# On Securitization, market completion and equilibrium Risk transfer

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Humboldt University / École Polytechnique - CMAP

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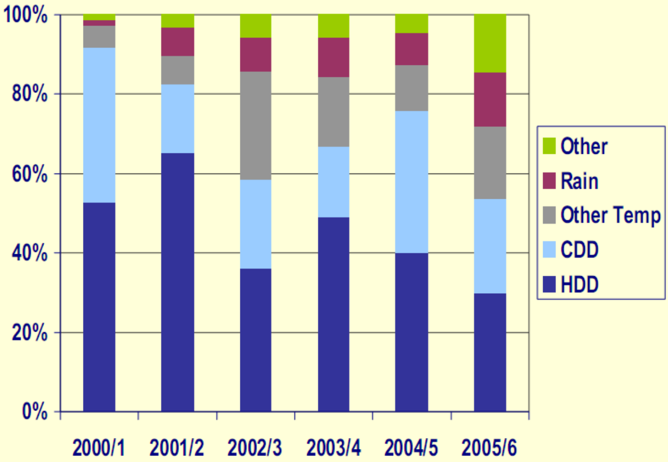
  - An example is a structured derivative issued to shift insurance risks to capital markets.

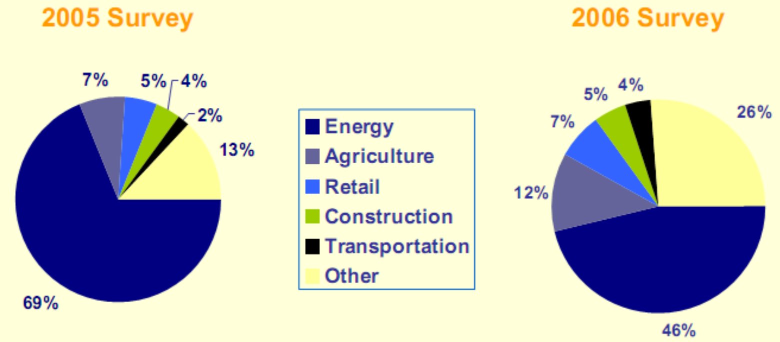
The question is how to price the structured derivative?

# What is wanted?




Source: Weather Risk Management Association 2006 Survey Results, Price Waterhouse Coopers

# Who wants what?




Reported values weighted by number of trades reported by respondent.

Source: Weather Risk Management Association 2006 Survey Results, Price Waterhouse Coopers

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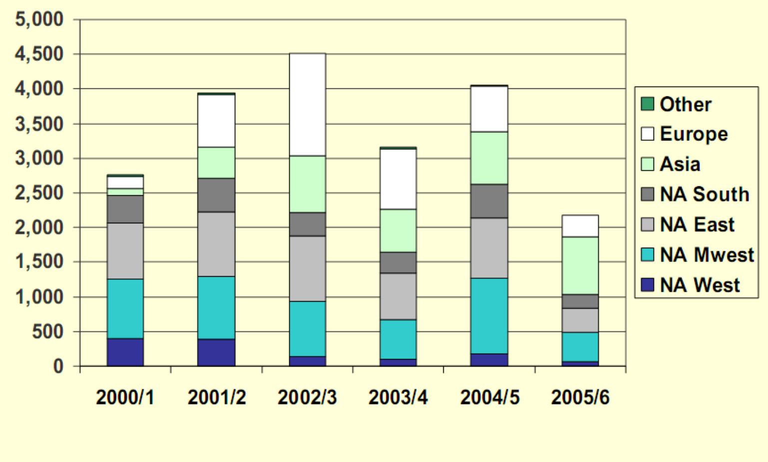


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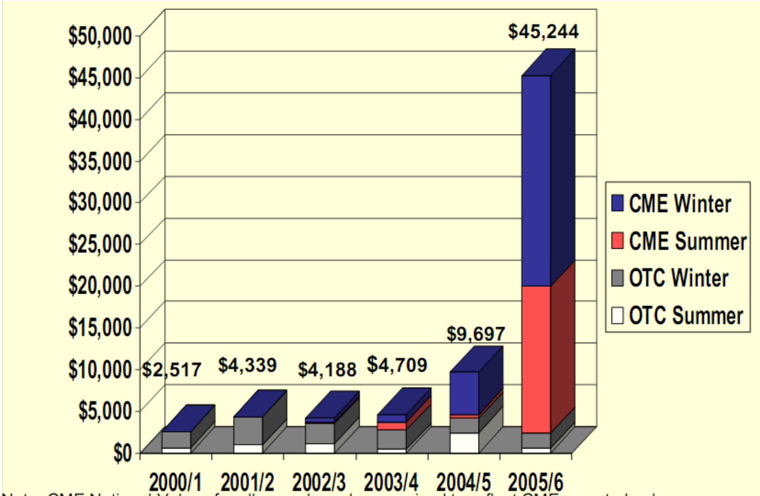
# How much?




