A stochastic target approach to the granularity problem

Journee de la Chaire

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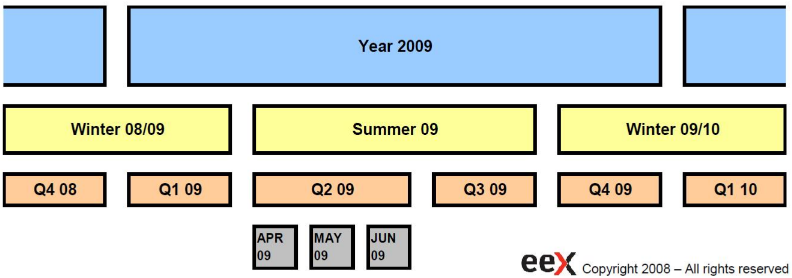
Problem

Definition

The granularity is the progressive apparition of traded futures contracts depending on their maturity and delivery period. The futures contracts 'split' in several contracts with shorter delivery period.



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Statement

- The price of X^M has a structural correlation with X^Q.
- The market is incomplete.

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- The market is incomplete.

Goal : quantify the premium for a level of loss.

Approach : the stochastic target problem of Bouchard, Elie and Touzi (2009)

Introduction of the shaping factor

- By absence of arbitrage :

$$X_t^Q = \frac{h^{M_1} X_t^{M_1} + h^{M_2} X_t^{M_2} + h^{M_3} X_t^{M_3}}{h^{M_1} + h^{M_2} + h^{M_3}}$$

with h^{M_i} = number of hours in month i
and X^{M_i} = price of contract covering month i.

- Let $[0, T]$ be the time interval of hedging,
and $t_0 \in (0, T)$ the time of apparition of the month contract.
We suppose that

$$X_{t_0}^M = \lambda X_{t_0}^Q$$

with λ a $\mathcal{F}_{t_0}$-measurable variable of law J and support E
independent of $X_{t_0}^Q$.

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Empirical analysis of the shaping factor

A study on EEX Power Derivatives market (mid-price) :

- Month and Quarter contracts on 7 years (78 occurrences)
- Daily mid-price (non continuous trading strategy)
- Goal : hedge and price a call option on month with quarter.

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Relevant results in the Black-Scholes framework :

- λ is a random parameter (no clear correlation with the price $X^Q_{t_0}$).
- The hedging error due to the estimation is greater than model or discretization error with an error $> 10\%$.
- A reasonably wrong estimation of λ impacts the price significantly.

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Conclusion : operational need to take the incompleteness into account

Dynamics and portfolio

The dynamics :

- We denote by $X_{t,x}$ the price process on $[0, T]$ starting at (t, x).
  $Y^{\nu}_{t,x,y}$ is the portfolio process starting at (t, y) with the strategy ν.

$$\begin{cases} X_{t,x}(r) & = x + \int_t^r \mu_s X_{t,x}(s) ds + \int_t^r \sigma_s X_{t,x}(s) dW_s \\ Y^{\nu}_{t,x,y}(r) & = y + \int_t^r \nu_s dX_{t,x}(s) \end{cases}$$

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The agent's objective :

- We are endowed with a claim $g(\lambda X_{t,x}(T))$.
- We introduce the loss function $\ell(z) = \dfrac{(z^-)^n}{n}$

The stochastic target problem

Mathematical formulation :

- Let $\mathcal{U}_t$ be the set of controls ν starting at time t.

  For a given threshold $p \leq 0$, we want to solve the following problem :

$$v(t,x,p) := \inf \left\{ \begin{array}{c} y \in \mathbb{R} \ : \ \exists \nu \in \mathcal{U}_t \text{ such that} \\ \mathbb{E}\left[-\ell\Big(Y^{\nu}_{t,x,y}(T) - g(\lambda X_{t,x}(T)) \Big) \right] \geq p \end{array} \right\}$$

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Interpretation :

- Find the minimal capital controlling the level p of loss.
- Focus only on loss with a n moment function (e.g. quadratic asymmetrical error).
- We do NOT hedge the claim but control its impact (if$p < 0$).
- Advantage : we take into account the risk of λ.

