Stochastic Target Problems with Controlled Loss in Jump Diffusion Models

GT EDF

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Sommaire

Introduction

We want to give a PDE characterization to the following problem

$$v(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p \text{ for some } \nu\right\}$$

introduced by Bouchard, Elie, Touzi (2009) for Brownian controlled SDEs, **in the case of jump diffusion processes $\mathbf{X}^{\nu}$ and $\mathbf{Y}^{\nu}$.**

$$dX = \mu_X(X,\nu)ds + \sigma_X(X,\nu)dW + \int_E \beta_X(X,\nu,e)J(de,ds)$$

$$dY = \mu_Y(Z,\nu)ds + \sigma_Y(Z,\nu)dW + \int_E \beta_Y(Z,\nu,e)J(de,ds)$$

where Z stands for (X,Y).

Notations : The controls ν are in $\mathcal{U}$ and take values in U.

What has been done ?

- Soner and Touzi : Brownian filtration and bounded controls $\mathbb{P}-$ a.s. criteria.
- Bouchard : Jump diffusion with bounded control and locally bounded jumps. $\mathbb{P}-$ a.s. criteria.
- Bouchard, Elie and Touzi : Brownian filtration with unbounded controls. Criteria in expectation (concentrating on the case of a criteria in expectation).
- Bouchard and Vu : "American" case.
- Bouchard, Elie and Imbert : Optimal control under stochastic target constraints
- Bouchard and Vu : Multidimensional target
- Bouchard and Dang : Optimal Control vs Stochastic target

Sommaire

Examples

Financial Market

X^ν : Stocks (possibly influenced by a large investor strategy ν)
Y^ν : Portfolio process of the (large) investor

The market is incomplete
We do not treat the dual problem, but directly the primal

Examples

Insurance Market

X^{ν} : Sources of risks
Y^{ν} : Portfolio process of an insurance company

Examples

"Hybrid" Market

$X^{\nu,(d),(n)}$: Sources of risks (d stocks + n random variables)
Y^{ν} : Portfolio process of an insurance company

Examples

Probability of Ruin

$$\Psi(x,y) := -\mathbb{1}_{\{y \le g(x)\}},$$

$$v(t,x,p) = \inf\left\{y \ge -\kappa : \exists\ \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \le g\left(X^{\nu}_{t,x}(T)\right)\right] \le p\right\}.$$

Remark

- *In general settings, but no jumps, Bouchard, Elie and Touzi (2009)*

Examples

Superhedging

$\Psi(x,y) := \mathbb{1}_{\{y \geq g(x)\}}$,

$$v(t,x,1) = \inf\left\{y \geq -\kappa : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right)\right] = 1\right\}.$$

Remark

- *For U compact and no jumps, Soner and Touzi (2002)*
- *For U compact and bounded jumps, Bouchard (2002)*
- *For the American case, Bouchard and Vu (2009)*

Example : *If $g(x) = (x-K)^+$, then*

$$v(t,x,1) := \inf\left\{y \geq -\kappa : \exists\, \nu \in \mathcal{U} \textit{ s.t. } Y^{\nu}_{t,x,y}(T) \geq \left(X^{\nu}_{t,x}(T) - K\right)^+ \mathbb{P}\text{-a.s.}\right\}.$$

Hedging a European call option with finite credit line.

Examples

Quantile Hedging

$$\Psi(x,y) := \mathbb{1}_{\{y \geq g(x)\}},$$

$$v(t,x,p) = \inf\left\{y \geq -\kappa : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right)\right] \geq p\right\}.$$

Remark

- *In "standard" financial models, Follmer and Leukert (1999)*
- *In general settings, but no jumps, Bouchard, Elie and Touzi (2009)*

Examples

Loss Function

$\Psi(x,y) := -\rho\left((y-g(x))^-\right)$, with ρ convex non-decreasing

$$v(t,x,p) = \inf\left\{y \geq -\kappa : \exists\ \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\rho\left(\left(Y^\nu_{t,x,y}(T) - g(X^\nu_{t,x}(T))\right)^-\right)\right] \leq p\right\}.$$

