Méthodes de différences finies et au delà

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Séminaire “Méthodes numériques pour la commande optimale”
EDF R & D, Clamart, 9 avril 2009

Stochastic optimal control problem

(P_x)

$$
\begin{cases}
\displaystyle \operatorname*{Min}_{u\in\mathcal{U}} \boldsymbol{E} \int_0^\infty \ell(y(t),u(t))e^{-\lambda t}\mathrm{d}t; \\[2ex]
\begin{cases} \mathrm{d}y(t) = f(y(t),u(t))\mathrm{d}t + \sigma(y(t),u(t))\mathrm{d}w(t), \\ y(0) = x, \end{cases} \\[2ex]
u(t) \in U, \ \text{a.s., for all } t \in [0,\infty[,\ \mathcal{U} \text{ space of } \mathcal{F}_t \text{ adapted functions.}
\end{cases}
$$

- $\lambda \geq 0$: discounting factor,
- $\ell : \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}$: distributed cost,
- $f : \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$: trend,
- $\sigma(\cdot) : \boldsymbol{R}^n \times \boldsymbol{R}^m \to$ space of $n \times r$ matrices
- w :standard r dimensional Brownian motion.

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Dynamic programming principle: DPP

Denote by y_x the state trajectory with initial condition x.

For each $\tau > 0$:

$$v(x) = \operatorname*{Min}_{u \in \mathcal{U}} \boldsymbol{E} \left[\int_0^\tau \ell(y_x(t), u(t)) e^{-\lambda t} \mathrm{d}t + e^{-\lambda \tau} v(y_x(\tau)) \right];$$

with $\mathcal{F}_\tau$ space of adapted functions $[0, \tau] \to \boldsymbol{R}$.

HJB equation

$$\lambda v(x) = \inf_{u \in U} \{\ell(x,u) + f(x,u) \cdot v_x(x) + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x,u)\, v_{x_i x_j}(x)\}, \quad \text{for all } x \in \boldsymbol{R}^n.$$

- Covariance matrix:
  $a(x,u) := \sigma(x,u)\sigma(x,u)^T, \ \forall \ (x,u) \in \boldsymbol{R}^n \times \boldsymbol{R}^m.$
- All functions Lipschitz continuous and bounded: V unique bounded *viscosity solution* of HJB.

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Deterministic problems: $\sigma = 0$

$D_k^u v_k$: upwind spatial finite difference

$$(D_k^u v_k)_i = \begin{cases} \dfrac{v_{k+e_i} - v_k}{h_i} & \text{if} \quad f(x_k, u)_i \geq 0, \\ \dfrac{v_k - v_{k-e_i}}{h_i} & \text{if not.} \end{cases}$$

The classical finite differences approximation of (HJB) is

$$\lambda v_k = \inf_{u \in U} \{ \ell(x_k, u) + f(x_k, u) \cdot D_k^u v_k \} \quad \text{for all} \quad k \in \mathbb{Z}^n.$$

Error estimate: order $h^{1/2}$.

Contracting fixed point form

Fictitious time step: $h_0 > 0$.
Equivalent scheme:

$$v_k := (1 + \lambda h_0)^{-1} \inf_{u \in U} \{ v_k + h_0 \ell(x_k, u) + h_0 f(x_k, u) \cdot D_k^u v_k \}.$$

Monotonicity for any arbitrary control: coefficient of v_k

$$1 - h_0 \sum_{i=1}^{n} \frac{|f_i(x, u)|}{h_i} \geq 0.$$

We recover the classical CFL condition for hyperbolic systems.

Second order case: Classical FD

Shift operator: $\delta_\xi \varphi_k := \varphi_{\xi+k}$.

Φ: C^2 function over $\boldsymbol{R}^n$,

$\varphi_k := \Phi(x_k)$ for all k.

$$\Phi_{x_i x_j}(x_k) = \frac{\delta_{\xi+e_i+e_j} - \delta_{\xi+e_i} - \delta_{\xi+e_j} + \delta_\xi}{h_i h_j} \varphi_k + o(1).$$

Denote the corresponding operators as follows:

$$d_{ij}^{\xi} := \frac{\delta_{\xi+e_i+e_j} - \delta_{\xi+e_i} - \delta_{\xi+e_j} + \delta_\xi}{h_i h_j}.$$

d_{ij}^{ξ}: the right upper approximation of $\Phi_{x_i x_j}$.