The complete and half complete market cases

(HCM) : we suppose that if λ is known, the market is complete.

Standard case : If $t \geq t_0$, we know λ and the market is complete.

- We keep the controlled loss approach (consistency of the strategy).
- What link with perfect replication of $g(\lambda X_{t,x}(T))$?

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The half-complete market : λ is $\mathcal{F}_{t_0}$-independent and $t < t_0$.

- We study a specific case of claim $f(X_{t,x}(T), \lambda)$
- Other applications : volume risk, mix finance-insurance claims,...

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Question : can we join the two separated problems ?

The complete market case

- If $t \geq t_0$, we can write $\lambda X_{t,x}(r) = X_{t,\lambda x}(r)$.
- Following Bouchard and al. (2009), the stochastic target problem becomes

$$v(t,x,p) := \inf \left\{ \begin{array}{c} y \in \mathbb{R} \;:\; \exists(\nu,\alpha) \in \mathcal{U}_t \times \mathcal{A}_t \text{ such that} \\ -\ell\Big(Y^{\nu}_{t,x,y}(T) - g(\lambda X_{t,x}(T))\Big) \geq P^{\alpha}_{t,x,p}(T) \end{array} \right\}$$

  where $P^{\alpha}_{t,x,p}(r) = p + \int_t^r \alpha_s P^{\alpha}_{t,x,p}(s) dW_s$ and $\mathcal{A}_t$ is the set of controls α independent of $\mathcal{F}_t$.
- $P^{\alpha}_{t,p}$ being a martingale, we have

$$\mathbb{E}\left[-\ell\Big(Y^{\nu}_{t,x,y}(T) - g(\lambda X_{t,x}(T))\Big)\right] \geq p$$

- Similar to a superreplication problem of function

$$\Psi^{-1}(X_{t,x}(T), P^{\alpha}_{t,p}(T)) = g(X_{t,x}(T)) - (nP^{\alpha}_{t,p}(T))^{1/n}$$

The complete market case

- Following Bouchard and al. (2009), we have that v_* is a viscosity supersolution on $[t_0, T) \times \mathbb{R}_+ \times \mathbb{R}_-$ of

$$\begin{cases} -\partial_t \varphi + \sup_{(u,a)\in\mathcal{N}_0} \mathcal{L}^{u,a}\varphi \geq 0 \,, & \text{on } [t_0, T) \times \mathbb{R}_+ \times \mathbb{R}_- \\ v_* - [g(x) - (np)^{1/n}] \geq 0 \,, & \text{on } \{T\} \times \mathbb{R}_+ \times \mathbb{R}_- \end{cases}$$

  with $\mathcal{N}_0 := \{(u,a) \ : \ \sigma_t x u - \sigma_t x \partial_x \varphi - a p \partial_p \varphi = 0\}$
  and $\mathcal{L}^{u,a}\varphi := u x \mu_t - [\mu_t x \partial_x \varphi + \frac{1}{2}\sigma_t^2 x^2 \partial_{xx}\varphi + \frac{1}{2} a^2 p^2 \partial_{pp}\varphi + \sigma_t x a \partial_{xp}\varphi]$
- v_* being convex in p, we can explicit (u,a) and obtain :

$$-\partial_t \varphi - \frac{1}{2}\sigma_t^2 x^2 \partial_{xx}\varphi + \frac{(\mu_t/\sigma_t \partial_p \varphi - \sigma_t x \partial_{xp}\varphi)^2}{2\partial_{pp}\varphi} \geq 0$$

The complete market case

- Using the Fenchel transform $u(t,x,q) = \sup_p(pq - v(t,x,p))$ we have that u is a subsolution of

$$\begin{cases} \varphi_t + \frac{1}{2}\sigma_t^2 x^2 \partial_{xx}\varphi + \frac{1}{2}\frac{\mu_t}{\sigma_t} q^2 \partial_{qq}\varphi + \mu_t xq\partial_{xq}\varphi \leq 0 \ , & \text{on } [t_0, T) \\ \varphi(T,x,q) - (1-\frac{1}{n})q^{\frac{1}{1-n}} - g(x) \leq 0 \ , & \text{on } \{T\} \end{cases}$$

