Méthodes de différences finies et au delà

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Stochastic optimal control problem

$$(P_x) \quad \begin{cases} \displaystyle \operatorname*{Min}_{u\in\mathcal{U}} \boldsymbol{E} \int_0^\infty \ell(y(t),u(t))e^{-\lambda t}\mathrm{d}t; \\ \begin{cases} \mathrm{d}y(t) = f(y(t),u(t))\mathrm{d}t + \sigma(y(t),u(t))\mathrm{d}w(t), \\ y(0) = x, \end{cases} \\ u(t) \in U, \ \text{a.s., for all } t \in [0,\infty[, \ \mathcal{U} \text{ space of } \mathcal{F}_t \text{ adapted functions.} \end{cases}$$

- $\lambda \geq 0$: discounting factor,
- $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$: distributed cost,
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$: trend,
- $\sigma(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to$ space of $n \times r$ matrices
- w :standard r dimensional Brownian motion

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Smoothness of the value function

Theorem

Assume that U is compact and that ℓ, f and σ are Lipschitz w.r.t. y, uniformly in $u \in U$.
Then the state equation is well-posed, and the value function $V(x)$ is Lipschitz if λ is large enough, and Hölder otherwise with constant say μ_V.

Dynamic programming principle: DPP

Denote by y_x the state trajectory with initial condition x.

For each $\tau > 0$:

$$v(x) = \operatorname*{Min}_{u \in \mathcal{U}} \boldsymbol{E} \left[\int_0^\tau \ell(y_x(t), u(t)) e^{-\lambda t} \mathrm{d}t + e^{-\lambda \tau} v(y_x(\tau)) \right];$$

with $\mathcal{F}_\tau$ space of adapted functions $[0, \tau] \to \mathbb{R}$.

Itô formula applied to v

Skipping the dependence in u, we have that:

$$\mathrm{d}y = f\mathrm{d}t + \sigma\mathrm{d}w$$

If $v(\cdot)$ is smooth, setting $a := \sigma\sigma^\top$:

$$\mathrm{d}v(y_x(t)) = v'(y_x(t))\mathrm{d}y + \tfrac{1}{2}v''(y_x(t)) \cdot a$$

In integrated form:

$$v(y_x(t)) = v(x) + \int_0^T \left[v'(y_x(t))(f\mathrm{d}t + \sigma\mathrm{d}w) + \tfrac{1}{2}v''(y_x(t)) \cdot a\right]\mathrm{d}t$$

Note that when taking expectations the term $\int_0^T \sigma\mathrm{d}w$ vanishes.

Back to DPP with small time τ

$$
\begin{aligned}
v(x) &= \operatorname*{Min}_{u\in\mathcal{U}} \boldsymbol{E}\left[\int_0^\tau \ell(y_x(t),u(t))e^{-\lambda t}\mathrm{d}t + e^{-\lambda\tau} v(y_x(\tau))\right] \\
&= e^{-\lambda\tau} v(x) + e^{-\lambda\tau} \operatorname*{Min}_{u\in\mathcal{U}} \boldsymbol{E}\Big[\\
&\quad \int_0^\tau \ell(y_x(t),u(t))e^{\lambda(\tau-t)} + v'(y_x(t))f + \tfrac{1}{2}v''(y_x(t))\cdot a\Big]\mathrm{d}t
\end{aligned}
$$

Or using $e^{\lambda\tau} = 1 + \lambda\tau + O(\tau^2)$:

$$
\lambda\tau v(x) = \operatorname*{Min}_{u\in\mathcal{U}} \boldsymbol{E}\int_0^\tau H(y_x(t),u(t),v'(y_x(t)),v''(y_x(t)))\mathrm{d}t,
$$

where for $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $p\in\mathbb{R}^n$, $Q\in\mathbb{R}^{n\times n}$:

$$
H(x,u,p,Q) := \ell(x,u) + v'(x)f(x,u) + \tfrac{1}{2}v''(x)\cdot a(x,u).
$$

From DPP to HJB

$$
\begin{aligned}
\lambda v(x) &= \lim_{\tau \downarrow 0} \frac{1}{\tau} \operatorname*{Min}_{u \in \mathcal{U}} \boldsymbol{E} \int_0^\tau H(y_x(t), u(t), v'(y_x(t)), v''(y_x(t))) \mathrm{d}t, \\
&= \lim_{\tau \downarrow 0} \frac{1}{\tau} \operatorname*{Min}_{u \in \mathcal{U}} \boldsymbol{E} \int_0^\tau H(x, u(t), v'(x), v''(x)) \mathrm{d}t, \\
&= \operatorname*{Min}_{u \in U} H(x, u(t), v'(x), v''(x))
\end{aligned}
$$

HJB equation

$$\lambda v(x) = \inf_{u \in U} \{ \ell(x,u) + f(x,u) \cdot v_x(x) + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x,u)\, v_{x_i x_j}(x) \}, \quad \text{for all} \quad x \in \mathbb{R}^n.$$

- Covariance matrix: $a(x,u) := \sigma(x,u)\sigma(x,u)^T, \ \forall \ (x,u) \in \mathbb{R}^n \times \mathbb{R}^m.$
- All functions Lipschitz continuous and bounded: V unique bounded *viscosity solution* of HJB.