Left upper, right lower, and left lower approximations of $\Phi_{x_i x_j}$: take ξ equal to $-e_i$, $-e_j$, and $-e_i - e_j$.

Centered approximations of cross second derivatives

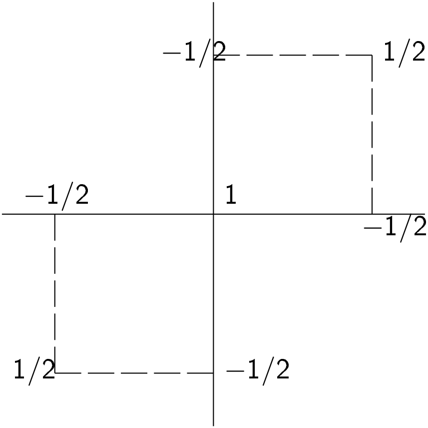
Centered approximations along main and second diagonals:

$$D_{ij}^{+} := \frac{1}{2}(d_{ij}^{0} + d_{ij}^{-e_i-e_j}), \qquad D_{ij}^{-} := \frac{1}{2}(d_{ij}^{-e_i} + d_{ij}^{-e_j}).$$

In other words,

$$\begin{aligned} D_{ij}^{+} &= \frac{1}{2h_ih_j}(\delta_{e_i+e_j} + \delta_{-e_i-e_j} + 2\delta_0 \\ &\qquad -\delta_{e_i} - \delta_{-e_i} - \delta_{e_j} - \delta_{-e_j}), \\ D_{ij}^{-} &= \frac{1}{2h_ih_j}(\delta_{e_i} + \delta_{-e_i} + \delta_{e_j} + \delta_{-e_j} \\ &\qquad -\delta_{e_i-e_j} - \delta_{e_j-e_i} - 2\delta_0). \end{aligned}$$

Second derivatives: figure



Resulting scheme

Diagonal second order derivatives:

$$D_{ii} := \frac{\delta_{e_i} + \delta_{-e_i} - 2\delta_0}{h_i h_i}.$$

The classical FD approximation of HJB:

$$\begin{aligned} \lambda v_k \quad = \quad & \inf_{u \in U} \{ \ell(x_k, u) + f(x_k, u) \cdot D_k^u v_k + \\ & \qquad \tfrac{1}{2} \textstyle\sum_{i,j=1}^n a_{ij}(x_k, u) D_{ij} v_k \}, \\ & \qquad\qquad \text{for all} \quad k \in \mathbb{Z}^n, \end{aligned}$$

where if $i \neq j$, D_{ij} equal to D_{ij}^+ or D_{ij}^-;
D_k^u upwind spatial finite difference.

Contracting fixed point form

Fictitious time step: $h_0 > 0$.
Equivalent scheme:

$$\begin{aligned} v_k := \; & (1+\lambda h_0)^{-1} \inf_{u \in U} \{ v_k + h_0 \ell(x_k, u) \\ & + h_0 f(x_k, u) \cdot D_k^u v_k + \tfrac{1}{2} h_0 \sum_{i,j=1}^{n} a_{ij}(x_k, u) D_{ij} v_k \}. \end{aligned}$$

Monotonicity of the scheme ? restrictive assumptions : diagonal dominant a.
Uniform monotonicity: Monotonicity for any arbitrary control.
Let a^h denote the scaled covariance matrix $\{a_{ij}/h_i h_j\}$.

Monotonicity

Lemma

Uniform monotonicity satisfied iff the following 3 conditions hold:
(i) *If $i \neq j$ is such that $a_{ij}(x_k, u) \neq 0$, then $D_{ij} = D_{ij}^+$ if $a_{ij}(x_k, u) > 0$, and $D_{ij} = D_{ij}^-$ if $a_{ij}(x_k, u) < 0$,*
(ii) *The matrix $a^h(x_k, u)$ is dominant diagonal, or equivalently,*

$$\frac{a_{ii}(x_k, u)}{h_i} \geq \sum_{j \neq i} \frac{|a_{ij}(x_k, u)|}{h_j} \quad \text{for all} \quad i = 1, \dots, n,$$

(iii) *The time step h_0 satisfies the following condition*

$$\sum_{i=1}^{n} \frac{|f(x_k, u)_i|}{h_i} + \sum_{i=1}^{n} \left(2\frac{a_{ii}(x_k, u)}{h_i^2} - \sum_{j \neq i} \frac{|a_{ij}(x_k, u)|}{h_i h_j} \right) \leq \frac{1}{h_0}.$$

Size of time step

Last condition on time step implies if $a() \neq 0$

$$h_0 = O(\inf_i h_i^2).$$

By contrast, if $a() = 0$ then we may take

$$h_0 = O(\inf_i h_i).$$

Local maps

It is possible to perform a change of variables in order to satisfy $a^h(x)$ dominant diagonal ?