- By the Feynman-Kac formula, u is smaller than

$$\bar{u} := \mathbb{E}^{Q_{t,x,q}}[(1-\frac{1}{n})q^{\frac{1}{1-n}} - g(X_{t,x}(T))]$$

$$\text{with} \begin{cases} dX_{t,x}(s) = \sigma_t X_{t,x}(s) dW_s^{Q_{t,x,q}} \\ dQ_{t,x,q}(s) = \frac{\mu_t}{\sigma_t} Q_{t,x,q}(s) dW_s^{Q_{t,x,q}} \ \text{ and } \ Q_{t,x,q}(t) = q \end{cases}$$

- Using the Fenchel again, we obtain

$$v(t,x,p) = \mathbb{E}^{Q_{t,x,1}}[g(X_{t,x}(T)] - (-np)^{\frac{1}{n}} \exp\left\{\frac{1}{2(n-1)}\int_t^T \frac{\mu_s^2}{\sigma_s^2} ds\right\}$$

The complete market case

- The complete market case allows to obtain explicit formulae for μ and σ constant :

$$P^{\alpha}_{t,p}(s) = p(\frac{X_{t,x}}{x})^{-\frac{n}{\mu}}(n-1)\sigma^2 \exp(\frac{n}{(n-1)}(\frac{n-2}{n-1} - \mu)(s-t))$$

$$\nu_s = \Delta(s, X_{t,x}(s)) + \frac{\mu}{(n-1)x\sigma^2}(-np)^{1/n} \exp(\frac{1}{2(n-1)}(\frac{\mu^2}{\sigma^2}(T-s))$$

- With a moment loss function, we separate the claim and the level of loss.
- $\Rightarrow$: the diffusion of $P^{\alpha}_{t,p}$ depends only on $X_{t,x}$ (not the claim).
- $\Rightarrow$: the value function is the Black Scholes price minus a premium.
- When $\mu = 0$, the strategy is the Black Scholes delta hedging.
- The agent can choose judiciously $n \geq 2$ and $p \leq 0$.

When the contract appears

Idea : by definition, $v(t_0^-, x, p)$ shall verify

$$\mathbb{E}\left[-\ell(Y^{\nu}_{t_0^-,x,v(t_0^-,x,p)}(T) - g(\lambda X_{t_0^-,x}(T)))\right] \geq p \texttt{ for some } \nu \in \mathcal{U}_{t_0^-}$$

so that we shall find $\nu(\lambda) \in \mathcal{U}_{t_0}$ for each λ such that

$$\mathbb{E}\left[\int_E -\ell(Y^{\nu(\lambda)}_{t_0^-,x,v(t_0^-,x,p)}(T) - g(\lambda X_{t_0^-,x}(T))J(d\lambda)\right] \geq p$$

Proposition

For $(t, x, p) \in [0, t_0) \times \mathbb{R}_+ \times \mathbb{R}^-$, *we have*

$$v(t, x, p) = \inf \left\{ y \in \mathbb{R} \ : \ \exists \nu \in \mathcal{U}_t \ s.t. \ \mathbb{E}\left[\Xi(X_{t,x}(t_0), Y^{\nu}_{t,x,y}(t_0))\right] \geq p \right\}$$

with $\Xi(x, y) := \int_E \sup \{p \ : \ v(t_0, \lambda x, p) \leq y\} J(d\lambda)$

Interpretation : we have a piecewise problem. For $t < t_0$, we are in the **(HCM)** framework.

When the contract appears

Facts and hints

- If we compute Ξ from $v(t_0, .)$, the market is complete and we can use the DPP.
- Here, $\Xi(x, y)$ has not an explicit solution in general.
- If λ is constant, we retrieve the Black Scholes strategy.
- Otherwise, we weight the strategy by losses induced by λ.

Numerical procedure comes now...

- We suppose that λ follows a Beta law.
- We compute the expectation wrt $J(d\lambda)$ numerically by iid simulations.

