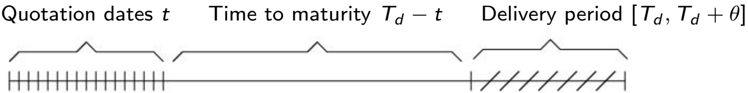
Plan

- Since electricity is not storable one has to hedge with forward or futures contracts

- Let F_t, the price quoted at time t for the delivery of one MWh on the period $[T_d, T_d + \theta]$, with $T_d \geq t$.



- Let V_t denote the value of a self-financed portfolio with a (short or long) forward position φ_t (positive or negative) at time t for delivery on $[T_d, T_d + \theta]$. Recall that entering in a forward contract is free, hence

$$V_{t+\Delta t} = \varphi_t (F_{t+\Delta t} - F_t) + e^{r\Delta t} V_t \ , \tag{1}$$

r being the (constant deterministic) interest rate. Then in a continuous time setting

$$d(e^{-rt} V_t) = \varphi_t e^{-rT_d} dF_t \ . \tag{2}$$

- Let H be a $\mathcal{F}_T$-measurable payoff e.g. $H = (F_{T_d} - K)_+$, then the hedging problem consists of finding an initial capital and a strategy (V_0, φ) st

$$e^{rT_d} V_0 + \int_0^{T_d} \varphi_u dF_u \approx H \ , \quad \text{in some sense} \ .$$

Forward and spot prices are characterized by

- the volatility term structure: the futures volatility increases when the time to maturity decreases
- the non gaussianity of log-returns on the short-term inducing huge spikes on the Spot.

[Benth (2003)] has proposed a modellize the dynamic of F as folllows

$$F_t = F_0 \exp\left(m_t + \underbrace{\int_0^t \sigma_s e^{-\lambda(T_d-u)} d\Lambda_u}_{\text{short-term factor}} + \underbrace{\sigma_l W_t}_{\text{long-term factor}} \right) , \quad \text{for all } t \in [0, T_d] \text{, where} \qquad (3)$$

- m is a real deterministic trend starting at 0 (a.c. wrt to Lebesgue);
- Λ is a Lévy process on $\mathbb{R}$ following a Normal Inverse Gaussian (NIG) distribution (with $\mathbb{E}[\Lambda_1] = 0$ and $Var[\Lambda_1] = 1$);
- W is a standard Brownian motion on $\mathbb{R}$;
- σ_s and σ_l stand respectively for the short-term and long-term volatility.

=> How to price and hedge contingent claims in such incomplete market ?

[Schweizer (1994)] proposed to minimize the expected quadratic distance between the hedging portfolio and the payoff.

Definition: The variance optimal or mean-variance hedging problem

Given a payoff $H \in \mathcal{L}^2$, an admissible strategy pair (V_0^*, φ^*) will be called **optimal** if it minimizes the expected squared hedging error

$$\mathbb{E}[(H - V_0 - G_T(\varphi))^2] \,, \quad \text{with} \quad G_t(\varphi) := \int_0^t \varphi_u dS_u \tag{4}$$

over all *admisible* strategy pairs $(V_0, \varphi) \in \mathbb{R} \times \Theta$.
Similarly, when the initial capital $c \in \mathbb{R}$ is given, we will consider $\varphi^{(c)}$ the admissible strategy minimizing the expected squared hedging error,

$$\mathbb{E}[(H - c - G_T(\varphi))^2] \,, \quad \text{with} \quad G_t(\varphi) := \int_0^t \varphi_u dS_u \tag{5}$$

over all *admisible* strategy $\varphi \in \Theta$.

Alternatively, [Cont-Tankov-Voltchkova (2007)] minimize the expected quadratic error under the pricing measure under which the underlying is martingale.

Laplace transform approach ([Hubalek-Kallsen-Krawczyk (2006)])

• In the specific case of Lévy log-prices, [Hubalek-Kallsen-Krawczyk (2006)] proposed to express the payoff as a linear combination of exponential payoffs (using generalized Laplace transform)

$$H = f(S_T) \quad \text{with} \quad f(s) = \int_{\mathbb{C}} s^z \Pi(dz) \ ,$$

for which the VO strategy can be expressed explicitely using the cumulant generating function of $\log(S_1)$.

They obtain quasi-explicit formula for

- the initial capital and the hedging strategy (V_0^*, φ^*)
- the variance optimal hedging error;

• Here we propose to extend this approach to the case where log-prices have independent but possibly non stationary increments.

Plan

The set of admissible strategies Θ

Let $S = (S_t)_{t\in[0,T]}$ be a real-valued special semimartingale with canonical decomposition $S = M + A$.

