The strategic exercise of options in incomplete markets

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Combining options and games

- A systematic application of both real options and game theory in strategic decisions has been proposed in the literature (see Smit and Trigeorgis (2004) for a review).
- The essential idea can be summarized in two rules:
  1. whenever the outcome of a given game involves a "wait–and–see" strategy, its pay-off should be calculated as the value of a real option;
  2. whenever the pay-off of a given involves a game, its value should calculated as the equilibrium solution to the game.
- In this way, option valuation and game theoretical equilibrium become dynamically related in a decision tree.
- In what follows, we denote the NE solution for a given game in bold face within the matrix of outcomes.
- For convenience of notation we will round all number to the nearest integer.

One–stage investment: single firm

- As a first example, suppose that a single firm can make an investment of $I = 90$ either at $t = 0$ or at $t = 1$.
- Let the underlying project values be $V_0 = 100$ at time $t = 0$, then either $\overline{V}^h = 120$ or $\overline{V}^\ell = 80$ at time $t = 1$ with equal probabilities.
- If V is perfectly correlated with a traded financial asset S, then the option to invest can be valued using standard risk–neutral pricing.
- For a one–period risk–free rate $R = 0.06$, the risk–neutral probability in this case is $q = \frac{(1+R)-\overline{h}}{\overline{h}-\overline{\ell}} = 0.65$.
- If the firm postpones investment until $t = 1$ it realizes an option value $c_0 = 18.40$.
- Since $c_0 \geq V_0 - I = 10$, a firm acting in isolation should postpone the investment.

One–stage investment: two firms

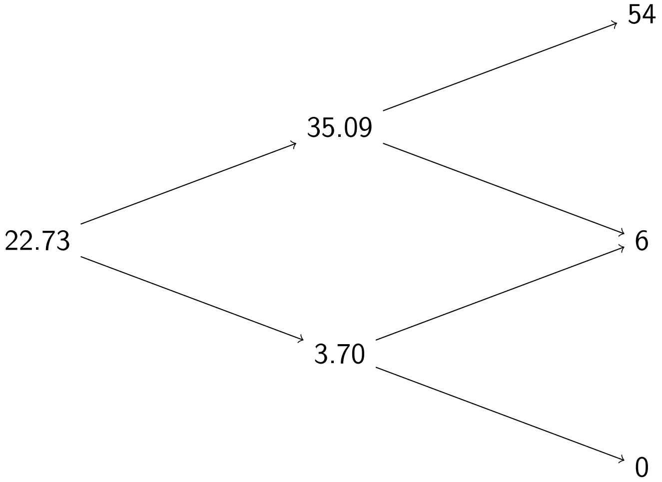
- Suppose now that two symmetric firms A and B face the same investment problem as before.
- Let us assume that if a firm invests in the project alone, then the payoff for the other firm is zero, whereas the payoff is divided equally between them if both firms reach the same decision.
- We then have the following matrix of outcomes:

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(9.20, 9.20)$

- Notice the “prisoner’s dilemma” character of this game.

Two–stage investment: one firm

- Using the same setting as in the previous example, let the project value be $V_0 = 100$ at time $t = 0$, then either 120 or 80 at time $t = 1$, and finally either 144, 96, or 64 at time $t = 2$, leading to the following option values :



Two–stage investment: two firms

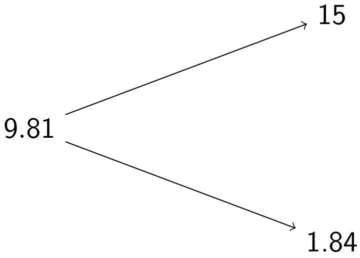
- Suppose now that two firms A and B face the same investment problem as before.
- The games played at time $t = 1$ are:

A \ B	invest	wait
invest	$(\mathbf{15}, \mathbf{15})$	$(30, 0)$
wait	$(0, 30)$	$(17.55, 17.55)$

A \ B	invest	wait
invest	$(-5, -5)$	$(-10, 0)$
wait	$(0, -10)$	$(\mathbf{1.84}, \mathbf{1.84})$

Two–stage investment: two firms (continued)

- Using the previous values to calculate the option value at time $t = 0$ leads to:



- Finally, the game played at time $t = 0$ is:

B / A	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(9.81, 9.81)$

Sensitivity to model parameters

- Using $R = 0.1$ leads to the following matrices of outcomes at time $t = 1$:

A \ B	invest	wait
invest	$(\mathbf{15}, \mathbf{15})$	$(30, 0)$
wait	$(0, 30)$	$(19.09, 19.09)$

A \ B	invest	wait
invest	$(-5, -5)$	$(-10, 0)$
wait	$(0, -10)$	$(\mathbf{2.05}, \mathbf{2.05})$

- This results in an option value of 10.69 at time $t = 0$, leading to:

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(\mathbf{10.69}, \mathbf{10.69})$

Incomplete Markets

- Consider the two–factor market where the *discounted* project value V and the *discounted* a correlated traded asset S follow:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \tag{1}$$

  where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, V_0 and historical probabilities p_1, p_2, p_3, p_4.
- Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$.
- An investment opportunity is model as an option with *discounted* payoff $C_t = (V - e^{-rt} I)^+$, for $t = 0, T$.

