

Pricing in Social Networks ^{*}

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Abstract

We analyze the problem of optimal monopoly pricing in social networks where agents care about consumption or prices of their neighbors. We characterize the relation between optimal prices and consumers' centrality in the social network. This relation depends on the market structure (monopoly vs. oligopoly) and on the type of externalities (consumption versus price). We identify two situations where the monopolist does not discriminate across nodes in the network (linear monopoly with consumption externalities and local monopolies with price externalities). We also analyze the robustness of the analysis with respect to changes in demand, and the introduction of bargaining between the monopolist and the consumer.

JEL Classification Numbers: D85, D43, C69

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1 Introduction

This paper analyzes the optimal pricing strategy of a monopoly in a social network. Our objective is to understand how discriminatory prices reflect (or not) the centrality of consumers in the social network. Marketing techniques to discriminate among consumers based on their social connections have long been in use. When selling new products or creating an installed base for products with network externalities, it is not uncommon for firms to offer “referral bonuses” – discounts or cash to consumers who bring new friends into the network. In doing so, the firm rewards agents with a large number of friends, and price discriminates according to the consumer’s number of neighbors, or degree centrality. In a more systematic fashion, following MCI in 1990, telecommunication companies have introduced “friends and family plans” as a way to discriminate among consumers based on their number of friends and pattern of calls (Shi, 2003).

With the spectacular emergence of online social networks like Facebook, Orkut and MySpace, new possibilities for large scale social network based discriminatory pricing have emerged. Due to a combination of privacy and technical reasons, this possibility has not yet been exploited, and most of the monetization of online social networks stems from targeted advertising using data on consumer characteristics rather than their social connections. However, the discrepancy between the current revenue of Facebook (between 1.2\$ and 2\$ billion in 2010) and its value (82.9\$ billion reported as of January 29, 2011)(Levy, 2011) suggests that new marketing opportunities based on social network data will likely be exploited in the near future. In fact, the agreement between Facebook and the group buying platform Groupon which allows consumers to sign up on Groupon on their Facebook page points in that direction. Groupon may exploit the social network of Facebook to attract new customers, offering deals and coupons to consumers who bring in new friends, thereby discriminating in favor of consumers with higher degree centrality in the network.¹

While the current social-network based price discrimination strategies only make use of the consumer’s number of neighbors, it is very likely that more detailed data on social networks will soon be used in pricing and marketing strategies (Arthur et al. (2009), Hartline et al. (2008)). An important

¹The drop in Facebook’s share prices following the initial public offering on May 18, 2012 casts a shadow on the future profitability of Facebook and suggests that the exploitation of social networks in marketing may take longer than initially thought.

issue is to understand whether the number of neighbors is always the relevant measure of centrality that should be used for price discrimination. Even in case where this characteristic is relevant, it is necessary to assess its actual influence (positive or negative) on the prices that should be offered. In this paper, we consider price discrimination based on the entire social network, where each agent receives a price associated to her nodal characteristic. We consider two channels through which social networks influence a consumer's demand. In the first model of *local network externalities*, consumers benefit from the consumption of the same good by their direct neighbors. This model captures situations where agents receive discounts if they call friends who subscribe to the same network, share a common software with their colleagues or co-authors, or need to reach a critical mass of consumers to obtain a deal or launch a project. In the second model of *aspiration based reference price*, consumers construct a reference price for the good based on the price charged to their direct neighbors, and experience a positive utility if the price they receive is below their reference price. This model is applicable to situations where firms use discriminatory pricing that lacks transparency, like airline pricing and negotiated pricing.

In both models, our objective is to understand which measure of centrality is relevant to rank prices charged at different nodes. Are prices increasing or decreasing in the number of neighbors that a consumer has? Is the structure of the network at distance two (the number of neighbors of neighbors) a relevant information for optimal monopoly pricing? When does the monopoly charge uniform prices across nodes? To answer these questions, we consider a linear model, where consumers pick a random valuation for the object according to a uniform distribution. In the model of local network externalities, a consumer's utility is positively affected by the consumption of her direct neighbors ; in the model of aspiration-based price reference, a consumer's utility is positively affected by the average price charged to her direct neighbors. Using the analysis pioneered by Ballester, Calvó-Armengol and Zenou (2006), we characterize the demand of every consumer as a function of her centrality in the network. We then consider two different market structures: one where a single monopoly serves all the consumers in the network and one where oligopolistic firms control a fraction of the nodes in the network.²

In the local network externalities model, we first obtain a *network irrele-*

²An example of an oligopoly where firms control a fraction of the nodes in the social network is given by Apple and Microsoft. Both firms compete to establish exclusive partnerships with universities. Researchers from two different universities may be forced to use two different operating systems even though they interact and share files repeatedly.

vance result: in the linear model, the monopoly optimally chooses a uniform price in the network. This striking result can be explained as follows. There are two countervailing effects of the centrality of a node on the optimal price. On the one hand, a more central node generates more positive externalities on its neighbors and hence should be subsidized (the classical effect by which more central agents receive lower prices) ; on the other hand, more central agents benefit more from the object, and have a higher valuation which can be captured by the monopolist. In the linear model, these two effects are exactly balanced, giving rise to a uniform pricing strategy. However, this exact balance disappears as soon as one moves away from the linear model. When costs are quadratic, the price at each node is proportional to the Katz-Bonacich centrality. When influence is directed, so that the social network is represented by a directed graph, prices are higher for nodes which receive more influence than they provide. Finally, in an oligopolistic model, the optimal price depends both on the node centrality and on the competition structure in the node's neighborhood. Higher prices are charged to more central nodes whose neighbors are controlled by competitors.

