Hotelling Game on Networks

Alberto A. Pinto (with J.P. Almeida and T. Parreira)

LIAAD-INESC TEC, UNIVERSITY of PORTO, PORTUGAL

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Introduction

- Since the seminal work of **Hotelling in 1929**, the model of spatial competition has been seen by many researchers as an attractive framework for analyzing oligopoly markets (Spokes, Graitson, Osborne, Salop, Tabuchi, ...).
- Hotelling presented a village represented by a line segment where a uniformly distributed continuum of consumers buy a single commodity.
- The consumers have to support linear transportation costs when buying the commodity in one of the two firms of the village.
- The firms compete in a two-stage location-price game:
  1. in the first stage firms choose their location;
  2. in the second stage firms choose their prices.
- The **goal** of Hotelling's model is to study the **location** and **price** strategies that maximizes the profit of the competing firms.

Introduction

- Hotelling concluded that firms would agglomerate at the center of the line, an observation referred as the **Principle of Minimum Differentiation**.
- In 1979, D'Aspremont et al. showed that the Principle of Minimum Differentiation is invalid, since there is **no** price equilibrium solution for all possible locations of the firms, in particular when they are not far enough from each other.

The Hotelling model on a line

Brief description

Hotelling model

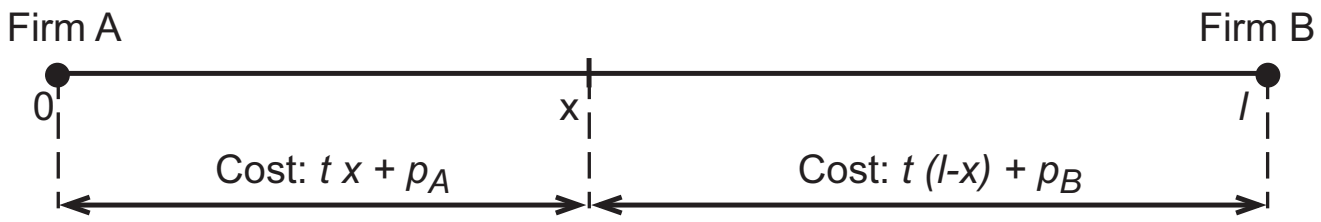


Figure: Hotelling's linear village

- The **buyers** of a commodity will be supposed uniformly distributed along a line with length l.
- In the two ends of the line there are two **firms** A and B selling the same commodity with unitary **production costs** c_A and c_B.
- No customer has any preference for either seller except on the ground of **price** plus **transportation cost** t.

Hotelling model

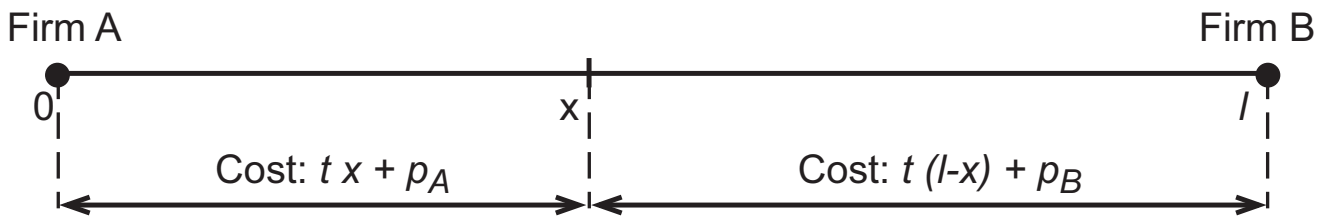


Figure: Hotelling's linear village

- We will assume that each consumer buys a **single unit** of the commodity, in each unit of time and in each unit of length of the line.
- Denote A's **price** by p_A and B's **price** by p_B.
- The **point of division** $x = x(p_A, p_B) \in]0, l[$ between the regions served by the two entrepreneurs is determined by the condition that at this place it is a matter of indifference whether one buys from A or from B