European Indifference Price

- Without the opportunity to invest in the project V, a rational agent with initial wealth x will try to solve the optimization problem
$$u^0(x) = \max_H E[U(X_T^{x,H})], \tag{2}$$
where
$$X_T^{x,H} = \xi + HS_T = x + H(S_T - S_0). \tag{3}$$
is the wealth obtained by keeping ξ dollars in a risk–free cash account and holding H units of the traded asset S.
- An agent with initial wealth x who pays a price π for the opportunity to invest in the project will try to solve the modified optimization problem
$$u^C(x - \pi) = \max_H E[U(X_T^{x-\pi,H} + C_T)] \tag{4}$$
- The *indifference price* for the option to invest in the final period as the amount π^C that solves the equation
$$u^0(x) = u^C(x - \pi). \tag{5}$$

Explicit solution

Denoting the two possible pay-offs at the terminal time by C_h and C_ℓ, the European indifference price defined in (5) is given by

$$\pi^C = g(C_h, C_\ell) \tag{6}$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} g(x_1, x_2) \quad = \quad & \frac{q}{\gamma} \log\left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}}\right) \\ & + \frac{1-q}{\gamma} \log\left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}}\right), \end{aligned} \tag{7}$$

with

$$q = \frac{1-d}{u-d}.$$

Early exercise

- When investment at time $t = 0$ is allowed, it is clear that immediate exercise of this option will occur whenever its *exercise value* $(V_0 - I)^+$ is larger than its *continuation value* given by π^C.
- That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT} I)^+, (\ell V_0 - e^{-rT} I)^+)\}.$$

One–period investment revisited

- As a first example, consider again the one–period setting with $I = 90,\ V_0 = 100,\ R = 0.06$.
- For the dynamics of S we choose $u = 1.2/1.06$, $d = 0.8/1.06$ (so that $q = 0.65$ as before) and $p_1 = p_4 = 0.4$, $p_2 = p_3 = 0.1$.
- Finally, let us set $\gamma = 0.01$.
- Therefore, using the function g to calculate the *option value* for the "wait–and–see" strategy, we have the matrix of outcomes for this game shown in Table 13.

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(8.02, 8.02)$

- As expected, the utility–based option value is smaller than the one obtained under risk–neutrality.

Two–period investment revisited

- For the two–period investment game we find

A \ B	invest	wait
invest	$(\mathbf{15}, \mathbf{15})$	$(30, 0)$
wait	$(0, 30)$	$(15.39, 15.39)$

A \ B	invest	wait
invest	$(-5, -5)$	$(-10, 0)$
wait	$(0, -10)$	$(\mathbf{1.66}, \mathbf{1.66})$

- This gives an *indifference* option value of 8.86 at time $t = 0$, leading to

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(8.86, 8.86)$

One–period expansion option under monopoly

- Suppose now that a firm faces the decision to expand capacity for a product with uncertain demand:

$$Y_1 = \begin{cases} hY_0 & \text{with probability } p \\ \ell Y_0 & \text{with probability } 1-p \end{cases}, \tag{8}$$

  correlated with a traded asset
- The expansion requires a discounted sunk cost I.
- The state of the firm after the investment decision at time t_i is

$$x(i) = \begin{cases} 1 & \text{if the firm invests at time } t_i \\ 0 & \text{if the does not invest at time } t_i \end{cases} \tag{9}$$

- The discounted cash flow per unit demand for the firm is denoted by $D_{x(i)}$.

Definition of project values

- We denote by $V^{(x(i))}(i+1, Y_{i+1})$ the project value at time t_{i+1} given that the state of the firm at time t_i was $x(i)$ and that the firm will act optimally from time t_{i+1} onwards.
- Next, denote by $v^{(x(i))}(i, Y_i)$ the sum of the discounted cash flow from time t_i to t_{i+1} plus the indifference value of the project at time t_{i+1}, that is

$$v^{(x(i))}(i, Y_i) = D_{x(i)} Y_i + g(V^{(x(i))}(i+1, hY_i), V^{(x(i))}(i+1, \ell Y_i))$$

- For simplicity, we assume in this section that the project terminates one period after time t_1 so that

$$v^{(x(1))}(1, Y_1) = D_{x(1)} Y_1.$$

The NPV solution

- Assume first that the decision has to be taken at time t_0.
- If no expansion occurs, then $V^{(0)}(1, Y_1) = D_0 Y_1$ and

$$v^{(0)}(0, Y_0) = D_0 Y_0 + g(D_0 h Y_0, D_0 \ell Y_0).$$

- If expansion occurs, then $V^{(1)}(1, Y_1) = D_1 Y_1$ and

$$v^{(1)}(0, Y_0) = D_1 Y_0 + g(D_1 h Y_0, D_1 \ell Y_0).$$

- Accordingly, the firm should expand provided $v^{(1)} - I \geq v^{(0)}$, that is, provided $Y_0 \geq Y^{NPV}$ where Y^{NPV} solves