In the aspiration-based price reference model, we obtain a second *network irrelevance result*, this time when every node is served by a different firm. This irrelevance result, which is robust to changes in the model, stems from the following observation. If all other firms charge the optimal monopoly price, a local monopoly cannot benefit from charging a different price. When all nodes are served by a single monopolist, this reasoning fails as the monopolist may want to increase the price at some node in order to increase demand at the neighboring nodes. For example, in a star, the monopoly has an obvious incentive to charge a high price at the hub in order to increase demand at peripheral nodes.

We finally discuss two extensions of the model. In the first extension, we consider general demand schedules and analyze the robustness of our results. In the second extension, we compute the consumer surplus accruing at each node. This enables us to analyze the agent's incentives to form links in the social network and the formation of prices as a result of a bargaining process between the monopoly and the consumer.

We now discuss briefly the related literature. The model of local network externalities finds its origin in the seminal work of Farrell and Saloner (1985) and Katz and Shapiro (1985) on network externalities. These early papers eschew the "network" dimension of network externalities and implicitly assume that consumers are affected by the global consumption of all other

consumers. Models of local network externalities which explicitly take into account the graph theoretic structure of social networks have been proposed by Jullien (2001), Sundarajan (2006), Saaskhilati (2007) and Banerji and Dutta (2009). Jullien (2001) and Banerji and Dutta (2009) analyze competition between two price-setting firms. While Banerji and Dutta (2009) consider uniform prices, Jullien (2001) allows for discriminatory pricing at different nodes, and provides partial results suggesting that firms set lower prices at nodes with higher degree. Sundarajan (2006) studies monopoly pricing in a model where consumers make a deterministic choice between adopting the new product or not. Ghigliano and Goyal (2010) focus instead on a model of conspicuous consumption, where agents compare their consumption with that of their neighbors and suffer a negative consumption externality. In the same linear model as the one we consider, they characterize the competitive equilibrium prices and allocations and show that identical consumers located in asymmetric positions in the network choose to trade and end up at different equilibrium allocations. Finally, in a work which is independent from ours, Saaskhilati (2007) studies uniform monopoly pricing on social networks. His main focus is not on discriminatory pricing but on the relation between the network topology and the uniform price charged by the monopoly, and he computes optimal prices and consumer surplus for some specific network structures like symmetric networks and stars.

The study of optimal pricing and marketing strategies in social networks has recently received attention in the computer science literature. Following the work on influence maximization of Domingos and Richardson (2001) and Kempe, Kleinberg and Tardos (2003) which aimed at identifying influential agents in a network without any reference to price and revenue maximization, recent work by Hartline et al. (2008) and Arthur et al. (2009) compute optimal pricing strategies. They show that a simple two-price strategy (the "Influence and Exploit Marketing", where the seller chooses a set of consumers to which the product is sold for free – or at a cashback "referral bonus" –) performs very well compared to the optimal marketing strategy which is NP-hard to compute. The main difference between these approaches and ours stem from the timing of purchases. Both Hartline et al. (2008) and Arthur et al. (2009) consider sequential purchases where myopic consumers base their consumption decision on the number of consumers who have *already bought* the product. We consider instead a simultaneous consumption decision for all consumers in the network who are fully rational.

The model of aspiration based reference price has been studied in marketing (Xia, Monroe and Cox (2004), Mazumdar et al. (2005)) along lines

developed in social psychology. The theory of social comparison (see Suls and Wheeler (2000) for a detailed account) posits that most outcomes (like prices and salaries) are perceived in comparison to other agents' outcomes, so that prices are deemed fair or unfair in reference to prices paid by other consumers in a similar situation. Hence, consumers construct reference prices based on what their neighbors have been charged, and evaluate the price they receive by comparison to this reference price.

The rest of the paper is organized as follows. We start by introducing preliminary definitions and notations in Section 2. In section 3, we discuss the model of local network externalities. Section 4 is devoted to the model of aspiration based reference price. Section 5 contains a discussion of the robustness of the analysis and an extension to bargaining over total surplus. Section 6 concludes. Most proofs are relegated to the Appendix.

2 Preliminaries

In this Section, we introduce the basic notions on social networks and matrix algebra which will be used in the analysis. The definitions and notations of this preliminary section apply both to the models of consumption and price externalities.

