Switching problems and related BSDE approximation

Romuald ELIE

CEREMADE, Université Paris-Dauphine

Joint work with : I. Kharroubi & J.-F. Chassagneux

Outline of the talk

- Starting and stopping problem (d=2)
- Numerical resolution of BSDE
- Numerical resolution of BSDE with oblique reflections
- An alternative approach : Constrained BSDEs with jumps

Starting and Stopping problem

Hamadene and Jeanblanc (07) :

- Consider e.g. a power station producing electricity whose price is given by a diffusion process X : $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$
- Two modes for the power station :
  mode **1** : operating, with running profit $f_1(X_t)dt$ and terminal one $g_1(X_T)$
  mode **0** : closed, with running profit $f_0(X_t)dt$ and terminal one $g_0(X_T)$

  $\hookrightarrow$ switching from one mode to another has a cost : $c > 0$
- Management decide to produce electricity only when it is profitable enough.
- The management strategy is (θ_j, α_j) : θ_j is a sequence of stopping times representing switching times from mode α_{j-1} to α_j.

  $(a_t)_{0 \leq t \leq T}$ is the state process, i.e. the management strategy.

Starting and Stopping problem

- Following a strategy a from t up to T, gives

$$J(a,t) = g_{a_T} + \int_t^T f_{a_s}(X_s)ds - \sum_{j\geq 0} c\mathbf{1}_{\{t\leq\theta_j\leq T\}}$$

- The value processes starting respectively at time 0 in mode 1 and 2 are

$$Y_0^0 := \sup_{\{a \text{ s.t. } a_0=1\}} \mathbb{E}\left[J(a,0)\right] \qquad \text{and} \qquad Y_0^1 := \sup_{\{a \text{ s.t. } a_0=2\}} \mathbb{E}\left[J(a,0)\right]$$

- At any date $t \in [0,T]$ in state $i \in \{0,1\}$, the value function is Y_t^i.

- The solutions in both modes of production are interconnected

Starting and Stopping problem

- Y is solution of a coupled optimal stopping problem

$$Y_t^0 = \operatorname{ess}\sup_{t \le \tau \le T} \mathbb{E}\left[\int_t^\tau f_0(X_s)ds + (Y_\tau^1 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

$$Y_t^1 = \operatorname{ess}\sup_{t \le \tau \le T} \mathbb{E}\left[\int_t^\tau f_1(X_s)ds + (Y_\tau^0 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

- The optimal strategy (θ_j^*, α_j^*) is given by

$$\alpha_{j+1}^* := 1 - \alpha_j^* \quad \text{and} \quad \theta_{j+1}^* := \inf\{s \ge \theta_j^* \mid Y_s^{\alpha_j^*} = Y_s^{\alpha_{j+1}^*} - c\}$$

System of reflected BSDEs

Y is the solution of the following system of reflected BSDEs :

$$Y_t^i = g_i(X_T) + \int_t^T f_i(X_s)ds - \int_t^T Z_s^i dW_s + \int_t^T dK_s^i \ , \ i \in \{0,1\} \ ,$$

with (the coupling...)

$$Y_t^1 \geq Y_t^0 - c \text{ and } Y_t^0 \geq Y_t^1 - c \ , \ \forall t \in [0,T]$$

and ('optimality' of K)

$$\int_0^T \Big(Y_s^1 - (Y_s^0 - c)\Big) dK_s^1 = 0 \text{ and } \int_0^T \Big(Y_s^0 - (Y_s^1 - c)\Big) dK_s^0 = 0$$

- Problem : Oblique reflections.
- Idea : Interpret $Y^1 - Y^0$ as the solution of a doubly reflected BSDE.