Definition: Θ space

For a given local martingale M, the space $L^2(M)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[\int_0^T |v_u|^2 d\left\langle M\right\rangle_u\right] < \infty \; .$$

For a given predictable bounded variation process A, the space $L^2(A)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[(\int_0^T |v_u| d||A||_u)^2\right] < \infty \; .$$

Finally, we set

$$\Theta := L^2(M) \cap L^2(A) \; .$$

Let $S = (S_t)_{t\in[0,T]}$ be a real-valued special semimartingale:

$$S = M + A \ .$$

Definition: Structure Condition and Mean-Variance Tradeoff Process

S is said to satisfy the **structure condition (SC)** if there is a predictable $\mathbb{R}$-valued process $\alpha = (\alpha_t)_{t\in[0,T]}$ st for all $t \in [0, T]$,

1. $A_t = \int_0^t \alpha_u d\langle M\rangle_u$, so that $dA \ll d\langle M\rangle$.
2. $K_t := \int_0^t \alpha_u^2 d\langle M\rangle_u < \infty \ , \quad P-$a.s.

(K) is called the **mean-variance tradeoff** (MVT) process.

Definition: Föllmer-Schweizer (FS) decomposition

We say that a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ admits a **Föllmer-Schweizer (FS) decomposition**, if it can be written as

$$H = H_0 + \int_0^T \xi_u^H dS_u + L_T^H \ , \quad P - a.s. \ , \tag{6}$$

where

- $H_0 \in \mathbb{R}$ is a constant,
- $\xi^H \in \Theta$,
- $L^H = (L_t^H)_{t \in [0,T]}$ is a square integrable martingale, strongly orthogonal to M with $\mathbb{E}[L_0^H] = 0$.

Related approaches

- When S is a square integrable martingale the FS decomposition coincides with the Kunita-Watanabe decomposition.
- Minimizing the quadratic error under the minimal martingal measure.
- BSDE: finding a triple (V, ξ, L) where

$$V_t = H - \int_t^T \xi_s dM_s - \int_t^T f(\omega, s, V_s, \xi_s) d\langle M \rangle_s - (L_T - L_t) ,$$

with $f(\omega, s, y, z) = z\alpha_s(\omega)$ and

1. $E(V_t^2) < \infty$ for any $t \in [0, T]$.
2. $E(\int_0^T \xi_s^2 sd < M > s) < \infty$.
3. L is an $(\mathcal{F}_t)$-local martingale orthogonal to M.

Existence and unicity of FS decomposition

Theorem 1: Theorem 3.4 of Monat, P. and Stricker, C. (1995)

Under Assumption SC, if the MVT process K is uniformly bounded in t and ω, every random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ admits a FS decomposition. Moreover, this decomposition is unique i.e. if $\quad H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H = H_0' + \int_0^T \xi_s'^H dX_s + L_T'^H$, where (H_0, ξ^H, L^H) and (H_0', ξ'^H, L'^H) satisfy the conditions of the FS decomposition, then

$$\begin{cases} H_0 & = & H_0' \ , & P-a.s. \ , \\ \xi^H & = & \xi'^H & \text{in } L^2(M) \ , \\ L_T^H & = & L_T'^H, & P-a.s. \ . \end{cases}$$

Link between FS decomposition and VO hedging

Theorem 2: Theorem 3 of [Schweizer (1994)]

Suppose that S satisifies (SC) and that the MVT process K of X is deterministic. If $H \in \mathcal{L}^2$ admits a FS decomposition of type (6), then the minimization problem (5) has a solution $\varphi^{(c)} \in \Theta$ for any $c \in \mathbb{R}$, such that

$$\varphi_t^{(c)} = \xi_t^H + \frac{\alpha_t}{1 + \Delta K_t}(H_{t-} - c - G_{t-}(\varphi^{(c)})) \ , \quad \text{for all } t \in [0, T] \qquad (7)$$

where the process $(H_t)_{t\in[0,T]}$ is defined by

$$H_t := H_0 + \int_0^t \xi_s^H dX_s + L_t^H \ . \qquad (8)$$

Corollary 1: Corollary 10 of [Schweizer (1994)]

Under the assumptions of Theorem 2, the solution of the minimization problem (4) is given by the pair $(H_0, \varphi^{(H_0)})$.

Expression of the variance optimal error

Corollary 2: Theorem of [Schweizer (1994)]

Under the assumptions of Theorem 2, for any $c \in \mathbb{R}$, if $\langle M, M \rangle$ is continuous, we have

$$\min_{v \in \Theta} \mathbb{E}[(H - c - G_T(v))^2] = e^{-K_T}\left((H_0 - c)^2 + \mathbb{E}[(L_0^H)^2]\right) + \mathbb{E}\left[\int_0^T e^{-(K_T - K_s)} d\left\langle L^H \right\rangle_s\right] \quad (9)$$

Plan

Definition: PII

$X = (X_t)_{t \in [0,T]}$ is a (real) **process with independent increments (PII)** iff

1. X has cadlag paths.
2. $X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \leq s < t \leq T$ where $(\mathcal{F}_t)$ is the canonical filtration associated with X.
   Moreover we will also suppose
3. X is continuous in probability, i.e. X has no fixed time of discontinuties.

We recall Theorem II.4.15 of Jacod, J. and Shiryaev, A. (2003).

Theorem 4:

Let $(X_t)_{t \in [0,T]}$ be a real-valued special semimartingale, with $X_0 = 0$. Then, X is a process with independent increments, iff there is a version (b, c, ν) of its characteristics that is deterministic.