Remark

- *In "standard" financial models, Follmer and Leukert (1999)*
- *In general settings, but no jumps, Bouchard, Elie and Touzi (2009)*

Examples

Success Ratio

$\Psi(x,y) := \mathbb{1}_{\{g(x)\le y\}} + \frac{y}{g(x)}\mathbb{1}_{\{g(x)>y\}}$, for $y \ge 0$,

$$v(t,x,p) = \inf\left\{ y \ge 0 : \exists\ \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\frac{Y^{\nu}_{t,x,y}(T)}{g(X^{\nu}_{t,x}(T))} \wedge 1\right] \ge p \right\}.$$

Remark

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- *In general settings, but no jumps, Bouchard, Elie and Touzi (2009)*

Examples

Utility indifference Price in incomplete Markets

$\Psi(x,y) := U\left(y - g(x)\right)$, with U concave non-decreasing,

$$v(t,x,p) = \inf\left\{y \geq -\kappa : \exists\ \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[U\left(Y^{\nu}_{t,x,y_0+y}(T) - g(X^{\nu}_{t,x}(T))\right)\right] \geq p\right\}.$$

$$\left(p := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[U\left(Y^{\nu}_{t,x,y_0}(T)\right)\right]\right)$$

Life Insurance

In one of the previous cases, we sell the claim $G^{(n)}(x)$:

$$\Psi(x,y) = \mathbb{1}_{\left\{y - G^{(n)}(x) \geq 0\right\}} \text{ or } -\rho\left(\left(y - G^{(n)}(x)\right)^{-}\right) \text{ or } U\left(y - G^{(n)}(x)\right)$$

For $x =: (x^0, x^1) \in \mathbb{R}^2$, where x^0 stands for the stock and x^1 stands for some non tradable asset :

$$G(x) := g(x^0)\mathbb{1}_{\{x^1 = 0\}},$$

and $X^1(\cdot) = N.$ is a Poisson process of intensity $\lambda(\cdot)$ indicating if the customer is still alive $(N_T = 0)$ or not $(N_T \neq 0)$.

Remark We may consider a more intuitive case where we sell n contracts based on d stocks :

$$x^0 \in \mathbb{R}^d \quad \text{and} \quad G^{(n)}(x) := \sum_{i=1}^{n} g(x^0)\mathbb{1}_{\{x^i = 0\}}.$$

Volume and price Insurance

In one of the previous cases, we sell the claim $G^{(n)}(x)$:

$$\Psi(x,y) = \mathbb{1}_{\left\{y-G^{(n)}(x)\geq 0\right\}} \text{ or } -\rho\left(\left(y-G^{(n)}(x)\right)^{-}\right) \text{ or } U\left(y-G^{(n)}(x)\right)$$

For $x =: (x^0, x^1) \in \mathbb{R}^2$, where x^0 stands for the stock and x^1 stands for a volume :

$$G(x) := \left(k^0 \times k^1 - x^0 \times x^1\right)^+,$$

and $X^1(\cdot)$ is a bounded pure jump process living in $[0, V_{\max}]$ indicating the volume produced by the customer at time T.

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$$G^{(n)}(x) := \sum_{i=1}^{n} \left(k^0 \times k^i - x^0 \times x^i\right)^+.$$

Volume, price and production costs Insurance

In one of the previous cases, we sell the claim $G^{(n)}(x)$:

$$\Psi(x,y) = \mathbb{1}_{\left\{y-G^{(n)}(x)\geq 0\right\}} \text{ or } -\rho\left(\left(y-G^{(n)}(x)\right)^{-}\right) \text{ or } U\left(y-G^{(n)}(x)\right)$$

For $x =: (x^1, \cdots, x^d; r) \in \mathbb{R}^{d+1}$, where $(x^1, \cdot, x^d)$ stands for d stocks and r stands for a volume :

$$G(x) := \left(\left(k^1 \times k^r - x^1 \times r\right) + k^r \times \alpha \cdot \left(x^{2,(d)} - k^{2,(d)}\right)\right)^{+},$$

and $R(\cdot) = N(\cdot)$ is a bounded pure jump process living in $[0, V_{\max}]$ indicating the volume produced by the customer at time T.