Related PDE

Link with coupled system of PDE

- on $\mathbb{R} \times [0, T)$

$$\min\Big(-\partial_t u^0 - \mathcal{L}u^0 - f^0, u^0 - u^1 + c\Big) = 0$$
$$\min\Big(-\partial_t u^1 - \mathcal{L}u^1 - f^1, u^1 - u^0 + c\Big) = 0$$

- Terminal conditions

$$u^0(T,.) = g^0(.) \qquad \text{and} \qquad u^1(T,.) = g^1(.)$$

- Link via

$$Y_t^0 = u^0(t, X_t) \qquad \text{and} \qquad Y_t^1 = u^1(t, X_t)$$

Literature

Literature on optimal switching :

- Hamadène & Jeanblanc 07 : starting and stopping problem ($m = 2$).
- Djehiche, Hamadène & Popier 07 : studied the multidimentional case.

Link with Backward SDE :

- Hu & Tang 07 "multi-dimentional BSDEs with oblique reflection" BSDE representation for optimal switching in the case where X uncontrolled or at most partially controlled : $dX_t^a = \sigma(X_t^a)\Big[b_a(X_t^a)dt + dW_t\Big]$.
- Hamadène & Zhang 08 Generalization of Hu & Tang's BSDEs but still with an uncontrolled underlying diffusion.

Literature on control :

- Bouchard 09 : Relation with stochastic target problems with jumps.

Multi-dimensional reflected BSDE

- Multi-dimensional reflected BSDE (see Hamadène & Zhang 08) :
Find m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{A}^2)^{\mathcal{I}}$ satisfying

$$\begin{cases} Y^i_t = g(X_T) + \int_t^T f_i(s, X_s, Y^1_s, \ldots, Y^m_s, Z^i_s)ds - \int_t^T Z^i_s dW_s + K^i_T - K^i_t \\ Y^i_t \geq Y^j_t - c_{i,j}(X_t) \\ \int_0^T [Y^i_t - \max_{j\in\mathcal{I}}\{h_{i,j}(t, Y^j_t)\}]dK^i_t = 0 \end{cases}$$

- The reflections are oblique with respect to the domain of definition of Y.

FBSDE system

- **FSDE**
- **BSDE**

$$\begin{cases} X_t &= x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t &= g(X_1) + \int_t^1 h(s, X_s, Y_s, Z_s)\, ds - \int_t^1 Z_s\, dW_s \end{cases}$$

- **Solution and link with PDE** (Pardoux & Peng, 90 & 92) ;

$$\|Y\|_{\mathcal{S}^2} := \mathbb{E}\left[\sup_{0\le r\le 1} |Y_r|^2\right]^{\frac{1}{2}} < \infty\,, \qquad \|Z\|_{\mathcal{L}^2} := \mathbb{E}\left[\int_0^1 |Z_r|^2 dr\right]^{\frac{1}{2}} < \infty\,,$$

- **The Four step scheme** (Ma Protter & Yong, 94) ;
- **Approximation of the BM** (Chevance 97, Briand 01, Ma 02) ;
- **Discrete time scheme based on the path regularity of Z** (Zhang) ;

Discrete time scheme (Zhang 02)

- **FSDE**
- **BSDE**

$$\begin{cases} X_t &= x + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s \\ Y_t &= g(X_1) + \int_t^1 h(s,X_s,Y_s,Z_s)\,ds - \int_t^1 Z_s\,dW_s \end{cases}$$

- Regular time grid $(t_i)_{i\leq n}$ on $[0,1]$
- **Forward Euler approximation X^π of X**

Initial value : $\mathbf{X}_0^\pi := x$

From t_i to t_{i+1} : $\mathbf{X}_{t_{i+1}}^\pi := X_{t_i}^\pi + \frac{1}{n}\mu(t_i, X_{t_i}^\pi) + \sigma(t_i, X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i})$

- **Backward approximation $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ of (Y,Z)**