Definition: Lévy process (PIIS)

We add the stationnary increments property to the PII definition:
The distribution of $X_t - X_s$ depends only on $t - s$ for $0 \leq s \leq t \leq T$.

Set $T > 0$ a fixed terminal time. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ a filtered probability space and $X = (X_t)_{t\in[0,T]}$ be a real valued stochastic process.

Definition: cumulant generating function

The **cumulant generating function** of (the law of) X_t is the mapping $z \mapsto \mathrm{Log}(\mathbb{E}[e^{zX_t}])$ where $\mathrm{Log}(w) = \log(|w|) + i\mathrm{Arg}(\mathrm{w})$ where $\mathrm{Arg}(\mathrm{w})$ is the Argument of w, chosen in $]-\pi, \pi]$; Log is the principal value logarithm. In particular we have

$$\kappa_t : D \to \mathbb{C} \quad \text{with} \quad e^{\kappa_t(z)} = \mathbb{E}[e^{zX_t}] \ ,$$

where $D := \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_t}] < \infty, \ \forall t \in [0, T]\}$.

Proposition 1:

Suppose that (X_t) is a semimartingale with independent increments. Then the function $(t, z) \mapsto \kappa_t(z)$ is continuous. In particular, $(t, z) \mapsto \kappa_t(z)$, $t \in [0, T]$, z belonging to a compact real subset, is bounded.

We come back to the main optimization problem

$$\min_{V_0,\varphi} \mathbb{E}[(H - V_0 - G_T(\varphi))^2] , \quad \text{with} \quad G_t(\varphi) := \int_0^t \varphi_u dS_u$$

We assume that the process S is the discounted price of the non-dividend paying stock st,

$$S_t = s_0 \exp(X_t) , \quad \text{for all } t \in [0, T] ,$$

where

- s_0 is a strictly positive constant;
- X is a semimartingale process with independent increments (PII) but not necessarily with stationary increments.

A significant reference measure

Definition: ρ

For any $t \in [0, T]$, let ρ_t denote the complex valued function st

$$\rho_t(z, y) = \kappa_t(z + y) - \kappa_t(z) - \kappa_t(y) \ , \quad \text{for all } z, y \in \frac{D}{2} \ . \quad (10)$$

To shorten notations ρ_t will also denote the real valued function st,

$$\rho_t(z) = \rho_t(z, \bar{z}) = \kappa_t(2Re(z)) - 2Re(\kappa_t(z)) \ , \quad \text{for all } z \in \frac{D}{2} \ .$$

Lemma 1

Let $z \in \frac{D}{2}$, with $z \neq 0$, then, $t \mapsto \rho_t(z)$ is strictly increasing if and only if X has no deterministic increments.

A reference measure

Assumption NDI2

1. (X_t) has no deterministic increments.
2. $2 \in D$.

We will note $d\rho_t = \rho_{dt}$ the measure

$$d\rho_t = \rho_{dt}(1) = d(\kappa_t(2) - 2\kappa_t(1)) \ . \qquad (11)$$

It is a positive measure which is strictly positive on each interval.

Proposition 2:

Under Assumption NDI2

$$d(\kappa_t(z)) \ll d\rho_t \ , \quad \text{for all } z \in D \ . \qquad (12)$$

Expression of the canonical decomposition of S^z

Proposition 3:

Let $y, z \in \frac{D}{2}$, then S^z is a special semimartingale whose canonical decomposition

$$S^z_t = M(z)_t + A(z)_t$$

satisfies $M(z)_0 = s^z_0$ and

$$A(z)_t = \int_0^t S^z_{u-} \kappa_{du}(z) \ , \quad \langle M(y), M(z) \rangle_t = \int_0^t S^{y+z}_{u-} \rho_{du}(z, y) \ , \tag{13}$$

In particular we have the following:

1. $\langle M(z), M \rangle_t = \int_0^t S^{z+1}_{u-} \rho_{du}(z, 1)$
2. $\langle M(z), M(\bar{z}) \rangle_t = \int_0^t S^{2Re(z)}_{u-} \rho_{du}(z)$.

Expression of the Mean Variance Tradeoff

If we apply Proposition 3 with $y = z = 1$, we obtain

Proposition 4:

Under Assumption NDI2, we have

$$A_t = \int_0^t \alpha_u d\langle M\rangle_u \ , \quad \text{where} \quad \alpha_u := \frac{1}{S_{u-}} \frac{d\kappa_u(1)}{d\rho_u} \ . \tag{14}$$

Moreover the MVT process is given by

$$K_t = \int_0^t \left(\frac{d(\kappa_u(1))}{d\rho_u} \right)^2 d\rho_u \ . \tag{15}$$

Then if K_T is bounded, according to Theorem 2 of existence of the FS decomposition, there will exist a unique FS decomposition for any $H \in \mathcal{L}^2$ and since K is deterministc the minimization problem (4) will have a unique solution, by Theorem 2.