Hotelling model

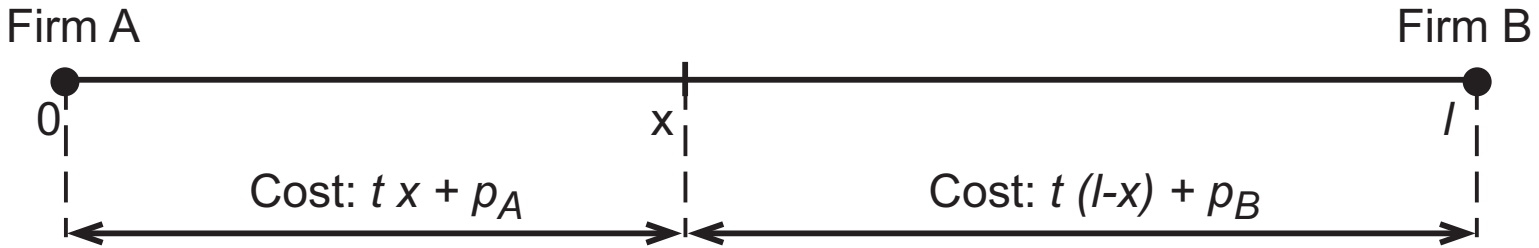


Figure: Hotelling's linear village

- Hence, the point of division x is the location of the consumer indifferent to buy from firm A or firm B, if

$$p_A + t\,x = p_B + t\,(l - x)$$

Solving for x, we obtain

$$x = \frac{p_B - p_A + t\,l}{2\,t}.$$

- Both firms have a **non-empty market share** if, and only if,

$$0 < x < l.$$

Prices and profits

- Assuming that both firms A and B have a non-empty demand, the **profit** of the two firms are given by

$$\pi_A = (p_A - c_A)\, x = (p_A - c_A)\, \frac{p_B - p_A + t\, l}{2\, t}$$

$$\pi_B = (p_B - c_B)\, (l - x) = (p_B - c_B)\, \frac{p_A - p_B + t\, l}{2\, t}$$

- Hence, the first and second order conditions imply that the **optimal prices** are

$$p_A = t\, l + \frac{1}{3}\, (2\, c_A + c_B)$$

and

$$p_B = t\, l + \frac{1}{3}\, (c_A + 2\, c_B).$$

Equilibrium Prices and profits

Hence, the prices (p_A, p_B) are a **local strategic optimum equilibrium**, if the indiferent consumer x satisfies

$$0 < x = \frac{p_B - p_A + t\,l}{2\,t} = \frac{l}{2} + \frac{c_B - c_A}{6\,t} < l. \tag{1}$$

Definition

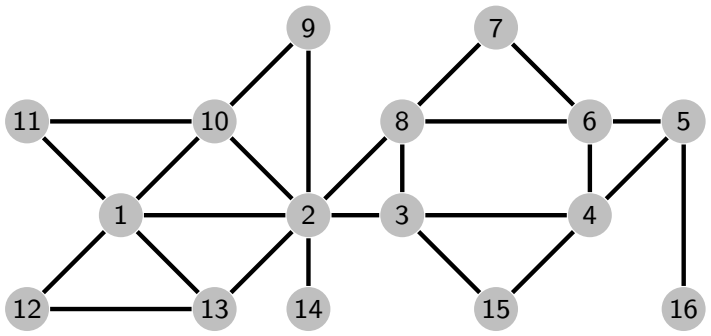
The Hotelling model satisfies the **bounded costs** (BC) condition, if

$$|c_A - c_B| < 3\,t\,l.$$

Notting that inequality (1) is equivalent to the BC condition, we obtain the following result.

