with ℓ':

$$v'_h - v_h \leq \frac{1}{\lambda} \sup(\ell' - \ell).$$

In particular

$$\|v_h\|_\infty \leq \frac{1}{\lambda} \|\ell\|_\infty.$$

Truncation errors for smooth solutions of the HJB equation

Hölder constants for the solution V:

$$\left| V^{(k)}(x+h) - V^{(k)}(x) \right| \leq C_k h^{\mu_k}.$$

Then

$$\left| \frac{V(x+h) - V(x)}{h} - V'(x) \right| \leq C_1 h^{\mu_1}.$$

$$\left| \Delta_{i,u} v(x) - V''(x)(\eta_i(x,u), \eta_i(x,u)) \right| \leq C_{3,i} h^{1+\mu_3}.$$

Set

$$C_3 := \sum_i C_{3,i}.$$

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Put V in the numerical scheme

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and hence,

$$\|V - v_h\|_\infty \leq \frac{1}{\lambda} \left(\|f\|_\infty C_1 h^{\mu_1} + \|a\|_\infty C_3 h^{1+\mu_3} \right).$$

Case of smooth subsolutions

(Viscosity) subsolutions of the HJB equation:

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Monotonicity: $v \leq V$

V_ε ε-subsolution: $V - \varepsilon \leq V_\varepsilon \leq V$.

$$\left| V_\varepsilon^{(k)}(x+h) - V_\varepsilon^{(k)}(x) \right| \leq C_k h^{\mu_k}.$$

Alors

$$\sup (V - v_h) \leq \varepsilon + \sup (V_\varepsilon - v_h) \leq \varepsilon + \frac{1}{\lambda} \left(\|f\|_\infty C_1 h^{\mu_1} + \|a\|_\infty C_3 h^{1+\mu_3} \right).$$

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$$\sup (V - v_h) \leq \varepsilon + \sup (V_\varepsilon - v_h) \leq \varepsilon + \frac{1}{\lambda} \left(\|f\|_\infty C_1 h^{\mu_1} + \|a\|_\infty C_3 h^{1+\mu_3} \right).$$

Shaking coefficients (Krylov 2000)

$$\lambda v(x) = \inf_{u \in U, |e| \leq 1} \{\ell(x - \varepsilon e, u) + f(x - \varepsilon e, u) \cdot v_x(x) + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x - \varepsilon e, u)\, v_{x_i x_j}(x)\}, \quad \text{for all } x \in \mathbb{R}^n.$$

Solution denoted V^ε. Lipschitz coefficient

$$|\ell(x,u) - \ell(x - \varepsilon e, u)| + |f(x,u) - f(x - \varepsilon e, u)| + |\sigma(x,u) - \sigma(x - \varepsilon e, u)| = O(\varepsilon)$$

Monotonicity+ stability of solutions of HJB equations if λ large enough (e.g. Jakobsen-Karlsen)

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Convexity of the set of subsolutions

Write the HJB equation

$$F(x, v(x), v'(x), v''(x)) = 0, \quad \text{for all } \; x \in \mathbb{R}^n,$$

$$F(x, r, p, Q) := \lambda r + \sup_{u \in U} \left(-\ell(x, u) - f(x, u) \cdot p - \tfrac{1}{2} a(x, u) \cdot Q \right)$$

F convex as supremum of affine functions

Therefore the following set is convex:

$$\{(r, p, Q);\; F(x, r(x), p(x), Q(x)) \leq 0 \quad \text{for all } \; x \in \mathbb{R}^n\}$$

It follows that the set of classical subsolutions is convex

Proof for viscosity subsolutions based on Ishii's lemma, see Barles and Jakobsen.

Smoothing of subsolutions

- $x \mapsto V^\varepsilon(x+\varepsilon e)$ subsolution if $|e| \leq 1$.
- $\rho \mathbb{R}^n \to \mathbb{R}_+$, C^∞ with support in $B(0,1)$;
- $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$.
- $V_\varepsilon(x) := \int_{\mathbb{R}^n} V^\varepsilon(y)\rho_\varepsilon(x-y)\mathrm{d}y = V^\varepsilon * \rho_\varepsilon$.
- $V_\varepsilon(x)$ subsolution since we have a “convex Hamiltonian”.
- $V_\varepsilon^{(k)}(x) := \int_{\mathbb{R}^n} V^\varepsilon(x-y)\rho_\varepsilon^{(k)}(y)\mathrm{d}y$
- $\left\| \rho_\varepsilon^{(k)} \right\|_\infty \leq C_{\rho,k}\varepsilon^{-n-k}$,
- $\left| V_\varepsilon^{(k)}(x) - V_\varepsilon^{(k)}(y) \right| \leq \varepsilon^{-k} C_{\rho,k} C_V |x-y|^{\mu_V}$.

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Lower estimate of the solution of the scheme

- Since $C_k = O(\varepsilon^{-k})$ and $\mu_k = \mu_V$, obtain

$$C_1 h^{\mu_1} = O(\varepsilon^{-1} h^{\mu_V}); \quad C_3 h^{1+\mu_3} = O(\varepsilon^{-3} h^{\mu_V}).$$

For $\varepsilon = h^{\mu_V/(1+\mu_V)}$, obtain

$$\sup(V - v_h) \leq O(h^{\mu_V^2/(1+\mu_V)}).$$

If λ large enough, $\mu_V = 1$ and

$$\sup(V - v_h) \leq O(h^{1/2}).$$

Same estimate as for deterministic problems ($\sigma = 0$).

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Upper estimate of the solution of the scheme

- Symmetrix procedure valid when v_h Hölder.
- Shake coefficients of the scheme: solution v_h^ε.
- $v_{h,\varepsilon} := v_h^\varepsilon * \rho_\varepsilon$ is a smooth subsolution of the scheme.
- $v_{h,\varepsilon}$ subsolution of a perturbed HJB equation with some $\ell_\varepsilon(x, u)$

$$\ell_\varepsilon(x, u) \geq \ell - \varepsilon^{-k} C_{\rho,k} C_V |x - y|^{\mu_V}.$$

Error estimates of the same order on the other side: $h^{1/2}$ in the Lipschitz case.

Regularity of the solution of the scheme

Theorem

Krylov (2005): if λ large enough, v_h Lipschitz uniformly in h.

Proof: based on similar result for the parabolic case, and let $t \longrightarrow \infty$.

Open problems

- Simple proof for previous theorem
- Hölder regularity for the solution of the scheme if λ is small (as for the solution of HJB equation).
- Extension to nonconvex Hamiltonians (games ?)
  See B-Maroso-Zidani (adverse stopping).
- Combinations with Monte-Carlo algorithms.

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