Corollary 3:

Under Assumption NDI2, the structure condition (SC) is verified if and only if

$$K_T = \int_0^T \left(\frac{d(\kappa_u(1))}{d\rho_u} \right)^2 d\rho_u < \infty \ .$$

In particular, (K_t) is deterministic therefore bounded.

Definition: $\mathcal{D}$

We denote by $\mathcal{D}$ the set of $z \in D$ such that

$$\int_0^T \left| \frac{d\kappa_u(z)}{d\rho_u} \right|^2 d\rho_u < \infty. \tag{16}$$

Assumption K

$1 \in \mathcal{D}$.

Proposition 5: FS Decomposition in the exponential PII case

Let $z \in \mathcal{D} \cap \frac{D}{2}$ with $z+1 \in \mathcal{D}$. Then $S_T^z \in \mathcal{L}^2(\Omega, \mathcal{F}_T)$.
Assume NDI2-K. Then

- $\displaystyle\int_0^T \left| \frac{d(\rho_t(z,1))}{d\rho_t} \right|^2 \rho_{dt} < \infty$
- $\displaystyle\eta(z,t) := \kappa_t(z) - \int_0^t \frac{d\rho_u(z,1)}{d\rho_u} \kappa_{du}(1)$ is ac wrt ρ_{ds} and therefore bounded.
- $H(z) = S_T^z$ admits a FS decomposition
  $H(z) = H(z)_0 + \int_0^T \xi(z)_t dS_t + L(z)_T$, where

$$H(z)_t := e^{\int_t^T \eta(z,du)} S_t^z \ , \tag{17}$$

$$\xi(z)_t := \frac{d\rho_t(z,1)}{d\rho_t} e^{\int_t^T \eta(z,du)} S_{t-}^{z-1} \ , \tag{18}$$

$$L(z)_t := H(z)_t - H(z)_0 - \int_0^t \xi(z)_u dS_u \ . \tag{19}$$

Contingent claims

Now, we will proceed to the FS decomposition of more general contingent claims. We consider now options of the type

$$H = f(S_T) \quad \text{with} \quad f(s) = \int_{\mathbb{C}} s^z \Pi(dz) \ , \tag{20}$$

where Π is a (finite) complex measure in the sense of Rudin (1987), Section 6.1.

Assumption Π

Let $I_0 = \mathrm{supp}\Pi \cap \mathbb{R}$. We denote $I := 2I_0 \cup \{1\}$.

1. I_0 is compact.
2. $\forall z \in \mathrm{supp}\Pi, \quad z, z+1 \in \mathcal{D}$.
3. $I_0 \subset \frac{D}{2}$.
4. $\sup_{x \in I} \left\| \frac{d(\kappa_t(x))}{d\rho_t} \right\|_\infty < \infty$.

Remark:

1. Point 4. of Assumption Π implies $\sup_{z \in I + i\mathbb{R}} \|\kappa_{dt}(Re(z))\|_T < \infty$.
2. Under Assumption Π, $H = f(S_T)$ is square integrable. In particular it admits a FS decomposition.
3. Because of (12), the Radon-Nykodim derivative at Point 4. of Assumption Π, always exists.

Theorem 5:

Let Π be a finite complex-valued Borel measure on $\mathbb{C}$. Suppose Assumptions NDI2, K and Π. Any complex-valued contingent claim $H = f(S_T)$, where f is of the form (20), and $H \in \mathcal{L}^2$, admits a unique FS decomposition $H = H_0 + \int_0^T \xi_t dS_t + L_T$ with the following properties.

1. $H \in \mathcal{L}^2$ and
   - $H_t = \int H(z)_t \Pi(dz)$,
   - $\xi_t = \int \xi(z)_t \Pi(dz)$,
   - $L_t = \int L(z)_t \Pi(dz)$,

   where for $z \in \mathrm{supp}(\Pi)$, $H(z), \xi(z)$ and $L(z)$ are the same as those introduced before and we convene that they vanish if $z \notin \mathrm{supp}(\Pi)$.
2. Previous decomposition is real-valued if f is real-valued.

Lemma 2: European Call and Put case.

Let $K > 0$, the European Call option $H = (S_T - K)_+$ has two representations of the form (20):

1. For arbitrary $R > 1$, $s > 0$, we have

$$(s - K)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{21}$$

2. For arbitrary $0 < R < 1$, $s > 0$, we have

$$(s - K)_+ - s = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{22}$$

Let $K > 0$, the European Put option $H = (K - S_T)_+$ gives for an arbitrary $R < 0$, $s > 0$

$$(K - s)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{23}$$

Plan

Theorem 6: The exponential of PII Case

Let $X = (X_t)_{t\in[0,T]}$ be a PII with cumulant generating function κ.
Let $H = f(e^{X_T})$ where f is of the form (20).
Suppose Assumptions NDI2, K and Π.
The variance-optimal capital and strategy (V_0, φ), are given by

$$V_0 = H_0 \ , \quad \text{and} \quad \varphi_t = \xi_t + \frac{1}{S_{t-}} \frac{d\kappa_t(1)}{d\rho_t}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s) \ , \tag{24}$$

where the processes (H_t) and (ξ_t) are defined by

$$\begin{aligned} H_t &:= \int_{\mathbb{C}} e^{\int_t^T \eta(z,du)} S_t^z \Pi(dz) \ , \\ \xi_t &:= \int_{\mathbb{C}} \frac{d\rho_t(z,1)}{d\rho_t} e^{\int_t^T \eta(z,du)} S_{t-}^{z-1} \Pi(dz) \ , \\ \eta(z,dt) &:= \kappa_{dt}(z) - \frac{d\rho_t(z,1)}{d\rho_t} \kappa_{dt}(1) \ . \end{aligned}$$

