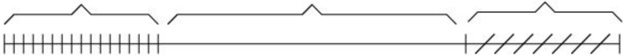
Plan

• Since electricity is not storable one has to hedge with forward or futures contracts.

• Let us consider, F_t, the price quoted at time t for the delivery of one MWh on the period $[T_d, T_d + \theta]$, with $T_d \geq t$.

Quotation dates t Time to maturity $T_d - t$ Delivery period $[T_d, T_d + \theta]$



• Let V_t denote the value of a self-financed portfolio with a (short or long) forward position φ_t (positive or negative) at time t for delivery on $[T_d, T_d + \theta]$. Recall that entering in a forward contract is free, hence

$$V_{t+\Delta t} = \varphi_t(F_{t+\Delta t} - F_t) + e^{r\Delta t} V_t \ , \tag{1}$$

r being the (constant deterministic) interest rate. Then in a continuous time setting

$$d(e^{-rt} V_t) = \varphi_t e^{-rT_d} dF_t \ . \tag{2}$$

Let H be a payoff e.g. $H = (F_{T_d} - K)_+$, then the hedging problem consists of finding an initial capital and a strategy (V_0, φ) st

$$V_0 + \int_0^T \varphi_u dF_u \approx H \ , \quad \text{in some sense} \ .$$

A non Gaussian and non stationnary model for forward prices log-returns [Benth (2003)]

[Benth (2003)] has proposed a model to represent two specific stylized features:

- the volatility term structure: the futures volatility increases when the time to maturity decreases
- the non gaussianity of log-returns inducing huge spikes on the Spot.

$$F_t = F_0 \exp\left(m_t + \underbrace{\int_0^t \sigma_s e^{-\lambda(T_d-u)} d\Lambda_u}_{\text{long-term factor}} + \underbrace{\sigma_l W_t}_{\text{short-term factor}} \right) , \quad \text{for all } t \in [0, T_d] \text{ , where} \tag{3}$$

- m is a real deterministic trend starting at 0 (a.c. wrt to Lebesgue);
- Λ is a Lévy process on $\mathbb{R}$ following a Normal Inverse Gaussian (NIG) distribution (with $\mathbb{E}[\Lambda_1] = 0$ and $Var[\Lambda_1] = 1$);
- W is a standard Brownian motion on $\mathbb{R}$;
- σ_s and σ_l standing respectively for the short-term and long-term volatility.

=> How to price and hedge contingent claims in such incomplete market ?

The quadratic or mean-variance hedging appproach ([Schweizer (1994)])

[Schweizer (1994)] proposed to minimize the expected quadratic distance between the hedging portfolio and the payoff.

Definition: The mean-variance hedging problem

Given a payoff $H \in \mathcal{L}^2$, an admissible strategy pair (V_0^*, φ^*) will be called **optimal** if it minimizes the expected squared hedging error

$$\mathbb{E}[(H - V_0 - G_T(\varphi))^2] \,, \quad \text{with} \quad G_t(\varphi) := \int_0^t \varphi_s dS_s \tag{4}$$

over all *admisible* strategy pairs $(V_0, \varphi) \in \mathbb{R} \times \Theta$.

- Related approaches
  - minimizing the expected quadratic hedging error under the pricing measure under which the underlying is martingale ([Cont-Tankov-Voltchkova (2007)] Via integro-differential equations).
  - minimizing the quadratic error under the minimal martingal measure ...
  - BSDE ...

Laplace transform approach ([Hubalek-Kallsen-Krawczyk (2006)])

• In the specific case of Lévy log-prices, [Hubalek-Kallsen-Krawczyk (2006)] proposed to express the payoff as a linear combination of exponential payoffs (using generalized Laplace transform) for which the VO strategy can be expressed explicitely.

Then they obtain quasi-explicit formula for

- the initial capital and the hedging strategy (V_0^*, φ^*)
- the variance optimal hedging error;

• Here we propose to extend this approach to the case where log-prices have independent but possibly non stationary increments.

Plan

The set of admissible strategies Θ

Let $X = (X_t)_{t\in[0,T]}$ be a real-valued special semimartingale with canonical decomposition $X = M + A$.

Definition: Θ space

For a given local martingale M, the space $L^2(M)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[\int_0^T |v_s|^2 d\left\langle M\right\rangle_s\right] < \infty \ .$$

For a given predictable bounded variation process A, the space $L^2(A)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[(\int_0^T |v_s| d||A||_s)^2\right] < \infty \ .$$

Finally, we set

$$\Theta := L^2(M) \cap L^2(A) \ .$$

The structure condition and the Mean-Variance Tradeoff Process

Let $X = (X_t)_{t\in[0,T]}$ be a real-valued special semimartingale with canonical decomposition

$$X = M + A\ .$$

Definition: Structure Condition and Mean-Variance Tradeoff Process

X is said to satisfy the **structure condition (SC)** if there is a predictable $\mathbb{R}$-valued process $\alpha = (\alpha_t)_{t\in[0,T]}$ such that

1. $A_t = \int_0^t \alpha_s d\langle M\rangle_s\ , \quad \text{for all } t \in [0,T]$, so that $dA \ll d\langle M\rangle$.
2. $K_t := \int_0^T \alpha_s^2 d\langle M\rangle_s < \infty\ , \quad P-$a.s.

$K = (K_t)_{t\in[0,T]}$ is called the **mean-variance tradeoff** (MVT) process.

Definition: Föllmer-Schweizer (FS) decomposition

We say that a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ admits a **Föllmer-Schweizer (FS) decomposition**, if it can be written as

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \ , \quad P - a.s. \ , \tag{5}$$

where

- $H_0 \in \mathbb{R}$ is a constant,
- $\xi^H \in \Theta$,
- $L^H = (L_t^H)_{t \in [0,T]}$ is a square integrable martingale, with $\mathbb{E}[L_0^H] = 0$ and strongly orthogonal to M.