Terminal value : $\mathbf{Y}_1^\pi := g(X_1^\pi)$

From t_{i+1} to t_i :

$$\begin{cases} \mathbf{Z}_{t_i}^\pi &:= n\,\mathbb{E}\left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \mathbf{Y}_{t_i}^\pi &:= \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} h\left(t_i, X_{t_i}^\pi, \mathbf{Y}_{t_i}^\pi, \mathbf{Z}_{t_i}^\pi\right) \end{cases}$$

Intuition of the scheme

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} h(r, X_r, Y_r, Z_r)\, dr - \int_{t_i}^{t_{i+1}} Z_r \cdot dW_r$$

Step 1 : Constant step driver (Z^π given by the representation of $Y^\pi_{t_{i+1}}$)

$$\mathbf{Y}^\pi_{\mathbf{t_i}} = \mathbf{Y}^\pi_{\mathbf{t_{i+1}}} + \frac{1}{n} h\left(t_i, X^\pi_{t_i}, \mathbf{Y}^\pi_{\mathbf{t_i}}, \mathbf{Z}^\pi_{\mathbf{t_i}}\right) - \int_{t_i}^{t_{i+1}} Z^\pi_r \cdot dW_r$$

Step 2 : Best $\mathcal{L}^2(\Omega \times [t_i, t_{i+1}])$ approximation of Z^π by $\mathcal{F}_{t_i}$-meas. r.v.

$$\mathbf{Z}^\pi_{\mathbf{t_i}} := n\,\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z^\pi_r\, dr \mid \mathcal{F}_{t_i}\right] = \mathbf{n}\,\mathbb{E}\left[\mathbf{Y}^\pi_{\mathbf{t_{i+1}}}(\mathbf{W}_{\mathbf{t_{i+1}}} - \mathbf{W}_{\mathbf{t_i}}) \mid \mathcal{F}_{\mathbf{t_i}}\right]$$

Step 3 : Conditioning the first expression

$$\mathbf{Y}^\pi_{\mathbf{t_i}} = \mathbb{E}\left[\mathbf{Y}^\pi_{\mathbf{t_{i+1}}} \mid \mathcal{F}_{\mathbf{t_i}}\right] + \frac{\mathbf{1}}{\mathbf{n}} \mathbf{h}\left(\mathbf{t_i}, \mathbf{X}^\pi_{\mathbf{t_i}}, \mathbf{Y}^\pi_{\mathbf{t_i}}, \mathbf{Z}^\pi_{\mathbf{t_i}}\right).$$

Approximation Error (Zhang 02) $\mathbf{Y_t = y(t, X_t)}$

- **PDE** $\mathcal{L}^X[y] + h(., y, \sigma\nabla y) = 0 \qquad y(1,.) = g(.)$

- **Forward Euler approximation** X^π **of** X

$$\mathbf{X}^\pi_0 := x \quad \text{and} \quad \mathbf{X}^\pi_{t_i+1} := X^\pi_{t_i} + \frac{1}{n}\mu(t_i, X^\pi_{t_i}) + \sigma(t_i, X^\pi_{t_i})(W_{t_{i+1}} - W_{t_i})$$

- **Backward approximation** $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ **of** (Y, Z)

$$\mathbf{Y}^\pi_1 := g(X^\pi_1) \ \& \ \begin{cases} \mathbf{Z}^\pi_{t_i} := n\,\mathbb{E}\left[Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \mathbf{Y}^\pi_{t_i} := \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} h\left(t_i, X^\pi_{t_i}, \mathbf{Y}^\pi_{t_i}, Z^\pi_{t_i}\right) \end{cases}$$

- **Approximation Error**

$$\mathrm{Err}_n(\mathbf{Y}, \mathbf{Z}) := \sup_{t_i} \mathbb{E}\left[|Y_{t_i} - Y^\pi_{t_i}|^2\right] + \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[|Z_{t_i} - Z^\pi_{t_i}|^2\right] \le \frac{\mathbf{K}}{\mathbf{n}}$$