Theorem 7:

Under the assumptions NDI2, K and Π, the variance of the hedging error equals

$$J_0 := \mathbb{E}[(V_0^* + G_T(\varphi^*) - H)^2] = \left(\int_{\mathbb{C}} \int_{\mathbb{C}} J_0(y,z)\Pi(dy)\Pi(dz) \right) ,$$

where

$$J_0(y,z) := \begin{cases} s_0^{y+z} \int_0^T \beta(y,z,t) e^{\kappa_t(y+z)+\alpha(y,z,t)} d\rho(t) & : \quad y,z \in \operatorname{supp}\Pi \\ 0 & : \quad \textit{otherwise}. \end{cases}$$

and

$$\alpha(y,z,t) := \eta(z,T) - \eta(z,t) - (\eta(y,T) - \eta(y,t)) - \int_t^T \left(\frac{d\kappa_s(1)}{d\rho_s} \right)^2 d\rho_s ,$$

$$\beta(y,z,t) := \frac{d\rho_t(y,z)}{d\rho_t} - \frac{d\rho_t(y,1)}{d\rho_t} \frac{d\rho_t(z,1)}{d\rho_t} .$$

Plan

If (X_t) is a Lévy process (Λ_t).
Using the fact that (Λ_t) is a process with independent stationary increments it is not difficult to show that

$$\kappa_t(z) = t\kappa^{\Lambda}(z)\ , \quad \text{where} \quad \kappa^{\Lambda}(z) = \kappa_1(z)\ , \tag{25}$$

$\kappa^{\Lambda} : D \to \mathbb{C}$. Since for every $z \in D$, $t \mapsto \kappa_t(z)$ has bounded variation then $X = \Lambda$ is a semimartingale and a previous Proposition says that $(t, z) \mapsto \kappa_t(z)$ is continuous.

We make the following hypothesis.

Assumption L

1. $2 \in D$;
2. $\kappa^{\Lambda}(2) - 2\kappa^{\Lambda}(1) \neq 0$.

1. $\rho_{dt} = \left(\kappa^{\Lambda}(2) - 2\kappa^{\Lambda}(1)\right) dt$;
2. $\frac{d\kappa_t(z)}{d\rho_t} = \frac{1}{\kappa^{\Lambda}(2) - 2\kappa^{\Lambda}(1)} \kappa^{\Lambda}(z)$ for any $t \in [0, T], z \in D$; so $D = \mathcal{D}$.
3. Assumptions NDI2 and K are verified.

Again we denote the process S as

$$S_t = s_0 \exp(X_t) = s_0 \exp(\Lambda_t) \ .$$

It remains to verify Assumption Π which of course depends on the contingent claim.

Verification of assumption Π: Call and Put examples

1. $H = (S_T - K)_+$. We choose the second representation for the call. So, for $0 < R < 1$,

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R, 1\}\,.$$

   Then, Assumption Π.3 becomes $2R \in D$. This is always satisfied since $D \supset [0, 2]$ and it is convex.
   Assumption Π.4 is always verified because I is compact and κ^Λ is continuous.

2. $H = (K - S_T)_+$. We recall that $R < 0$ and so

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R\}.$$

   Asuuption Π.3 requires $2R \in D$.
   This is not a restriction provided that D contains some negative values since we have the degree of freedom for choosing R.

Examples of cumulant generating functions

1. Poisson Case: If X is a Poisson process with intensity λ, we have $D = \mathbb{C}$ and

$$\kappa^{\Lambda}(z) = \lambda(e^{z} - 1) \ .$$

2. NIG Case: This process was introduced by Barndorff-Nielsen. Then X is a Lévy process with $X_1 \sim NIG(\alpha, \beta, \delta, \mu)$, with $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. We have $D =]-\alpha-\beta, \alpha-\beta[+i\mathbb{R}$, and

$$\kappa^{\Lambda}(z) = \mu z + \delta(\gamma_0 - \gamma_z) \ , \quad \text{with} \quad \gamma_z = \sqrt{\alpha^2 - (\beta+z)^2} \ .$$

Assumption NDI2 is verified if $2 \in D$. This happens in the following situations:

1. always in the Poisson case;
2. if $\Lambda = X$ is a NIG process and if $2 < \alpha - \beta$;

Hedging error for the Poisson process

If X is a Poisson process with parameter $\lambda > 0$ then the quadratic error is zero. In fact, the quantities

$$\begin{aligned} \kappa^{\Lambda}(z) &= \lambda(e^z - 1)) \\ \rho_t(y,z) &= \lambda t(e^y - 1)(e^z - 1) \end{aligned}$$

Hence,

$$\beta(y,z,t) = 0 \ , \quad \text{for every} \quad y,z \in \mathbb{C}, t \in [0,T] \ .$$

Therefore $J_0(y,z,t) \equiv 0$. In particular all the options of type (20) are perfectly hedgeable.

