Cross hedging with stochastic correlation

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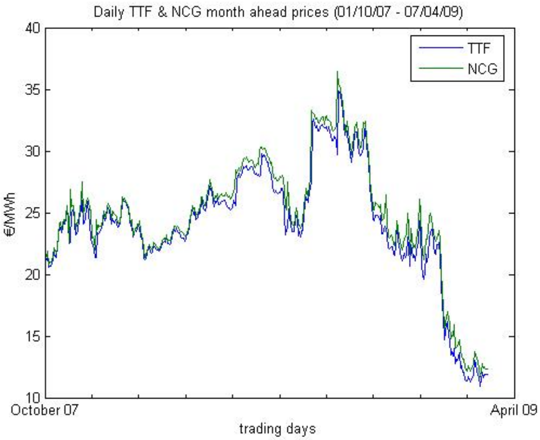
Hedging with futures entails basis risk



Basis risk

Basis = price of the underlying - futures price

More generally: Basis = asset/income to be hedged - price of hedging instrument

Question: With how many TTF futures shall the gas power plant protect itself against increasing prices?

$$\begin{aligned} \Delta G_T &= \text{gas price change in first area} \\ \Delta H_T &= \text{gas price change in second area} \end{aligned}$$

Optimal static hedging

- $\sigma_G =$ st. deviation of ΔG_T
- $\sigma_H =$ st. deviation of ΔH_T
- Correlation

$$\rho = \frac{\text{Cov}(\Delta G_T, \Delta H_T)}{\sigma_G \sigma_H}$$

The variance minimizing hedge is given by

$$a = \frac{N_G}{N_H} \times \rho \frac{\sigma_G}{\sigma_H}$$

- 1. factor $\frac{N_G}{N_H}$ adjusts the units
- 2. factor so-called

$$\text{hedge ratio} = \rho \frac{\sigma_G}{\sigma_H}$$

(see John C. Hull 'Options, Futures and other derivatives')

Local versus global risk minimization

	Bachelier model	Black-Scholes model
natural gas price change	$\Delta G_T = \sigma_G W_T$	$\Delta G_T = G_0(e^{\sigma_G W_T - \frac{\sigma^2}{2}T} - 1)$
hedge ratio	$\rho \frac{\sigma_G}{\sigma_H}$	$\rho \frac{G_0 \sqrt{e^{\sigma_G^2 T} - 1}}{H_0 \sqrt{e^{\sigma_H^2 T} - 1}}$

Convergence

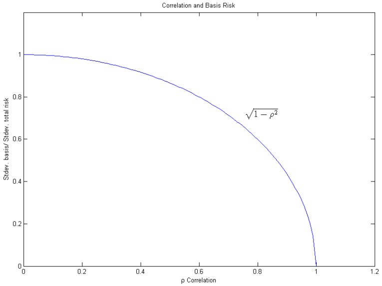
$$\lim_{T \downarrow 0} \rho \frac{G_0 \sqrt{e^{\sigma_G^2 T} - 1}}{H_0 \sqrt{e^{\sigma_H^2 T} - 1}} = \rho \frac{G_0}{H_0} \frac{\sigma_G}{\sigma_H}$$

$$= \text{'local variance hedge ratio'}$$

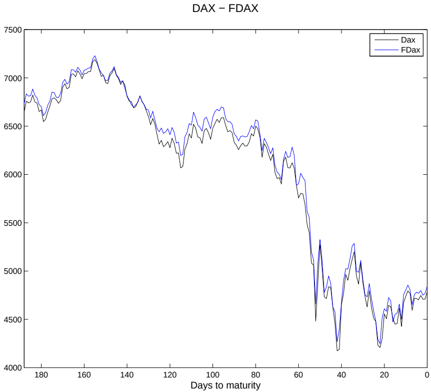
Basis risk matters

Hedging error in the Bachelier model:

- Hedging error $= \Delta G_T - \text{ var minimizing hedge } \times \Delta H_T$
- Std. deviation
$$\text{std}(\text{error}) = \sqrt{1-\rho^2}\sqrt{T}\sigma_G$$
- Example: If $\rho = 90\%$ then $\sqrt{1-\rho^2} \approx 44\%$ of price risk remains!