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$$G^{(n)}(x) := \sum_{i=1}^{n} \left(\left(k^1 \times k^{i,(r)} - x^1 \times r^i\right) + k^{i,(r)} \times \alpha \cdot \left(x^{2,(d)} - k^{2,(d)}\right)\right)^{+}.$$

On the "hybrid" case

In the case where we sell n contracts based on d stocks $\left(X^\nu = X^{(d),(n),\nu} = \left(X^{(d),\nu}; R^1, \cdots, R^n\right)\right)$:

1. The dimension of the PDE is at least $n + d$, but...
2. ... if the n random variables R^i_T are independent of the market, and i.i.d. conditionally to the market information, the dimension of the PDE is then at least d.

The problem

$$v(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^{(d),(n),\nu}_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\right] \geq p \text{ for some } \nu\right\}$$

becomes indeed

$$v(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\widetilde{\Psi}\left(X^{(d),\nu}_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\right] \geq p \text{ for some } \nu\right\}$$

with

$$\widetilde{\Psi}\left(X^{(d),\nu}_{t,x}(T), Y^\nu_{t,x,y}(T)\right) := \mathbb{E}\left[\Psi\left(X^{(d),(n),\nu}_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\middle| \mathcal{F}^0_T\right]$$

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$$\widetilde{\Psi}\left(X^{(d),\nu}_{t,x}(T), Y^\nu_{t,x,y}(T)\right) := \mathbb{E}\left[\Psi\left(X^{(d),(n),\nu}_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\middle|\mathcal{F}^0_T\right]$$

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Geometric Dynamic Programming Principle

(Soner Touzi (2002) , Bouchard Vu (2009))

Fix (t,x) and $\{\theta^\nu, \nu \in \mathcal{U}\}$ a family of $[t,T]$-valued stopping times,

<u>(GDP1) :</u> $y > v(t,x,1) \Rightarrow \exists\ \nu \in \mathcal{U}$ s.t.

$$Y^\nu_{t,x,y}(\theta^\nu) \geq v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu), 1\right).$$

<u>(GDP2) :</u> For every $-\kappa \leq y < v(t,x,1), \nu \in \mathcal{U}$

$$\mathbb{P}\left[Y^\nu_{t,x,y}(\theta^\nu) > v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu), 1\right)\right] < 1.$$

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Formal GDPP with Jumps

(Bouchard Elie Touzi (2009) , Bouchard Vu (2009))

$$y > v(t,x,p) \not\Rightarrow \exists\ \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y}(\theta^{\nu}) \geq v\left(\theta^{\nu}, X^{\nu}_{t,x}(\theta^{\nu}), p\right),$$

but

$$y > v(t,x,p) \Rightarrow \exists\ \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y}(\theta^{\nu}) \geq v\left(\theta^{\nu}, X^{\nu}_{t,x}(\theta^{\nu}), P\right)$$

where $P := \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\Big|\,\mathcal{F}_t\right]$, and $\mathbb{E}[P] = p$, i.e.

$$P_{t,p}(\cdot) := p + \int_t^{\cdot} \alpha_s \cdot dW_s + \int_t^{\cdot}\int_E \chi_s(e)\tilde{J}(de,ds).$$

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Formal GDPP with Jumps

(Bouchard Elie Touzi (2009) , Bouchard Vu (2009))

Main difficulties in Bouchard Elie Touzi (2009) :

- α possibly unbounded $\Rightarrow$ unbounded controls

$\Rightarrow$ Local relaxation.