The half complete case

In the case of $t < t_0$, we have in our case

- $P^{\alpha}_{t,p}$ is an explicit function of $X_{t,x}$ ($P^{\alpha}_{t,p}$ is markov if μ, σ constant).
- $\Rightarrow v(s, X_{t,x}(s), P^{\alpha}_{t,p}(s)) = u_0(s, X_{t,x}(s))$.
- regularity assumptions

Dynamic Programming Principle + martingale representation theorem
$\Rightarrow v(t, x, p)$ is the superhedging price of $u_0(t_0^-, X_{t,x}(t_0^-))$.

$$\begin{cases} v(s, X_{t,x}(s), P^{\alpha}_{t,p}(s)) = \mathbb{E}[u_0(t_0^-, X_{t,x}(t_0^-)) \,|\, X_{t,x}(s)] \\ \\ \nu(s, X_{t,x}(s)) = \dfrac{\partial \mathbb{E}[u_0(t_0^-, X_{t,x}(t_0^-)) \,|\, X_{t,x}(s)]}{\partial X} \end{cases} .$$

The half complete case

Procedure

- We compute numerically $X_{t,x}(t_0^-)$ and $v(t_0, \lambda X_{t,x}(t_0), P_{t,p}(t_0))$
- We then obtain $u_0(t_0^-, X_{t,x}(t_0^-))$ (optimization).
- We compute ν (the derivative) with tangent processes.
- We compute the conditional expectations with regressions (on 3 monomials).

Benchmark : we compare $v(0, x, p)$ to the Black-Scholes approach.

- on call options price and loss (OTM, ATM, ITM)
- Black Scholes price is computed with $\mathbb{E}[\lambda]$.

Numerical procedure

Main initial data :

- λ has a standard deviation of 0.081.
- $S_0 = 50.89$.
- Loss function $\ell(Y - g(X)) = ((Y - g(x))^-)^2$.

Simulations :

- 10000 trajectories
- 10000 simulations of λ (Calibrated Beta law).

Price Vs Loss :ITM

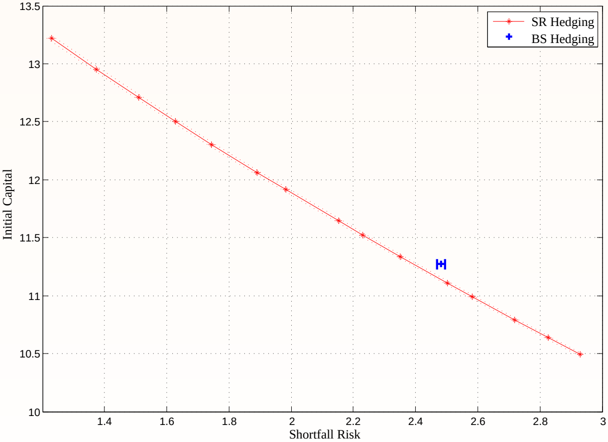


FIGURE: For $K = 0.80 \times S_0$. BS price $= 11.27$ eur.

Price Vs Loss :ATM

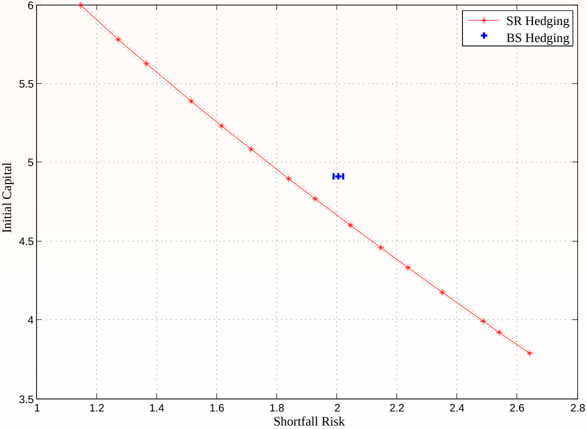


FIGURE: For $K = S_0$. BS price= 4.91 eur.

