Stochastic Targets in Law: Super-Hedging for Quantile Hedging

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Problem Formulation

- **Stock price:** (with large investor's strategy π)

$$\frac{dS^\pi(u)}{S^\pi(u)} = \mu\left(u, S^\pi(u), \pi_u\right) du + \sigma\left(u, S^\pi(u), \pi_u\right) dW_u$$

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- **Wealth process:** (risk free interest rate $r = 0$)

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- **Quantile Hedging problem:** Given $p \in (0,1)$, find

$$v(t,s;p) := \inf\left\{x \geq 0 :\ \mathbb{P}\left[X^\pi_{t,s,x}(T) \geq g\left(S^\pi_{t,s}(T)\right)\right] \geq p \text{ for some } \pi \in \mathcal{A}\right\} .$$

Explicit Solution (Complete Market)

- **Stock price under the (unique) Risk Neutral Measure $\mathbb{Q}$:**

$$\frac{dS(u)}{S(u)} = \sigma\left(u, S(u)\right) dW_u^{\mathbb{Q}} \quad \text{(independent on } \pi\text{)}$$

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$$\max \mathbb{P}[A] \quad \text{under} \quad \mathbb{P}^{g}[A] := \mathbb{E}^{\mathbb{Q}}\left[\frac{g\left(S_{t,s}(T)\right)}{\mathbb{E}^{\mathbb{Q}}\left[g\left(S_{t,s}(T)\right)\right]}\mathbf{1}_A\right] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}\left[g\left(S_{t,s}(T)\right)\right]}\,.$$

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- Relies heavily on the duality between super-hedgeable claims and risk neutral measures. How to extend this to large investor's problems, non financial problems,... ?

Comparison with the super-hedging problem

- **Dual approach:**

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v(t,s;1) &:= \inf\left\{x \geq 0 : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \geq g\left(S_{t,s}(T)\right)\right] = 1\right\} \\
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- **Direct approach of Soner and Touzi:**

- (DP1): $x > v(t,s;1) \Rightarrow \exists\, \pi \in \mathcal{A}$ s.t. for all stopping time $\tau \leq T$

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\- (DP2): $x < v(t,s;1) \Rightarrow$ for all stopping time $\tau \leq T$ and $\pi \in \mathcal{A}$

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$\Rightarrow$ is sufficient to derive PDEs associated to $v(\cdot;1)$.

Direct approach for quantile hedging ?

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and $\mathbb{E}[P] = p$ i.e.

$$P = p + \int_t^{\tau} \alpha_u dW_u$$

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for all stopping time $\tau \leq T$.

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PDE derivation (formally)

- Take $x = v(t, s; p)$. There is $\pi \in \mathcal{A}$ and $\alpha \in L^2(dt \times d\mathbb{P})$ s.t.

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This leads to

$$\max_{(\pi,\alpha)\in\mathcal{G}(t,s,p)} \pi\mu\left(t,s,\pi\right) - \mathcal{L}^{\pi,\alpha} v(t,s;p) = 0$$

where $\mathcal{G}(t,s,p) := \{(\pi,\alpha) :\ \pi\sigma\left(t,s,\pi\right) = D_s v(t,s;p)\sigma\left(t,s,\pi\right) + D_p v(t,s;p)\alpha\}$

Extensions

- **On the Dynamics:**

$$
\begin{aligned}
S^\pi &= s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u \\
X^\pi &= x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u
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- **On the Problems:** Given ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$ and $p \in \mathrm{Im}(\ell)$,

$$v(t,s;p) \;:=\; \inf\left\{x \in \mathbb{R} \;:\; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right] \geq p\right\}\,.$$

Examples: Super Hedging

- **Model specification**

- S^π: stocks (possibly influenced by a large investor's strategy π).

- X^π: portfolio process of the (large) investor.

- $\ell(x,s) = \mathbf{1}\{x \geq g(s)\}$

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- **In "standard" financial models:** Dual formulation of Foellmer and Leukert.

Examples: Loss Functions

- **Model specification**

- S^π: stocks (possibly influenced by a large investor's strategy π).

- X^π: portfolio process of the (large) investor.

- $\ell(x,s) = -V([x-g(s)]^-)$ with V convex non decreasing

$$v(t,s;-p) := \inf\left\{x \in \mathbb{R} : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[V\left(\left[X^\pi_{t,x,s}(T) - g(S^\pi_{t,s}(T))\right]^-\right)\right] \le p\right\}$$

- **In "standard" financial models:** Dual formulation of Foellmer and Leukert.