Existence and unicity of FS decomposition

Assumption SC

We assume that X satisfies (SC) and that the MVT process K is uniformly bounded in t and ω.

Theorem 1: Theorem 3.4 of Monat, P. and Stricker, C. (1995)

Under Assumption SC, every random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ admits a FS decomposition. Moreover, this decomposition is unique in the following sense:
If

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H = H_0' + \int_0^T \xi_s'^H dX_s + L_T'^H,$$

where (H_0, ξ^H, L^H) and (H_0', ξ'^H, L'^H) satisfy the conditions of the FS decomposition, then

$$\begin{cases} H_0 &= H_0', \quad P - a.s. \ , \\ \xi^H &= \xi'^H \quad \text{in } L^2(M) \ , \\ L_T^H &= L_T'^H, \quad P - a.s. \ . \end{cases}$$

Link between FS decomposition and VO hedging

Theorem 2: Theorem 3 of [Schweizer (1994)]

Suppose that X satisifies (SC) and that the MVT process K of X is deterministic. If $H \in \mathcal{L}^2$ admits a FS decomposition of type (5), then the minimization problem has a solution $\varphi^{(c)} \in \Theta$ for any $c \in \mathbb{R}$, such that

$$\varphi_t^{(c)} = \xi_t^H + \frac{\alpha_t}{1 + \Delta K_t}(H_{t-} - c - G_{t-}(\varphi^{(c)})) \ , \quad \text{for all } t \in [0, T] \tag{6}$$

where the process $(H_t)_{t\in[0,T]}$ is defined by

$$H_t := H_0 + \int_0^t \xi_s^H dX_s + L_t^H \ , \tag{7}$$

and the process α is the process appearing in Definition of the (SC).

Corollary 1: Corollary 10 of [Schweizer (1994)]

Under the assumptions of Theorem 2, the solution of the minimization problem is given by the pair $(H_0, \varphi^{(H_0)})$.

Expression of the variance optimal error

Theorem 3: Theorem of [Schweizer (1994)]

Under the assumptions of Theorem 2, for any $c \in \mathbb{R}$, we have

$$\min_{v\in\Theta} \mathbb{E}[(H - c - G_T(v))^2] = \mathcal{E}(-\tilde{K}_T)\left((H_0 - c)^2 + \mathbb{E}[(L_0^H)^2] + \int_0^T \frac{1}{\mathcal{E}(-\tilde{K}_s)} d\left(\mathbb{E}[\left\langle L^H\right\rangle_s]\right)\right) .$$

Corollary 2: Theorem of [Schweizer (1994)]

If $\langle M, M\rangle$ is continuous

$$\begin{aligned}\min_{v\in\Theta} \mathbb{E}[(H - c - G_T(v))^2] &= \exp(-K_T)\left((H_0 - c)^2 + \mathbb{E}[(L_0^H)^2]\right) \\ &+\mathbb{E}\left[\int_0^T \exp\{-(K_T - K_s)\} d\left\langle L^H\right\rangle_s\right] . \qquad (9)\end{aligned}$$

Plan

Definition: PII

$X = (X_t)_{t\in[0,T]}$ is a (real) **process with independent increments (PII)** iff

1. X has cadlag paths.
2. $X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \leq s < t \leq T$ where $(\mathcal{F}_t)$ is the canonical filtration associated with X.
   Moreover we will also suppose
3. X is continuous in probability, i.e. X has no fixed time of discontinuities.

We recall Theorem II.4.15 of Jacod, J. and Shiryaev, A. (2003).

Theorem 4:

Let $(X_t)_{t\in[0,T]}$ be a real-valued special semimartingale, with $X_0 = 0$. Then, X is a process with independent increments, iff there is a version (b, c, ν) of its characteristics that is deterministic.

Definition: Lévy process (PIIS)

We add the stationnary increments property to the PII definition:
The distribution of $X_t - X_s$ depends only on $t - s$ for $0 \leq s \leq t \leq T$.

Set $T > 0$ a fixed terminal time. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ a filtered probability space and $X = (X_t)_{t\in[0,T]}$ be a real valued stochastic process.

Definition: cumulant generating function

The **cumulant generating function** of (the law of) X_t is the mapping $z \mapsto \mathrm{Log}(\mathbb{E}[e^{zX_t}])$ where $\mathrm{Log}(w) = \log(|w|) + i\mathrm{Arg(w)}$ where $\mathrm{Arg(w)}$ is the Argument of w, chosen in $]-\pi, \pi]$; Log is the principal value logarithm. In particular we have

$$\kappa_t : D \to \mathbb{C} \quad \text{with} \quad e^{\kappa_t(z)} = \mathbb{E}[e^{zX_t}] \ ,$$

where $D := \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_t}] < \infty, \ \forall t \in [0, T]\}$.

Proposition 5:

Suppose that (X_t) is a semimartingale with independent increments. Then the function $(t, z) \mapsto \kappa_t(z)$ is continuous. In particular, $(t, z) \mapsto \kappa_t(z)$, $t \in [0, T]$, z belonging to a compact real subset, is bounded.

We come back to the main optimization problem which was formulated in the beginning of this presentation. We assume that the process S is the discounted price of the non-dividend paying stock which is supposed to be of the form,

$$S_t = s_0 \exp(X_t) \ , \quad \text{for all } t \in [0, T] \ ,$$

where s_0 is a strictly positive constant and X is a semimartingale process with independent increments (PII) but not necessarily with stationary increments.

A significant reference measure

Definition

For any $t \in [0, T]$, let ρ_t denote the complex valued function such that for all $z, y \in \frac{D}{2}$

$$\rho_t(z, y) = \kappa_t(z + y) - \kappa_t(z) - \kappa_t(y) \ . \tag{10}$$

For all $z, \bar{z} \in \frac{D}{2}$, $\rho_t(z, \bar{z})$ is well defined. To shorten notations ρ_t will also denote the real valued function defined on D such that,

$$\rho_t(z) = \rho_t(z, \bar{z}) = \kappa_t(2Re(z)) - 2Re(\kappa_t(z)) \ . \tag{11}$$

An important technical lemma follows below.