Hedging of basket options entails basis risk



Two approches of hedging options with basis risk

Two optimality criteria when cross hedging options:

- Minimization of the quadratic hedging error
  - globally: mean variance hedging
  - locally: local risk minimization
- Maximization of exponential utility

Related literature

- Henderson, Valuation of claims on nontraded assets using utility maximization, 2002
- Musiela, Zariphopoulou, An example of indifference prices under exponential preferences, 2008
- Davis, Optimal hedging with basis risk, 2006
- A., Imkeller and dos Reis, Pricing and hedging of derivatives based on non-tradable underlyings, 2008
- Monoyios, Performance of utility-based strategies for hedging basis risk, 2004
- Hulley, McWalter, Quadratic hedging of basis risk, 2008

A GBM model

- $h(I_T)$ = derivative of index I with pay-off function h
- X correlated traded asset
- ρ = correlation

discounted processes:

$$dX_t = X_t((\mu_X - r)dt + \sigma_X dW_t^1)$$
$$dI_t = I_t((\mu_I - r)dt + \sigma_I(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2))$$

Local risk minimizing cross hedge

$$
\begin{aligned}
dX_t &= X_t((\mu_X - r)dt + \sigma_X dW_t^1) \\
dI_t &= I_t((\mu_I - r)dt + \sigma_I(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2))
\end{aligned}
$$

Theorem

The locally risk minimizing strategy ξ for the derivative $h(I_T)$ is given by

$$
\begin{aligned}
\xi(t,x,y) &= \rho\frac{y\sigma_I}{x\sigma_X}\frac{d}{dy}\tilde{E}[h(I_T^{t,y})] \\
&= \textit{local variance hedge ratio} \times \textit{asset delta}
\end{aligned}
$$

where $\tilde{P}$ denotes the minimal martingale measure *of X. Recall that*

$$
\frac{d\tilde{P}}{dP} = \mathcal{E}\left(-\frac{\mu_X - r}{\sigma_X}\cdot W^1\right)_T
$$

Utility-based cross hedge

- $U(x) = -e^{-\eta x}$ exponential utility preferences
- $\pi =$ optimal investment if no derivative in the investor's portfolio
- $\widehat{\pi} =$ optimal investment if derivative $h(I_T)$ in the investor's portfolio

$$\text{utility-based hedge } = \widehat{\pi} - \pi$$

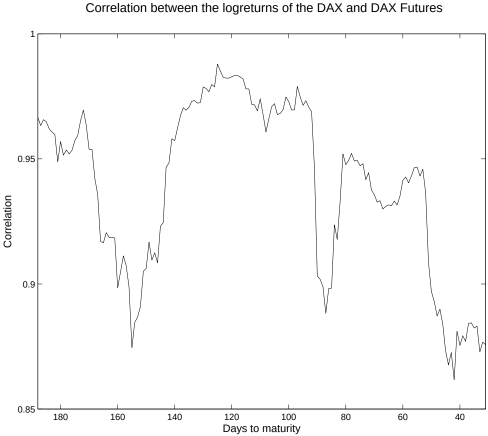
Theorem

The utility-based hedge for the derivative $h(I_T)$ is given by

$$\begin{aligned} \xi(t,x,y) &= \rho \frac{y\sigma_I}{x\sigma_X} \frac{d}{dy} p(t,y) \\ &= \textit{local variance hedge ratio} \times \textit{indifference price delta} \end{aligned}$$

	utility-based approach	**local risk minimization**
Pros	only downside risk	quadratic integrability
Cons	exponential integrability	up and downside risk

Correlation is random



Correlation is mean reverting

Direct modelling as mean reverting process:

$$d\rho_t = \kappa(\vartheta - \rho_t)dt + \alpha\sqrt{1-(\rho_t)^2}\hat{W}_t, \text{ with } \alpha, \kappa > 0,\ \vartheta \in (-1, 1)$$

Indirect modelling as mean reverting process:

- mapping of the correlation onto $\mathbb{R}$ via a continuous bijection $b : (-1, 1) \to \mathbb{R}$
- $\rho_t = b^{-1}(U_t)$ and U is f.ex. a generalised Ornstein-Uhlenbeck

$$dU_t = a(\vartheta - U_t)dt + \sigma_U d\hat{W}_t,$$

where $a > 0$, $\vartheta \in \mathbb{R}$, $\sigma_U > 0$.