2.1 Social Networks

We consider a set N of consumers, $i = 1, 2, \dots, n$ who are distributed along a social network g with adjacency matrix \mathbf{G} . For any pair of agents i, j , $g_{ij} = 1$ if there exists an edge between i and j , and $g_{ij} = 0$ otherwise. For most of the analysis, we assume that the graph is *undirected*, $g_{ij} = g_{ji}$, so that \mathbf{G} is a symmetric matrix.

The vector $\mathbf{G}\mathbf{1}$ measures the number of edges at each node of the social network. The *degree* of a node i is the number of edges at i , $\text{deg}_i = (\mathbf{G}\mathbf{1})_i = \sum_j g_{ij}$. When the network is directed, we distinguish between the number of nodes which point towards node i (the *in-degree*) and the number of nodes to which i points (the *out-degree*). Formally, $\text{indeg}_i = (\mathbf{1}^T \mathbf{G})_i^T = \sum_j g_{ji}$ and $\text{outdeg}_i = (\mathbf{G}\mathbf{1})_i = \sum_j g_{ij}$.

Consider a scalar α such that the matrix $\mathbf{I} - \alpha\mathbf{G}$ is invertible. The Katz-Bonacich notion of network centrality (Katz (1953), Bonacich (1987)) is defined as:

$$\mathbf{b}(\mathbf{G}, \alpha) = [\mathbf{I} - \alpha\mathbf{G}]^{-1}\mathbf{1}.$$

In order to interpret this definition, we use the power series expansion $[\mathbf{I} - \alpha\mathbf{G}]^{-1} = \sum_{k=0}^{\infty} \alpha^k \mathbf{G}^k$, to rewrite $\mathbf{b}_i = \sum_j \sum_k \alpha_k \mu_{ij}^k$, where μ_{ij}^k , the ij entry of the matrix \mathbf{G}^k , counts the number of paths of length k between i and j . The Katz-Bonacich centrality thus measures the discounted number of paths originating from any node in the social network.

2.2 Matrix Algebra

Let \mathbf{A} be a square $n \times n$ matrix. The matrix \mathbf{A} is a *P-matrix* if all its principal minors are positive. The matrix \mathbf{A} is a *non-singular M-matrix* if $\mathbf{A} = \mathbf{I} - \mathbf{B}$ for a positive matrix \mathbf{B} with largest eigenvalue $\rho(\mathbf{B}) < 1$. The matrix \mathbf{A} is *positive definite* if $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for any vector \mathbf{x} . The matrix \mathbf{A} is *strictly row diagonally dominant* if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all $i = 1, 2, \dots, n$.

By well-known results in matrix algebra, a matrix \mathbf{A} is an M-matrix if and only if it is a P-matrix and $a_{ij} \leq 0$ for all $i \neq j$ and a matrix \mathbf{A} is a P-matrix if and only if it is a symmetric positive definite matrix (Berman and Plemmons (1994), Theorem 6.2.3 and Section 10.2). Furthermore, if $a_{ii} > 0$ for all i , and \mathbf{A} is a strictly row diagonally dominant matrix, then it is a P-matrix (Tsatsomeros (2002)).

For any matrix \mathbf{A} , and any index set S of rows and columns, we let \mathbf{A}_S denote the submatrix formed by the rows and columns in S . For any vector $\mathbf{x} \in \mathfrak{R}^n$, we denote by \mathbf{x}^S the vector defined by $x_i^S = x_i$ for all $i \in S$ and $x_i = 0$ for all $i \in N \setminus S$ and by x_S the subvector of x formed by the components in S . For two vectors \mathbf{x} and \mathbf{x}' , we let $\mathbf{x} \leq \mathbf{x}'$ denote the weak vector order, where $x_i \leq x'_i$ for all i .

Finally, we recall that, for two matrices \mathbf{A} and \mathbf{B} of the same dimension, the *Hadamard product* is defined by: $\mathbf{C} = \mathbf{A} \circ \mathbf{B}$ where $c_{ij} = a_{ij} b_{ij}$. By the Schur product theorem (Horn and Johnson (1985), Theorem 7.5.3), the Hadamard product of two positive definite matrices is positive definite.

3 Local Network Externalities

3.1 The Model

As in Jullien (2001), Saaskhilati (2007) and Banerji and Dutta (2009), we construct a model of network externalities where consumers only care about the consumption of a subset of agents determined by an exogenous social network. Each consumer i has a unit demand for the good, and draws an intrinsic value θ_i from the uniform distribution F over $[0, 1]$. We suppose

that intrinsic values are independently distributed. Consumers experience *local network externalities* in the sense that their value for the good increases by the constant value $\alpha > 0$ whenever one of their neighbors consumes the good. Finally, consumers have positive linear utility for money, so that the utility of consumer i is expressed by:

$$U_i = \theta_i - p_i + \alpha \sum_j g_{ij} \Pr[j \text{ buys the good}]. \quad (1)$$