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Two “opposite” approaches

A Probabilistic point of view: evaluate the expectation in the DPP and solve the resulting contracting fixed-poinr equation

$$v(x) = \operatorname*{Min}_{u\in\mathcal{U}} \boldsymbol{E}\left[\int_0^\tau \ell(y_x(t),u(t))e^{-\lambda t}\mathrm{d}t + e^{-\lambda\tau}v(y_x(\tau))\right];$$

B Ignore the stochastic aspects, and solve the HJB partial differential equation

$$\lambda v(x) \quad = \quad \inf_{u\in U}\left[\ell(x,u) + f(x,u)\cdot v_x(x) + \tfrac{1}{2}v''(x)\cdot a(x,u)\right],$$

Semi Lagrangian algorithms 1: Euler scheme

1) Take constant control over $[0, \tau]$

$$v(x) = \operatorname*{Min}_{u \in U} \boldsymbol{E} \left[\int_0^\tau \ell(y_x(t), u) e^{-\lambda t} \mathrm{d}t + e^{-\lambda \tau} v(y_x(\tau)) \right];$$

2) Approximate the integral by an Euler scheme

$$v(x) = \operatorname*{Min}_{u \in U} \boldsymbol{E} \left[\tau \ell(x, u) + e^{-\lambda \tau} v(y_x(\tau)) \right];$$

3) Approximate $y_x(\tau)$ using an Euler scheme: $\hat{w}$ q dimensional standard Gaussian

$$v(x) = \operatorname*{Min}_{u \in U} \boldsymbol{E} \left[\tau \ell(x, u) + e^{-\lambda \tau} v(x + \tau f(x, u) + \sqrt{\tau} \sigma \hat{w}) \right];$$

Semi-Lagrangian algorithms 2: discrete expectation

Simplest idea: approximate each components of $\hat{w}$ with value $\pm$ with probability $\frac{1}{2}$

$$v(x) = \operatorname*{Min}_{u \in U} \mathbb{E} \left[\tau \ell(x,u) + \frac{e^{-\lambda\tau}}{2^q} v \left(x + \tau f(x,u) \pm \sqrt{\tau} \sum_{k=1}^{q} \sigma_k \right) \right];$$

This is the *conceptual* semi-Lagrangian scheme.

The spatial representation of functions being not provided, the conceptual algorithm cannot be implemented.

Error estimate: order $\tau^{1/4}$ (perhaps non optimal ?).

Proof: similar arguments as for FD schemes (see next week)

Spatial triangulation

Space of continuous functions affine on each triangle of a regular triangulation of "spatial size" h, DPP written on vertices.

Standard result (easy proof): error estimate of order $\tau^{1/4} + \frac{h}{\tau}$.

Perhaps direct proof (not two steps ?) (triangular inequality)

This leads to choose $h = O(\tau^{5/4})$!

In any case : h small w.r.t. τ: large spatial grid.

Opposite to usual rule: $\tau = O(h^2)$ for FD schemes

Cost: in time: linear in τ; in space: at least h^n

Big space steps: a one dimensional discussion

CFL principle: distance run over time τ must be less than h

v continuous, piecewise affine over grip $[kh, (k+1)h]$, $k \in \mathbb{Z}$, DPP written over grid points.

$$v(x) = \underset{u \in U}{\mathrm{Min}}\, \boldsymbol{E} \left[\tau \ell(x,u) + \frac{e^{-\lambda\tau}}{2} [v(x + \tau f(x,u) + \sqrt{\tau}\sigma(x,u)) \right. \\ \left. + v(x + \tau f(x,u) - \sqrt{\tau}\sigma(x,u))] \right];$$

Assume for simplicity $f \geq 0$ and set

$$x^{\pm} := x + \tau f(x,u) \pm \sqrt{\tau}\sigma(x,u).$$

We have $x^{-} < x$ iff $\sigma(x,u) > \sqrt{\tau} f(x,u)$.