$$(D_1 - D_0) y = g(D_0 h y, D_0 \ell y) - g(D_1 h y, D_1 \ell y) + I.$$

The Real Options solution

- Assume now that the decision can be taken either at t_0 or t_1.
- If expansion occurs at t_0, then we still have

$$v^{(1)}(0, Y_0) = D_1 Y_0 + g(D_1 h Y_0, D_1 \ell Y_0).$$

- Conversely, if no expansion occur at t_0, then $V^{(0)}(1, Y_1) = \max\{D_1 Y_1 - I, D_0 Y_1\}$ and

$$v^{(0)}(0, Y_0) = D_0 Y_0 + g(V^{(0)}(1, hY_0), V^{(0)}(1, \ell Y_0)).$$

- Accordingly, the firm should expand provided $Y_0 \geq Y^{RO}$ where Y^{RO} solves

$$\begin{aligned}(D_1 - D_0)y = {} & g(\max\{D_1 hy - I, D_0 hy\}, \max\{D_1 \ell y - I, D_0 \ell y\}) \\ & - g(D_1 hy, D_1 \ell y) + I.\end{aligned}$$

- It is easy to show that $Y^{RO} \geq Y^{NPV}$, so that the firm is less likely to expand at time t_0.

One–period expansion game under duopoly

- Consider now two firms A and B facing the same decision as before.
- The state of the firm m after the investment decision at time t_i is

$$x_m(i) = \begin{cases} 1 & \text{if firm } m \text{ invests at time } t_i \\ 0 & \text{if firm } m \text{ does not invest at time } t_i \end{cases} \tag{10}$$

- Let $D_{x_A(t_i)x_B(t_i)}$ denote the cash–flow per unit of demand of firm A and $D_{x_B(t_i)x_A(t_i)}$ the cash–flow per unit of demand of firm B.
- Assume that $D_{10} > D_{11} > D_{00} > D_{01}$.
- We say that there is FMA is $(D_{10} - D_{00}) > (D_{11} - D_{01})$ and that there is SMA otherwise.

Definition of project values

- $V_m^{(x_A(i),x_B(i))}(i+1, Y_{i+1})$ the value of the project for firm m at time t_{i+1} given that the state of the firms at time t_i was $(x_A(i), x_B(i))$ and assuming that both firms will follow an equilibrium strategy from t_{i+1} onwards.
- Next denote by $v_m^{(x_A(i),x_B(i))}(i, Y_i)$ the sum of the cash–flows for firm m from time t_i to time t_{i+1} with the indifference value of the project at time t_{i+1}, that is

$$v_m^{(x_A(i),x_B(i))}(i,Y_i)=D_{x_{m(i)}x_{m'(i)}}Y_i\Delta t+g\left(V_m^{(x_A(i),x_B(i))}(i+1,hY_i),V_m^{(x_A(i),x_B(i))}(i+1,\ell Y_i)\right),$$

  where $m' = B$ whenever $m = A$ and vice-versa.
- For simplicity, we still assume that the project terminates one period after time t_1 so that

$$v_m^{(x_A(1),x_B(1))}(1, Y_1) = D_{x_{m(1)}x_{m'(1)}}Y_1.$$

NPV analysis

- Assume for now that firm A decides first and firm B observes the decision of A before reaching it own (this will be dropped later !).
- If firm A invests at t_0 we have that

$$v_B^{(1,1)}(0, Y_0) = D_{11} Y_0 + g(D_{11} h Y_0, D_{11} \ell Y_0),$$

  and

$$v_B^{(1,0)}(0, Y_0) = D_{01} Y_0 + g(D_{01} h Y_0, D_{01} \ell Y_0).$$

- Therefore, firm B should also invest provided $Y_0 \geq Y_B^{NPV}$, where Y_B^{VPN} solves

$$(D_{11} - D_{01}) y = g(D_{01} hy, D_{01} \ell y) - g(D_{11} hy, D_{11} \ell y) + I$$

- Similarly, if firm A does not invest t_0, then firm B should invest provided $Y_0 \geq Y_A^{NPV}$, where Y_A^{NPV} solves

$$(D_{10} - D_{00}) y = g(D_{00} hy, D_{00} \ell y) - g(D_{10} hy, D_{10} \ell y) + I$$

NPV equilibirum

Proposition

Under first mover advantage (FMA) and assuming that the investment decision can only be made at time t_0, we have that $Y_A^{NPV} \leq Y_B^{NPV}$ and:

1. *If $Y_0 \geq Y_B^{NPV}$, then the optimal strategy at time zero is $(x_A(0), x_B(0)) = (1, 1)$.*
2. *If $Y_A^{NPV} \leq Y_0 < Y_B^{NPV}$, then the optimal strategy at time zero is $(x_A(0), x_B(0)) = (1, 0)$.*
3. *If $Y_0 < Y_A^{NPV}$, then the optimal strategy at time zero is $(x_A(0), x_B(0)) = (0, 0)$.*

In other words, under FMA, the demand thresholds for firms A and B are Y_A^{NPV} and Y_B^{NPV}, respectively.

