Arbitrage Pricing in electricity Markets

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GT FiME
IHP, 05.11.10

Joint work with alumni.

Phenomenology of Electricity markets

- Non Storability of the underlying asset

  Arbitrage pricing theory is no longer valid :

$$F_t(T) \neq \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] = S_t e^{r(T-t)}$$

  $\Rightarrow$ no pure link between spot and future prices

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  Prices as *physical* supply demand equilibrium

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  Prices as *physical* supply demand equilibrium $+$ inflexible demand $\Rightarrow$ price as a production cost (Barlow)

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  Prices as *physical* supply demand equilibrium

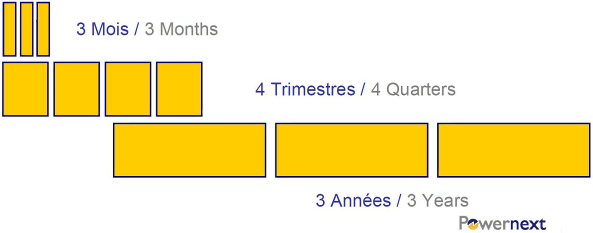
  Non storability + market rules $\Rightarrow$ possible specific prices

  (negative prices, bounded, spikes, seasonality).

Phenomenology of Electricity markets

- Non Storability of the underlying asset
- Price formation
- Granularity of term structure

  Availability of assets on the French future market :



$\Rightarrow$ no hedging without a more subtile term structure (incomplete market)

Phenomenology of Electricity markets

- Non Storability of the underlying asset
- Price formation
- Granularity of term structure
- Illiquidity and transaction costs
    - Specific market (and minor against OTC)
    - Localized selling (national, regional)
    - Unflexibility of production (fuel, gaz, coal,...)

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⇒ How can we link spot prices and future prices with classical financial paradigm ?

⇒ How do we price and hedge claims ?

Open problems

- Future contract pricing
- Options on non available granularity
- Spread options on electricity and combustible
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Previous work and literature

- Plethora of spot models (Barlow, Cartea, Benth, Hepperger,...) : levy processes, regime switching models, processes in Hilbert space.
- Plethora of Future models : two factor model, integrals of spot prices.
- Statistical models : link with commodities, weather, production capacities, demand.
- Calibration on future prices, option prices, spot prices, specific knowledge.

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Still : $F(t,T) \neq \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t]$

A structural model of electricity prices

with R. Aid, L. Campi, N. Touzi

Idea

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Consider

- N production means depending with respective costs (S_t^k), $k \leq N$.
- their N capacities of production Δ_t^k, $k \leq N$.
- the permutation $\pi_t(k)$ s.t. $S_t^{\pi_t(1)} \leq \cdots \leq S_t^{\pi_t(N)}$
- an independent positive demand process D_t.

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Then the electricity spot price/cost is :

$$P_t = \sum_{k \leq N} S_t^k \mathbf{1}_{D_t \in I_t^k} \quad \text{where} \quad I_t^k := \left[\sum_{i=1}^{k-1} \Delta^{\pi_t(i)}, \sum_{i=1}^{k} \Delta^{\pi_t(i)} \right]$$

Pricing methodology

Situation :

- Commodities are storable : classical (yet difficult) arbitrage pricing
- Parameters Δ_t^k are supposed to be known (by the producer).
- D is non-tradable asset ($\Rightarrow$ incomplete market)

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$\Rightarrow$ "Some" risk neutral pricing in Electricity markets :

$$F_t(T) = \sum_{i=1}^{n} \sum_{\pi \in \Pi} F_t^{\pi(i)}(T) \mathbb{Q}[D_T \in I_T^i | \mathcal{F}_t] \mathbb{Q}^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t]$$

Novelty and limits of the model

- Reproduce price stylized facts.
- Allows pricing and hedging of claims (Aid, Campi, Langrenet)

but

- A structural approach to the optimal behaviour of the producer
- The production function is a (tractable) approximation
- Production issues and Financial issues are separated

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No Marginal Arbitrage for High Production Regime
in discrete time investment-production models
With proportional transaction costs.