With the two factors model, the forward price F_t is then given as the exponential of a PII,

$$F_t = F_0 exp(X_t) \ , \quad \text{with} \quad X_t = m_t + \sigma_s \int_0^t e^{-\lambda(T_d - u)} d\Lambda_u + \sigma_l W_t \ . \tag{26}$$

For this model, we formulate the following assumption.

Assumption ELEC

1. $2\sigma_s \in D_\Lambda$.
2. If $\sigma_l = 0$, we require Λ not to have deterministic increments.
3. $f : \mathbb{C} \to \mathbb{C}$ is of the type (20) fulfilling Point 4.
4. Then $I := 2I_0 \cup \{1\} \subset D_\Lambda^{\sigma_s} := \{x \in \mathbb{R} \,|\, x\sigma_s \in D_\Lambda\}$.

Proposition 6:

1. The cumulant generating function of X is $\kappa : [0, T_d] \times D \to \mathbb{C}$ is such that for all $z \in D_{\varepsilon,r} := \{x \in \mathbb{R} \,|\, x\sigma_s \in D_\Lambda\} + i\mathbb{R}$, then for all $t \in [0, T_d]$,

$$\kappa_t(z) = zm_t + \frac{z^2\sigma_l^2 t}{2} + \int_0^t \kappa^\Lambda(z\sigma_s e^{-\lambda(T_d-u)})du \ . \qquad (27)$$

   In particular for fixed $z \in D_\Lambda^{\sigma_s}$, $t \mapsto \kappa_t(z)$ is absolutely continuous with respect to Lebesgue measure.
2. Assumption ELEC implies assumptions SC-NDI2 and Π.

Theorem 8: Electricity Model

We suppose Assumption ELEC. The variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem (4), are given by

$$V_0 = H_0 \tag{28}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s), \tag{29}$$

where the processes (H_t),(ξ_t) and (λ_t) are defined as follows:

$$\begin{aligned} \widetilde{z}_t : &= \sigma_s e^{-\lambda(T_d - t)}, \\ \gamma(z,t) : &= \frac{z\sigma_l^2 + \kappa^\Lambda((z+1)\widetilde{z}) - \kappa^\Lambda(z\widetilde{z}) - \kappa^\Lambda(\widetilde{z})}{\sigma_l^2 + \kappa^\Lambda(2\widetilde{z}) - 2\kappa^\Lambda(\widetilde{z})} \end{aligned}$$

Theorem 8 (following):

$$\eta(z,t) : \quad = \quad \left[z m_t + \frac{z^2 \sigma_l^2}{2} + \kappa^{\Lambda}(z\tilde{z}) - \gamma(z,t)\left(m_t + \frac{\sigma_l^2}{2} + \kappa^{\Lambda}(\tilde{z})\right)\right] dt$$

$$\lambda_t = \frac{m_t + \frac{\sigma_l^2}{2} + \kappa^{\Lambda}(\tilde{z})}{\sigma_l^2 + \kappa^{\Lambda}(2\tilde{z}) - 2\kappa^{\Lambda}(\tilde{z})},$$

$$H_t = \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz),$$

$$\xi_t = \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) .$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega) \otimes dt)$-null set.

Example of a call

The last condition to be checked is

$$2\sigma_s \in D_\Lambda. \tag{30}$$

1. Λ_1 is a Normal Inverse Gaussian random variable. If $\sigma_s \leq \frac{\alpha-\beta}{2}$ then (30) is verified.
2. Λ_1 is a Variance Gamma random variable then (30) is verified. if for instance $\sigma_s \leq \frac{-\beta+\sqrt{\beta^2+2\alpha}}{2}$.

Plan

• We consider the problem of pricing a European call, with payoff $(S_T - K)_+$, where the underlying process S is given as the exponential of a NIG Lévy process i.e. for all $t \in [0, T]$,

$$S_t = s_0 e^{X_t} \ , \quad \text{where } X \text{ is a Lévy process with } \ X_1 \sim NIG(\alpha, \beta, \delta, \mu) \ .$$

$s_0 = 100$ Euros, $T = 0.25$ year (i.e. three months from now).
• Five different sets of parameters for the NIG distribution have been considered, going from the case of *almost Gaussian* returns corresponding to standard equities, to the case of *highly non Gaussian* returns.
• The standard set of parameters is estimated on the *Month-ahead base* forward prices of the French Power market in 2007:

$$\alpha = 38.46 \, , \ \beta = -3.85 \, , \ \delta = 6.40 \, , \ \mu = 0.64 \ . \tag{31}$$

Coefficient	$C = 0.08$	$C = 0.14$	$C = 0.2$	$C = 1$	$C = 2$
α	3.08	5.38	7.69	38.46	76.92
Excess kurtosis	1.87	0.61	0.30	0.01	$4.\,10^{-3}$

Figure: Excess kurtosis of X_1 for different values of α, (β, δ, μ) insuring the same three first moments.