A GBM model

- $h(I_T)$ = derivative of index I with pay-off function h
- X correlated traded asset

$$
\begin{aligned}
dX_t &= X_t((\mu_X - r)dt + \sigma_X dW_t^1) \\
dI_t &= I_t((\mu_I - r)dt + \sigma_I(\rho_t dW_t^1 + \sqrt{1-\rho_t^2}dW_t^2)) \\
d\rho_t &= a(\rho_t)dt + g(\rho_t)(\gamma dW_t^1 + \delta dW_t^2 + \sqrt{1-\gamma^2-\delta^2}dW_t^3)
\end{aligned}
$$

Optimality criterion: Local risk minimization

Local risk minimizing cross hedge

Theorem

Under some conditions, the locally risk minimizing strategy ξ for the derivative $h(I_T)$ is given by

$$\xi(t,x,y,v) = \rho \frac{y\sigma_I}{x\sigma_X} \frac{d}{dy} \tilde{\mathbb{E}}[h(I_T^{t,y,v})] + \gamma \frac{g(v)}{x\sigma_X} \frac{d}{dv} \tilde{\mathbb{E}}[h(I_T^{t,y,v})]$$

where $\tilde{P}$ denotes the minimal martingale measure *of X.*

Interpretation:

$$\begin{aligned} \text{optimal hedge} \quad &= \quad \text{local variance hedge ratio} \times \text{ asset delta} \\ &\quad + \text{correlation hedge ratio} \times \text{ correlation delta} \end{aligned}$$

Questions:

- How to get the representation?
- Under which conditions may we differentiate under the integral?

FS decomposition via BSDEs

Standard method for deriving the local risk minimizing strategy is based on the **Föllmer-Schweizer decomposition**:

$$h(I_T) = C + \int_0^T \xi_s dX_s + L_T,$$

where

- L is a square-int. martingale w.r.t. P and $L_0 = 0$,
- $\langle L, W^1 \rangle = 0$, i.e. L is orthogonal w.r.t the martingale driving X

Let Y and Z be the solution of the BSDE

$$Y_t = h(I_T) - \int_t^T Z_s dW_s - \int_t^T Z_s^1 \frac{\mu_X - r}{\sigma_X} ds,$$

for $0 \leq t \leq T$.

Then the FS decomposition of $h(I_T)$ is given by

$$h(I_T) = Y_0 + \int_0^T \frac{Z_s^1}{\sigma_X X_s} dX_s + \int_0^T Z_s^2 dW_s^2 + \int_0^T Z_s^3 dW_s^3.$$

Explicit solution of linear BSDE

The solution of

$$Y_t = h(I_T) - \int_t^T Z_s dW_s - \int_t^T Z_s^1 \frac{\mu_X - r}{\sigma_X} ds, \tag{1}$$

is given by

$$Y_t = \tilde{\mathbb{E}}[h(I_T)|\mathcal{F}_t]$$

We will show that

$$Z_t = \sigma(t, I_t, \rho_t)^* \begin{pmatrix} \partial_y \psi(t, I_t, \rho_t) \\ \partial_v \psi(t, I_t, \rho_t) \end{pmatrix},$$

where

$$\psi(t, y, v) = \tilde{\mathbb{E}}[h(I_T^{t,y,v})].$$

Technical details

$$\psi(t, y, v) = \tilde{\mathbb{E}}[h(I_T^{t,y,v})]$$

1. Under which conditions is $\psi(t, y, v)$ differentiable with respect to y and v?
2. Under which conditions are $I^{t,y,v}$ and $\rho^{t,v}$ differentiable with respect to y and v?
3. Under which conditions may we differentiate under the expectation?