Lemma 1

Let $z \in \frac{D}{2}$, with $z \neq 0$, then, $t \mapsto \rho_t(z)$ is strictly increasing if and only if X has no deterministic increments.

A reference measure

From now on, we will always suppose the following assumption.

Assumption NDI-L2

1. (X_t) has no deterministic increments.
2. $2 \in D$.

We will note $d\rho_t = \rho_{dt}$ the measure

$$d\rho_t = \rho_{dt}(1) = d(\kappa_t(2) - 2\kappa_t(1)) \ . \tag{12}$$

It is a positive measure which is strictly positive on each interval.

Proposition 6:

Under Assumption NDI-L2

$$d(\kappa_t(z)) \ll d\rho_t \ , \quad \text{for all } z \in D \ . \tag{13}$$

Expression of the canonical decomposition of S^z

Proposition 7:

Let $y, z \in \frac{D}{2}$, then S^z is a special semimartingale whose canonical decomposition

$$S_t^z = M(z)_t + A(z)_t$$

satisfies

$$A(z)_t = \int_0^t S_{u-}^z \kappa_{du}(z) \ , \quad \langle M(y), M(z) \rangle_t = \int_0^t S_{u-}^{y+z} \rho_{du}(z, y) \ , \quad M(z)_0 = s_0^z, \quad (14)$$

where $d\rho_u(z)$ is defined by equation (11). In particular we have the following:

1. $\langle M(z), M \rangle_t = \int_0^t S_{u-}^{z+1} \rho_{du}(z, 1)$
2. $\langle M(z), M(\bar{z}) \rangle_t = \int_0^t S_{u-}^{2Re(z)} \rho_{du}(z)$.

Expression of the Mean Variance Tradeoff

If we apply Proposition 7 with $y = z = 1$, we obtain

Proposition 8:

Under Assumption NDI-L2, we have

$$A_t = \int_0^t \alpha_u d\langle M\rangle_u \ , \quad \text{where} \quad \alpha_u := \frac{\lambda_u}{S_{u-}} \quad \text{with} \quad \lambda_u := \frac{d\kappa_u(1)}{d\rho_u} \ . \tag{15}$$

Moreover the MVT process is given by

$$K_t = \int_0^t \left(\frac{d(\kappa_u(1))}{d\rho_u}\right)^2 d\rho_u \ . \tag{16}$$

Then if K_T is bounded, according to Theorem of existence of the FS decomposition, there will exist a unique FS decomposition for any $H \in \mathcal{L}^2$ and since K is deterministc the minimization problem (4) will have a unique solution, by Theorem 2.

Corollary 3:

Under Assumption NDI-L2, the structure condition (SC) is verified if and only if

$$K_T = \int_0^T \left(\frac{d(\kappa_u(1))}{d\rho_u} \right)^2 d\rho_u < \infty \ .$$

In particular, (K_t) is deterministic therefore bounded.

We denote by $\mathcal{D}$ the set of $z \in D$ such that

$$\int_0^T \left| \frac{d\kappa_u(z)}{d\rho_u} \right|^2 d\rho_u < \infty. \quad (17)$$

From now on, we formulate another assumption which will be in force for the whole presentation.

Assumption K

$1 \in \mathcal{D}$.

Proposition 9: FS Decomposition in exponential PII case

Let $z \in \mathcal{D} \cap \frac{D}{2}$ with $z+1 \in \mathcal{D}$. Then $S_T^z \in \mathcal{L}^2(\Omega, \mathcal{F}_T)$.
Moreover, we suppose Assumptions NDI-L2 and K. Then we can define

- $\gamma(z,t) := \frac{d(\rho_t(z,1))}{d\rho_t}$, $t \in [0,T]$. such that $\int_0^T |\gamma(z,t)|^2 \rho_{dt} < \infty$
- $\eta(z,t) := \kappa_t(z) - \int_0^t \gamma(z,s)\kappa_{ds}(1)$ which is well-defined and st $\eta(z,\cdot)$ is ac wrt ρ_{ds} and therefore bounded.

Under those assumptions, $H(z) = S_T^z$ admits a FS decomposition
$H(z) = H(z)_0 + \int_0^T \xi(z)_t dS_t + L(z)_T$ where

$$H(z)_t := e^{\int_t^T \eta(z,ds)} S_t^z \, , \tag{18}$$

$$\xi(z)_t := \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \, , \tag{19}$$

$$L(z)_t := H(z)_t - H(z)_0 - \int_0^t \xi(z)_u dS_u \, . \tag{20}$$

Contingent claims

Now, we will proceed to the FS decomposition of more general contingent claims. We consider now options of the type

$$H = f(S_T) \quad \text{with} \quad f(s) = \int_{\mathbb{C}} s^z \Pi(dz) \ , \tag{21}$$

where Π is a (finite) complex measure in the sense of Rudin (1987), Section 6.1.

Assumption 8*Pi*

Let $I_0 = \mathrm{supp}\Pi \cap \mathbb{R}$. We denote $I := 2I_0 \cup \{1\}$.

1. $\forall z \in \mathrm{supp}\Pi, \quad z, z+1 \in \mathcal{D}$.
2. $I_0 \subset \frac{D}{2}$ and $\sup_{x \in I \cup \{1\}} \left\| \frac{d(\kappa_t(x))}{d\rho_t} \right\|_\infty < \infty$.

Remark:

1. Point 2. of Assumption Π implies $\sup_{z \in I + i\mathbb{R}} \|\kappa_{dt}(Re(z))\|_T < \infty$.
2. Under Assumption Π, $H = f(S_T)$ is square integrable. In particular it admits an FS decomposition.
3. Because of (13), the Radon-Nykodim derivative at Point 2. of Assumption Π, always exists.