Theorem

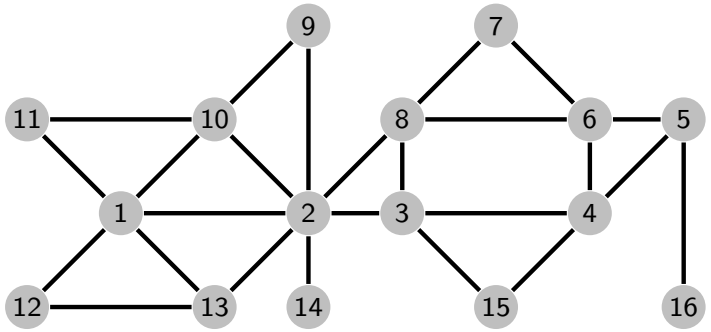
Under the BC condition, the pair of prices (p_A, p_B) is a local strategic optimum equilibrium.



The Hotelling town

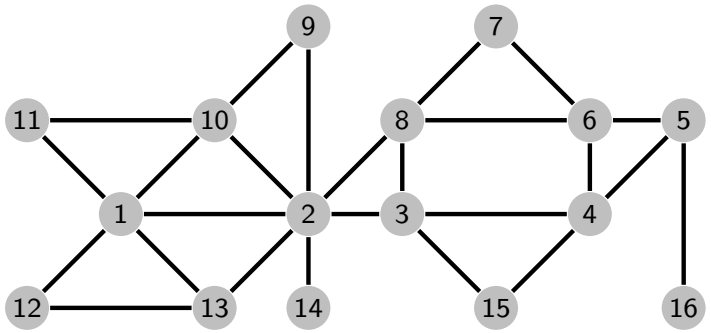
Extending the Hotelling model from a line to a network

Introduction



Other models have been developed replacing the line in the Hotelling model by **other topologies**, as for example in the Salop Model where the line is replaced by the circle.

Introduction



- In this work, we introduce the **Hotelling town model**, that extends the Hotelling model from a **line** (village) to a **network** (town).
- Trying to mimic a real town, the roads of the town are the **edges** of the network, the crossroad are the **vertices** with degree higher than two and the ends of no-exit roads are the vertices with degree one.
- The firms are spread over the town and the consumers are uniformly distributed along the **roads** (similar size houses).
- The roads can have different lengths (**market sizes**) and the firms can have different productions costs (**firm's heterogeneity**).

Introduction

- We note that our model is **different** from the usual network games (Bramoulle, Gal and Vega, Goyal, ...), because the consumers are assumed uniformly distributed along the edges of the network and not at the nodes.
- A strategy is a **price vector** that associates to each firm its selling price for the commodity.
- As in the original Hotelling model, the **expenditure** of a consumer that chooses to buy in a firm (shop) consists in the **sum** of the **price** practiced by that firm plus the **transportation cost** that is proportional to the minimal distance between his house (position at the network) and the firm.
- Again, the firms compete in a two-stage location-price game:
  1. in the first stage firms choose their location;
  2. in the second stage firms choose their prices.
- The **goal** is to study the **location** and **price** strategies that maximizes the profit of the competing firms.

Introduction

- In the **price subgame**, our main goal is to compute the price strategy that has the following two essential economic properties:
  1. **Local strategic optimum:** any small deviation of a price of a firm provokes a decrease in its own profit;
  2. **Local market structure:** all firms have a non-empty market.
- Property (i) stabilizes the prices because **no** firm has an incentive to do small changes in their prices.
- Property (ii) stabilizes the set of competing firms because every firm has a non-empty market (**positive profit**).
  Hence, using a sufficiently high entry cost, no firms enter or leave the market.
- We call a price strategy satisfying these two properties a **local market optimum price strategy**.

Introduction

- We introduce the **weak bounded costs (WBC) condition** on the **exogenous** parameters of the model that gives a bound in terms of the transportation cost and the minimal road length:
  - on the maximum difference between the production costs; and
  - on the maximum difference between the road length.
- Under the WBC condition, we **prove** that the price subgame has a **local market optimum price strategy**. Furthermore, the local market optimum price strategy is unique.
- We note that, under the WBC condition, the firms can not be located too close, and so D'Aspremont et al. 1979's objection to the existence of a local market optimum price equilibrium does not apply.