The timing of events is as follows: the monopoly first chooses a price vector $\mathbf{p} = (p_1, \dots, p_n)$. Each consumer learns her valuation θ_i and makes her consumption decision at the interim stage, knowing p_i and θ_i , but not the valuations θ_{-i} drawn by other consumers. Clearly, if a consumer of type θ_i buys the good, so does any consumer of type $\theta'_i > \theta_i$. Hence, consumer i 's optimal purchasing decision is characterized by a threshold rule, $\tilde{\theta}_i$. As other consumers also adopt a threshold consumption rule, we can compute the threshold of consumer i using the following expression:

$$\tilde{\theta}_i = p_i - \alpha \sum_j g_{ij} (1 - F(\tilde{\theta}_j)). \quad (2)$$

where $(1 - F(\tilde{\theta}_j))$ denotes the probability that agent j draws a valuation above the threshold $\tilde{\theta}_j$. Alternatively, if we let $x_i = 1 - F(\tilde{\theta}_i)$, we characterize a system of interdependent demands:

$$x_i = \begin{cases} 0 & \text{if } 1 - p_i + \alpha \sum_j g_{ij} x_j < 0 \\ 1 & \text{if } -p_i + \alpha \sum_j g_{ij} x_j > 0, \\ 1 - p_i + \alpha \sum_j g_{ij} x_j & \text{otherwise} \end{cases} \quad (3)$$

We now solve this system of interdependent demands in order to obtain the demand of a consumer at node i as a function of the vector of prices, $\mathbf{p} = (p_1, \dots, p_n)$ charged at different nodes. This amounts to finding a vector $\mathbf{x} \in [0, 1]^n$ such that for each $i \in \{1, 2, \dots, n\}$, one of the following holds:

$$\begin{aligned} x_i = 0, & \quad [(1 - \mathbf{p}) - (\mathbf{I} - \alpha \mathbf{G})\mathbf{x}]_i \leq 0, \\ 0 < x_i < 1, & \quad [(1 - \mathbf{p}) - (\mathbf{I} - \alpha \mathbf{G})\mathbf{x}]_i = 0, \\ x_i = 1, & \quad [(1 - \mathbf{p}) - (\mathbf{I} - \alpha \mathbf{G})\mathbf{x}]_i \geq 0. \end{aligned}$$

This problem is known as the *bounded linear complementarity problem* and is the linear instance of the general mixed complementarity problems discussed by Simsek, Ozdaglar and Acemoglu (2005). A well-known sufficient

condition for the existence and uniqueness of a solution to the problem is that the matrix $(\mathbf{I} - \alpha\mathbf{G})$ be a P-matrix. (See also Ballester Calvó-Armengol and Zenou (2006), Ballester and Calvó-Armengol (2010), and Bramoullé, Kranton and d'Amours (2011).) In very recent work, Belhaj, Bramoullé and Deroian (2012) show that, if $p_i < 1$ for all i , the system of interdependent equations has a unique solution, without any restriction on the matrix $\mathbf{I} - \alpha\mathbf{G}$. However, their proof does not extend to the case $p_i \leq 1$ for some i , which we allow here.

Proposition 3.1 *If $\alpha\rho(\mathbf{G}) < 1$, for any vector of prices $\mathbf{p} = (p_1, \dots, p_n)$, there exists a unique set of demands satisfying equation (3). In this solution, the set of consumers is partitioned into three sets S_0, S_1 and $S = N \setminus (S_0 \cup S_1)$ such that:*

$$x_i = \begin{cases} 0 & \text{if } i \in S_0 \\ 1 & \text{if } i \in S_1 \\ \sum_{j \in S} a_{ij,S}(1 - p_j + \alpha \sum_{k \in S_1} g_{jk}) & \text{if } i \in S \end{cases} \quad (4)$$

where $a_{ij,S}$ is the ij entry of the matrix $\mathbf{A}_S = [(\mathbf{I} - \alpha\mathbf{G})_S]^{-1}$.

Proposition 3.1 shows that, if the externality parameter α and the largest eigenvalue of the adjacency matrix, $\rho(\mathbf{G})$, are not too large, the interdependence between consumer demands at different points in the network results in a *unique* system of demands. For an arbitrary price vector, \mathbf{p} , as demands must belong to the bounded interval $[0, 1]$, the description of equilibrium demands involves a partition of the set of nodes into (i) nodes with zero demand, (ii) nodes where consumers buy with probability one and (iii) nodes where consumers buy with a probability $x_i \in (0, 1)$. For consumers at these last nodes, the coefficients of the demand system $\frac{\partial x_i}{\partial p_j} = a_{ij}$ are exactly the entries of the matrix $[(\mathbf{I} - \alpha\mathbf{G})_S]^{-1}$, which can be interpreted as the Katz-Bonacich centrality of consumers in S .

3.2 Monopoly pricing

We now analyze the optimal price vector \mathbf{p} chosen by a monopolist. We first consider the baseline linear model, then discuss two variants: (i) one where the monopoly faces a quadratic cost at each node, and (ii) one where consumption externalities are directed.