Theorem 5:

Let Π be a finite complex-valued Borel measure on $\mathbb{C}$. Suppose Assumptions NDI-L2, K and Π. Any complex-valued contingent claim $H = f(S_T)$, where f is of the form (21), and $H \in \mathcal{L}^2$, admits a unique FS decomposition $H = H_0 + \int_0^T \xi_t dS_t + L_T$ with the following properties.

1. $H \in \mathcal{L}^2$ and
   - $H_t = \int H(z)_t \Pi(dz)$,
   - $\xi_t = \int \xi(z)_t \Pi(dz)$,
   - $L_t = \int L(z)_t \Pi(dz)$,

   where for $z \in \mathrm{supp}(\Pi)$, $H(z), \xi(z)$ and $L(z)$ are the same as those introduced before and we convene that they vanish if $z \notin \mathrm{supp}(\Pi)$.
2. Previous decomposition is real-valued if f is real-valued.

Lemma 2: European Call and Put case.

Let $K > 0$, the European Call option $H = (S_T - K)_+$ has two representations of the form (21):

1. For arbitrary $R > 1$, $s > 0$, we have

$$(s - K)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{22}$$

2. For arbitrary $0 < R < 1$, $s > 0$, we have

$$(s - K)_+ - s = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{23}$$

Let $K > 0$, the European Put option $H = (K - S_T)_+$ gives for an arbitrary $R < 0$, $s > 0$

$$(K - s)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{24}$$

Plan

Theorem 6: The exponential of PII Case

Let $X = (X_t)_{t\in[0,T]}$ be a process with independent increments with cumulant generating function κ. Let $H = f(e^{X_T})$ where f is of the form (21). We assume the validity of Assumptions NDI-L2, K and Π. The variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem (4), are given by

$$V_0 = H_0 \tag{25}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s) \ , \tag{26}$$

where the processes (H_t), (ξ_t) and (λ_t) are defined by

$$\gamma(z,t) := \frac{d\rho_t(z,1)}{d\rho_t} \quad \text{with} \quad \rho_t(z,y) = \kappa_t(z+y) - \kappa_t(z) - \kappa_t(y) \ , \tag{27}$$

Theorem 6 (following):

$$\eta(z, dt) := \kappa_{dt}(z) - \gamma(z, t)\kappa_{dt}(1) \ , \tag{28}$$

$$\lambda_t := \frac{d(\kappa_t(1))}{d\rho_t} \ , \tag{29}$$

$$H_t := \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz) \ , \tag{30}$$

$$\xi_t := \int_{\mathbb{C}} \gamma(z, t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) \ . \tag{31}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega) \otimes dt)$-null set.

Theorem 7:

Under the assumptions NDI-L2, K and Π, the variance of the hedging error equals

$$J_0 := \mathbb{E}[(V_0^* + G_T(\varphi^*) - H)^2] = \left(\int_{\mathbb{C}} \int_{\mathbb{C}} J_0(y,z)\Pi(dy)\Pi(dz) \right) ,$$

where

$$J_0(y,z) := \begin{cases} s_0^{y+z} \int_0^T \beta(y,z,dt) e^{\kappa_t(y+z)+\alpha(y,z,t)} dt & : \quad y,z \in \mathrm{supp}\Pi \\ 0 & : \quad otherwise. \end{cases}$$

and

$$\begin{aligned} \alpha(y,z,t) &:= \eta(z,T) - \eta(z,t) - (\eta(y,T) - \eta(y,t)) - \int_t^T \left(\frac{d\kappa_s(1)}{d\rho_s} \right)^2 d\rho_s , \\ \beta(y,z,t) &:= \rho_t(y,z) - \int_0^t \gamma(z,s) \rho_{ds}(y,1) . \end{aligned}$$

Plan

In this section, we specify rapidly the results concerning FS decomposition and the minimization problem when (X_t) is a Lévy process (Λ_t). Using the fact that (Λ_t) is a process with independent stationary increments it is not difficult to show that

$$\kappa_t(z) = t\kappa^\Lambda(z) \ , \tag{32}$$

where $\kappa^\Lambda(z) = \kappa_1(z)$, $\kappa^\Lambda : D \to \mathbb{C}$. Since for every $z \in D$, $t \mapsto \kappa_t(z)$ has bounded variation then $X = \Lambda$ is a semimartingale and a previous Proposition says that $(t, z) \mapsto \kappa_t(z)$ is continuous.

We make the following hypothesis.

Assumption 6

1. $2 \in D$;
2. $\kappa^{\Lambda}(2) - 2\kappa^{\Lambda}(1) \neq 0$.

1. $\rho_{dt} = \left(\kappa^{\Lambda}(2) - 2\kappa^{\Lambda}(1)\right) dt$;
2. $\frac{d\kappa_t}{d\rho_t}(z) = \frac{1}{\kappa^{\Lambda}(2) - 2\kappa^{\Lambda}(1)}\kappa^{\Lambda}(z)$ for any $t \in [0, T], z \in D$; so $D = \mathcal{D}$.
3. Assumptions 3 and 4 are verified.

Again we denote the process S as

$$S_t = s_0 \exp(X_t) = s_0 \exp(\Lambda_t) \ .$$

It remains to verify Assumption 5 which of course depends on the contingent claim.

Examples

1. $H = (S_T - K)_+$. We choose the second representation for the call. So, for $0 < R < 1$,

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R, 1\}\,.$$

In this case Assumption 5.1 becomes $I = [R, R+1] \subset D$. This is always satisfied since $D \supset [0,2]$ and it is convex. Assumption 5.2 is always verified because I is compact and κ^{Λ} is continuous.

2. $H = (K - S_T)_+$. We recall that $R < 0$ and so

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R\}.$$

In this case, Assumption 5.1, gives again $I = [2R, 1] \subset D$. Since $[0,2]$ is always included in D, we need to suppose here that $2R$ (which is a negative value) belongs to D.
This is not a restriction provided that D contains some negative values since we have the degree of freedom for choosing R.

Examples

We recall some cumulant generating functions of some typical Lévy processes.