Introduction

- For the local market optimum price strategy, we present an explicit **closed formula** and an explicit **series expansion formula**.
- The closed and series expansion formulas **explicitly exhibit** how the local market optimum price strategy of a firm depends:
  1. on the production costs;
  2. road market sizes; and
  3. firms locations.
- Hence, the **static analysis** of the relevant economic quantities like profit of firms, consumer surplus and welfare can be done rigorously.

Introduction

Assuming that the firms might not know the entire network:

- We say that a firm has **n-space bounded information** if the firm knows the network structure up to n roads from its location, in terms of production costs, node degrees and road sizes.
- We **prove** that a firm with n-space bounded information is able to compute an **n-approximation price** with the property that the n-aproximation price converges to the local market optimum price exponentially fast with n.
- Hence, the **influence** of a firm in the local market optimum price strategy of other firm decreases exponentially fast with the distance between these two firms.

Introduction

A local market optimum price strategy might not be a Nash equilibrium price.

- Hence, we introduce the **strong bounded (SB) condition** that in comparison with the weak bounded condition has the additional feature of its **bound** to depend also on the **maximum node degree** of the network.
- Under the SB condition, we **prove** that the local market optimum price strategy is a **Nash price equilibrium strategy**.

Introduction

- In the **location-subgame**, we consider that a location for the firms is **admissible** if it satisfies the weak bounded condition.
- We **assume** that firms, after choosing an admissable location, will practice the local market optimum price strategy.
- Similarly to the original Hotteling model in the line, we prove that the firms located at the ends of **no-exit** roads have an incentive to deviate from their localization.
- However, we prove that the firms located at **crossroads** do not have an incentive to deviate from their localization. In this sense, firms prefer **maximum differentiation**, in contrast with the original Hotteling model in the line.
- We note that this result has some **empirical evidence**, because in real towns the owners of the shops usually prefer to have them located at crossroads.
- Finally, we extend all the previous results to the case of **incomplete information** on the production costs of the firms.

The Hotelling town model

Price formation

Hotelling town network

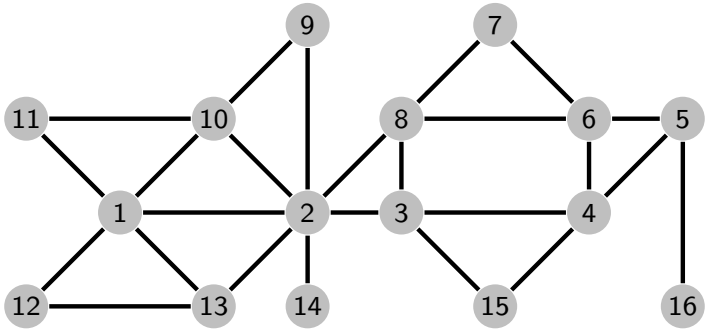


Figure: Network example with $y_i = 0$ for all $i \in V$

- The **Hotelling town network** consists of a group E of roads (**edges** of the network) where the consumers live and a group V of firms (**nodes** of the network);
- For simplicity, every firm F_i is located at a position y_i in a neighborhood of a **vertex** (**node**) $i \in V$.
- Every road has two vertices and in a neighborhood of every vertex is located a single firm;
- The consumers can buy one unit of the commodity in any firm of the town;
- The firms compete in prices à la Hotelling in the town (network).