3.2.1 Optimal monopoly pricing

Suppose that a monopolist chooses the vector of prices \mathbf{p} in order to maximize profit. We assume that the monopoly produces each unit of the good at a constant cost $c < 1$, so that the problem of the monopolist is given by:

$$\max_{\mathbf{p}} \Pi = (\mathbf{p} - c\mathbf{1})^T \mathbf{x}(\mathbf{p}).$$

As the system of demand is invertible by Proposition 3.1, we can equivalently consider the problem where the monopolist chooses quantities,

$$\max_{\mathbf{x}} \Pi = (\mathbf{p}(\mathbf{x}) - c\mathbf{1})^T \mathbf{x}. \quad (5)$$

It is easy to check³ that the monopolist always chooses interior values $0 \leq x_i \leq 1$ for which $\mathbf{p}(\mathbf{x}) = \mathbf{1} - (\mathbf{I} - \alpha\mathbf{G})\mathbf{x}$. We thus rewrite the profit of the monopolist as:

$$\Pi = [(1 - c)\mathbf{1} - (\mathbf{I} - \alpha\mathbf{G})\mathbf{x}]^T \mathbf{x},$$

and compute the gradient as

$$\nabla \Pi = (1 - c)\mathbf{1} - 2(\mathbf{I} - \alpha\mathbf{G})\mathbf{x}.$$

Because the matrix $(\mathbf{I} - \alpha\mathbf{G})$ is a symmetric P-matrix, it is positive definite, and hence the Hessian matrix $\nabla^2 \Pi$ is negative semi-definite, so that the first-order conditions are necessary and sufficient, and the optimal quantities of the monopolist is characterized by:

$$\mathbf{x}^* = \frac{1 - c}{2}(\mathbf{I} - \alpha\mathbf{G})^{-1}\mathbf{1}.$$

with the corresponding optimal prices:

$$\begin{aligned} \mathbf{p}^* &= \mathbf{1} - \frac{1 - c}{2}(\mathbf{I} - \alpha\mathbf{G})(\mathbf{I} - \alpha\mathbf{G})^{-1}\mathbf{1}, \\ &= \frac{1 + c}{2}\mathbf{1}. \end{aligned}$$

Proposition 3.2 *Let $\alpha\rho(\mathbf{G}) < 1$. The pricing strategy of the monopoly is to charge a uniform price $p^* = \frac{1+c}{2}$ at each node. Given this pricing strategy, the expected demand of consumers are given by*

$$\mathbf{x}^* = \frac{1 - c}{2}(\mathbf{I} - \alpha\mathbf{G})^{-1}\mathbf{1}$$

³The formal proof is in the Appendix.

and are proportional to the Katz-Bonacich centrality.

The striking result of Proposition 3.2 is that the monopoly does not exploit differences in consumer’s centralities to charge discriminatory prices, but chooses instead a uniform monopoly price at each node. She lets demand adjust at each node in the network according to consumer’s centralities, with consumers with higher levels of Katz-Bonacich centrality having a higher probability of purchasing the good.

The network irrelevance result of Proposition 3.2 is supported by the following intuition. When choosing the price at node i , the monopoly balances two effects: a price increase at node i raises profit at that node, but also reduces demand and profits at all other nodes in the network. This is a trade-off between raising the price at more central nodes to ”exploit” the nodes centrality or lowering the price at more central nodes to maximize ”influence” on other consumers.⁴ In the linear model we analyze, this trade-off, measured by a positive effect $\sum a_{ij}(1-p_j)$ and a negative effect $\sum a_{ji}(c-p_j)$, is independent of a node’s centrality. Hence, the monopoly faces the same trade-off at every node and optimally chooses a uniform pricing rule.

In order to assess the robustness of the network irrelevance result and to study the trade-off between exploitation and influence at central nodes, we consider two tractable variants of the basic model. In the first variant, production costs are quadratic ; in the second variant, the social network is directed.

3.2.2 Monopoly pricing with quadratic costs

Suppose that the production cost is quadratic at every node, $c_i(x_i) = cx_i^2$. By the same argument as in Proposition 3.2, the monopolist will always choose interior quantities. The profit of the monopolist is thus given by:

$$\Pi = [\mathbf{1} - ((1 + c)\mathbf{I} - \alpha\mathbf{G})\mathbf{x}]^T \mathbf{x}. \quad (6)$$

and the gradient of profit is

$$\nabla\Pi = \mathbf{1} - 2((1 + c)\mathbf{I} - \alpha\mathbf{G})\mathbf{x}. \quad (7)$$

⁴This definition of ”influence and exploit” is different from the definition in the computer science literature (Hartline et al. (2008) and Arthur et al. (2009)) where prices are lower at more central nodes to maximize influence and higher at less central nodes which are exploited.