Real Option analysis at time t_1

- Suppose now that both firms can either invest at time t_0 or postpone investment to time t_1 and are perfectly symmetric.
- We start with time t_1, where

$$V_A^{(1,1)}(1, Y_1) = V_B^{(1,1)}(1, Y_1) = D_{11} Y_1 \quad (11)$$

$$V_B^{(1,0)}(1, Y_1) = V_A^{(0,1)}(1, Y_1) = \max\{D_{11} Y_1 - I, D_{01} Y_1\} \quad (12)$$

$$V_A^{(1,0)}(1, Y_1) = V_B^{(0,1)}(1, Y_1) = \begin{cases} D_{11} Y_1 & \text{if } D_{11} Y_1 - I \geq D_{01} Y_1 \\ D_{10} Y_1 & \text{otherwise} \end{cases} . \quad (13)$$

- Finally, the values $V_m^{(0,0)}(1, Y_1)$ corresponds to the game:

A \ B	invest	wait
invest	$(D_{11} Y_1 - I, D_{11} Y_1 - I)$	$(D_{10} Y_1 - I, D_{01} Y_1)$
wait	$(D_{01} Y_1, D_{10} Y_1 - I)$	$(D_{00} Y_1, D_{00} Y_1)$

- When multiple equilibria occur, we select one at random with equal probabilities.

Real Option analysis at time t_0

- The conditional values at time t_0 are

$$v_m^{(1,1)}(0, Y_0) = D_{11} Y_0 + g\left(V_m^{(1,1)}(1, hY_0), V_m^{(1,1)}(1, \ell Y_0)\right)$$

$$v_B^{(1,0)}(0, Y_0) = v_A^{(0,1)}(0, Y_0) = D_{01} Y_0 + g\left(V_B^{(1,0)}(1, hY_0), V_B^{(1,0)}(1, \ell Y_0)\right)$$

$$v_A^{(1,0)}(0, Y_0) = v_B^{(0,1)}(0, Y_0) = D_{10} Y_0 + g\left(V_A^{(1,0)}(1, hY_0), V_A^{(1,0)}(1, \ell Y_0)\right)$$

$$v_m^{(0,0)}(0, Y_0) = D_{00} Y_0 + g\left(V_m^{(0,0)}(1, hY_0), V_m^{(0,0)}(1, \ell Y_0)\right)$$

- Since by definition both firms still have the option to invest at time t_0, they play the game

A \ B	invest	wait
invest	$(v_A^{(1,1)} - I, v_B^{(1,1)} - I)$	$(v_A^{(1,0)} - I, v_B^{(1,0)})$
wait	$(v_A^{(0,1)}, v_B^{(0,1)} - I)$	$(v_A^{(0,0)}, v_B^{(0,0)})$

- Again, when multiple equilibria occur, we select one at random with equal probabilities.

The N–period game

- Consider now a continuous-time model of the form

$$\begin{aligned} dP_t &= (\mu_1 - r)P_t dt + \sigma_1 P_t dW \\ dY_t &= (\mu_2 - r)Y_t dt + \sigma_2 Y_t(\rho dW + \sqrt{1-\rho^2} dZ). \end{aligned}$$

- Next take $\Delta t = \frac{T}{N}$ and

$$p_1 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(\frac{\nu_1}{\sigma_1} + \frac{\nu_2}{\sigma_2}\right)\right] \tag{14}$$

$$p_2 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(\frac{\nu_1}{\sigma_1} - \frac{\nu_2}{\sigma_2}\right)\right] \tag{15}$$

$$p_3 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(-\frac{\nu_1}{\sigma_1} + \frac{\nu_2}{\sigma_2}\right)\right] \tag{16}$$

$$p_4 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(-\frac{\nu_1}{\sigma_1} - \frac{\nu_2}{\sigma_2}\right)\right] \tag{17}$$

$$u = e^{\Delta y_1} = e^{\sigma_1\sqrt{\Delta t}}, \qquad d = 1/u = e^{-\sigma_1\sqrt{\Delta t}} \tag{18}$$

$$h = e^{\Delta y_2} = e^{\sigma_2\sqrt{\Delta t}}, \qquad \ell = 1/h = e^{-\sigma_2\sqrt{\Delta t}}, \tag{19}$$

where $\nu_i = \mu_i - r - \sigma_i^2/2$.

Numerical experiments

- In what follows, we use $I = 200$, $r = 0.03$, $T = 1$, $N = 500$.
- For the dynamics of S_t we choose $\mu_1 = 0.10$ and $\sigma_1 = 0.30$.
- For the demand Y_t we fix $\sigma_2 = 0.20$ and calculate μ_2 as

$$\mu_2 = \overline{\mu}_2 - \delta, \tag{20}$$

  where $\overline{\mu}_2$ is an equilibrium expected rate of return on the non-traded asset and $\delta = 0.04$ is the *below-equilibirum shortfall rate*

- For the equilibrium rate $\overline{\mu}_2$ we use the CAPM relation

$$\lambda = \frac{\mu_1 - r}{\sigma_1} \tag{21}$$

$$\overline{\mu}_2 = r + \lambda \rho \sigma_2 \tag{22}$$

- Finally we consider FMA with $D_{10} = 8$, $D_{00} = 3$, $D_{01} = 0$.