with B. Bouchard

Motivation

From an electricity producer point of view :

- how to optimize the production with a general production function ?
- how to consider both production and financial strategies ?
- What is then a No Arbitrage Condition ?
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$$K_t(\omega) := \left\{ x \in \mathbb{R}^d : \exists a^{ij} \geq 0 \text{ s.t. } x^i + \sum_{j\neq i} a^{ji} - a^{ij}\pi_t^{ij}(\omega) \geq 0 \ \forall i \right\}$$

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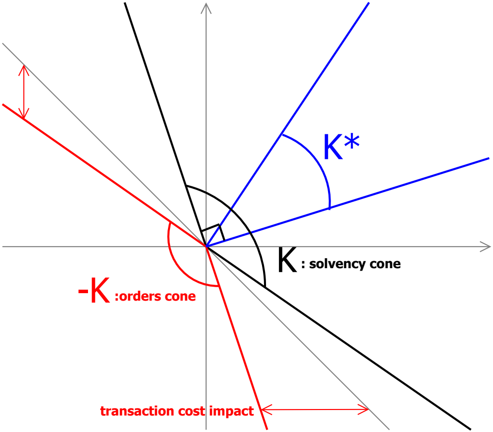
- Set of self-financed exchanges at time t : $-K_t(\omega)$.

A comprehensive geometrical interpretation



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- $R_{t+1}(\beta)$ depends only on β^3
- $R^i_{t+1}(\beta) = 0$ for $i = 2, 3$.

Model description - Wealth process

- Strategies

$$(\xi, \beta) \in \mathcal{A}_0 := L^0((-K) \times \mathbb{R}^d_+, \mathbb{F}),$$

i.e. s.t. $(\xi_t, \beta_t) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$ for all $0 \leq t \leq T$.

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- Set of portfolio holdings that are attainable at time T by trading and producing from time t with zero initial holding

$$A^R_t(T) := \left\{ \sum_{s=t}^{T} \xi_s - \beta_s + R_s(\beta_{s-1}) \mathbf{1}_{s \geq t+1},\ (\xi, \beta) \in \mathcal{A}_0 \right\}.$$

Back to the structural model

- Let π_t be the bid-ask prices of assets : 1 =cash, 2 . . . n =commodities.
- c_t^i the conversion factor from 1 unit of asset i to 1 MWh.
- π_t^e the spot price of electricity in cash.
- Δ_t^i, $i = 2 \dots n$ the maximum capacity of production from the ith commodity.

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Then we can write :

$$R^1_{t+1}(\beta) = \pi^e_{t+1}(\beta) \times \left(\sum_{i>2} c^i_{t+1} \min(\beta^i, \Delta^i_{t+1}) \right)$$

with

$$\pi^e_{t+1}(\beta) = \max_i (\pi^{1i}_{t+1} c^i_{t+1} \mathbf{1}_{\beta^i>0})$$

Beyond the structural model

Some advantages...

- Additional features on the production function (starting costs, various conversion factors for different plants).
- Possibility to chose another electricity spot price π^e.

... and difficulties :

- an additional optimization problem (with possible no solution)
- Possible no explicit solutions for pricing claims.

No-arbitrage of the second kind with $R \equiv 0$

- **NA2** (Rasonyi, 2009) : for $\zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $t < T$,

$$(\zeta + A_t^{K,0}(T)) \cap L^0(K_T, \mathcal{F}_T) \neq \{0\} \quad \Rightarrow \quad \zeta \in L^0(K_t, \mathcal{F}_t) \,.$$

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- **EF** : there is efficient friction if

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Under **EF**,

$$\textbf{NA2} \ : \ \zeta \in L^0(K_{t+1}, \mathcal{F}) \Rightarrow \ \zeta \in L^0(K_t, \mathcal{F}),\ t < T,$$

for all $\zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$.

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- **EF**
- $\mathcal{M}_t^T$: set of martingale selectors Z on $[t, T]$ of the random sets $\mathrm{int} K_s^*$, for $t \le s \le T$, with

$$K_s^*(\omega) = \left\{ z \in \mathbb{R}^d \ : \ 0 \le z^j \le z^i \pi_s^{ij}(\omega), \ i, j \le d \right\}$$

the positive dual of $K_s(\omega)$.