Approximation Error (Gobet 05) $\quad Y_t = y(t, X_t)$

- **PDE** $\quad \mathcal{L}^X[y] + h(., y, \sigma\nabla y) = 0 \qquad y(1, .) = g(.)$

- **Forward Euler approximation X^π of X**

$$\mathbf{X}_0^\pi := x \quad \text{and} \quad \mathbf{X}^\pi_{t_i+1} := X^\pi_{t_i} + \frac{1}{n}\mu(t_i, X^\pi_{t_i}) + \sigma(t_i, X^\pi_{t_i})(W_{t_{i+1}} - W_{t_i})$$

- **Backward approximation $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ of (Y, Z)**

$$\mathbf{Y}_1^\pi := g(X_1^\pi) \;\; \& \begin{cases} \mathbf{Z}^\pi_{t_i} := n\,\mathbb{E}\left[Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \mathbf{Y}^\pi_{t_i} := \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n}\mathbb{E}\left[h\left(t_i, X^\pi_{t_i}, \mathbf{Y}^\pi_{t_{i+1}}, Z^\pi_{t_i}\right) \mid \mathcal{F}_{t_i}\right] \end{cases}$$

- **Approximation Error**

$$\mathrm{Err}_n(\mathbf{Y}, \mathbf{Z}) := \sup_{t_i} \mathbb{E}\left[|Y_{t_i} - Y^\pi_{t_i}|^2\right] + \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[|Z_{t_i} - Z^\pi_{t_i}|^2\right] \le \frac{\mathbf{K}}{\mathbf{n}}$$

Addition of reflections (Bouchard Chassagneux 08) $\quad Y_t = y(t, X_t)$

- **Reflected BSDE on a boundary** $l(X_t)$

$$Y_t = g(X_T) + \int_t^T f(X_t, Y_t, Z_t)\mathrm{d}t - \int_t^T (Z_t)'\mathrm{d}W_t + \int_t^T \mathrm{d}K_t$$

$$Y_t \geq l(X_t) \text{ and } \int_0^T \Big(Y_t - l(X_t)\Big)\mathrm{d}K_t = 0$$

- **Forward Euler approximation** X^π **of** X
- **Backward approximation** $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ **of** (Y, Z)

$$\mathbf{Y}_1^\pi := g(X_1^\pi) \;\; \& \quad \begin{cases} \mathbf{Z}_{\mathbf{t_i}}^\pi & := \; n\,\mathbb{E}\left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \tilde{\mathbf{Y}}_{\mathbf{t_i}}^\pi & := \; \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} h\left(t_i, X_{t_i}^\pi, \mathbf{Y}_{\mathbf{t_i}}^\pi, Z_{t_i}^\pi\right) \\ \mathbf{Y}_{\mathbf{t_i}}^\pi & := \; \max[\tilde{Y}_{t_i}^\pi \,;\, l(X_{t_i}^\pi)]\, \mathbf{1}_{\{t_i \in \Re\}} \end{cases}$$

with $\Re \subset \pi$ the **reflection grid** to be chosen properly.

- **Approximation Error**

$$\mathrm{Err}_n(\mathbf{Y}, \mathbf{Z}) := \sup_{t_i} \mathbb{E}\left[|Y_{t_i} - Y_{t_i}^\pi|^2\right] + \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[|Z_{t_i} - Z_{t_i}^\pi|^2\right] \;\leq\; \frac{K}{\sqrt{n}}$$

Obliquely reflected BSDEs

- **Multidimensional value process** contrained in a domain ($d \geq 2$)

  $$\mathcal{C} = \{y \in \mathbb{R}^d | y^i \geq \mathcal{P}^i(y) := \max_j (y_j - c_{ij})\}$$

  with $c_{ii} = 0$, $\inf_{i \neq j} c_{ij} > 0$, $c_{ij} + c_{jk} > c_{ik}$

  $\hookrightarrow \mathcal{P}$ (oblique projection) is L-lipschitz with $L > 1$ (euclidean norm)