Project values

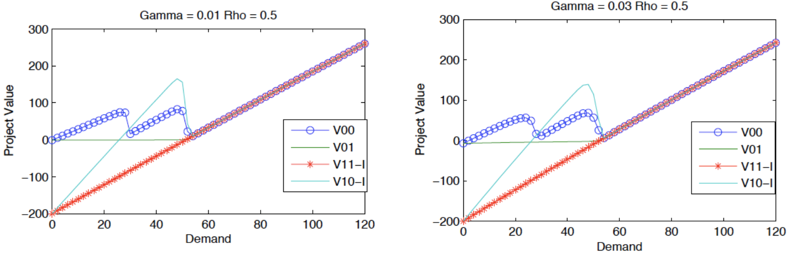


Figure: Project values in FMA case for different risk aversions.

Competition in continuous times

- Consider the model of Grenadier (2000), where two firms contemplating the decision to pay a cost K to invest in a project leading to instantaneous cash flows $Y_t D_Q$ where

$$\frac{dY_t}{Y_t} = \nu dt + \eta dW_t = rdt + \eta(dW_t + \xi dt), \quad \xi = \frac{\nu - r}{\eta} \quad (23)$$

- Assume both market completeness and infinite maturity.
- More specifically, assume that Y_t is perfectly correlated with a traded financial asset

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t = rdt + \sigma(dW_t + \lambda dt) \quad \lambda = \frac{\mu - r}{\sigma}. \quad (24)$$

- After both firms have invested, the value of the project is given by the expected value of all discounted future cash flows, that is

$$E^Q\left[\int_t^\infty e^{-r(s-t)} Y_s D_2 ds | Y_t = y\right] = \frac{yD_2}{\delta},$$

where $\delta = \eta(\lambda - \xi)$.

Follower value

- Given that a leader has already invested, the follower has an option to invest with value $F(y)$ satisfying

$$\frac{1}{2}\sigma^2 Y^2 F'' + (r-\delta)YF = rF, \quad 0 \leq Y \leq Y_F,$$

subject to the boundary conditions

$$F(0) = 0, \quad F(Y_F) = \frac{Y_F D(2)}{\delta} - K, \quad F'(Y_F) = \frac{D(2)}{\delta}.$$

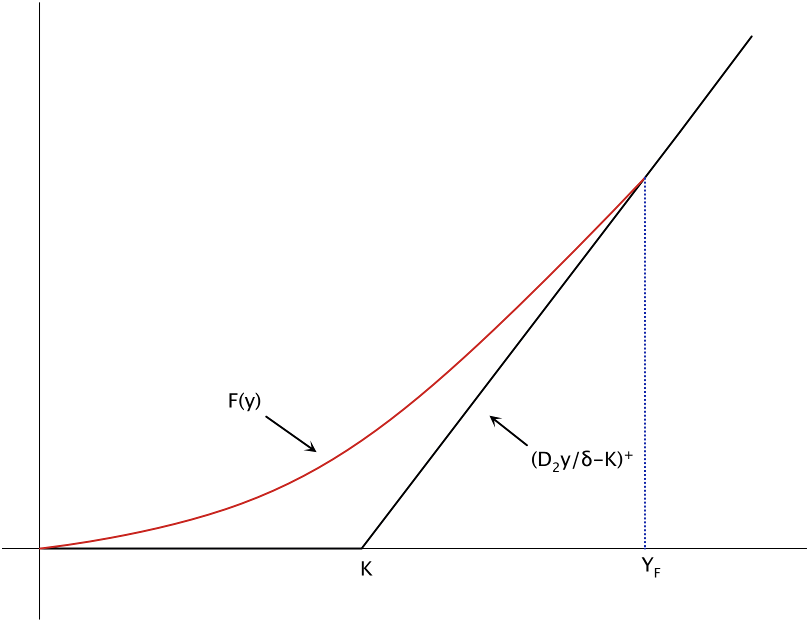
- The solution to this equation is

$$F(y) = \begin{cases} \frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^{\beta}, & Y \leq Y_F \\ \frac{yD(2)}{\delta} - K, & y \geq Y_F \end{cases}$$

where $Y_F = \frac{\delta K \beta}{D_2(\beta-1)}$ and $\beta > 1$ is a solution of

$$\frac{1}{2}\eta^2\beta(\beta-1) + (r-\delta)\beta = r.$$

Follower value



Leader value and simultaneous exercise

- After investing, the leader has no more options to exercise. As a result, the value of being a leader can be obtained entirely by expected value of future cash flow at a rate $Y_t D_1$ until the process Y reaches Y_F and $Y_t D_2$ thereafter.
- The solution to this simple first–passage time problem is

$$L(y) = \begin{cases} \frac{yD(1)}{\delta} - \frac{D_1 - D_2}{D_2}\beta\frac{K}{\beta-1}\left(\frac{y}{Y_F}\right)^{\beta}, & Y < Y_F \\ \frac{yD_2}{\delta}, & y \geq Y_F \end{cases}$$