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Formal GDPP with Jumps

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Main difficulties here :

- α and χ possibly unbounded $\Rightarrow$ unbounded controls and **unbounded jumps**

$\Rightarrow$ Non-local Relaxation.

- The control χ is a measurable function

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Reduction of the Problem

We then reduce to the problem :

$$v(t,x,p) = \inf \{ \, y \geq -\kappa : \exists \, (\nu,\alpha,\chi) \in \mathcal{U} \times \mathbb{L}^2 \times \mathbb{H}^2_\lambda \text{ s.t. } \Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \geq P^{\alpha,\chi}_{t,p}(T) \, \}$$

where $\mathbb{H}^2_\lambda$ denotes the set of maps $\chi : \Omega \times [0,T] \times E \to \mathbb{R}$ s.t.

$$\mathbb{E}\left[\int_0^T \int_E (\chi_t(e))^2 \, \lambda(de)dt\right] < \infty,$$

and $\lambda(de)dt$ is the intensity of $J(de,dt)$.

Geometric Dynamic Programming Principle

Set

$$P_{t,p}^{\alpha,\chi}(\cdot) := p + \int_t^{\cdot} \alpha_s \cdot dW_s + \int_0^{\cdot} \int_E \chi_s(e) \widetilde{J}(de, ds).$$

(GDP1) : $y > v(t,x,p) \Rightarrow \exists\ (\nu, \alpha, \chi) \in \mathcal{U} \times \mathbb{L}^2 \times \mathbb{H}_\lambda^2$ s.t.

$$Y_{t,x,y}^{\nu}(\theta^\nu) \geq v\left(\theta^\nu, X_{t,x}^{\nu}(\theta^\nu), P_{t,p}^{\alpha,\chi}(\theta^\nu)\right)$$

for all stopping times θ^ν.

(GDP2) : $y < v(t,x,p) \Rightarrow$ for all $\theta^\nu \leq T, (\nu, \alpha, \chi) \in \mathcal{U} \times \mathbb{L}^2 \times \mathbb{H}_\lambda^2$

$$\mathbb{P}\left[Y_{t,x,y}^{\nu}(\theta^\nu) > v\left(\theta^\nu, X_{t,x}^{\nu}(\theta^\nu), P_{t,p}^{\alpha,\chi}(\theta^\nu)\right)\right] < 1.$$

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$$\mathbb{P}\left[Y_{t,x,y}^{\nu}(\theta^\nu) > v\left(\theta^\nu, X_{t,x}^{\nu}(\theta^\nu), P_{t,p}^{\alpha,\chi}(\theta^\nu)\right)\right] < 1.$$

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Formal PDE Derivation

We hence study the problem

$$v(t,x) := \inf\left\{ y \geq -\kappa : \widehat{\Psi}\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U} \right\}$$

with

$$dX = \mu_X(X,\nu)ds + \sigma_X(X,\nu)dW + \int_E \beta_X(X,\nu,e)J(de,ds)$$
$$dY = \mu_Y(Z,\nu)ds + \sigma_Y(Z,\nu)dW + \int_E \beta_Y(Z,\nu,e)J(de,ds)$$

where Z stands for (X,Y).

Notations : The controls ν are in $\mathcal{U}$ and take values in U...

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$$dY = \mu_Y(Z,\nu)ds + \sigma_Y(Z,\nu)dW + \int_E \beta_Y(Z,\nu(e),e)J(de,ds)$$

where Z stands for (X,Y).