1. Poisson Case: If X is a Poisson process with intensity λ, we have that $\kappa^{\Lambda}(z) = \lambda(e^{z} - 1)$. and we have $D = \mathbb{C}$.
2. NIG Case: This process was introduced by Barndorff-Nielsen. Then X is a Lévy process with $X_1 \sim NIG(\alpha, \beta, \delta, \mu)$, with $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. We have $\kappa^{\Lambda}(z) = \mu z + \delta(\gamma_0 - \gamma_z)$ and $\gamma_z = \sqrt{\alpha^2 - (\beta + z)^2}$, $D =]-\alpha - \beta, \alpha - \beta[+ i\mathbb{R}$.

Assumption 6 is verified if $2 \in D$. This happens in the following situations:

1. always in the Poisson case;
2. if $\Lambda = X$ is a NIG process and if $2 < \alpha - \beta$;

Examples Poisson process

If X is a Poisson process with parameter $\lambda > 0$ then the quadratic error is zero. In fact, the quantities

$$\begin{aligned}
\kappa^{\Lambda}(z) &= \lambda(\exp(z) - 1)) \\
\rho_t(y, z) &= \lambda t(\exp(y) - 1)(\exp(z) - 1) \\
\gamma(z, t) &= \frac{\kappa^{\Lambda}(z+1) - \kappa^{\Lambda}(z) - \kappa^{\Lambda}(1)}{\kappa^{\Lambda}(2) - 2\kappa^{\Lambda}(1)} t = \frac{\exp(z) - 1}{e - 1}
\end{aligned}$$

imply that $\beta(y, z, t) = 0$ for every $y, z \in \mathbb{C}, t \in [0, T]$.
Therefore $J_0(y, z, t) \equiv 0$. In particular all the options of type (21) are perfectly hedgeable.

We consider the PII process $X_t = \int_0^t l_s d\Lambda_s$.
Let Λ be a Lévy process. The cumulant function of Λ_t equals $\kappa_t^\Lambda(z) = t\kappa_1^\Lambda(z)$ for $\kappa_1^\Lambda = \kappa^\Lambda : D_\Lambda \to \mathbb{C}$. We formulate the following hypothesis:

Assumption 7

1. There is $r > 0$ such that $r \in D_\Lambda$.
2. $\kappa^\Lambda(2) - 2\kappa^\Lambda(1) \neq 0$.
3. Let ε such that $2\varepsilon \leq r$ and $l : [0, T] \to [\varepsilon, r/2]$ be a (deterministic continuous) function.

Remarks:

According to Lemma 1 for every $\gamma > 0$, such that $\gamma \in D$,

$$\kappa^{\Lambda}(2\gamma) - 2\kappa^{\Lambda}(\gamma) > 0 \ . \tag{33}$$

1. D contains $D_{\varepsilon,r} := \left\{x \in \mathbb{R} \,|\, \varepsilon x, \frac{rx}{2} \in D_{\Lambda}\right\} + i\mathbb{R}$, and $\kappa_t(z) = \int_0^t \kappa^{\Lambda}(zl_s)ds$.
2. $\rho_t = \int_0^t \left(\kappa^{\Lambda}(2l_s) - 2\kappa^{\Lambda}(l_s)\right) ds$;
3. $2 \in D$; X is a PII semimartingale since $t \mapsto \kappa_t(2)$ has bounded variation because $t \mapsto \kappa_t(z)$ is continuous.
4. $1 \in D_{\varepsilon,r}$ since $0, r \in D_{\Lambda}$.

If $l \equiv 1$ then $X = \Lambda$ and the validity of Assumption 7 is equivalent to the validity of Assumption 9. In fact if Assumption 7 is verified then, setting $r = 2, \varepsilon = 1$, Assumption 6 is verified. The converse is a consequence of Remark just before.

Proposition 10:

Assumptions 3 and 4 are verified. Moreover $D_{\varepsilon,r} \subset \mathcal{D}$.

Remarks:

1. Point 1. of Assumption 5 is also verified if we show that $I \subset D_{\varepsilon,r}$; in fact $D_{\varepsilon,r} \subset \mathcal{D}$ and $I_0 \cup (I_0 + 1) \subset I$.
2. From previous proof it follows that

$$\frac{d\kappa_t(z)}{d\rho_t} = \frac{\kappa^\Lambda(zl_t)}{\kappa^\Lambda(2l_t) - 2\kappa^\Lambda(l_t)} .$$

3. Since I is compact and $t \mapsto \frac{d\kappa_t(z)}{d\rho_t}$ is continuous, point 2. of Assumption 5 would be verified again for all cases provided that $I \subset D_{\varepsilon,r}$.

It remains to verify Assumption 5 for the same class of options as in previous subsections. The only point to establish will be to show

$$I \subset \{x | \varepsilon x, \frac{rx}{2} \in D_\Lambda\}. \tag{34}$$

1. $H = (S_T - K)_+$. Similarly to the case where X is a Lévy process, we take the second representation of the European Call. In this case $I = [R, R+1]$ and (34) is verified.
2. $H = (K - S_T)_+$. Again, here $R < 0$, $I = [2R, R+1]$. We only have to require that D_Λ contains some negative values, which is the case for the two Lévy processes examples introduced before. Selecting R in a proper way, (34) is fulfilled.

We provide now the FS decomposition and the solution to the minimization problem under Assumption 7.

Corollary 4:

We consider a process X of the form $X_t = \int_0^t l_s d\Lambda_s$ under Assumption 7. We consider an option H of the type (21). For $z \in \mathrm{supp}\Pi$, $t \in [0, T]$ we set

$$\begin{aligned}
\lambda(s) &= \frac{\kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(ls)}, \\
\gamma(z,s) &= \frac{\kappa^\Lambda((z+1)l_s) - \kappa^\Lambda(zl_s) - \kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)}, \\
\eta(z,s) &= \kappa^\Lambda(zl_s) - \frac{\kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)} \left(\kappa^\Lambda((z+1)l_s) - \kappa^\Lambda(zl_s) - \kappa^\Lambda(l_s)\right).
\end{aligned}$$

For convenience, if $z \notin \mathrm{supp}\Pi$ then we define

$$\gamma(z,\cdot) = \eta(z,\cdot) \equiv 0.$$

Corollary 4(following):

The following properties hold true.