- System of reflected BSDEs

  $$Y_t^i = g^i(X_T) + \int_t^T f^i(X_u, Y_u^i, Z_u^i)\mathrm{d}u - \int_t^T Z_u^i \mathrm{d}W_u + K_T^i - K_t^i$$

  $Y_t \in \mathcal{C}$ **(constrained by K)** $\qquad \int_0^T \left(Y_t^i - \mathcal{P}^i(Y_t) \right) \mathrm{d}K_t^i = 0$

Goal and method

Goal : approximation scheme for Continuously Obliquely Reflected BSDE (COR) and convergence results...

Method :

(*i*) **Discretize the reflection** along a grid $\Re$

$\Longrightarrow$ Discretely Obliquely Reflected BSDE' **(DOR)**

(*ii*) **Approximation scheme for the DOR** along a grid $\pi \supset \Re$

$\Longrightarrow$ Convergence of the scheme, via regularity of DOR

(*iii*) **Convergence of the DOR to the COR** when $\Re$ is refined.

$\Longrightarrow$ The scheme converges to the DOR ($\Re$ and π well chosen)

Discretely reflected BSDEs

- Grid $\Re = \{0 = r_0 < \dots < r_k < \dots < r_\kappa = T\}$ given.
- A DOR is a triplet $(Y^{d\Re}, \widetilde{Y}^{d\Re}, Z^{d\Re})$ satisfying $Y_T^{d\Re} = \widetilde{Y}_T^{d\Re} := g(X_T)$ and, for $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\left\{ \begin{array}{lcl} \widetilde{Y}_t^{d\Re} & = & Y_{r_{j+1}}^{d\Re} + \int_t^{r_{j+1}} f(X, \widetilde{Y}^{d\Re}, Z^{d\Re})\mathrm{d}u - \int_t^{r_{j+1}} Z_u^{d\Re}\mathrm{d}W_u \quad , \\ Y_t^{d\Re} & = & \widetilde{Y}_t^{d\Re}\mathbf{1}_{\{t\notin\Re\}} + \mathcal{P}(\widetilde{Y}_t^{d\Re})\mathbf{1}_{\{t\in\Re\}}. \end{array} \right.$$

- It can be rewritten as

$$\widetilde{Y}_t^{d\Re} = g(X_T) + \int_t^T f(X_s, \widetilde{Y}_s^{d\Re}, Z_s^{d\Re})\mathrm{d}s - \int_t^T (Z_s^{d\Re})'\mathrm{d}W_s + \tilde{K}_T^d - \tilde{K}_T^d$$

$$\tilde{K}_t^d = \sum_{r\in\Re\setminus\{0\}} \Delta\tilde{K}_r^d 1_{t\ge r} \;,\; \Delta\tilde{K}_r^d = Y_r^{d\Re} - \widetilde{Y}_r^{d\Re} = \mathcal{P}(\widetilde{Y}_r^{d\Re}) - \widetilde{Y}_r^{d\Re}$$

- Same switching representation property as the COR with switching times restricted to the grid $\Re$.
- Regularity results on $\widetilde{Y}^{d\Re}$ and $Z^{d\Re}$

Approximation Scheme

- Discretization grid $\pi \supset \Re$
- Start from the terminal condition $Y_T^\pi := g(X_T^\pi) \in \mathcal{C}$
- Compute at each step

$$\left\{ \begin{array}{rcl} \bar{Z}_{t_i}^\pi & = & (t_{i+1} - t_i)^{-1}\, \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i}) \dot{Y^\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] \\ \widetilde{Y}_{t_i}^\pi & = & \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \\ Y_{t_i}^\pi & = & \widetilde{Y}_{t_i}^\pi \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(\widetilde{Y}_{t_i}^\pi) \mathbf{1}_{\{t_i \in \Re\}} \end{array} \right.$$