Barycentric coordinates

If $\tau^2 \leq ch$, $c > 0$ large enough: $x^+ \in [x, x+h]$ and

$$x^+ = \frac{h - (x^+ - x)}{h} x + \frac{x^+ - x}{h}(x+h)$$

$$x^- = \frac{h - (x - x^-)}{h} x + \frac{x - x^-}{h}(x-h)$$

and hence the DDP written at point x of grid reads

$$\begin{aligned} v(x) = \quad \operatorname*{Min}_{u \in U} \mathbb{E} \Big[& \tau \ell(x,u) + \tfrac{1}{2} e^{-\lambda\tau} \frac{\sqrt{\tau}\sigma(x,u) + \tau f(x,u)}{h} v(x+h)) \\ & + \tfrac{1}{2} e^{-\lambda\tau} \frac{\sqrt{\tau}\sigma(x,u) - \tau f(x,u)}{h} v(x-h) \\ & e^{-\lambda\tau} \left(1 - \frac{\sqrt{\tau}\sigma(x,u)}{h} \right) v(x)) \Big]; \end{aligned}$$

Finite difference interpretation

$$
\begin{aligned}
v(x) = \quad & \tau \operatorname*{Min}_{u \in U} \boldsymbol{E} \left[\ell(x,u) + e^{-\lambda\tau} f(x,u) \frac{v(x+h) - v(x-h)}{2h}) \right. \\
& \left. + \frac{e^{-\lambda\tau}}{\tau} v(x) + \tfrac{1}{2} e^{-\lambda\tau} \frac{h}{\sqrt{\tau}} \sigma(x,u) \frac{v(x-h) + v(x-h) - 2v(x)}{h^2} \right] ;
\end{aligned}
$$

Link with FD methods

Using $e^{\lambda\tau} = 1 + \lambda\tau$, and $\tau e^{\lambda\tau} = \tau$, obtain

$$\lambda v(x) = \operatorname*{Min}_{u \in U} \mathbb{E}\left[\ell(x,u) + f(x,u)\frac{v(x+h) - v(x-h)}{2h} \right. \\ \left. + \frac{1}{2}\frac{h}{\sqrt{\tau}}\sigma(x,u)\frac{v(x-h) + v(x-h) - 2v(x)}{h^2}\right];$$

We can take τ depending on the space; choosing

$$\sqrt{\tau} = \frac{h}{\sigma(x,u)}$$

we recover a FD discretization of the HJB equation.

HJB equation again

$$\lambda v(x) = \inf_{u\in U}\{\ell(x,u) + f(x,u)\cdot v_x(x) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,u)\, v_{x_i x_j}(x)\},$$
$$\text{for all} \quad x \in \mathbb{R}^n.$$

- Covariance matrix: $a(x,u) := \sigma(x,u)\sigma(x,u)^T, \ \forall \ (x,u) \in \mathbb{R}^n \times \mathbb{R}^m.$
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Deterministic problems: $\sigma = 0$

$D_k^u v_k$: upwind spatial finite difference

$$(D_k^u v_k)_i = \begin{cases} \dfrac{v_{k+e_i} - v_k}{h_i} & \text{if} \quad f(x_k, u)_i \geq 0, \\ \dfrac{v_k - v_{k-e_i}}{h_i} & \text{if not.} \end{cases}$$

The classical finite differences approximation of (HJB) is

$$\lambda v_k = \inf_{u \in U} \{\ell(x_k, u) + f(x_k, u) \cdot D_k^u v_k\} \qquad \text{for all} \quad k \in \mathbb{Z}^n.$$

Error estimate: $O(h^{1/2}$.

Classical FD

Shift operator: $\delta_\xi \varphi_k := \varphi_{\xi+k}$.

Φ: C^2 function over $\mathbb{R}^n$,

$\varphi_k := \Phi(x_k)$ for all k.

$$\Phi_{x_i x_j}(x_k) = \frac{\delta_{\xi+e_i+e_j} - \delta_{\xi+e_i} - \delta_{\xi+e_j} + \delta_\xi}{h_i h_j} \varphi_k + o(1).$$

Denote the corresponding operators as follows:

$$d_{ij}^{\xi} := \frac{\delta_{\xi+e_i+e_j} - \delta_{\xi+e_i} - \delta_{\xi+e_j} + \delta_\xi}{h_i h_j}.$$

d_{ij}^{ξ}: the right upper approximation of $\Phi_{x_i x_j}$.

Left upper, right lower, and left lower approximations of $\Phi_{x_i x_j}$: take ξ equal to $-e_i$, $-e_j$, and $-e_i - e_j$.