Hotelling town network

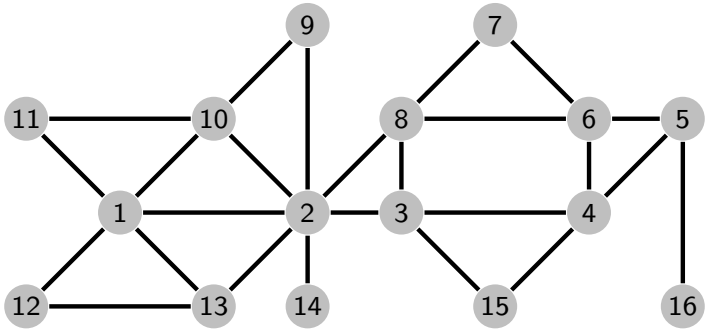


Figure: $l_{1,2} = 2$; $l_{2,3} = 1$; $N_2 = \{1, 3, 8, 9, 10, 13, 14\}$

- The firm F_i has a **unit cost** c_i and charges the same unit price p_i to all its customers;
- The road (edge) $R_{i,j}$ has **length** $l_{i,j}$ and the consumers are uniformly distributed along the road;
- N_i is the set of all neighboring vertices j for which there is a road $R_{i,j}$ connecting the vertices;
- The cardinal k_i of N_i is the **degree** of the node i.

Node deviation

- Let $\tilde{l}_{i,j} = d(y_i, y_j)$ be the distance between the locations of firms F_i and F_j.
- Let $\epsilon = \max_{i \in V} d(i, y_i)$ be the maximum deviation between the node i and the location od the firm F_i.
- Hence, for every road $R_{i,j}$, the length of the road $l_{i,j}$ is close to the distance $\tilde{l}_{i,j}$ between the firms F_i and F_j,

$$l_{i,j} - 2\,\epsilon \leq \tilde{l}_{i,j} \leq l_{i,j} + 2\,\epsilon.$$

Price competition

- A **price strategy P** consists in a vector whose coordinates are the competitive prices p_i of each firm F_i;
- A consumer located at a point x of the network who decides to buy at firm F_i **spends**

$$E(x; i, \mathbf{P}) = p_i + t\, d(x, y_i)$$

  the price p_i charged by the firm F_i plus the **transportation cost** that is proportional t to the minimal distance measured in the network between the position y_i of the firm F_i and the position x of the consumer;
- Given a price strategy **P**, the consumer will choose to buy in the firm $F_{v(x,\mathbf{P})}$ that minimizes his expenditure

$$v(x, \mathbf{P}) = argmin_{i \in V} E(x; i, \mathbf{P}).$$

Markets and Profits

- The **market**

$$M(i, \mathbf{P}) = \{x : v(x, \mathbf{P}) = i\}$$

  of firm F_i consists of all consumers who minimize their expenditures by opting to buy in firm F_i;
- The market size $S_i(\mathbf{P})$ of firm F_i is given by the Lebesgue measure of $M_i(\mathbf{P})$;
- The profit of the firm F_i is

$$\pi_i(\mathbf{P}, \mathbf{C}) = (p_i - c_i)S_i(\mathbf{P})$$

  where p_i is the competitive price of firm F_i determined by the price strategy $\mathbf{P}$, and c_i is the production cost of firm F_i determined by the cost vector $\mathbf{C}$.

Local market structure

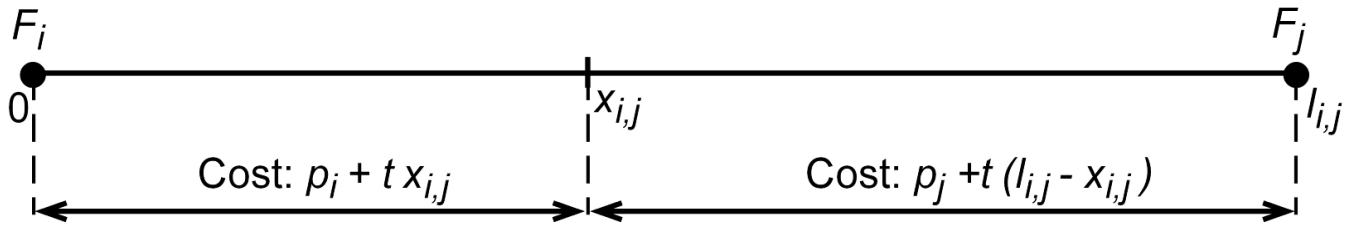


Figure: Indifferent consumer in road $R_{i,j}$

- The **local firms** of a consumer located at a point x in a road $R_{i,j}$ with vertices i and j are the firms F_i and F_j.
- A price strategy **P** determines a **local market structure** if every consumer buys from one of his local firms;