Examples: Indifference price

- **Model specification**

- S^π: stocks (possibly influenced by a large investor's strategy π).

- X^π: portfolio process of the (large) investor.

- $\ell(x, s) = U(x - g(s))$ with U concave non decreasing

$$v(t, s; p) := \inf \left\{ x \in \mathbb{R} : \exists \, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[U(X^\pi_{t,x_0+x,s}(T) - g(S^\pi_{t,x_0+x}(T)) \right] \geq p \right\}$$

Extensions

- **On the Dynamics:**

$$
\begin{aligned}
S^\pi &= s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u \\
X^\pi &= x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u
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- **On the Problems:** Given ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$ and $p \in \mathrm{Im}(\ell)$,

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- **DP based on the reformulation:**

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v(t,s;p) = \inf\left\{x \in \mathbb{R} : \;\exists\, (\pi,\alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq P^\alpha_{t,p}(T)\right\}\,.
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$\Rightarrow$ **Back to Stochastic Target Problems !!!**

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$\Rightarrow$ **Back to Stochastic Target Problems !!!**

$\Rightarrow$ **Main difficulty: unbounded controls (new technics...)**

Conclusion

- **The Problem:**

$$v(t,s;p) \;:=\; \inf\left\{x\in\mathbb{R} \;:\; \exists\,\pi\in\mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell(X^{\pi}_{t,x,s}(T),S^{\pi}_{t,s}(T))\right]\ge p\right\}\;.$$

- **Either :** Compute

$$u(t,s;x)=\sup_{\pi}\mathbb{E}\left[\ell(X^{\pi}_{t,x,s}(T),S^{\pi}_{t,s}(T))\right]$$

and then find $\hat{x}$ such that $u(t,s;\hat{x})=p$ so that $v(t,s;p)=\hat{x}$.

- **Or :** Directly compute $v(t,s;p)$.

- **Evolution of P^{α}:** If we have a verification result for the PDE, then one constructs $\hat{\alpha}$ and $\hat{\pi}$. It provides the evolution of

$$\mathbb{E}\left[\ell(X^{\hat{\pi}}_{t,x,s}(T),S^{\hat{\pi}}_{t,s}(T)) \mid \mathcal{F}_t\right]=P^{\hat{\alpha}}_{t,p}\;.$$

$\Rightarrow$ Evolution of the level of “reachability level” P according to different path of W.

Verification in the quantile hedging problem

- **The Problem:**

$$v(t, s; p) \;:=\; \inf\left\{x \in \mathbb{R}_+ \;:\; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,x,s}(T) \geq g(S_{t,s}(T))\right] \geq p\right\} .$$

where

$$dS_{t,s}(r) = S_{t,s}(r)\left(\mu dt + \sigma dW_r\right) \quad \text{and} \quad dX^{\pi}_{t,x,s}(r) = \pi_r dS_{t,s}(r)$$

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$$0 = \sup_{\pi\sigma s = \sigma s v_s + \alpha v_p} \left(\pi\mu s - \mu s v_s - \frac{1}{2}\sigma^2 s^2 v_{ss} - \alpha\sigma s v_{sp} - \alpha^2 v_{pp}\right)$$

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- This implies that $v_{pp} \geq 0$ and

$$\begin{aligned} 0 \;&=\; \sup_{\alpha}\left(\frac{\mu}{\sigma}\,\alpha v_p - v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} - \alpha\sigma s v_{sp} - \alpha^2 v_{pp}\right) \\ &=\; -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}} \end{aligned}$$

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c- Feynman-Kac:

$$u(t,s,q) = \mathbb{E}^{\mathbb{Q}}\left[\left(Q_{t,q}(T) - g\left(S_{t,s}(T)\right)\right)^+\right] \quad \text{where} \quad \frac{dQ(r)}{Q(r)} = (\mu/\sigma) dW_r^{\mathbb{Q}}$$

Verification in the quantile hedging problem

- Optimal controls: solution to

$$0 = \sup_{\pi\sigma s = \sigma s v_s + \alpha v_p} \left(\pi \mu s - \mu s v_s - \frac{1}{2} \sigma^2 s^2 v_{ss} - \alpha \sigma s v_{sp} - \alpha^2 v_{pp} \right)$$

is given by

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$\Rightarrow$ Retrieve also the dynamics of the probability of hedging $P^{\hat{\alpha}}$!