If $\sigma(x,u)$ independent on u
Write $a = a(x)$; fix $x_0 \in \boldsymbol{R}^n$:
Let $w_1, \dots, w_n$ orthonormal basis of eigenvectors of $a(x_0)$.
Take $w_1, \dots, w_n$ as new basis.
In this new basis: $a(x_0)$ is diagonal.

Case 1 $a(x_0)$ is positive definite.
Locally: $a(x)$ positive definite as well, and dominant diagonal.

Case 2 $a(x_0)$ has eigenvector 0 with multiplicity r.
- If $\mathrm{Ker}\, a(x)$ is constant: $a(x)$ still diagonal dominant.
- Otherwise ???

Except for (interesting) special cases, when σ depends on u, we cannot have $a(x)$ or $a^h(x)$ dominant diagonal.

Simplified HJB equation

Null drift f and data not depending on control variable u:

$$\lambda v(x) \quad = \quad \ell(x) + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x)\, v_{x_i x_j}(x), \text{ for all } \; x \in \boldsymbol{R}^n.$$

Regular grid discretization: $n = 2$

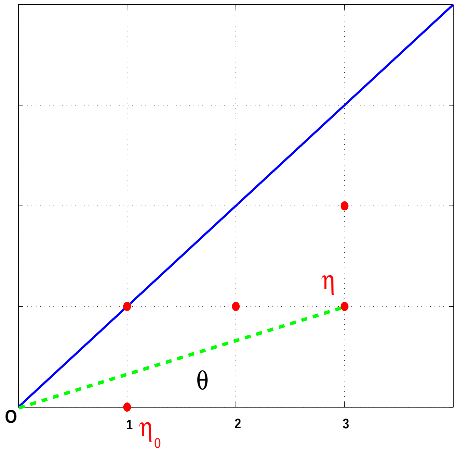
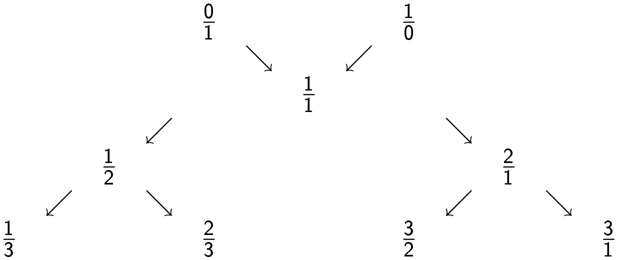


Figure: **Directions in regular grid**

Stern-Brocot tree



Family relations

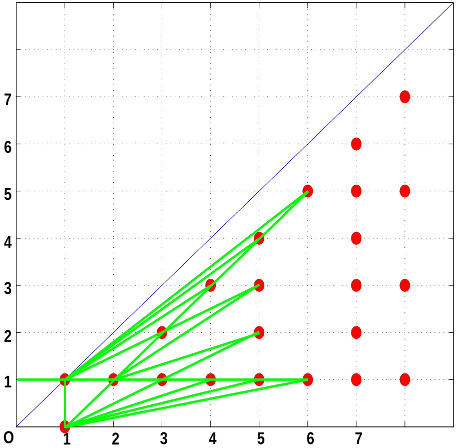


Figure: **Family relations in regular grid**

Covariances ($n = 2$)

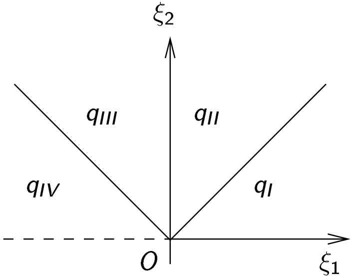
Cone $\mathcal{C}$ of symmetric semidefinite positive matrices. Extreme points:

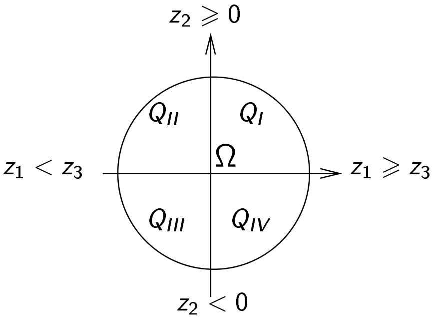
$$\sigma\sigma^T = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \text{where } \sigma \in \mathbb{R}^2.$$

View $\mathcal{C}$ as a subset of $\mathbb{R}^3$:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} \\ \sqrt{2}a_{12} \\ a_{22} \end{pmatrix}$$

Isometry: illustration





View of SDP cone $\mathcal{C}$

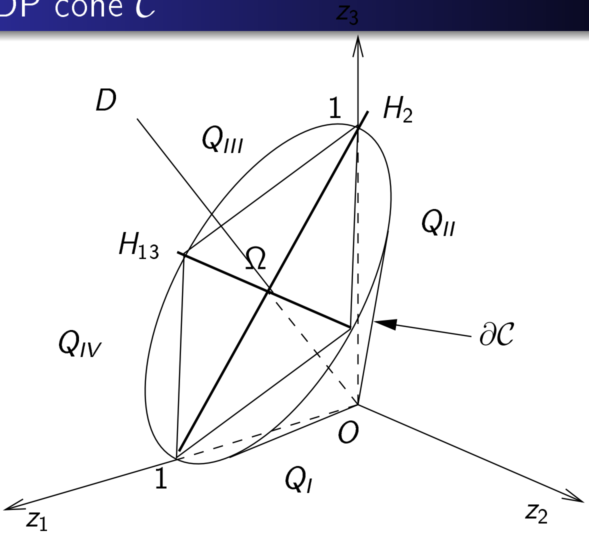
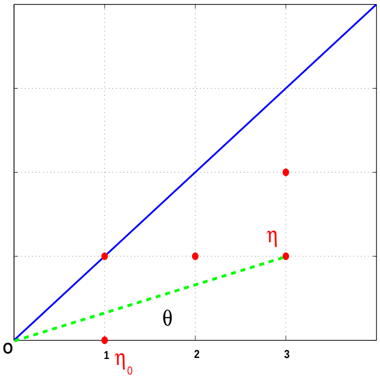
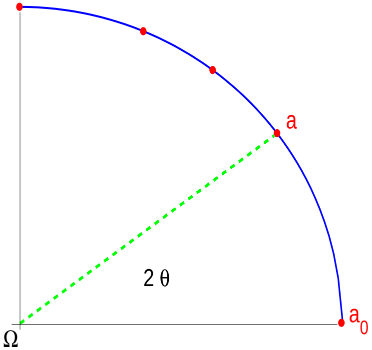


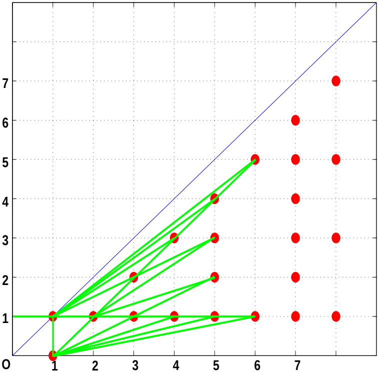
Figure: **Cone of semidefinite positive matrices**

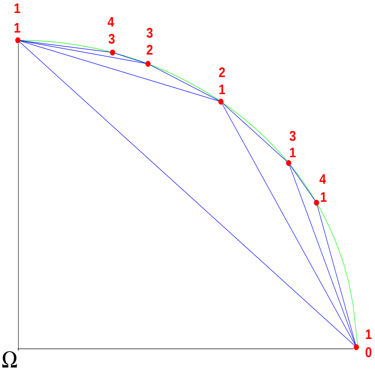
Correspondance grid / section of cone





Intersection with trace $x = 1$





Generalized finite differences

- Discretization steps $h_1, \ldots, h_n$, points $x_k := (k_1 h_1, \ldots, k_n h_n)$.
- Given $\varphi = \{\varphi_k\}$: real valued function over $\mathbb{Z}^n$.
- Second order finite difference operator

$$\Delta_\xi \varphi_k := \varphi_{k+\xi} + \varphi_{k-\xi} - 2\varphi_k = \varphi_{k+\xi} - \varphi_k - (\varphi_k - \varphi_{k-\xi}).$$