Price Vs Loss :OTM

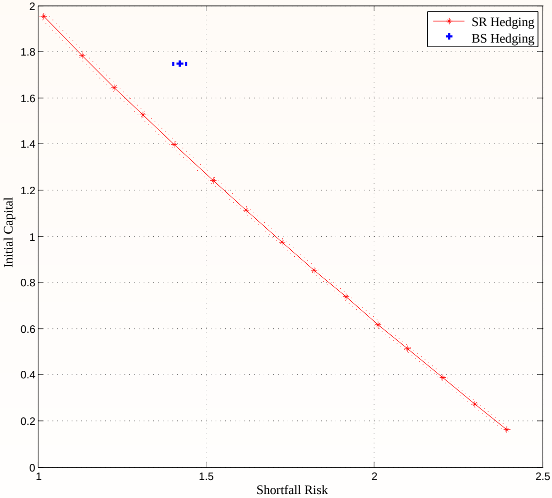


FIGURE: For $K = 1.20 \times S_0$. BS price= 1.75 eur.

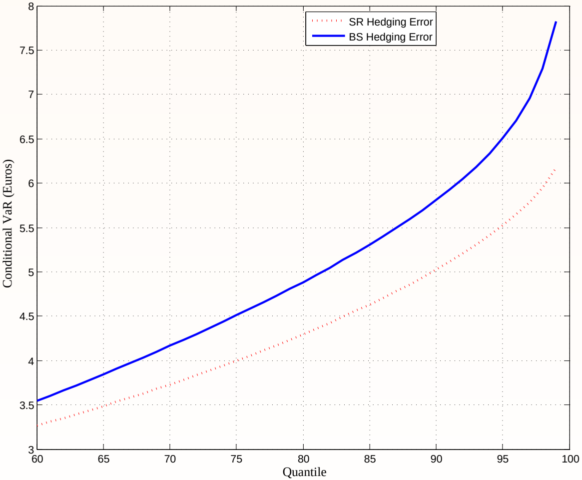
Suming up the results

Loss = % of Loss Reduction with BS Price ($C^{BS}(X) = v(0, X, p)$)

Price = % of Price Reduction compared to BS Loss (same p).

Strike	$(1-20\%)S_0$	S_0	$(1+20\%)S_0$
BS price (eur)	11.27	4.91	1.75
Loss	3.49%	8.70%	18.68%
Price	1.16%	5.12%	21.30%

CVar comparison :ITM



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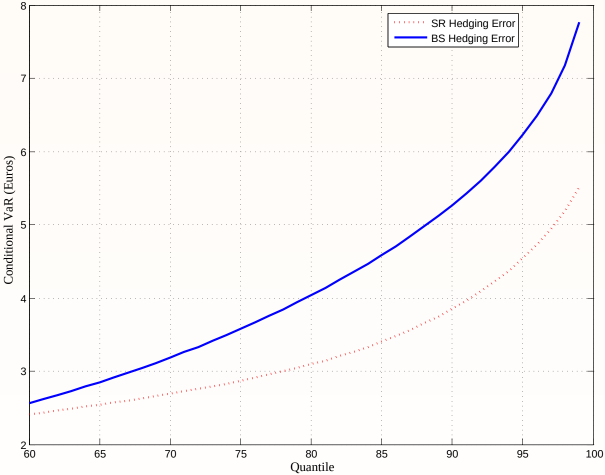
CVar comparison :ATM



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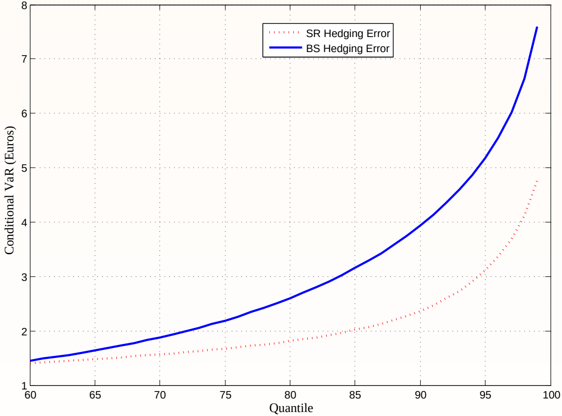

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Conclusions

The stochastic target approach :

- minimizes a given criteria till a threshold p.
- reduces the CVaR Risk with the same initial capital as BS.
- reduces the initial capital needed to achieve the same Loss as BS.

What remains to be done :

- Present results on real data.
- Develop the general continuous semimartingale framework.
- give a comprehensive interpretation of the differences between SR and BS strategy.
- calibrate new objects : p, n (preferences) and λ (statistics).