Note: CME Notional Values for all years have been revised to reflect CME-reported values.
Source: Weather Risk Management Association 2006 Survey Results, Price Waterhouse Coopers

Our market

# The underlyings

- A set $\mathcal{A}$ of agents are exposed to tradable and non-tradable risk factors:
  - The non-tradable risk process follows a diffusion with additive noise:

$$dR_t = \mu^R(t, R_t)dt + \sigma^R(t, R_t)dW_t^R,$$

  - A tradable asset (a stock or a commodity) whose price follows a positive diffusion process:

$$\frac{dS_t}{S_t} = \mu^S(t, R_t, S_t)dt + \sigma^S(t, R_t, S_t)dW_t^S.$$

# The payoffs

- The agents receive random incomes at some terminal horizon $T$:

$$H^a = h^a(X_T) + \int_0^T \varphi_s^a(X_s)ds, \quad a \in \mathcal{A}$$

where $X_t = (R_t, S_t)$ is the state (forward process).

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The market is a priori incomplete!

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Assumption: All payoffs are bounded.

# Pricing schemes

- The structured derivative is priced by market forces and hence by an arbitrage-free pricing scheme.
- Each such pricing scheme can be identified with a measure $\mathbb{Q} \approx \mathbb{P}$.
- The agents have no impact on the tradable asset price $\Rightarrow$ Hence stock prices must be $\mathbb{Q}$-martingales.
- In equilibrium the structured derivative price is given by

$$B_t^* = \mathbb{E}_{\mathbb{Q}^*}[H^I|\mathcal{F}_t] \quad \text{w.r.t. an endogenous measure } \mathbb{Q}^*.$$

# Pricing rules

→look for a linear pricing rule.

Look for a predictable $\theta = (\theta^S, \theta^R) \in L^2$ such that

$$\mathbb{Q} = \exp\Big( - \int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T |\theta_s|^2 ds \Big)\mathbb{P}$$

defines a measure and $dW_t^\theta = dW_t + \theta_t dt$ are $\mathbb{Q}$-Brownian motion

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- $\theta = (\theta^S, \theta^R)$ is the market price of risk
  - $\theta^S = \mu^S/\sigma^S$ is exogenously given
  - $\theta^R$ is endogenously given by an equilibrium condition.

# The aim of the pricing rule

- $\theta^S = \mu^S/\sigma^S$ is exogenously given
- $\theta^R$ is endogenously given by an equilibrium condition.

Characterize the equilibrium market price of external risk, i.e. look for a pricing measure $\mathbb{Q}$ such that, when all agents minimize their risk exposures by trading in the financial market, then the aggregate demand for the derivative equals its supply.

# The bond price process

- For a given market price of risk $\theta$ the derivative price process

$$B_t^\theta = \mathbb{E}_{\mathbb{Q}_\theta}[H^I|\mathcal{F}_t] = \mathbb{E}^\theta[H^I] + \int_0^t \kappa_s^\theta dW_s^\theta$$

$$= \mathbb{E}^\theta[H^I] + \int_0^t \kappa_s^{\theta,R}(dW_s^R + \theta_s^R ds) + \int_0^t \kappa_s^{\theta,S}(dW_s^S + \theta_s^S ds).$$

- Structured derivative volatility is $\kappa^\theta = (\kappa^{\theta,S}, \kappa^{\theta,R})$ and is endogenously given by equilibrium. We assume that

$$\kappa^{\theta,R} \neq 0.$$

fluctuations of the external risk translate into fluctuations of the bond price

# The wealth process

- The gains or losses from trading according to $\pi^{a,\theta} = (\pi^{a,\theta,1}, \pi^{a,\theta,2})$ are

$$V_t^{a,\theta}(\pi^{a,\theta}) = \int_0^t \pi_s^{a,\theta,1} dS_s + \int_0^t \pi_s^{a,\theta,2} dB_s^\theta$$

and agent's *a* payoff at terminal horizon *T* from trading according to $\pi^{a,\theta}$ is $H^a + V_T^{a,\theta}(\pi^{a,\theta})$.