Notations : The controls ν are in $\mathcal{U}$ and take values in U... is a space of unbounded measurable functions

Formal PDE Derivation

$$dY^{\nu}_{t,x,y} = \mu_Y(X,Y,\nu)ds + \sigma_Y(X,Y,\nu)dW_s + \int_E \beta_Y(X,Y,\nu(e),e)J(de,ds)$$

$$\geq dv(s,X(s))$$

$$= \mathcal{L}^{\nu}v(\cdot)ds + D_x v(\cdot)\sigma_X(\cdot)dW_s + \int_E \left[v\left(\cdot + \beta_X(\cdot)\right) - v(\cdot)\right] J(de,ds)$$

which leads to

$$\sup_{u \in \mathcal{N}_{0,0}} \left\{\mu_Y(x, v(t,x), u) - \mathcal{L}^u v(t,x)\right\} = 0$$

where

$$\mathcal{N}_{\varepsilon,\eta} := \{u \in U \text{ s.t. } |\sigma_Y(x,y,u) - Dv(t,x)\sigma_X(x,u)| \leq \varepsilon \text{ and } \mathcal{G}^{u,e}v(t,x) \geq \eta \text{ for } \lambda\text{-a.e. } e \in E\}.$$

and

$$\mathcal{G}^{u,e}v(t,x) := \beta_Y(\cdot, v(\cdot), u(e), e) - v\left(\cdot + \beta_X(\cdot, u(e), e)\right) + v(\cdot)$$

Formal PDE Derivation

$$dY^{\nu}_{t,x,y} = \mu_Y(X,Y,\nu)ds + \sigma_Y(X,Y,\nu)dW_s + \int_E \beta_Y(X,Y,\nu(e),e)J(de,ds)$$

$$\geq dv(s,X(s))$$

$$= \mathcal{L}^{\nu} v(\cdot)ds + D_x v(\cdot)\sigma_X(\cdot)dW_s + \int_E \left[v\left(\cdot + \beta_X(\cdot)\right) - v(\cdot)\right] J(de,ds)$$

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$$\sup_{u \in \mathcal{N}_{0,0}} \left\{\mu_Y(x, v(t,x), u) - \mathcal{L}^u v(t,x)\right\} = 0$$

where

$$\mathcal{N}_{\varepsilon,\eta} := \{u \in U \text{ s.t. } |\sigma_Y(x,y,u) - Dv(t,x)\sigma_X(x,u)| \leq \varepsilon \text{ and } \mathcal{G}^{u,e}v(t,x) \geq \eta \text{ for } \lambda\text{-a.e. } e \in E\}.$$

and

$$\mathcal{G}^{u,e}v(t,x) := \beta_Y(\cdot, v(\cdot), u(e), e) - v\left(\cdot + \beta_X(\cdot, u(e), e)\right) + v(\cdot)$$

Formal PDE Derivation

$$dY^{\nu}_{t,x,y} = \mu_Y(X,Y,\nu)ds + \sigma_Y(X,Y,\nu)dW_s + \int_E \beta_Y(X,Y,\nu(e),e)J(de,ds)$$

$$\geq dv(s,X(s))$$

$$= \mathcal{L}^{\nu}v(\cdot)ds + D_x v(\cdot)\sigma_X(\cdot)dW_s + \int_E \left[v\left(\cdot + \beta_X(\cdot)\right) - v(\cdot)\right] J(de,ds)$$

which leads to

$$\sup_{u \in \mathcal{N}_{0,0}} \left\{\mu_Y(x, v(t,x), u) - \mathcal{L}^u v(t,x)\right\} = 0$$

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Sommaire

The (local) Relaxation of Bouchard Elie Touzi (2009)

$$H^*(\Theta) = \limsup_{\varepsilon \searrow 0, \Theta' \to \Theta} H_\varepsilon(\Theta') \qquad H_*(\Theta) = \liminf_{\varepsilon \searrow 0, \Theta' \to \Theta} H_\varepsilon(\Theta'),$$

with $\Theta' = (t', x', y, k, q, A)$, $\Theta = \left(\cdot, \varphi(\cdot), \partial_t \varphi(\cdot), D\varphi(\cdot), D^2\varphi(\cdot)\right)(t, x)$
and

$$H_\varepsilon(\Theta) = \sup_{u \in \mathcal{N}_\varepsilon} \left\{ \mu_Y(z, u) - k - \mu_X(x, u) \cdot q - \frac{1}{2} \mathsf{Tr} \left[\sigma_X \sigma_X^T(x, u) A \right] \right\}$$

and

$$\mathcal{N}_\varepsilon(x, y, q) := \{ u \in U \text{ s.t. } |\sigma_Y(x, y, u) - q\sigma_X(x, u)| \le \varepsilon \}.$$