Clearly, as $\alpha\rho(\mathbf{G}) < 1$, the matrix $((1+c)\mathbf{I} - \alpha\mathbf{G})$ is a symmetric P-matrix so that the Hessian matrix $\nabla^2\Pi$ is negative definite, and the optimal quantities are given by the first order condition:

$$\mathbf{x}^* = \frac{1}{2}((1+c)\mathbf{I} - \alpha\mathbf{G})^{-1}\mathbf{1},$$

resulting in optimal prices:

$$\mathbf{p}^* = \mathbf{1} - \frac{1}{2}(\mathbf{I} - \alpha\mathbf{G})((1+c)\mathbf{I} - \alpha\mathbf{G})^{-1}\mathbf{1}.$$

Proposition 3.3 *Let $\alpha\rho(\mathbf{G}) < 1$. In the model with quadratic costs, the optimal pricing strategy of the monopoly is given by:*

$$\mathbf{p}^* = \mathbf{1} - \frac{1}{2}(\mathbf{I} - \alpha\mathbf{G})((1+c)\mathbf{I} - \alpha\mathbf{G})^{-1}\mathbf{1}.$$

or

$$\mathbf{p}^* = \frac{2c+1}{2(1+c)}\mathbf{1} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{c\alpha^k}{(1+c)^{k+1}}\mathbf{G}^k.$$

Proposition 3.3 shows that, when costs are quadratic, prices are proportional to the Katz-Bonacich centrality measure, and nodes which are more central according to this measure will be charged higher prices. In order to gain additional intuition, we take α to be small, and approximate the price at node i by:

$$\mathbf{p}_i^* = \frac{2c+1}{2(1+c)} + \alpha\frac{c}{2(1+c)^2}\text{deg}_i + \alpha^2\frac{c}{2(1+c)^3}\sum_j\mu_{ij}^2 + \mathcal{O}(\alpha^3),$$

showing that nodes with higher degree will be charged higher prices, and if two nodes have the same degree, prices will be higher for the node which has the most paths of length 2. This characterization is not surprising. With quadratic costs, the monopolist has less incentives to increase quantities at central nodes in order to influence neighboring consumers than in the linear cost model. Hence, the trade-off between influence and exploitation is resolved in favor of exploitation, and more central nodes are charged higher prices.

3.2.3 Monopoly pricing with directed influence

Suppose that the matrix of externalities \mathbf{G} is *not symmetric* so that influences in the social network are directed. This assumption enables us to consider other types of social interactions. In the literature on networks, directed graphs allow for instance to consider situations described as Royal family types of social interactions (that is, where there exists a sub-set of agents who are observed by everyone).

With an asymmetric matrix of externalities, the coefficient $g_{ij} \in \{0, 1\}$ denotes the positive externality enjoyed by player i because of j 's consumption of the good. As the derivation of the demand system $\mathbf{x}(\mathbf{p})$ does not rely on the symmetry of the matrix \mathbf{G} , we can still write the profit of the monopolist as:

$$\Pi = [(1 - c)\mathbf{1} - (\mathbf{I} - \alpha\mathbf{G})\mathbf{x}]^T \mathbf{x};$$

The gradient of the profit is given by

$$\nabla \Pi = (1 - c)\mathbf{1} - (2\mathbf{I} - \alpha(\mathbf{G} + \mathbf{G}^T))\mathbf{x}.$$

If $\alpha\rho(\mathbf{G} + \mathbf{G}^T) < 2$, the matrix $(2\mathbf{I} - \alpha(\mathbf{G} + \mathbf{G}^T))$ is a symmetric P-matrix, which implies that the Hessian $\nabla^2 \Pi$ is negative definite, and the first order condition is necessary and sufficient, resulting in optimal quantities

$$\mathbf{x}^* = (1 - c)(2\mathbf{I} - \alpha(\mathbf{G} + \mathbf{G}^T))^{-1}\mathbf{1},$$

and equilibrium prices

$$\mathbf{p}^* = \mathbf{1} - (1 - c)(\mathbf{I} - \alpha\mathbf{G})(2\mathbf{I} - \alpha(\mathbf{G} + \mathbf{G}^T))^{-1}\mathbf{1}.$$

Proposition 3.4 *Let $\alpha\rho(\mathbf{G}) < 1$ and $\alpha\rho(\mathbf{G} + \mathbf{G}^T) < 2$. In the model with directed influence, the optimal pricing strategy of the monopoly is given by:*

$$\mathbf{p}^* = \mathbf{1} - (1 - c)(\mathbf{I} - \alpha\mathbf{G})(2\mathbf{I} - \alpha(\mathbf{G} + \mathbf{G}^T))^{-1}\mathbf{1}.$$

or

$$\mathbf{p}^* = \frac{1 + c}{2}\mathbf{1} + \frac{1 - c}{2} \sum_{k=1}^{\infty} \frac{\alpha^k}{2^k} (\mathbf{G} - \mathbf{G}^T)(\mathbf{G} + \mathbf{G}^T)^{k-1}\mathbf{1}.$$

Proposition 3.4 shows that whenever the matrix \mathbf{G} is not symmetric ($\mathbf{G} \neq \mathbf{G}^T$), optimal prices are not uniform across nodes, and prices depend on the

difference between the two matrices, $\mathbf{G} - \mathbf{G}^T$. When α is small, we can compute an approximation of the price charged at node i as

$$p_i = \frac{1+c}{2} + \frac{\alpha}{4}(1-c)(\text{indeg}_i - \text{outdeg}_i) + \mathcal{O}(\alpha^2).$$

We thus observe that the price charged to node i depends on the difference between the in-degree and out-degree. This difference reflects the difference between the number of neighbors who influence agent i and the number of neighbors that agent i influences. More influential agents, for whom this difference is negative, face lower prices, reflecting the fact that the monopolist reduces his price in order to increase the consumption of influential agents. Agents who have more influential neighbors will be charged lower prices. This result suggests that the monopolist should charge low prices to critics, gurus and "royal families" – agents who are likely to influence many other agents, without being influenced themselves.