1. The FS decomposition is given by $H_T = H_0 + \int_0^T \xi_t dS_t + L_T$ where

$$
\begin{aligned}
H_t &= \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz), \\
\xi_t &= \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz), \\
L_t &= H_t - H_0 - \int_0^t \xi_u dS_u.
\end{aligned}
$$

2. The solution of the minimization problem is given by a pair (V_0, φ) where

$$
V_0 = H_0 \quad \text{and} \quad \varphi_t = \xi_t + \frac{\lambda(t)}{S_{t-}}(H_{t-} - V_0 - G_{t-}(\varphi)).
$$

Model

We suppose that the forward price F follows the two factors model

$$F_t = F_0 \exp(m_t + X_t^1 + X_t^2) , \quad \text{for all } t \in [0, T_d] \text{ , where} \quad (35)$$

- m is a real deterministic trend starting at 0. It is supposed to be absolutely continuous with respect to Lebesgue;
- $X_t^1 = \int_0^t \sigma_s e^{-\lambda(T_d - u)} d\Lambda_u$, where Λ is a Lévy process on $\mathbb{R}$ with Λ following a Normal Inverse Gaussian (NIG) distribution. Moreover, we will assume that $\mathbb{E}[\Lambda_1] = 0$ and $Var[\Lambda_1] = 1$;
- $X^2 = \sigma_l W$ where W is a standard Brownian motion on $\mathbb{R}$;
- Λ and W are independent.
- σ_s and σ_l standing respectively for the short-term volatilty and long-term volatility.

The result below helps to extend Theorem of the solution of the mean-variance hedging to the case where X is a finite sum of independent PII semimartingales, each one verifying Assumptions 3,4 and 5 for a given payoff $H = f(s_0 e^{X_T})$.

Lemma 3:

Let X^1, X^2 be two independent PII semimartingales with cumulant generating functions κ^i and related domains $D^i, \mathcal{D}^i, i = 1, 2$. Let $f : \mathbb{C} \to \mathbb{C}$ of the form (21).
For $X = X^1 + X^2$ with related domains $D, \mathcal{D}$ and cumulant generating function κ, we have the following.

1. $D = D^1 \cap D^2$.
2. $\mathcal{D}^1 \cap \mathcal{D}^2 \subset \mathcal{D}$.
3. If X^1, X^2 verify Assumptions 3, 4 and 5, then X has the same property.

With the two factors model, the forward price F is then given as the exponential of a PII, X, such that for all $t \in [0, T_d]$,

$$X_t = m_t + X_t^1 + X_t^2 = m_t + \sigma_s \int_0^t e^{-\lambda(T_d - u)} d\Lambda_u + \sigma_l W_t \ . \quad (36)$$

For this model, we formulate the following assumption.

Assumption 11

1. $2\sigma_s \in D_\Lambda$.
2. If $\sigma_l = 0$, we require Λ not to have deterministic increments.
3. We define $\varepsilon = \sigma_s e^{-\lambda T_d}, \quad r = 2\sigma_s$.
4. $f : \mathbb{C} \to \mathbb{C}$ is of the type (21) fulfilling (34).

Proposition 11:

1. The cumulant generating function of X is $\kappa : [0, T_d] \times D \to \mathbb{C}$ is such that for all $z \in D_{\varepsilon,r} := \{x \in \mathbb{R} \,|\, x\sigma_s \in D_\Lambda\} + i\mathbb{R}$, then for all $t \in [0, T_d]$,

$$\kappa_t(z) = zm_t + \frac{z^2\sigma_l^2 t}{2} + \int_0^t \kappa^\Lambda(z\sigma_s e^{-\lambda(T_d-u)})du \ . \qquad (37)$$

   In particular for fixed $z \in D_{\varepsilon,r}$, $t \mapsto \kappa_t(z)$ is absolutely continuous with respect to Lebesgue measure.
2. Assumptions 3, 4 and 5 are verified.

Theorem 8: Electricity Model

We suppose Assumption 11. The variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem (4), are given by

$$V_0 = H_0 \tag{38}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s), \tag{39}$$

where the processes (H_t),(ξ_t) and (λ_t) are defined as follows:

$$\begin{aligned} \widetilde{z}_t : &= \sigma_s e^{-\lambda(T_d - t)}, \\ \gamma(z,t) : &= \frac{z\sigma_l^2 + \kappa^\Lambda((z+1)\widetilde{z}) - \kappa^\Lambda(z\widetilde{z}) - \kappa^\Lambda(\widetilde{z})}{\sigma_l^2 + \kappa^\Lambda(2\widetilde{z}) - 2\kappa^\Lambda(\widetilde{z})} \end{aligned}$$

Theorem 8 (following):

$$\eta(z,t): \quad = \quad \left[zm_t + \frac{z^2\sigma_l^2}{2} + \kappa^\Lambda(z\tilde{z}) - \gamma(z,t)\big(m_t + \frac{\sigma_l^2}{2} + \kappa^\Lambda(\tilde{z})\big)\right] dt$$

$$\begin{aligned}
\lambda_t &= \frac{m_t + \frac{\sigma_l^2}{2} + \kappa^\Lambda(\tilde{z})}{\sigma_l^2 + \kappa^\Lambda(2\tilde{z}) - 2\kappa^\Lambda(\tilde{z})}, \\
H_t &= \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz), \\
\xi_t &= \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) \ .
\end{aligned}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega)\otimes dt)$-null set.

Remarks

Previous formulae are practically exploitable numerically. The last condition to be checked is

$$2\sigma_s \in D_\Lambda. \tag{40}$$

In our classical examples, this is always verified.