Our (Non-Local) Relaxation

The relaxation of is no longer sufficient to ensure the upper (resp. lower) semi continuity of H^* (resp. H_*) in the non-local term $\mathcal{G}^{u,e}v(t,x,p)$.

$$H^*(\Theta,\varphi) = \limsup_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta \\ \eta\to 0,\psi\underset{u.c.}{\longrightarrow}\varphi}} H_{\varepsilon,\eta}(\Theta',\psi) \qquad H_*(\Theta,\varphi) = \liminf_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta \\ \eta\to 0,\psi\underset{u.c.}{\longrightarrow}\varphi}} H_{\varepsilon,\eta}(\Theta',\psi),$$

with $\Theta' = (t',x',y,k,q,A)$, $\Theta = \left(\cdot,\varphi(\cdot),\partial_t\varphi(\cdot),D\varphi(\cdot),D^2\varphi(\cdot)\right)(t,x)$ and, for $\varepsilon \geq 0$ and $\eta \in [-1,1]$

$$H_{\varepsilon,\eta}(\Theta,\psi) = \sup_{u\in\mathcal{N}_{\varepsilon,\eta}} \left\{\mu_Y(z,u) - k - \mu_X(x,u)\cdot q - \frac{1}{2}\text{Tr}\left[\sigma_X\sigma_X^T(x,u)A\right]\right\}$$

where $\psi \underset{u.c.}{\longrightarrow} \varphi$ has to be understood in the sense that ψ converges uniformly on compact sets towards φ, and

$$\mathcal{N}_{\varepsilon,\eta}(t,x,y,q,\psi) := \{u\in U \text{ s.t. } |\sigma_Y(x,y,u) - q\sigma_X(x,u)| \leq \varepsilon$$

$$\text{and} \quad \beta_Y(x,y,u(e),e) - \psi(t,x+\beta_X(x,u(e),e)) + y \geq \eta \quad \text{for } \lambda\text{-a.e. } e\in E\ \}$$

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Sommaire

Our main results

Theorem

The function v_* is viscosity supersolution on $[0,T)\times \mathbf{X}$ of

$$H^* v_* \geq 0.$$

Under some extra assumption of regularity of the set $\mathcal{N}_{0,\eta}(\cdot, f)$ for $f \in \mathcal{C}^0$ and $\eta \in [-1,1]$, the function v^* is a viscosity subsolution on $[0,T)\times \mathbf{X}$ of

$$\min\left\{H_* v^*, v^* + \kappa\right\} \leq 0.$$

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Sketch of the Proof (Supersolution) :

Let φ be a test function, and assume that

$$H^*\varphi(t_0, x_0) =: -2\eta < 0.$$

Define

$$\widetilde{\varphi}(t,x) := \varphi(t,x) - \iota\,|x - x_0|^4 \text{ for } \iota > 0.$$

By the definition of H^*, after possibly changing η, we may find $\varepsilon > 0$ and $\iota > 0$ small enough such that

$$\mu_Y(x, y, u) - \mathcal{L}^u \widetilde{\varphi}(t,x) \le -\eta$$

$$\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}(t, x, y, D\widetilde{\varphi}(t,x), \widetilde{\varphi})$$

$$\text{and } (t,x,y) \text{ s.t.} (t,x) \in B_\varepsilon(t_0, x_0) \text{ and } |y - \widetilde{\varphi}(t,x)| \le \varepsilon.$$