Local market structure

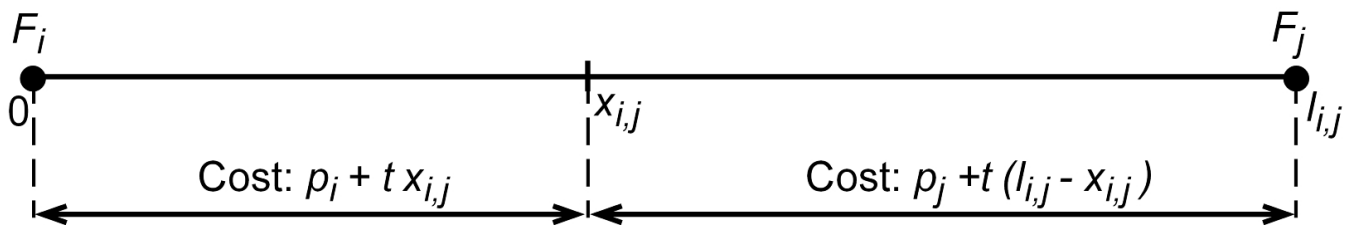


Figure: Indifferent consumer in road $R_{i,j}$

- If a price strategy **P** determines a local market structure then for every road $R_{i,j}$ there is one consumer located at a point $\mathbf{x}_{i,j} \in R_{i,j}$ who is **indifferent** to the local firm from which he is going to buy his commodity, i.e.

$$E(x; i, \mathbf{P}) = E(x; j, \mathbf{P}).$$

- Hence,

$$\pi_i(\mathbf{P}, \mathbf{C}) = (2\,t)^{-1}(p_i - c_i)\left(2\,t\,(2 - k_i)\,\epsilon_i + \sum_{j \in N_i}(p_j - p_i + t\,\tilde{l}_{i,j})\right).$$

Bounded length and costs

- Let k_M be the maximum node degree of the Hotelling town:

$$k_M = \max\{k_i : i \in V\}$$

- Let c_M be the maximum production cost of the Hotelling town and c_m be the minimum production cost of the Hotelling town:

$$c_M = \max\{c_i : i \in V\} \qquad c_m = \min\{c_i : i \in V\}.$$

  Furthermore,

$$\Delta(c) = c_M - c_m.$$

- Let l_M be the maximum road length of the Hotelling town and l_m be the minimum road length of the Hotelling town:

$$l_M = \max\{l_e : e \in E\} \qquad l_m = \min\{l_e : e \in E\}$$

  Furthermore,

$$\Delta(l) = l_M - l_m.$$

- A Hotelling town satisfies the **weak bounded length and costs** (WB) condition, if

$$\Delta(c) + t\Delta(l) < t(l_m - 6\epsilon)$$

Local market structure

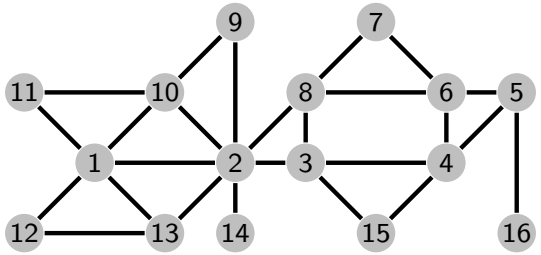


Figure: $L_3 = \frac{t}{4}\left(\widetilde{l}_{3,2} + \widetilde{l}_{3,4} + \widetilde{l}_{3,8} + \widetilde{l}_{3,15}\right)$

- The Hotelling town **admissible market size L** is the vector whose coordinates are the **admissible local firm market sizes**

$$L_i = k_i^{-1} \sum_{j \in N_i} \widetilde{l}_{i,j}.$$

- The Hotelling town **neighboring market structure K** is the matrix whose coordinates are
  1. $k_{i,j} = k_i^{-1}$, if there is a road $R_{i,j}$ between the firms F_i and F_j; and
  2. $k_{i,j} = 0$, if there is not a road $R_{i,j}$ between the firms F_i and F_j.