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- The set of admissible trading strategies is

$$\mathbb{E}\Big[\exp(-kV_T^{a,\theta}(\pi))\Big] < \infty \text{ for some positive } k$$

# The preferences

Assumption: Utilities of the agents generated by monetary dynamic convex risk measures $\rightarrow$ BSDE

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A BSDE is an equation of the type:

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds$$

- $T$, deterministic terminal time
- $\xi$, the terminal condition. An $\mathcal{F}_T$ adapted integrable R.V.
- $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ we call generator

In El Karoui & Peng & Quenez (1997) an overview is given

# The preferences

The agent's risk assessment dynamics

$$-dY_t^a = -Z_t^a dW_t - g^a(t, Z_t^a)dt, \quad Y_T^a = -[H^a + V_T^{a,\theta}(\pi^{a,\theta})]$$

with

- driver $g^a$ specifies the risk preference $\rightarrow$a convex function

# The entropic risk measures

- A special class of monetary utilities are the entropic ones. They lead to BSDEs with quadratic drivers

$$g^a(t,z) = \frac{1}{2\gamma_a}||z||^2$$

Here $\gamma_a > 0$ is the agent's coefficient of risk tolerance.

- It leads to the same risk criterion as the exponential Von-Newman Morgersten utility $U(x) = -\exp\left(\gamma_a^{-1} x\right)$.

# Agent's optimization problem

- Recall that for a given market price of risk $\theta$:
  the risk ( $Y_t^a$) of the agent's $a$ payoff $H^a + V_T^{a,\theta}(\pi^{a,\theta})$, is given by

$$-dY_t^a = g^a(t, Z_t)dt - Z_t^a dW_t \quad \text{with} \quad Y_T^a = -[H^a + V_T^{a,\theta}(\pi^{a,\theta})].$$

- Agent's $a$ goal is to pick a trading strategy $\tilde{\pi}^{a,\theta}$ to minimize the risk, i.e.,

$$\tilde{\pi}^{a,\theta} = \arg\min_{\pi^\theta} Y_0^a(\pi^{a,\theta}).$$

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- $\theta^*$ is an equilibrium market price of risk if

$$\sum_{a \in \mathcal{A}} \tilde{\pi}_t^{a,\theta^*,2} \equiv 1 \quad (0 \le t \le T).$$

# Agent's optimization problem

Under equilibrium and with some changes of variables:

$$\hat{Y}^a_t(\pi^a) = Y^a_t(\pi^a) + V^{a,\theta}_t(\pi^a)$$

$$\bar{Z}_s = Z_s + \pi^{a,1}_s \begin{pmatrix} \sigma^S_s S_s \\ 0 \end{pmatrix} + \pi^{a,2}_s \kappa^\theta_s$$

Agent's *a* BSDE

$$\hat{Y}^a_t = -H^a + \int_t^T G^a(\bar{Z})dt - \int_t^T \bar{Z}^a_t dW_t$$

with

$$G^a(Z) = -z^1\theta^S - z^2\theta^R - \frac{\gamma_a}{2}[(\theta^S)^2 + (\theta^R)^2]$$

# The representative agent I

In complete markets Pareto optimal allocation (hence competitive equilibria) can be supported by equilibria of the Representative agent

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In complete markets Pareto optimal allocation (hence competitive equilibria) can be supported by equilibria of the Representative agent

What is the Representative's agent risk measure?

Assume two agents $a$ and $b$; their risk profile is described by the BSDEs with drivers $g^a$ and $g^b$. Let

$$g^{ab}(t,z) = g^a \Box g^b(t,z) = \inf_x \{g^a(t, z-x) + g^b(t,x)\}.$$

(Inf-convolution - El Karoui & Barrieu 2005)

# The representative agent II

- The Rep. Ag. risk is given by

$$-dY^{ab}_t(\pi^\theta) = g^{ab}(t, Z_t)dt - Z_t dW_t \quad \text{with}$$

$$Y^{ab}_T(\pi^\theta) = -[H^a + H^b + H^I + V^{ab,\theta}_T(\pi^\theta)].$$

- Her goal is to minimize the risk:

$$\min_{\pi^\theta} \mathrm{Y}^{\mathrm{ab}}_0(\pi^\theta)$$

# Finding the equilibrium market price of risk

- Look for $\theta^* = (\theta^S, \theta^{*R})$ such that

$$\tilde{\pi}^{ab,\theta} \triangleq \arg\min_{\pi^\theta} Y_0^{ab}(\pi^{ab,\theta}) = (\tilde{\pi}^{ab,\theta,1}, 0).$$

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  Then $\theta^*$ is an equilibrium market price of risk characterized by a BSDE.