We then have

$$(v_* - \widetilde{\varphi})\,(t,x) \geq \zeta \wedge \iota\varepsilon^4 =: \xi > 0 \text{ for } (t,x) \in \mathcal{V}_\varepsilon(t_0,x_0),$$

with

$$\mathcal{V}_\varepsilon(t_0,x_0) := \partial_p B_\varepsilon(t_0,x_0) \cup [t_0, t_0+\varepsilon) \times B^c_\varepsilon(x_0).$$

Let $(t_n,x_n)_{n\geq 1} \to (t_0,x_0)$ s.t. $v(t_n,x_n) \to v_*(t_0,x_0)$ and set $y_n := v(t_n,x_n) + n^{-1}$.

For each $n \geq 1$, $y_n > v(t_n, x_n)$ together with (GDP1) : there exists some $\nu^n \in \mathcal{U}$ s.t.

$$Y^n(t \wedge \theta_n) \geq v\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) \geq \widetilde{\varphi}\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right), \quad t \geq t_n,$$

where

$$\begin{aligned} \theta_n^o &:= \{s \geq t_n : (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\} \\ \theta_n &:= \{s \geq t_n : |Y^n(s) - \widetilde{\varphi}\left(s, X^n(s)\right)| \geq \varepsilon\} \wedge \theta_n^o. \end{aligned}$$

We then have

$$\begin{aligned} Y^n(t \wedge \theta_n) - \widetilde{\varphi}\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) &\geq \left[\varepsilon \mathbb{1}_{\{\theta_n < \theta_n^o\}} + \xi \mathbb{1}_{\{\theta_n = \theta_n^o\}}\right] \mathbb{1}_{\{t \geq \theta_n\}}. \\ &\geq (\varepsilon \wedge \xi) \mathbb{1}_{\{t \geq \theta_n\}} \geq 0. \end{aligned}$$

We conclude by using Itô's lemma, and by making a "change of measure" to obtain a contradiction.
We need in order to do that to observe that

$$\mu_Y(x,y,u) - \mathcal{L}^u\widetilde{\varphi}(t,x) \leq -\eta$$

$$\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}\left(t,x,y,D\widetilde{\varphi}(t,x),\widetilde{\varphi}\right)$$

implies that, for $s \in [t_n, \theta_n]$ such that

$$\min\{\mu_Y\left(Z_s^n,\nu_s^n\right) - \mathcal{L}^{\nu_s^n}\widetilde{\varphi}\left(s,X_s^n\right),$$
$$\beta_Y\left(Z_{s-}^n,\nu_s^n(e),e\right) - \widetilde{\varphi}\left(s,X_{s-}^n + \beta_X\left(X_{s-}^n,\nu_s^n(e),e\right)\right) + \widetilde{\varphi}\left(s,X_{s-}^n\right)\} > -\eta,$$

we have

$$\sigma_Y\left(Z_s^n,\nu_s^n\right) - D\widetilde{\varphi}\left(s,X_s^n\right)\sigma_X\left(X_s^n,\nu_s^n\right) > \varepsilon.$$

□

Sommaire

On the terminal condition (formally)

In the expected loss case

$$v(t,x,p) := \inf\left\{y \ge -\kappa : \exists\ \nu : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \ge p\right\}$$

leads to

$$v(t,x,p) = \inf\left\{y \ge -\kappa : \exists\ \nu,\alpha,\chi : \Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \ge P^{\alpha,\chi}_{t,p}(T)\right\}.$$

Define

$$\psi(x,p) := \inf\left\{y : \Psi(x,y) \ge p\right\}.$$

We may expect that

$$v(T,x,p) = \psi(x,p).$$

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For the Quantile Hedging (Bouchard Elie Touzi (2009))

$$\Psi(x,y) := \mathbb{1}_{\{y \geq g(x)\}}$$

leads to

$$\psi(x,p) = g(x)\mathbb{1}_{\{p>0\}}.$$

Discontinuous in p, we hedge or not ! !
$\Rightarrow$ If v is convex in its p-variable

$$v(T,x,p) = \mathsf{Conv}\left(\psi(x,p)\right) = pg(x).$$

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On the terminal condition (formally)