3.3 Oligopoly pricing

In this subsection, we suppose that K firms compete in the network. Each firm k controls a subset of nodes N_k , and prices arise as an *equilibrium* of a non-cooperative pricing game rather than as the optimal choice of a multi-product monopolist. Let \mathbf{D} be a square matrix which indicates whether two nodes i and j are controlled by the same firm, with $d_{ij} = 1$ if and only if i and j are controlled by the same firm, and $d_{ij} = 0$ otherwise.

As a first step in the analysis, we show that the solution to the demand system (3) is monotonic in prices. In order to prove monotonicity, we assume that $\mathbf{I} - \alpha\mathbf{G}$ satisfies strict row diagonal dominance, $1 > \alpha \sum_j g_{ij}$ for all i , a condition which is stronger than assuming that $\mathbf{I} - \alpha\mathbf{G}$ is a P-matrix.⁵ We prove:

Lemma 3.5 *Assume that $\mathbf{I} - \alpha\mathbf{G}$ satisfies strict row diagonal dominance. Suppose that $\mathbf{p}' \geq \mathbf{p}$, and let \mathbf{x}^* (respectively, $\mathbf{x}^{*'}$) be the solution to the system of equations (3) for price \mathbf{p} (respectively, price \mathbf{p}'). Then $\mathbf{x}^* \geq \mathbf{x}^{*'}$.*

When Lemma 3.5 holds, the demand vectors \mathbf{x} are non-increasing in prices \mathbf{p} . This implies that firms will never choose prices which lead to zero demand,

⁵To the best of our knowledge, there does not exist any general monotonicity result for the solutions of linear complementarity problems. Murty (1972) shows that an increase in p_i results in a decrease in x_i , but his result does not extend to the entire vector \mathbf{x} . Cottle (1972) considers a different problem, and shows that, starting from any price vector $\mathbf{p} \leq \mathbf{1}$, any move along a particular direction \mathbf{q} results in a monotonic change in the solution \mathbf{x} .

as a decrease in any price results in a weak increase in demand at all nodes. Hence, under monotonicity of demand, an oligopolistic firm will optimally set prices so that demands are interior. The maximization problem of firm k can thus be expressed as:

$$\max_{\mathbf{p}_{N(k)}} \Pi_k = ((\mathbf{p} - c\mathbf{1})^{N(k)})^T (\mathbf{I} - \alpha\mathbf{G})^{-1} (\mathbf{1} - \mathbf{p}).$$

Computing the gradient of Π_k with respect to $\mathbf{p}_{N(k)}$,

$$\nabla \Pi_k = ((\mathbf{I} - \alpha\mathbf{G})^{-1} (\mathbf{1} - \mathbf{p}))_{N(k)} - ((\mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1}) (\mathbf{p} - c\mathbf{1}))_{N(k)},$$

And the Hessian is given by:

$$\nabla^2 \Pi_k = -2((\mathbf{I} - \alpha\mathbf{G})^{-1} + \mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1})_{N(k)}.$$

We prove in the Appendix that the Hessian is negative definite so that the best response function of firm k is given by the condition:

$$\mathbf{p}_{N(k)}^* = ((\mathbf{I} - \alpha\mathbf{G})^{-1} + \mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1})^{-1} ((\mathbf{I} - \alpha\mathbf{G})^{-1} \mathbf{1} + c(\mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1}) \mathbf{1})_{N(k)} \quad (8)$$

and the Nash equilibrium of the pricing game is characterized by

$$\mathbf{p}^* = ((\mathbf{I} - \alpha\mathbf{G})^{-1} + \mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1})^{-1} ((\mathbf{I} - \alpha\mathbf{G})^{-1} \mathbf{1} + c(\mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1}) \mathbf{1}) \quad (9)$$

Equation (9) gives an exact characterization of the equilibrium prices, but involves a complex transformation of the adjacency matrix \mathbf{G} , with two matrix inversions. In order to gain intuition on the relation between prices and the centrality of nodes in the social network, we compute an approximation formula which is valid for low values of the externalities parameter α .