Theorem

If the Hotelling town satisfies the WB condition, then there is unique local optimum price strategy given by

$$\begin{aligned} \mathbf{P}^L &= \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1}(\mathbf{C} + t(\mathbf{L})) \\ &= \sum_{m=0}^{\infty} 2^{-(m+1)}\mathbf{K}^m(\mathbf{C} + t(\mathbf{L})). \end{aligned}$$

where $\mathbf{1}$ *is the identity matrix.*

Nash equilibrium price strategy

Definition

A Hotelling town satisfies the **strong bounded length and costs (SB)** condition, if

$$\Delta(c) + t\Delta(l) \leq \frac{(2\,t\,l_m - \Delta(c) - 4\,t\,\epsilon)^2}{8\,t\,k_M\,(l_M + \epsilon)} - 3\,t\,\epsilon.$$

Theorem

If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Nash equilibrium price strategy $\mathbf{P}^* = \mathbf{P}^L$.

Hence, the Nash equilibrium price strategy for the Hotelling town satisfying the SB condition determines a local market structure, i.e. every consumer located at $x \in R_{i,j}$ spends less by shopping at his local firms F_i or F_j than in any other firm in the town and so the consumer at x will buy either at his local firm F_i or at his local firm F_j.

Strategic optimal location

- Let us explicit the dependence of the profit $\pi(\epsilon_i)$ of the firm F_i with the distance ϵ_i between its location y_i and the corresponding node i.
- We say that a firm F_i is **node local stable**, if

$$\pi_i(0) > \pi_i(\epsilon_i)$$

  with respect to the local optimum price strategy.
- A Hotelling network is **firm node local stable**, if every firm in the network is node local stable.

Strategic optimal location

- Let

$$Q_{i,j} = \sum_{m=0}^{\infty} 2^{-(m+1)} k_{i,j}^{m}.$$

- For every node i, with $k_i = 2$,
  - let $j(i)$ be the node with the property that y_i is at the road $R_{i,j(i)}$; and
  - let $v(i) \in N_i \setminus \{j(i)\}$ be the other node in N_i.
- Define the weighted balance

$$U_i = \frac{Q_{i,v(i)}}{k_{v(i)}} - \frac{Q_{i,j(i)}}{k_{j(i)}}.$$

Firm Position Stability

Theorem

- *Firms F_i, with node degree $k_i = 1$, are node local unstable.*
- *Firms F_i, with $k_i = 2$, are node local unstable, except for networks with zero weighted balance $U_i = 0$.*
- *Firms F_i, with $k_i = 3$ and whose neighboring firms have nodes degree greater or equal to 3, are node local stable.*
- *Firms F_i, with $k_i \geq 4$ and whose neighboring firms have nodes degree greater or equal to 2, are node local stable.*

Let $k_m = \min\{k_i : i \in V\}$ be the minimum node degree of the Hotelling town. Hence, A Hotelling town network satisfying the WB condition and with $k_m \geq 3$ is firm node local stable.

Hotelling Town: uncertainty on production costs

- For every $v \in V$, let I_v (finite, countable or uncountable) be the set of production costs types I_v of firm F_v.
- Let $\mathrm{d}q_v(z_v)$ be the probability of the **common believes** of the other firms on the production costs $c_v^{z_v}$ of the firm F_v.
- The Hotelling town **production cost C** is the vector $(c_1, \ldots, c_{N_v})$ whose coordinates
$$c_v : I_v \to [c_v^m, c_v^M] \subseteq [c_m, c_M] \subseteq \mathbb{R}_0^+$$
are measurable functions.
- The Hotelling town **average production cost** $E(\mathbf{C})$ is the vector $(E(c_1), \ldots, E(c_{N_v}))$ whose coordinates are the **expected production costs**
$$E(c_v) = \int_{I_v} c_v^{z_v} \mathrm{d}q_v(z_v) < \infty.$$