We may generalize it :
If v is convex in its p-variable

$$v(T, x, p) = \mathsf{Conv}\left(\psi(x, p)\right).$$

On the terminal condition

Proposition Assume that for all $(t_n, x_n, y_n, p_n, \nu_n)$ s.t. $(t_n, x_n, y_n, p_n) \to (T, x, y, p)$, there exists a sequence of $\mathbb{P}$-absolutely continuous probability measure $(\mathbb{Q}_n)_{n\geq 1}$ defined by $\frac{d\mathbb{Q}_n}{d\mathbb{P}} = H^n$ s.t.

$$\limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n}\left[Y^{\nu_n}_{t_n,x_n,y_n}\right] \leq y$$
$$\limsup_{n\to\infty} \mathbb{E}\left[\left|H^n D_p^+\overline{\psi}\left(X^{\nu_n}_{t_n,x_n}, p_n\right) - D_p^+\overline{\psi}(x_n, p_n)\right|\right] = 0$$
$$\liminf_{n\to\infty} \mathbb{E}\left[H^n\overline{\psi}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right)\right] \geq \overline{\psi}(x, p).$$

Then $v_*(T, x, p) \geq \overline{\psi}(x, p)$, with $\overline{\psi} = \mathrm{conv}\psi(x, p)$.

On the terminal condition

Proof We take $(t_n, x_n, p_n) \to (T, x, p)$ s.t. $v(t_n, x_n, p_n) \to v_*(T, x, p)$, and $y_n = v(t_n, x_n, p_n) + n^{-1}$. We may find ν_n, α_n, χ_n such that

$$Y^n(T) \geq \psi\left(X^n(T), P^n(T)\right)$$
$$H^n Y^n(T) \geq H^n \overline{\psi}\left(X^n(T), P^n(T)\right).$$

By convexity of $\overline{\psi}$ in its p variable (we omit the T)

$$\begin{aligned}
H^n Y^n &\geq H^n \overline{\psi}(X^n, p_n) + H^n \overline{\psi}_p^+(X^n, p_n)\left(P^n - p_n\right) \\
&\geq H^n \overline{\psi}(X^n, p_n) + H^n \overline{\psi}_p^+(X^n, p_n)\left(P^n - p_n\right) + \overline{\psi}_p^+(x_n, p_n)\left(P^n - P^n\right) \\
&\geq H^n \overline{\psi}(X^n, p_n) + P^n \left(H^n \overline{\psi}_p^+(X^n, p_n) - \overline{\psi}_p^+(x_n, p_n)\right) \\
&\qquad + \overline{\psi}_p^+(x_n, p_n) P^n - H^n \overline{\psi}_p^+(X^n, p_n) \\
&\geq H^n \overline{\psi}(X^n, p_n) - M \left| H^n \overline{\psi}_p^+(X^n, p_n) - \overline{\psi}_p^+(x_n, p_n) \right| \\
&\qquad + \overline{\psi}_p^+(x_n, p_n) P^n - H^n \overline{\psi}_p^+(X^n, p_n) p_n
\end{aligned}$$

Taking the expectation and sending $n \to \infty$ leads to the required result. □

Sommaire

Conclusion

- When the image of Ψ is of the form $[m, M]$, with m and/or M are finite, we proved boundary conditions at $p = m$ and/or $p = M$.
- In the B&S model and a complete market, using the Fenchel-Legendre transform of v with respect to the p-variable in the PDE, Bouchard, Elie and Touzi recover the dual problem, which is a control problem
  In incomplet markets, we recover in the same way a control problem, but we need a comparison theorem to conclude as they do.
  $\Rightarrow$ Specify a model ?
- There is work to do on the numerical scheme
- Some work has been done for a comparison theorem, in particular cases (Bouchard and Vu, Bouchard and Dang)

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