Strike impact on the pricing value

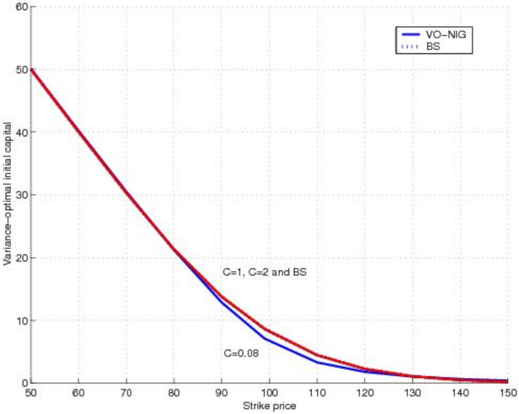


Figure: Initial capital w.r.t. the strike, for $C = 0.08\,,\ C = 1\,,\ C = 2$.

Strike impact on the pricing value

Strikes	$K = 50$	$K = 99$	$K = 150$
IC_{VO}	50.08	7.11	0.40
IC_{BS} (vs IC_{VO})	50.00 (99.56%)	8.65 (121.73%)	0.23 (57.30%)

Figure: Initial Capital of VO pricing (IC_{VO}) vs Initial Capital of BS pricing (IC_{BS}) for $C = 0.08$.

Strike impact on the hedging ratio

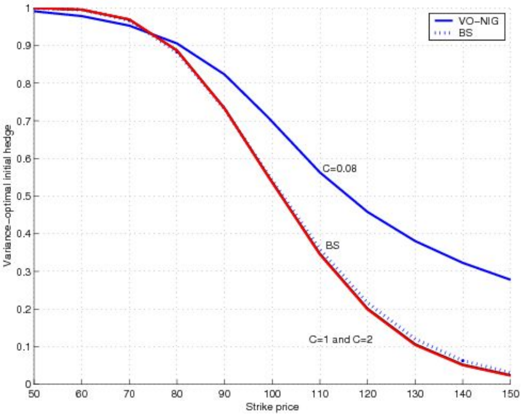


Figure: Initial Hedge w.r.t. the strike, for $C = 0.08\,,\ C = 1\,,\ C = 2.$

STD of the Hedging error and number of trading dates

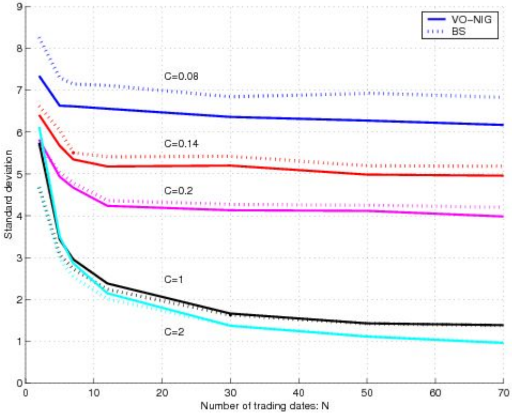


Figure: Std of the Hedging error w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros

Bias of the Hedging error and number of trading dates

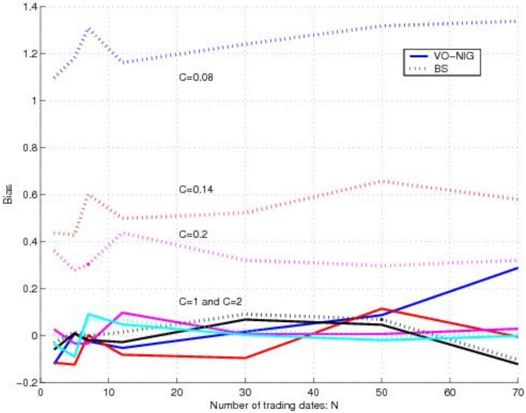


Figure: Bias of the Hedging error w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros

Hedging error and number of trading dates

Coef: C	0.08	0.14	0.2	1	2
$\mathrm{Std}_{VO}/\mathrm{Std}_{BS}$	91.19%	95.88%	97.63%	107.52%	109.39%
$\mathrm{Bias}_{BS} - \mathrm{Bias}_{VO}$	1.20	0.57	0.32	0.022	0.019
$\mathrm{IC}_{BS} - \mathrm{IC}_{VO}$	1.55	0.7	0.39	0.01	0

Figure: Variance optimal hedging error vs Black-Scholes hedging error for different values of C and for $K = 99$ Euros (averaged values for different numbers of trading dates).

Moments of the hedging error

Moments	Mean	Std	Ske	Kur
VO	-0.049	6.59	-3.50	31.51
BS	1.27	7.25	-7.65	152.09
VO with $\mathrm{IC}_{VO} = \mathrm{IC}_{BS}$	1.39	6.47	-2.37	10.70

Figure: Empirical moments of the hedging error for $C = 0.08$, $N = 12$ and $K = 99$ Euros (averaged values for different numbers of trading dates).