  - Strictly consistent price system : $Z \in \mathrm{int} K_s^*$: $Z_s^j / Z_s^i < \pi^{ij_s}$.
  - Z is a martingale fictitious price better than the market.

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- **EF** : $\mathrm{int} K_s^* \neq \emptyset$
- $\mathcal{M}_t^T$: set of Strictly consistent price systems
- **PCE** : Prices are consistently extendable if

$$\exists Z \in \mathcal{M}_t^T \text{ s.t. } Z_t = X, \ \forall t \leq T \ \text{ and } \ X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t).$$

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Theorem (Rasonyi, 2009) : Under **EF**, **NA2** $\Leftrightarrow$ **PCE**.

No-arbitrage of the second kind with linear production

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  - $\mathbb{E}\left[Z'_{s+1}(L_{s+1} - I)\beta_s \mid \mathcal{F}_s\right] < 0$ if Z is strictly more favorable than π.
  - or production arbitrage.

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Theorem : under **EF**, **NA2^L** $\Leftrightarrow$ **PCE^L**.

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- **NMA2** : $\exists (c, L) \in L^0(\mathbb{R}^d \times \mathbb{M}^d, \mathbb{F})$ s.t. $\mathbf{NA2}^L$ and

$$c_{t+1} + L_{t+1}\beta - R_{t+1}(\beta) \in L^0(K_{t+1}, \mathcal{F}_{t+1}),$$

$\forall \beta \in L^0(\mathbb{R}_+^d, \mathcal{F}_t), t < T.$

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Theorem : $A_0^L(T)$ is closed in probability under $\mathbf{NA2}^L$. The same holds for $A_0^R(T)$ under **NMA2** and **(USC)**, where

$$(\mathbf{USC}) \ : \ \limsup_{\beta \in \mathbb{R}_+^d, \beta \to \beta_0} R_t(\beta) - R_t(\beta_0) \in -K_t \text{ for all } \beta_0 \in \mathbb{R}_+^d.$$

and the lim sup is taken componentwise.

Application - Super Replication Theorem

Under some additional assumptions

- $\lambda R_t(\beta_1) + (1-\lambda)R_t(\beta_2) - R_t(\lambda\beta_1 + (1-\alpha)\beta_2) \in -K_t$,
- $R_t(\beta)^- \in L^\infty(\mathbb{R}^d, \mathcal{F})$ for $\beta \in L^\infty(\mathbb{R}^d_+, \mathcal{F})$.

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Proposition : Assume that **NMA2** holds. Let $V \in L^0(\mathbb{R}^d, \mathcal{F})$ be such that $V + \kappa \in L^0(K_T, \mathcal{F})$ for some $\kappa \in \mathbb{R}^d$. Then the following are equivalent :

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If moreover $\lim_{\eta \to \infty} R_{t+1}(\eta\beta)/\eta = L_{t+1}\beta$ then the following are equivalent :

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If $R = L$ then $\alpha^R = 0$.

Application - Portfolio optimization

Setting

- U a $\mathbb{P}-$a.s. upper continuous, concave, random map from $\mathbb{R}^d$ to $]-\infty,1]$,
- $U(V)=-\infty$ on $\{V\notin K_T\}$,
- $\mathcal{U}(x_0):=\left\{V\in A_0^R(T)\ :\ \mathbb{E}\left[|U(x_0+V)|\right]<\infty\right\}\neq\emptyset$.

Proposition : If **NMA2** , **(USC)** hold and $A_0^R(T)$ is convex, then $\exists V(x_0)\in A_0^R(T)$ such that

$$\mathbb{E}\left[U(x_0+V(x_0))\right]=\sup_{V\in\mathcal{U}(x_0)}\mathbb{E}\left[U(x_0+V)\right]\ .$$

Next steps

- Extension to continuous time ;
- Specification of a realistic production function ;
- Caracterization of $\mathcal{M}_0^T \cap \mathcal{L}_0^T$;
- Numerical implementation...