- Natural geometric scheme
- **Problem :** The projection operator is L-lipschitz with $L > 1$
- **Idea :** Monotonicity arguments and well chosen dominating BSDE

Convergence results

- Convergence of the scheme to the DOR (f independent of z)

$$\mathcal{E}rr(Y^{d\Re}, Y^{\pi}) \leq \frac{C}{\sqrt{n}} \quad \text{and} \quad \mathcal{E}rr(Z^{d\Re}, \bar{Z}^{\pi}) \;\leq\; C\Big(\sqrt{\frac{\kappa}{n}} + \frac{1}{n^{\frac{1}{4}}}\Big)$$

  where we used the regularity of $(\widetilde{Y}^{d\Re}, Z^{d\Re})$.

- Distance between the DOR and the COR (f bounded in z)

$$\mathcal{E}rr(Y, Y^{d\Re}) + \mathcal{E}rr(Z, Z^{d\Re}) \leq C\ |\kappa^{-\frac{1}{4}}$$

- **Always convergence of the scheme**
- If f independent of Z and $\Re = \pi$, we have

$$\mathcal{E}rr(Y, Y^{\pi}) \leq C\ n^{-\frac{1}{4}} \quad \text{and} \quad \mathcal{E}rr(Z, \bar{Z}^{\pi}) \leq C\ n^{-\frac{1}{6}}$$

General Multi-dimensional reflected BSDE

• Multi-dimensional reflected BSDE (see Hamadène & Zhang 08) :
Find m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{A}^2)^{\mathcal{I}}$ satisfying

$$\begin{cases} Y_t^i = \xi^i + \int_t^T \psi_i(s, Y_s^1, \ldots, Y_s^m, Z_s^i)ds - \int_t^T Z_s^i dW_s + K_T^i - K_t^i \\ Y_t^i \geq \max_{j\in\mathcal{I}} h_{i,j}(t, Y_t^j) \\ \int_0^T [Y_t^i - \max_{j\in\mathcal{I}}\{h_{i,j}(t, Y_t^j)\}]dK_t^i = 0 \end{cases}$$

where

- $(\xi^i)_{i\in\mathcal{I}} \in (\mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P}))^{\mathcal{I}}$,
- $h_{i,j} : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are a given constraint functions,
- $\psi_i : \ \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is an $\mathbb{F}$-progressively measurable map.

• The reflections are oblique with respect to the domain of definition of Y.
We reinterpret the solution in terms of the solution to a corresponding constrained BSDE with jumps.

Idea : Introduce an independent random switching regime allowing to jump between the components of the solution !

Alternative BSDE representation

- Introduce the random switching regime I defined by

$$I_t = I_0 + \int_0^t \int_{\mathcal{I}} (i - I_{s-})\mu(ds, di) \quad t \leq T.$$

- Consider the one-dimensional constrained BSDE with jumps :

$$\tilde{Y}_t = \xi^{I_T} + \int_t^T \psi_{I_s}(s, \tilde{Y}_s + \tilde{U}_s(1), \ldots, \tilde{Y}_s + \tilde{U}_s(m), \tilde{Z}_s)ds + \tilde{K}_T - \tilde{K}_t$$
$$- \int_t^T \tilde{Z}_s.dW_s - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i)\mu(ds, di), \quad 0 \leq t \leq T, \ a.s.$$

constrained by : $\tilde{Y}_{t-} - h_{I_{t-},j}(t, \tilde{Y}_{t-} + \tilde{U}_t(j)) \geq 0, d\mathbb{P} \otimes dt \otimes \lambda(dj) \ a.e.$

- Under technical assumptions, this constrained BSDE with jumps admits a unique minimal solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$, which relates to the solution $(Y^i, Z^i, K^i)_{i \in \mathcal{I}}$ of the multidimensional reflected BSDE via

$$\tilde{Y}_t = Y_t^{I_{t-}}, \qquad \tilde{Z}_t = Z_t^{I_{t-}} \quad \text{and} \qquad \tilde{U}_t = \left[Y_t^j - Y_{t-}^{I_{t-}} \right]_{j \in \mathcal{I}}.$$