Hedging error when $V_0 = V_0^{BS}$ is given

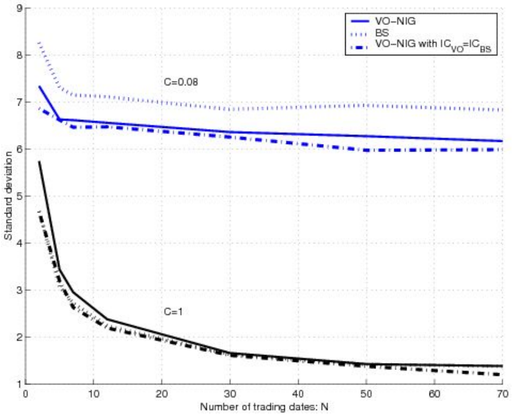


Figure: Std of the Hedging error of BS v.s. the VO strategy with the same initial capital as BS w.r.t. N for different C and $K = 99$

Hedging error when $V_0 = V_0^{BS}$ is given

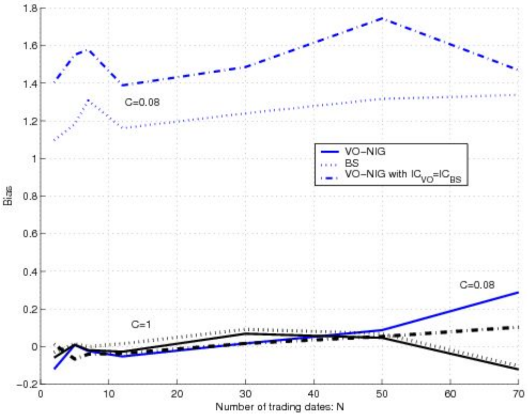


Figure: Bias of the Hedging error of BS v.s. the VO strategy with the same initial capital as BS w.r.t. N for different C and $K = 99$

• We consider the problem of hedging and pricing a European call on an electricity forward, with a maturity $T = T_d = 0.25$ of three month. The natural hedging instrument is the corresponding forward contract with value $S^0_t = e^{-r(T-t)}(F^T_t - F^T_0)$ for all $t \in [0, T]$.

• $F_t = e^{X_t}$ where $X_t = \int_0^t \sigma_s e^{-\lambda(T-u)} d\Lambda_u$ where Λ is a NIG process with $\Lambda_1 \sim NIG(\alpha, \beta, \delta, \mu)$

• The standard set of parameters ($C = 1$) for the distribution of Λ_1 is estimated on the same data as in the previous section (*Month-ahead base* forward prices of the French Power market in 2007):

$$\alpha = 15.81\,,\ \beta = -1.581\,,\ \delta = 15.57\,,\ \mu = 1.56\ .$$

Those parameters correspond to a standard and centered NIG distribution with a skewness of -0.019. The estimated annual short-term volatility and mean-reverting rate are $\sigma_s = 57.47\%$ and $\lambda = 3$.

Hedging error and number of trading dates (PII)

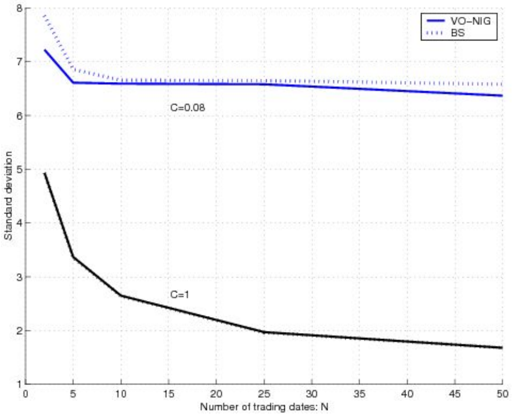


Figure: Std of the Hedging error w.r.t. the number of trading dates for $C = 0.08$ and $C = 1$, for $K = 99$ Euros

Hedging error and number of trading dates (PII)

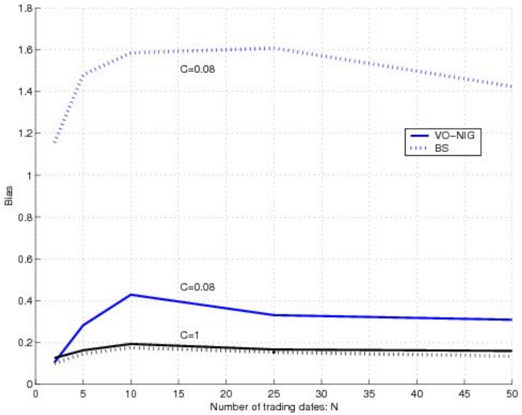


Figure: Bias of the Hedging error w.r.t. the number of trading dates for $C = 0.08$ and $C = 1$, for $K = 99$ Euros

Moments of the hedging error (PII)

Moments	Mean	Standard deviation	Skewness	Kurtosis
VO	0.43	6.59	−2.89	16.24
BS	1.58	6.65	−3.79	25.53

Figure: Empirical moments of the hedging error for $C = 0.08$, $N = 10$ and $K = 99$ Euros.

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