Differentiability of I and ρ

Denote $\bar\rho_s^{t,v} = \frac{d}{dv}\rho_s^{t,v}$ and $\bar I_s^{t,v} = \frac{d}{dv} I_s^{t,y,v}$

Assumption **(H1)** $\rho_t \in (-1,1), \text{ for all } t \le T$

Then

$$\bar\rho_s^{t,v} = 1 + \int_t^s a'(\rho_u^{t,v})\bar\rho_u^{t,v}du + \int_t^s g'(\rho_u^{t,v})\bar\rho_u^{t,v} d\hat W_u,$$

$$\begin{aligned}\bar I_s^{t,y,v} = &\int_t^s \bar I_u^{t,y,v}((\mu_I - r)du + \sigma_I(\rho_u^{t,v} dW_u^1 + \sqrt{1-(\rho_u^{t,v})^2} dW_u^2)) \\ &+ \int_t^s I_u^{t,y,v}\sigma_I(\bar\rho_u^{t,v} dW_u^1 - \frac{\rho_u^{t,v}}{\sqrt{1-(\rho_u^{t,v})^2}}\bar\rho_u^{t,v} dW_u^2).\end{aligned}$$

Differentiability of I and ρ

$$\bar{I}_s^{t,y,v} = \int_t^s \bar{I}_u^{t,y,v}((\mu_I - r)du + \sigma_I(\rho_u^{t,v} dW_u^1 + \sqrt{1-(\rho_u^{t,v})^2} dW_u^2))$$
$$+ \int_t^s I_u^{t,y,v} \sigma_I (\bar{\rho}_u^{t,v} dW_u^1 - \frac{\rho_u^{t,v}}{\sqrt{1-(\rho_u^{t,v})^2}} \bar{\rho}_u^{t,v} dW_u^2).$$

It must hold:

$$\int_t^T \frac{(\bar{\rho}_s^{t,v})^2}{1-(\rho_s^{t,v})^2} du < \infty, \qquad P-a.s.$$

(H2) There exists $p > 1$ such that for every $v_0 \in (-1, 1)$ there exists an open intervall $U \subset (-1, 1)$ of v_0 such that

$$\sup_{v \in U} \mathbb{E}\Big[\int_t^T \left| \frac{(\bar{\rho}_s^{t,v})^2}{1 - (\rho_s^{t,v})^2} \right|^p ds\Big] < \infty.$$

Lemma

Let h be Lipschitz such that the weak derivative h′ is Lebesgue-almost everywhere continuous. Under the Conditions (H1) and (H2) the partial derivative $\partial_v \psi(t, y, v)$ exists and is given by

$$\partial_v \psi(t, y, v) = \tilde{\mathbb{E}}[h'(I_T^{t,y,v}) \bar{I}_T^{t,y,v}].$$

Representation of Z

We want to justify

$$Z_s^{t,y,v} = \sigma(s, I_s^{t,y,v}, \rho_s^{t,v})^* \begin{pmatrix} \partial_y \psi(s, I_s^{t,y,v}, \rho_s^{t,v}) \\ \partial_v \psi(s, I_s^{t,y,v}, \rho_s^{t,v}) \end{pmatrix}.$$

Problem: $\psi(s,x,v) = \tilde{\mathbb{E}}[h(I_T^{t,x,v})]$ is only locally Lipschitz continuous
$\Longrightarrow$ no appeal to the chain rule of Malliavin calculus

Lemma

Assume that (H1) and (H2) hold, and that a and g are continuously differentiable on $(-1,1)$. Let h be Lipschitz such that the weak derivative h' is Lebesgue-almost everywhere continuous. Then

$$Z_s^{t,y,v} = \sigma(s, I_s^{t,y,v}, \rho_s^{t,v})^* \begin{pmatrix} \partial_y \psi(s, I_s^{t,y,v}, \rho_s^{t,v}) \\ \partial_v \psi(s, I_s^{t,y,v}, \rho_s^{t,v}) \end{pmatrix}. \tag{2}$$

By an argument of Imkeller, Reveillac, Richter (2009) one can show (2) without the use of Malliavin calculus!