Markovian Optimal Switching

Consider the optimal switching problem : $\sup_{a \in \mathcal{A}} J(a)$ with

$$J(a) := \mathbb{E}\Big[g_{a_T}(X_T^a) + \int_0^T \psi_{a_s}(X_s^a)ds - \sum_{0<\tau_k \le T} c_{a_{\tau_k^-}, a_{\tau_k}}(X_{\tau_k}^a)\Big].$$

where $\mathcal{A}$ is the set of strategies $a = (\tau_k, \alpha_k)_k$ with

- $(\tau_k)_k$ increasing sequence of stopping times,
- α_k is an $\mathcal{F}_{\tau_k}$-measurable r. v. taking values in $\mathcal{I} = \{1, \dots, m\}$.

• For $a \in \mathcal{A}$, the current regime is $a_t = \sum_{k \ge 0} \alpha_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t)$

• The underlying X^a, is the controlled diffusion defined by

$$X_t^a = X_0 + \int_0^t b_{a_s}(X_s^a)ds + \int_0^t \sigma_{a_s}(X_s^a)dW_s, \quad t \ge 0,$$

Hu & Tang 08 : Multidimensional reflected BSDE linked with Optimal Switching with uncontrolled diffusion.

Can we relate constrained BSDE with jumps
to switching problems with controlled diffusion ?

Related Constrained BSDE with Jumps

- Introduce the forward process (I, X^I) defined by

$$I_t = i_0 + \int_0^t \int_{\mathcal{I}} (i - I_{t-}) \mu(dt, di) \,, \quad X_t^I = x_0 + \int_0^t b_{I_s}(X_s^I) ds + \int_0^t \sigma_{I_s}(X_s^I) dWs$$

- Consider the constrained BSDE with jumps : find a minimal quadruple $(Y, Z, U, K) \in \mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2$ satisfying

$$Y_t = g_{I_T}(X_T^I) + \int_t^T \psi_{I_s}(X_s) ds - \int_t^T Z_s.dW_s - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di) + K_T - K_t,$$

on $[0, T]$, with the constraint : $U_t(i) \leq c(I_{s-}, i)$, $d\mathbb{P} \otimes dt \otimes \lambda(di)$ $a.e.$

- Y_0 is the solution of the **switching problem** starting in mode i_0 at time 0.

- **Numerical approximation** possible vie penalization of the constraint.

Conclusion

- Probabilistic numerical approximation of optimal switching problems.
  - via obliquely reflected BSDE
  - via constrained BSDE with jumps
- New BSDE Unifying and Generalizing approach for BSDE representation.
  - Constrained BSDE without jumps, Peng & Xu 07
  - BSDE with diffusion-transmutation process, Pardoux, Pradeilles & Rao 97
  - BSDE with constrained jumps, Kharroubi, Ma, Pham & Zhang 08
  - Multidimensional BSDE with oblic reflection, Hamadène & Zhang 08
- BSDE representation for coupled Systems of Variational Inequality

$$\min\Big[-\frac{\partial v_i}{\partial t} - \mathcal{L}^i v_i - f_i(., v_i, \sigma^\top D_x v_i, [v_j - v_i]_{j\in\mathcal{I}}), \min_{j\in\mathcal{I}} h_{i,j}(., v_i, \sigma^\top D_x v_i, v_j - v_i)\Big]$$

with the terminal condition on $\mathcal{I} \times \{T^-\} \times \mathbb{R}^d$

$$\min\Big[v_i - g_i \ , \min_{j\in\mathcal{I}} h_{i,j}(., v_i, \sigma^\top D_x v_i, v_j - v_i)\Big](T^-, x) = 0 \ .$$