- Finally, it is clear that the value obtained from simultaneous exercise is

$$S(y) = \frac{yD_2}{\delta}$$

Threshold for the leader

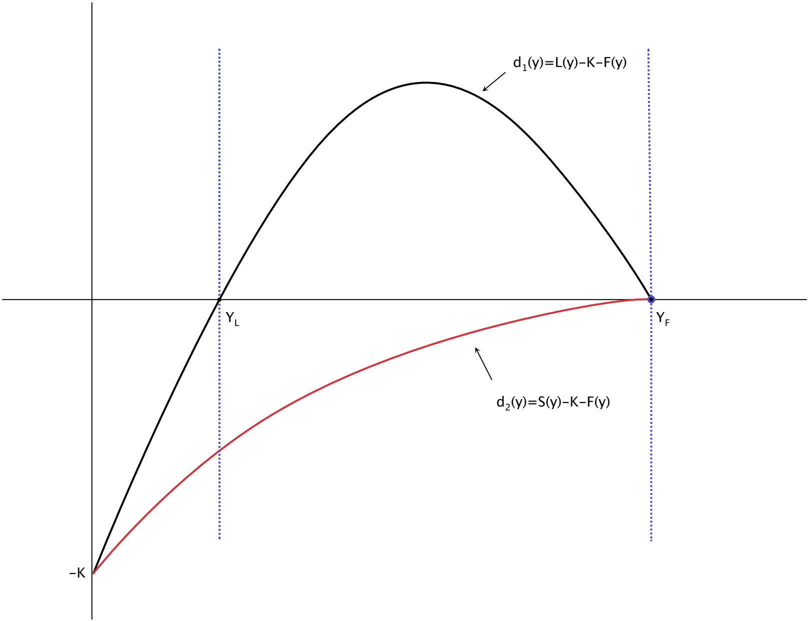
- It can be shown that there exists a unique point $Y_L \in (0, Y_F)$ such that

$$\begin{aligned} L(Y) - K &< F(Y), \quad Y < Y_L \\ L(Y) - K &= F(Y), \quad Y = Y_L \\ L(Y) - K &> F(Y), \quad Y_L < Y < Y_F \\ L(Y) - K &= F(Y), \quad Y \geq Y_F \end{aligned}$$

- In addition

$$\begin{aligned} S(Y) - K &< \min(L(Y) - K, F(Y), \quad Y < Y_F \\ S(K) - K &= L(Y) - K = F(Y), \quad Y \geq Y_F \end{aligned}$$

Threshold for the leader



Equilibrium strategies

- Consider a mixed–strategy game with

$$
\begin{aligned}
p_1(Y) &= \text{prob. of exercise for firm 1} \\
p_2(Y) &= \text{prob. of exercise for firm 2}
\end{aligned}
$$

- Assume that the game is played successively until one of the firms exercises.
- For $Y \geq Y_F$ we have that $p^*(Y) = p_1(Y) = p_2(Y) = 1$ is a Nash equilibrium.
- For $Y \leq Y_L$ we have that $p^*(Y) = p_1(Y) = p_2(Y) = 0$ is a Nash equilibrium.
- The interesting region is $Y_L < Y < Y_F$.

Equilibrium strategies (continued)

- For $Y_L < Y < Y_F$, the pay-off for firm i is

$$V_i = [p_i(1-p_j)(L(Y)-K) + p_i p_j(S(Y)-K) + (1-p_i)p_j F(Y)] \times \sum_{k=0}^{\infty}[(1-p_i)(1-p_j)]^k$$

$$\frac{p_i(1-p_j)(L(Y)-K) + p_i p_j(S(Y)-K) + (1-p_i)p_j F(Y)}{1-(1-p_i)(1-p_j)}$$

- Maximizing this expression with respect to p_i and using symmetry leads to

$$p^*(Y) = p_1(Y) = p_2(Y) = \frac{L(Y) - F(Y) - K}{L(Y) - S(Y)}.$$

Expected payoff

- Observe that the expected payoff for each firm is

$$V(y) = \begin{cases} F(y), & y < Y_L \\ (1 - p_S)\,\frac{F(y)+L(y)-K}{2} + p_S(S(y) - K), & y \in (Y_L, Y_F) \\ S(y) - K, & y > Y_F \end{cases} \tag{25}$$

- Using he expression for $\widehat{p}$ we find

$$p_S = \frac{L - K - F}{L + K + F - 2S}$$

  and

$$(1 - p_S) = \frac{2(K + F - S)}{L - 2S + K + F}.$$

- This gives $V(y) = F(y)$ for all y !