- If $\varphi_k = \Phi(x_k)$. Then

$$\begin{array}{rcl} \Delta_\xi \varphi_k & = & D^2\Phi(x_k)x_\xi x_\xi + o(\|x_\xi\|^2) \\ & = & \sum_{i,j=1}^n \xi_i h_i \xi_j h_j \Phi_{x_i x_j}(x_k) + o(h^2) \end{array}$$

Approximation of second-order term

- Scaled covariance: $a_{ij}^h(x) = a(x)/(h_i h_j)$.
- Strong consistency: $\sum_{\xi \in \mathcal{S}} \alpha_{k,\xi} \Delta_\xi \phi_k = \sum_{i,j=1}^n a_{ij}(x_k) \Phi_{x_i x_j}(x_k)$
- Characterization: $\sum_{\xi \in \mathcal{S}} \alpha_{k,\xi} \xi \xi^T = a^h(x_k), \quad \text{for all} \quad k \in \mathbb{Z}^n.$

Numerical scheme

- Explicit schemes: $\lambda v_k = \ell(x_k) + \sum_{\xi \in \mathcal{S}} \alpha_{k,\xi} \Delta_\xi v_k, \quad k \in \mathbb{Z}^n$
- *Fictitious* time step: $h_0 > 0$. Equivalent scheme:

$$v_k := (1 + \lambda h_0)^{-1} \left\{ v_k + h_0 \ell(x_k) + h_0 \sum_{\xi \in \mathcal{S}} \alpha_{k,\xi} \Delta_\xi v_k \right\}.$$

Monotonicity condition

- Nondecreasing mapping $v \to v_k + h_0 \ell(x_k) + h_0 \sum_{\xi \in \mathcal{S}} \alpha_{k,\xi} \Delta_\xi v_k$
- $$\begin{aligned} \alpha_{k,\xi} &\geq 0, \quad \forall\, (\xi, k) \in \mathcal{S} \times \mathbb{Z}^n. \\ 2 \sum_{\xi \in \mathcal{S}} \alpha_{k,\xi} &\leq h_0^{-1}, \quad \forall\, (k) \in \mathbb{Z}^n. \end{aligned}$$
- Second condition satisfied when h_0 small enough, once an estimate of $\sum_{\xi \in \mathcal{S}} \alpha_{k,\xi}$ is known (see below).

Convergence

- Monotonicity and consistency imply convergence.
- Strong consistency implies $\sum_{\xi \in \mathcal{S}} \alpha_{k,\xi} \leq \mathrm{trace}\, a^h(x_k)$
- Condition for time step: $h_0 = O(\min_i h_i^2)$

Explicit strong consistency conditions

- Characterization: $a^h(x_k)$ belongs to $\mathcal{C}(\mathcal{S}) := \left\{ \sum_{\xi \in \mathcal{S}} \alpha_\xi \xi \xi^T ; \alpha \in \boldsymbol{R}_+^{|\mathcal{S}|} \right\}$.
- Neighbours of order p: $\mathcal{S}^p = \{\xi \in \boldsymbol{R}^n; \ |\xi_i| \leq p\}$.
- Invariance conditions:
  (i) Permutation of coordinate
  (ii) Change of sign on one coordinate
- Qhull algorithm by Barbet et al., absolute precision of 10^{-10}

Some results for $n = 2$

- $\mathcal{C}(\mathcal{S}^1) = \{a; \quad a_{ii} \geq |a_{ij}|, \quad 1 \leq i \neq j \leq 2\}.$
- $$\mathcal{C}(\mathcal{S}^2) = \left\{ \begin{array}{l} 2a_{ii} \geq |a_{ij}| \\ 2a_{ii} + a_{jj} \geq 3|a_{ij}|. \end{array} \right. , \quad 1 \leq i \neq j \leq 2$$
- $$\mathcal{C}(\mathcal{S}^3) = \left\{ \begin{array}{l} 3a_{ii} \geq |a_{ij}| \\ 3a_{ii} + 2a_{jj} \geq 5|a_{ij}|. \\ 6a_{ii} + a_{jj} \geq 5|a_{ij}|. \\ 6a_{ii} + 2a_{jj} \geq 7|a_{ij}|. \end{array} \right. \quad 1 \leq i \neq j \leq 2$$