- We work under the standing assumption that derivative's price volatility (of $W^R$) under equilibrium pricing measure $\mathbb{Q}_{\theta^*}$ does not vanish,

$$\kappa^{\theta^*,R} \neq 0.$$

- This assumption is verified as long as structured derivative payoff is monotonic with respect to the non-tradable risk.

# Representative “real” BSDE

Again with some changes of variable, the representative agent’s BSDE

$$\hat{Y}_t^{ab} = -\frac{H^a + H^b + \beta H^l}{\gamma_a + \gamma_b} + \int_t^T G^{ab}(\bar{Z}^{ab})dt - \int_t^T \bar{Z}_t^{ab} dW_t$$

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**Theorem**

$$-\theta^{*R} = z^2/(\gamma_a + \gamma_b)$$

# Overview

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- Solve the BSDE for the Rep. Ag. (Quadratic growth BSDE)
- The $Z$ part of the Rep. Ag.'s BSDE will be the $\theta^R$
  - Verify the admissibility condition of $\theta^R$
- Knowing $\theta^S$ and $\theta^R$ the other quantities follow:
  - Derivative price
  - Agent's risks assessments

# A example

- Let $(R_t)$ be the temperature process and $(S_t)$ be the price of a share of an energy provider equity with dynamics

$$dR_t = 4t dt + 2.0\, dW_t^R,$$
$$\frac{dS_t}{S_t} = \mu^S dt + \frac{1}{\sqrt{\Gamma(t, R_t)}}\, dW_t^S,$$

where

$$\Gamma(t, R_t) = 8(\arctan(-R_t) + \pi/2).$$

- A bank holding the stock may chose to hedge its financial risk as measured by the stock volatility by issuing a structured derivative that pays yield

$$\varphi^I(t, S_t, R_t) = \exp\left\{ -M \left( \int_0^t a_s ds - R_t \right)^+ \right\}, \quad (M > 0).$$

# A example

- Two more agents $A$ and $B$ with risk preferences are described by entropic utilities $\gamma_a = 1.0$ and $\gamma_b = 2.0$, have the incomes

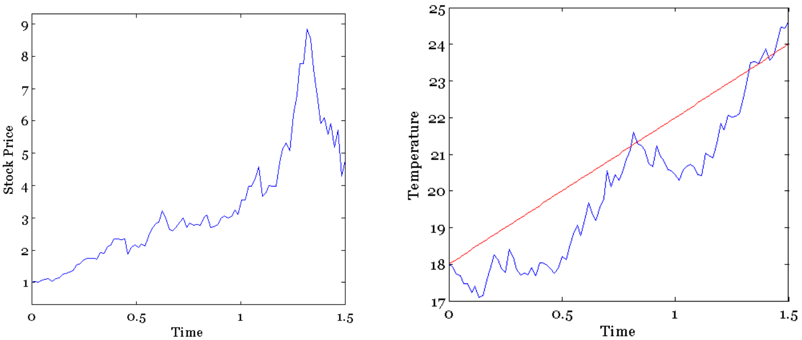
$$H^a = c^a S_T + \int_0^T \exp\{-M^a(R_t - R^a)^2\} dt,$$

$$H^b = c^b S_T + \int_0^T \exp\{-M^b(R_t - R^b)^2\} dt.$$

- The constants of our model are chosen as:

| $\gamma_a$ | $\gamma_b$ | $\gamma_R$ | $M$ | $M^a$ | $M^b$ | $c^a$ | $c^b$ | $R^a$ | $R^b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 2.0 | 3.0 | 2.0 | 0.5 | 0.5 | 0.5 | 0.5 | 4.0 | $-1.0$ |

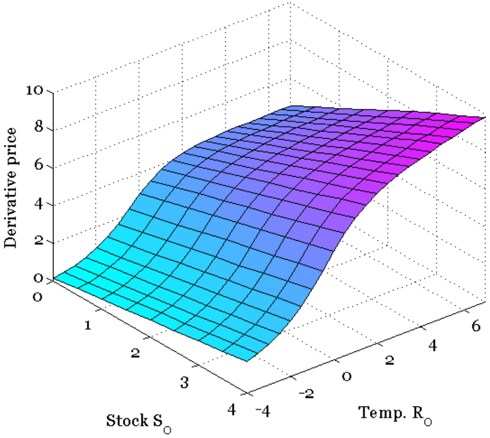
# A example
## The forward processes




Typical trajectories of the forward processes.