Predetermined roles

- Define $L^{\pi}(Y)$ as the project value for a firm that has been predetermined as the Leader.
- Following the same reasoning as before, this value is given by

$$L^{\pi}(y) = \sup_{\tau \geq 0} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} \Psi(Y_{\tau}) \mathbf{1}_{\{\tau < \infty\}} \middle| Y_0 = y \right], \tag{26}$$

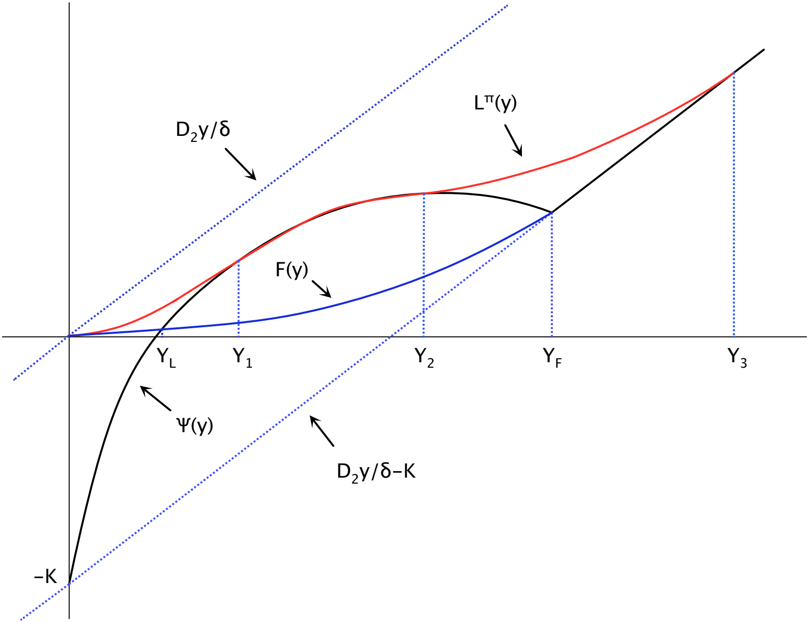
where τ is a stopping time, the payoff function is $\Psi(y) = L(y) - K$.

- Observe that

$$\Psi(y) = \begin{cases} \dfrac{D_1 y}{\delta} - \left(\dfrac{D_1 - D_2}{D_2} \right) \beta F(y) - K & \text{if } y < Y_F \\ \dfrac{D_2 y}{\delta} - K & \text{if } y \geq Y_F \end{cases}. \tag{27}$$

is not differentiable at Y_F.

Obstacle problem for the leader



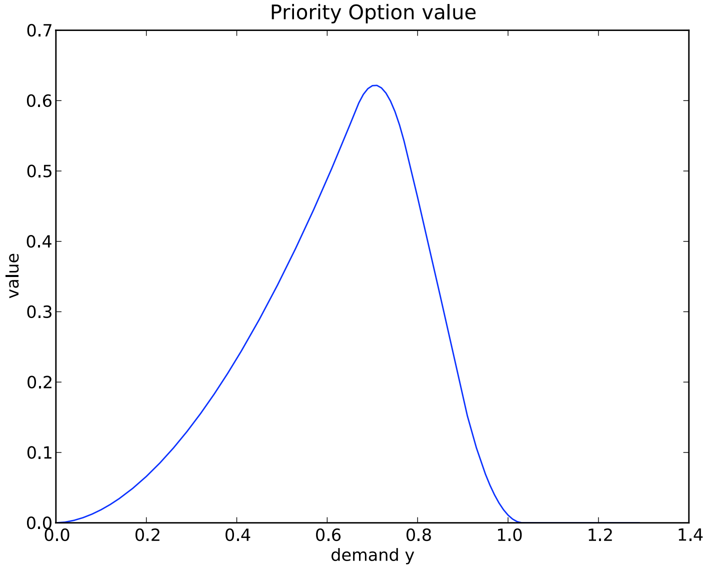
The value of the priority

- We conclude that

$$L^{\pi}(y) = \begin{cases} Ay^{\beta} & \text{if } 0 \leq y < Y_1 \\ L(y) - K & \text{if } Y_1 \leq y \leq Y_2 \\ By^{\beta} + Cy^{\beta_1} & \text{if } Y_2 < y < Y_3 \\ \dfrac{D_2 y}{\delta} - K & \text{if } y \geq Y_3, \end{cases} \tag{28}$$

- Observe that $Y_L < Y_1$, so the priority option delays investment.
- The value of the priority option is then given by $\pi(y) = L^{\pi}(y) - F(y)$.

Priority option value



Incomplete markets

- Suppose now that the stochastic demand Y_t is correlated with the market portfolio P_t as follows:

$$\begin{cases} \frac{dY}{Y} = \nu dt + \eta dW_t, & \xi := \frac{\nu}{\eta} \\ \frac{dP}{P} = \mu dt + \sigma dB_t, & \lambda := \frac{\mu}{\sigma} \end{cases},$$

  where W_t and B_t have instantaneous correlation ρ.
- For simplicity, take $r = 0$.
- According to CAPM, if Y *could* be traded its equilibrium rate of return $\bar{\nu}$ would satisfy

$$\frac{\bar{\nu}}{\eta} = \rho \frac{\mu}{\sigma}$$

- We then define $\delta := \bar{\nu} - \nu$ as the below–equilibrium–shortfall–rate, which plays the role of a dividend yield in this case.