1. Λ_1 is a Normal Inverse Gaussian random variable. If $\sigma_s \leq \frac{\alpha-\beta}{2}$ then (40) is verified.
2. Λ_1 is a Variance Gamma random variable then (40) is verified. if for instance $\sigma_s \leq \frac{-\beta+\sqrt{\beta^2+2\alpha}}{2}$.

Plan

We consider the problem of pricing a European call, with payoff $(S_T - K)_+$, where the underlying process S is given as the exponential of a NIG Lévy process i.e. for all $t \in [0, T]$,

$$S_t = s_0 e^{X_t} \ , \quad \text{where } X \text{ is a Lévy process with } \ X_1 \sim NIG(\alpha, \beta, \delta, \mu) \ .$$

The time unit is the year and the interest rate is zero in all our simulations. $s_0 = 100$ Euros, $T = 0.25$ i.e. three months from now. Five different sets of parameters for the NIG distribution have been considered, going from the case of *almost Gaussian* returns corresponding to standard equities, to the case of *highly non Gaussian* returns. The standard set of parameters is estimated on the *Month-ahead base* forward prices of the French Power market in 2007:

$$\alpha = 38.46\,,\ \beta = -3.85\,,\ \delta = 6.40\,,\ \mu = 0.64\ . \tag{41}$$

Those parameters imply a zero mean, a standard deviation of 41%, a skewness (measuring the asymmetry) of -0.02 and an excess kurtosis (measuring the *fatness* of the tails) of 0.01. The other sets of parameters are obtained by multiplying parameter α by a coefficient C, (β, δ, μ) being such that the first three moments are unchanged. Note that when C grows to infinity the tails of the NIG distribution get closer to the tails of the Gaussian distribution.

For instance, Table 1 shows how the excess kurtosis (which is zero for a Gaussian distribution) is modified with the five values of C chosen in our simulations.

Coefficient	$C = 0.08$	$C = 0.14$	$C = 0.2$	$C = 1$	$C = 2$
α	3.08	5.38	7.69	38.46	76.92
Excess kurtosis	1.87	0.61	0.30	0.01	$4.\,10^{-3}$

Figure: Excess kurtosis of X_1 for different values of α, (β, δ, μ) insuring the same three first moments.

We have compared on simulations the Variance Optimal strategy (VO) using the real NIG incomplete market model with the real values of parameters to the Black-Scholes strategy (BS) assuming Gaussian returns with the real values of mean and variance. Of course, the VO strategy is by definition theoritically optimal in continuous time, w.r.t. the quadratic norm. However, both strategies are implemented in discrete time, hence the performances observed in our simulations are spoiled w.r.t. the theoritical continuous rebalancing framework.

Strike impact on the pricing value and the hedging ratio

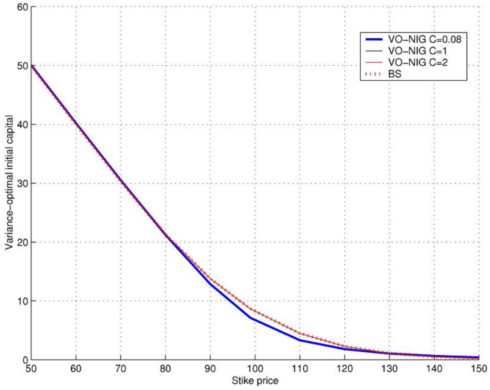


Figure: Initial capital w.r.t. the strike, for $C = 0.08\,,\ C = 1\,,\ C = 2.$

Strike impact on the pricing value and the hedging ratio

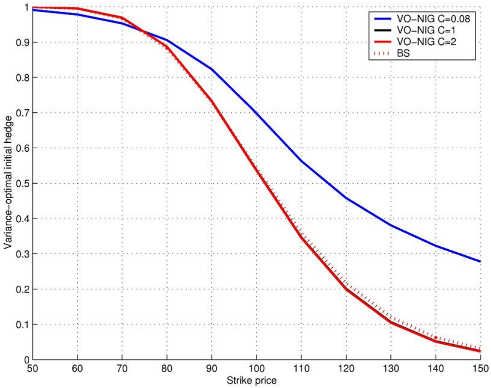


Figure: Initial Hedge w.r.t. the strike, for $C = 0.08\,,\;\; C = 1\,,\;\; C = 2.$

Strike impact on the pricing value and the hedging ratio

Strikes	$K = 50$	$K = 99$	$K = 150$
IC_{VO}	50.08	7.11	0.40
IC_{BS} (vs IC_{VO})	50.00 (99.56%)	8.65 (121.73%)	0.23 (57.30%)

Figure: Initial Capital of VO pricing (IC_{VO}) vs Initial Capital of BS pricing (IC_{BS}) for $C = 0.08$.

Hedging error and number of trading dates

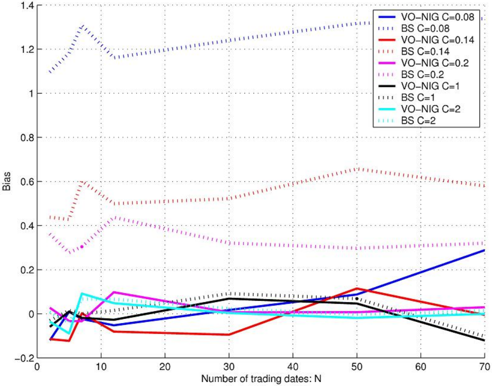


Figure: Bias of the Hedging error w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros

Hedging error and number of trading dates

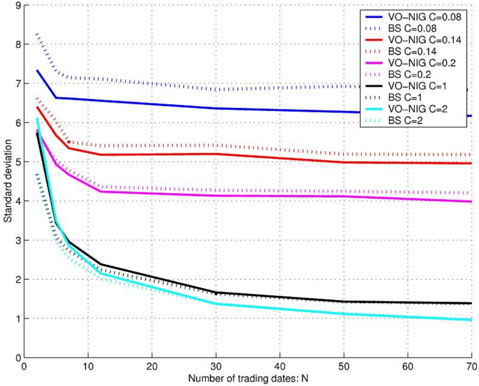


Figure: Std of the Hedging error w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros

Hedging error and number of trading dates

Coef: C	0.08	0.14	0.2	1	2
$\mathrm{Std}_{VO}/\mathrm{Std}_{BS}$	91.19%	95.88%	97.63%	107.52%	109.39%
$\mathrm{Bias}_{BS} - \mathrm{Bias}_{VO}$	1.20	0.57	0.32	0.022	0.019
$\mathrm{IC}_{BS} - \mathrm{IC}_{VO}$	1.55	0.7	0.39	0.01	0

Figure: Variance optimal hedging error vs Black-Scholes hedging error for different values of C and for $K = 99$ Euros (averaged values for different numbers of trading dates).