Overview of conditions: $n = 2$

Size of p	of generator of primal cone	Constraints defining $\mathcal{S}^*$	Constraints defining $\mathcal{C}$	Equiv. classes
1	4	6	4	1
2	8	13	8	2
3	16	27	16	4
4	24	39	24	6
5	40	67	40	10
6	48	87	48	12
7	72	123	72	18

Maximal error

$$\varepsilon = \sup_{\|a\|_F=1} \|a - P_{\mathcal{C}(S_p)}(a)\|_F = O(p^{-2})$$

p	ε
1	0.169102
2	0.055642
3	0.026325
4	0.015153
5	0.009804
15	0.001109

ε	p
10^{-1}	2
10^{-2}	5
10^{-3}	16
10^{-4}	20
10^{-5}	159
10^{-7}	1 582

Recursive approximations

- p: size of neighbourhood (as small as possible !)
- $P_p a :=$ projection of a onto $\mathcal{C}_p := \mathcal{C}(\mathcal{S}^p)$
- Desirable feature: fast computation of $P_{p+1}a$, having computed $P_p a$
- Stop when $\|a - P_p a\|_F \leq \varepsilon$.

Useful observations: $n = 2$

- Generators of $\mathcal{C}_p$ are extreme directions of SDP cone
- If $P_p a$ belongs to $\mathcal{C}_p$: stop
- Otherwise: projection on supporting hyperplane generated by two adjacent generators
- Update: add their son and check the next two supporting hyperplanes
- Cost of computation: $O(p)$

Numerical inconsistency I

Test function:

$$\begin{cases} W(t, x_1, x_2) = (1 + t)\sin(x_1)\sin(x_2) \\ W(0, x_1, x_2) = \sin(x_1)\sin(x_2) \\ 0 \leqslant x_1 \leqslant \pi; \quad 0 \leqslant x_2 \leqslant \pi; \quad 0 \leqslant t \leqslant 1 \end{cases}$$

$$\Delta x := \Delta x_1 = \Delta x_2$$

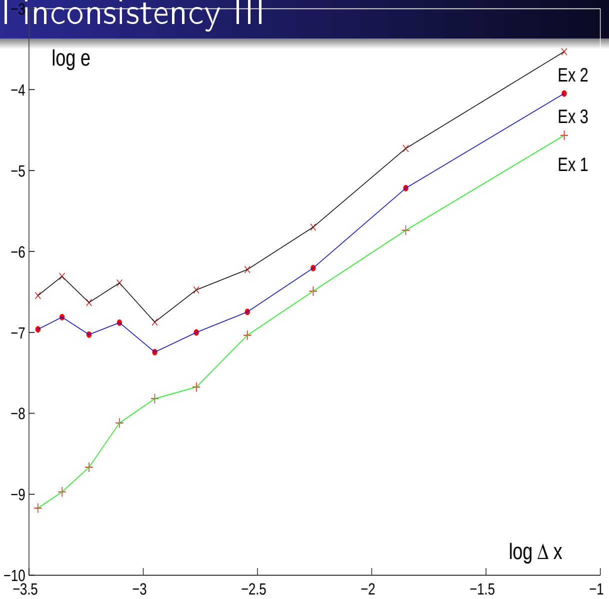
$$e := \frac{\|W_{approx} - W_{exact}\|_1}{N_1 \times N_2}.$$

Numerical inconsistency II

$$\begin{cases} \ell(t, x_1, x_2) = & \sin x_1 \sin x_2 [1 + \alpha(1 + t)] \\ & -2(1 + t) \; \cos x_1 \cos x_2 \\ & \sin(x_1 + x_2) \cos(x_1 + x_2) \\ & \\ f_1(x_1, x_2) = & f_2(x_1, x_2) = 0 \\ a_{11}(x_1, x_2) = & \sin^2(x_1 + x_2) + \beta \\ a_{12}(x_1, x_2) = & \sin(x_1 + x_2) \; \cos(x_1 + x_2) \\ a_{22}(x_1, x_2) = & \cos^2(x_1 + x_2) + \beta \end{cases}$$

Parameters: $\alpha = 1.2, 1, 1.2; \quad \beta = 0.1, 0, 0.$

Numerical inconsistency III



Challenge: $n = 3$

- Generators of $\mathcal{C}_p$ are generators of $\mathcal{C}_{p+1}$
- Relations between faces of $\mathcal{C}_p$ and $\mathcal{C}_{p+1}$?
- Possibility of projections on facets

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