Utility problem

- As before, we calculate the project value for a fixed level $D(Q)$ as

$$V^2(Y_t) = E\left[\int_t^{\infty} e^{-\bar{\nu}(s-t)} Y_s D(Q) ds\right] = \frac{Y_t D(Q)}{\bar{\nu} - \nu} = \frac{Y_t D(Q)}{\delta}.$$

- For a utility function $U(x) = -e^{-\gamma x}$, define

$$F(x, y) = \sup_{(\tau,\theta)} \mathbb{E}\left[e^{\frac{\lambda^2 \tau}{2}} U\left(X_\tau^\theta + \left(\frac{D_2 Y_\tau}{\delta(\rho)} - K\right)^+\right)\right].$$

Follower value function

- Using Henderson (2007), let

$$\beta(\rho) = 1 + \frac{2\delta(\rho)}{\eta^2} > 1$$

and define $Y_F(\rho)$ as the solution to

$$\frac{D_2 Y_F(\rho)}{\delta(\rho)} - K = \frac{1}{\gamma(1-\rho^2)} \log\left[1 + \frac{\gamma(1-\rho^2) D_2 Y_F(\rho)}{\beta(\rho)\delta(\rho)}\right],$$

- Then

$$F(x,y) = \begin{cases} -e^{-\gamma x}\left[1 - \left(\frac{\gamma(1-\rho^2) D_2 Y_F}{\delta\beta\gamma(1-\rho^2) D_2 Y_F}\right)\left(\frac{y}{Y_F}\right)^{\beta(\rho)}\right]^{\frac{1}{1-\rho^2}}, 0 \leq \\ -e^{-\gamma x} e^{-\gamma\left(\frac{D_2 y}{\delta(\rho)} - K\right)}, y > Y_F(\rho) \end{cases}$$

Leader's Expected Utility Value

- As before, the value for the Leader can be found by expected discounted cash–flows assuming that the Follower exercises optimally:

$$L(X,Y) = \begin{cases} -e^{-\gamma X} e^{-\gamma\left[\frac{D(1)}{\delta}Y + \left(\frac{D(2)-D(1)}{\delta}\right)Y_F\left(\frac{Y}{Y_F}\right)^{\Psi} - K\right]} & \text{if } Y \leq Y_F \\ -e^{-\gamma\left[X + \frac{D(2)}{\delta}Y - K\right]} & \text{if } Y \geq Y_F \end{cases},$$

where $\Psi = \left(\frac{1}{2} - \frac{\nu}{\eta^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\nu}{\eta^2}\right)^2 + \frac{2\bar{\nu}}{\eta^2}}$

- Similarly, the value for simultaneous exercise is

$$S(X,Y) = -e^{-\gamma\left[X + \frac{D(2)}{\delta}Y - K\right]}$$

Leader's threshold

- We can again show that, for each fixed X, there exists a unique point $Y_L \in (0, Y_F)$ such that

$$\begin{aligned} L(X,Y) &< F(X,Y), \quad Y < Y_L \\ L(X,Y) &= F(X,Y), \quad Y = Y_L \\ L(X,Y) &> F(X,Y), \quad Y_L < Y < Y_F \\ L(X,Y) &= F(X,Y), \quad Y \geq Y_F \end{aligned}$$

- In addition

$$\begin{aligned} S(X,Y) &< \min(L(X,Y), F(X,Y), \quad Y < Y_F \\ S(X,Y) &= L(X,Y) = F(X,Y), \quad Y \geq Y_F \end{aligned}$$

Equilibrium strategies

- Following the same arguments as before, we have that:
- For $Y \geq Y_F$, $p^*(X,Y) = p_1(X.Y) = p_2(X,Y) = 1$ is a Nash equilibrium.
- For $Y \leq Y_L$, $p^*(X,Y) = p_1(X,Y) = p_2(X,Y) = 0$ is a Nash equilibrium.
- For $Y_L < Y < Y_F$.

$$p^*(X,Y) = p_1(X,Y) = p_2(X,Y) = \frac{L(X,Y) - F(X,Y)}{L(X,Y) - S(X,Y)}.$$

- Moreover,

$$\begin{aligned} p_{sim}(Y) &= \frac{p^*(Y)}{2 - p^*(Y)} \\ p_{seq}^{(i)}(Y) &= \frac{1 - p^*(Y)}{2 - p^*(Y)} \end{aligned}$$

The priority option

- Define $L^{\pi}(Y)$ as the expected utility for a firm that has been given a priority option for choosing to be the Leader.
- Formally, this has the same type of two-interval solution as in the complete market, but a rigorous proof is still open.
- The value for the priority option can then be obtained by an indifference value argument comparing $L^{\pi}(X, Y)$ and the equilibrium value V^i without the priority option.

Conclusions

- Real options and game theory can be combined in a dynamic framework for decision making under uncertainty and competition.
- The effects of incompleteness and risk aversion can be incorporated using the concept of indifference pricing.
- Analytic expressions for exponential utility lead to numerical schemes with the same computational complexity as a binomial model.
- We have fully implemented a generic example of two firms and uncertain demand and finite maturity in discrete time.
- Continuous–time versions with infinite maturity are also possible (extensions of Grenadier (2000))
- We calculated the value of the priority option in complete markets and characterized it in incomplete markets (extension of Bensoussan et al (2010)).
- Much more work is necessary for a large number of firms.
- Merci !