Main results

Theorem

Assume (H1) and (H2). Suppose that the coefficients a and g in the dynamics of ρ are continuously differentiable on $(-1,1)$. Let h be Lipschitz with a.e. continuous weak derivative h'. Then, there exists a locally risk minimizing strategy ξ for the derivative $h(I_T)$, and

$$\xi(t,x,y,v) = v\frac{y\sigma_I}{x\sigma_X}\tilde{\mathbb{E}}[h'(I_T^{t,y,v})I_T^{t,1,v}] + \frac{g(v)\gamma}{x\sigma_X}\tilde{\mathbb{E}}[h'(I_T^{t,y,v})\bar{I}_T^{t,y,v}],$$

with $\tilde{P}$ denoting the minimal martingale measure of X.

Remark: Implementation via Monte-Carlo!

Main results

How strong is condition (H2)?

(H2) There exists $p > 1$ such that for every $v_0 \in (-1,1)$ there exists an open intervall $U \subset (-1,1)$ of v_0 such that

$$\sup_{v \in U} \mathbb{E}\Big[\int_t^T \left| \frac{(\bar{\rho}_s^{t,v})^2}{1-(\rho_s^{t,v})^2} \right|^p ds\Big] < \infty.$$

Theorem

Let a and g be bounded with bounded derivatives. We assume that $g(-1) = g(1) = 0$, and that g does not have any roots in $(-1,1)$. If

$$\limsup_{x \uparrow 1} \frac{2a(x)(1-x)}{g^2(x)} < 0 \text{ and } \liminf_{x \downarrow -1} \frac{2a(x)(1+x)}{g^2(x)} > 0,$$

then both Conditions (H1) and (H2) are satisfied.

Examples

- $d\rho_t = \kappa(\vartheta - \rho_t)dt + \alpha(1 - \rho_t^2)\hat{W}_t$, with $\alpha, \kappa > 0$, $\vartheta \in (-1, 1)$

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- Set $\rho_t = b^{-1}(U_t)$, with

$$dU_t = a(\vartheta - U_t)dt + \sigma_U d\hat{W}_t,$$

where $a > 0$, $\vartheta \in \mathbb{R}$, $\sigma_U > 0$ and $b(x) = \frac{x}{\sqrt{1-x^2}}$.

Examples

- $d\rho_t = \kappa(\vartheta - \rho_t)dt + \alpha(1-\rho_t^2)\hat{W}_t$, with $\alpha, \kappa > 0$, $\vartheta \in (-1,1)$
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where $a > 0$, $\vartheta \in \mathbb{R}$, $\sigma_U > 0$ and $b(x) = \frac{x}{\sqrt{1-x^2}}$.

- $d\rho_t = \kappa(\vartheta - \rho_t)dt + \alpha\sqrt{1-\rho_t^2}\hat{W}_t$, with

$$\kappa \geq \frac{\alpha^2}{1 \pm \vartheta} \text{ and } -1 < \frac{\vartheta}{2} \pm \sqrt{\frac{\vartheta^2}{4} + \frac{\alpha^2}{2\kappa}} < 1,$$

also fulfills Conditions (H1) and (H2)!

Conclusions

- Correlation is random.
- Explicit hedge formula under the assumption

  (H2) There exists $p > 1$ such that for every $v_0 \in (-1, 1)$ there exists an open intervall $U \subset (-1, 1)$ of v_0 such that

$$\sup_{v \in U} \mathbb{E}\Big[\int_t^T \left| \frac{(\bar{\rho}_s^{t,v})^2}{1 - (\rho_s^{t,v})^2} \right|^p ds\Big] < \infty.$$

- (H2) is fulfilled by a large class of models for correlation dynamics.

Thank you for your attention!