# A example

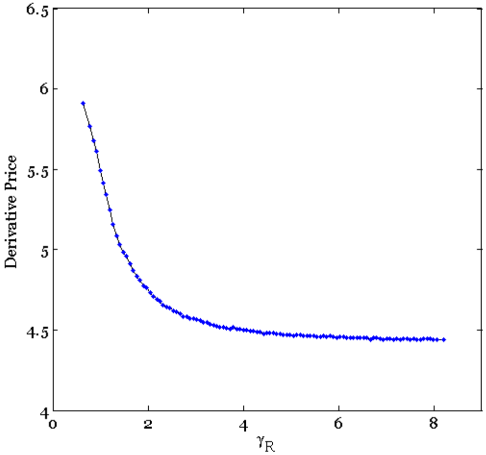
## Derivative prices as a function of the forward process




Derivative prices as a function of the forward process.

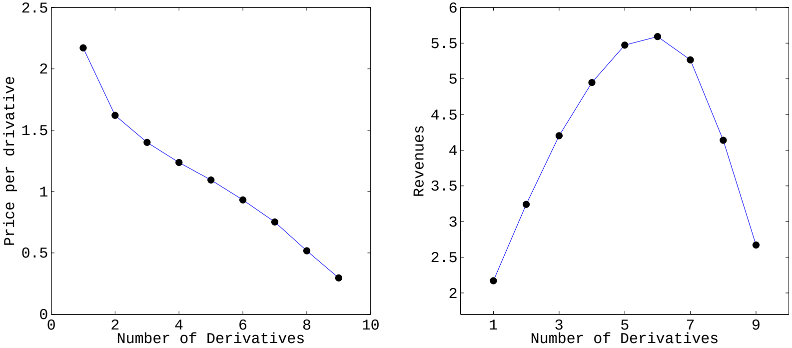
# A example

## Derivative price as function of the Tolerance factor $\gamma_R = \gamma_a + \gamma_b$




Derivative price as fct of $\gamma_R = \gamma_a + \gamma_b$ with $(s_0, r_0) = (1.0, 1.0)$.

# A example
## Price per share and Revenues



# A example
## Risk sharing Vs Risk Transfer

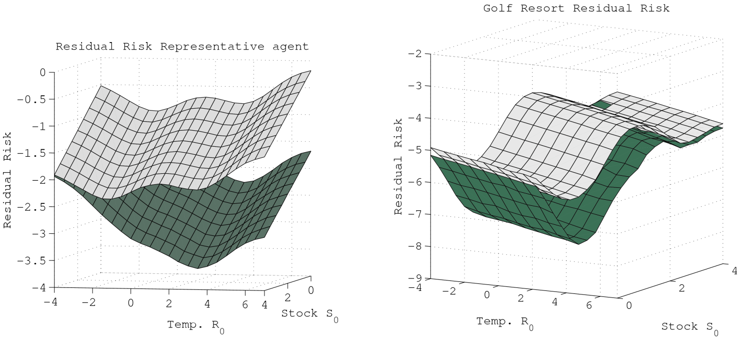



Figure: On the left the representative agent, on the right one of the agents.

# Conclusion and Outlook

- Recap:
  - We proposed an equilibrium approach to pricing structured derivatives .
  - We derived sufficient conditions for market completeness ( payoff's monotonicity with respect to the non-tradable risk).
  - Sensitivity analysis on the number of bonds and risk tolerance
  - We provide numerical results.

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- Topics for future research:
  - equilibrium models beyond the Markovian framework;
  - equilibrium pricing with jump processes;
  - optimal risk transfer and the design of derivatives.

# Thank you!

Thank you very much!

# For Further Reading I

U. Horst, T. Pirvu and G. d. R.
*On Securitization, Market Completion and Equilibrium Risk Transfer*
Preprint 2009

P. Imkeller and G. d. R.
*Path regularity and explicit convergence rate for BSDE with truncated quadratic growth*
Preprint 2009

N. Karoui, S. Peng and M. Quenez
BSDEs in finance
*Mathematical Finance*, Vol.7 (No. 1):1-71, 1997.