Hedging error and number of trading dates

Moments	Mean	Std	Ske	Kur
VO	-0.049	6.59	-3.50	31.51
BS	1.27	7.25	-7.65	152.09
VO with $\mathrm{IC}_{VO} = \mathrm{IC}_{BS}$	1.39	6.47	-2.37	10.70

Figure: Empirical moments of the hedging error for $C = 0.08$, $N = 12$ and $K = 99$ Euros (averaged values for different numbers of trading dates).

Hedging error and number of trading dates

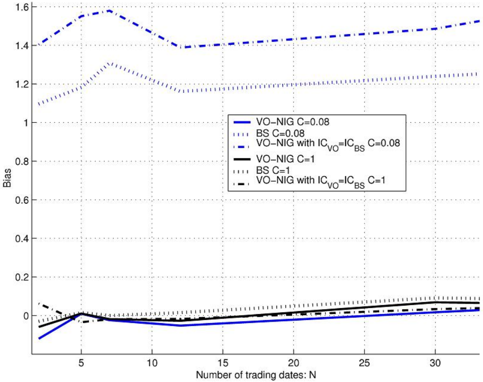


Figure: Bias of the Hedging error of BS v.s. the VO strategy with the same initial capital as BS w.r.t. N for different C and $K = 99$

Hedging error and number of trading dates

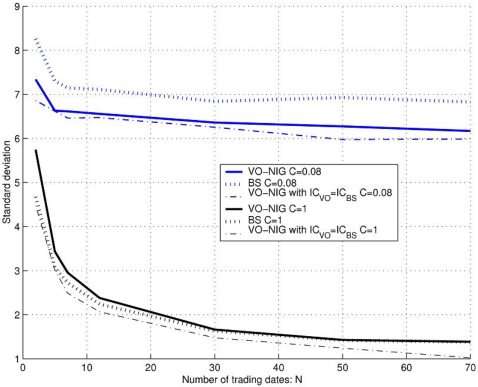


Figure: Std of the Hedging error of BS v.s. the VO strategy with the same initial capital as BS w.r.t. N for different C and $K = 99$

We consider the problem of hedging and pricing a European call on an electricity forward, with a maturity $T = T_d = 0.25$ of three month. The natural hedging instrument is the corresponding forward contract with value $S_t^0 = e^{-r(T-t)}(F_t^T - F_0^T)$ for all $t \in [0, T]$, where $F^T = F$ is supposed to follow the NIG one factor model: $F_t = e^{X_t}$ where $X_t = \int_0^t \sigma_s e^{-\lambda(T-u)} d\Lambda_u$ where Λ is a NIG process with $\Lambda_1 \sim NIG(\alpha, \beta, \delta, \mu)$
The standard set of parameters ($C = 1$) for the distribution of Λ_1 is estimated on the same data as in the previous section (*Month-ahead base* forward prices of the French Power market in 2007):

$$\alpha = 15.81\,,\ \beta = -1.581\,,\ \delta = 15.57\,,\ \mu = 1.56\ .$$

Those parameters correspond to a standard and centered NIG distribution with a skewness of -0.019. The estimated annual short-term volatility and mean-reverting rate are $\sigma_s = 57.47\%$ and $\lambda = 3$.

Hedging error and number of trading dates

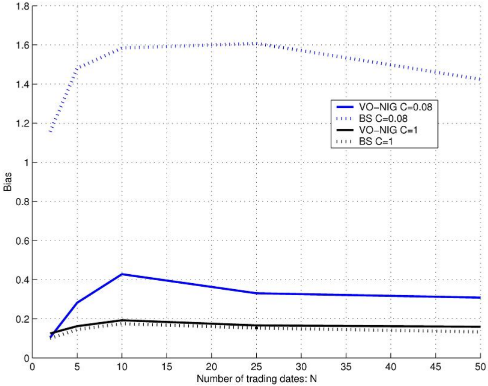


Figure: Bias of the Hedging error w.r.t. the number of trading dates for $C = 0.08$ and $C = 1$, for $K = 99$ Euros

Hedging error and number of trading dates

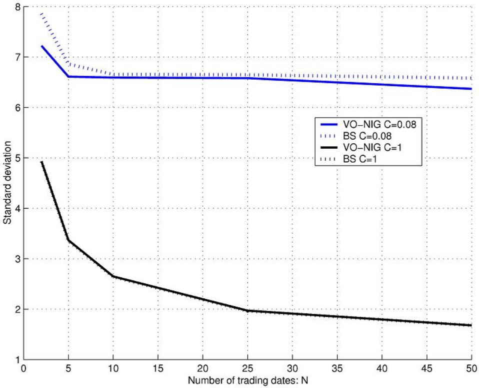


Figure: Std of the Hedging error w.r.t. the number of trading dates for $C = 0.08$ and $C = 1$, for $K = 99$ Euros

Hedging error and number of trading dates

Moments	Mean	Standard deviation	Skewness	Kurtosis
VO	0.43	6.59	-2.89	16.24
BS	1.58	6.65	-3.79	25.53

Figure: Empirical moments of the hedging error for $C = 0.08$, $N = 10$ and $K = 99$ Euros.