Hotelling Town: uncertainty on production costs

- A price strategy $\mathbf{P}$ is a vector $(p_1, \ldots, p_{N_v})$ whose coordinates

$$p_v : I_v \to \mathbb{R}_0^+$$

  are measurable functions.
- The average $E(\mathbf{P})$ of the price strategy $\mathbf{P}$ is the vector $(E(p_1), \ldots, E(p_{N_v}))$ whose coordinates are the expected prices

$$E(p_v) = \int_{I_v} p_v^{z_v} \mathrm{d}q_v(z_v).$$

Local optimum price strategy

Theorem

- *If the Hotelling town satisfies the WB condition, then there is unique local optimal equilibrium price strategy given by*

$$\mathbf{P}^E = \frac{1}{2}\left(\mathbf{C} + \mathbf{K}\, E(\mathbf{P}^E) + t\,\mathbf{L}\right) \tag{2}$$

where

$$E(\mathbf{P}^E) = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1}\left(E(\mathbf{C}) + t\,\mathbf{L}\right).$$

- *The local optimal equilibrium price* $\mathbf{P}^E$ *determines a local market structure.*
- *If a Hotelling town satisfies the SB condition, then there is a unique Hotelling town Bayesian-Nash equilibrium price strategy* $\mathbf{P}^* = \mathbf{P}^E$.

Thank you!

Hotelling Town: uncertainty on production costs

- Let the vector
$$\mathbf{Z}_{N_i} = (z_{i,1}, z_{i,2}, \ldots, z_{i,k_i})$$
be the type of the neighbours of a firm F_i of degree k_i, and let
$$I_{N_i} = I_{i,1} \times I_{i,2} \ldots \times I_{i,k_i}$$
be the set of all vectors $\mathbf{Z}_{N_i}$.
- Let
$$\mathrm{d}q_{N_i}(\mathbf{Z}_{N_i}) = \mathrm{d}q_{i,1}(z_{i,1})\,\mathrm{d}q_{i,2}(z_{i,2})\,\ldots\mathrm{d}q_{i,k_i}(z_{i,k_i}).$$
be the probability of the **belief** of the firm F_i on the production costs of its neighbours to be
$$\mathbf{C}_{N_i}^{z_{N_i}} = (c_{i,1}^{z_{i,1}}, c_{i,2}^{z_{i,2}}, \ldots, c_{i,k_i}^{z_{i,k_i}}).$$

Local market structure

- For each road $R_{i,j}$, the indifferent consumer

$$x_{i,j} : l_i \times l_j \to (0, l_{i,j})$$

is given by

$$x_{i,j}^{z_i,z_j} = \frac{p_j^{z_j} - p_i^{z_i} + t\, l_{i,j}}{2\, t}. \tag{3}$$

- Under a local market structure, the ex-post profit

$$\pi_i^{EP} : l_i \times l_{N_i} \to \mathbb{R}_0^+$$

of firm F_i is given by

$$\pi_i^{EP}(z_i, \mathbf{Z}_{N_i}) = (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} x_{i,j}^{z_i,z_j}.$$

Local market structure

- Under a local market structure, the ex-ante profit

$$\pi_i^{EA} : I_i \to \mathbb{R}_0^+$$

of firm F_i is given by

$$\begin{aligned} \pi_i^{EA}(z_i) &= \int_{I_{N_i}} \pi_i^{EP}(z_i, \mathbf{Z}_{N_i})\, \mathrm{d}q_{N_i}(\mathbf{Z}_{N_i}) \\ &= (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} \frac{E(p_j) - p_i^{z_i} + t\, l_{i,j}}{2\, t}. \end{aligned}$$