Centered approximations of cross second derivatives

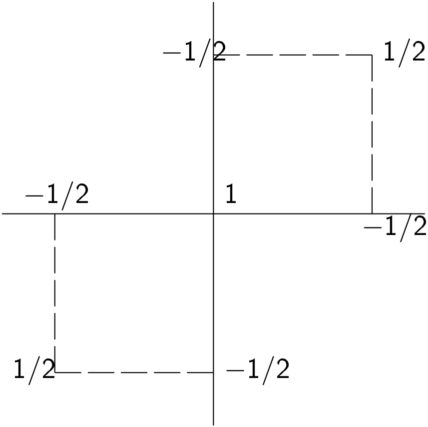
Centered approximations along main and second diagonals:

$$D_{ij}^{+} := \tfrac{1}{2}(d_{ij}^{0} + d_{ij}^{-e_i-e_j}), \qquad D_{ij}^{-} := \tfrac{1}{2}(d_{ij}^{-e_i} + d_{ij}^{-e_j}).$$

In other words,

$$\begin{aligned} D_{ij}^{+} &= \frac{1}{2h_ih_j}(\delta_{e_i+e_j} + \delta_{-e_i-e_j} + 2\delta_0 \\ &\qquad -\delta_{e_i} - \delta_{-e_i} - \delta_{e_j} - \delta_{-e_j}), \\ D_{ij}^{-} &= \frac{1}{2h_ih_j}(\delta_{e_i} + \delta_{-e_i} + \delta_{e_j} + \delta_{-e_j} \\ &\qquad -\delta_{e_i-e_j} - \delta_{e_j-e_i} - 2\delta_0). \end{aligned}$$

Second derivatives: figure



Resulting scheme

Diagonal second order derivatives:

$$D_{ii} := \frac{\delta_{e_i} + \delta_{-e_i} - 2\delta_0}{h_i h_i}.$$

The classical FD approximation of HJB:

$$\lambda v_k = \inf_{u \in U} \{\ell(x_k, u) + f(x_k, u) \cdot D_k^u v_k + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x_k, u) D_{ij} v_k\}, \quad \text{for all} \quad k \in \mathbb{Z}^n,$$

where if $i \neq j$, D_{ij} equal to D_{ij}^+ or D_{ij}^-;
D_k^u upwind spatial finite difference.

Contracting fixed point form

Fictitious time step: $h_0 > 0$.
Equivalent scheme:

$$\begin{aligned} v_k := \; & (1+\lambda h_0)^{-1} \inf_{u \in U} \{ v_k + h_0 \ell(x_k, u) \\ & + h_0 f(x_k, u) \cdot D_k^u v_k + \tfrac{1}{2} h_0 \sum_{i,j=1}^{n} a_{ij}(x_k, u) D_{ij} v_k \}. \end{aligned}$$

Monotonicity of the scheme ? restrictive assumptions : diagonal dominant a.
Uniform monotonicity: Monotonicity for any arbitrary control.
Let a^h denote the scaled covariance matrix $\{a_{ij}/h_i h_j\}$.

Monotonicity

Lemma

Uniform monotonicity satisfied iff the following 3 conditions hold:
(i) *If* $i \neq j$ *is such that* $a_{ij}(x_k, u) \neq 0$, *then* $D_{ij} = D_{ij}^+$ *if* $a_{ij}(x_k, u) > 0$, *and* $D_{ij} = D_{ij}^-$ *if* $a_{ij}(x_k, u) < 0$,
(ii) *The matrix* $a^h(x_k, u)$ *is dominant diagonal, or equivalently,*

$$\frac{a_{ii}(x_k, u)}{h_i} \geq \sum_{j \neq i} \frac{|a_{ij}(x_k, u)|}{h_j} \quad \text{for all} \quad i = 1, \dots, n,$$

(iii) *The time step* h_0 *satisfies the following condition*

$$\sum_{i=1}^{n} \frac{|f(x_k, u)_i|}{h_i} + \sum_{i=1}^{n} \left(2\frac{a_{ii}(x_k, u)}{h_i^2} - \sum_{j \neq i} \frac{|a_{ij}(x_k, u)|}{h_i h_j} \right) \leq \frac{1}{h_0}.$$

Size of time step

Last condition on time step implies if $a() \neq 0$

$$h_0 = O(\inf_i h_i^2).$$

By contrast, if $a() = 0$ then we may take

$$h_0 = O(\inf_i h_i).$$

Local maps

It is possible to perform a change of variables in order to satisfy $a^h(x)$ dominant diagonal ?

If $\sigma(x, u)$ independent on u
Write $a = a(x)$; fix $x_0 \in \mathbb{R}^n$:
Let $w_1, \ldots, w_n$ orthonormal basis of eigenvectors of $a(x_0)$.
Take $w_1, \ldots, w_n$ as new basis.
In this new basis: $a(x_0)$ is diagonal.

Case 1 $a(x_0)$ is positive definite.
Locally: $a(x)$ positive definite as well, and dominant diagonal.

Case 2 $a(x_0)$ has eigenvector 0 with multiplicity r.

- If $\mathrm{Ker}\, a(x)$ is constant: $a(x)$ still diagonal dominant.
- Otherwise ???

Except for (interesting) special cases, when σ depends on u, we cannot have $a(x)$ or $a^h(x)$ dominant diagonal.

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