Proposition 3.6 *Suppose that the matrix $\mathbf{I} - \alpha\mathbf{G}$ satisfies strict row diagonal dominance. The oligopolistic pricing game admits a unique equilibrium characterized by:*

$$\mathbf{p}^* = ((\mathbf{I} - \alpha\mathbf{G})^{-1} + \mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1})^{-1} ((\mathbf{I} - \alpha\mathbf{G})^{-1} \mathbf{1} + c(\mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1}) \mathbf{1})$$

In addition, there exists $\bar{\alpha}$ such that, for all $\alpha \leq \bar{\alpha}$,

$$\begin{aligned} \mathbf{p} &= \frac{1+c}{2} \mathbf{1} + \frac{\alpha}{4} (\mathbf{G} - \mathbf{D} \circ \mathbf{G}) \mathbf{1} \\ &\quad + \frac{\alpha^2}{8} (\mathbf{G}^2 - (\mathbf{D} \circ \mathbf{G}) \mathbf{G} + (\mathbf{D} \circ \mathbf{G})^2 - \mathbf{D} \circ \mathbf{G}^2) \mathbf{1} + \mathcal{O}(\alpha^3). \end{aligned}$$

Proposition 3.6 describes equilibrium prices for all possible market structures on the social network. If a single monopolist serves all nodes, the matrix \mathbf{D} is the identity matrix for the Hadamard product, and prices are uniform. If every node is served by a different firm, the matrix \mathbf{D} is the identity matrix, and $\mathbf{D} \circ \mathbf{G} = [0]$. Furthermore, the diagonal entries of \mathbf{G}^2 are the degrees of the nodes, so that $(\mathbf{D} \circ \mathbf{G}^2)\mathbf{1} = \mathbf{G}\mathbf{1}$. Hence, in that special case, the approximation formula reduces to

$$\mathbf{p}^* = c\mathbf{1} + \frac{1-c}{2}\mathbf{1} + \alpha\frac{1-c}{4}\mathbf{G}\mathbf{1} + \alpha^2\frac{1-c}{8}(\mathbf{G}^2\mathbf{1} - \mathbf{G}\mathbf{1}) + \mathcal{O}(\alpha^3).$$

We can also express the approximation formula at every node i :

$$p_i = \frac{1+c}{2} + \frac{\alpha}{4}(\text{deg}_i - \sum_{j|d_{ij}=1} g_{ij}) + \mathcal{O}(\alpha^2).$$

This formula shows that prices are high for nodes with high degree and whose neighbors are served by other firms, and low for nodes with low degree and whose neighbors are served by the same firm. In other words, firms have an incentive to reduce the prices at nodes which are surrounded by nodes that they control in order to internalize the consumption externalities. They have instead an incentive to increase prices at nodes which are surrounded by nodes controlled by their competitors. In order to judge the accuracy of the approximation formula, we ran a sensitivity analysis, by generating random networks and computing, for each network, the threshold value of α for which the ranking of equilibrium prices in our approximation coincides with the exact ranking of equilibrium prices for local monopolies when $\mathbf{D} = \mathbf{I}$.⁶ The results are given in the following table, which lists for different numbers of agents ($n = 6, 7, 8, 9, 10, 15$ and 20), the minimal, maximal and mean values of the threshold value of α over 1000 randomly generated networks.

n	6	7	8	9	10	15	20
$\bar{\alpha}_{min}$	0.19	0.14	0.01	0.01	0.005	0.01	0.01
$\bar{\alpha}_{max}$	1	1	0.38	0.38	0.305	0.15	0.11
$\bar{\alpha}_{mean}$	0.301	0.248	0.213	0.188	0.160	0.108	0.082

Table 1: Simulations for price rankings

⁶We are immensely grateful to Sebastian Bervoets who wrote the computer program and ran the simulations.

As expected, the threshold value of α decreases with the number of agents, but remains surprisingly high, showing that the approximation is reasonably accurate in order to compare equilibrium prices charged at different nodes.

4 Aspiration Based Reference Price

4.1 The model

We now consider a model where externalities do not result from consumption but from prices. Following the literature on social comparisons, we assume that agents compare the price they receive with the prices received by their neighbors, and enjoy positive utility if they receive a lower price than the prices charged to other consumers in their neighborhood. We assume that utilities are defined over the average price charged to a consumer's neighbor:

$$U_i = \theta_i - p_i + \alpha \frac{1}{d_i} \sum_j g_{ij} p_j. \quad (10)$$

where θ_i is a taste parameter uniformly distributed on $[0, 1]$. Prices are assumed to be bounded, $0 \leq p_i \leq \bar{p}$ for all i . As in the case of local network externalities, prices are announced before consumers learn their random valuation, and a consumer located at node i buys the good if and only if

$$\theta_i \geq p_i - \alpha \frac{1}{d_i} \sum_j g_{ij} p_j. \quad (11)$$

Notice that in the model of aspiration based reference price, a consumer's decision is independent of the consumption choices of other consumers, so agents do not need to learn the valuations of their neighbors. Furthermore, as opposed to the case of consumption externalities, we do not need to invert the demands to obtain the system of demands as a function of the price vector. Instead, the demand at node i is directly given by:

$$x_i = \begin{cases} 0 & \text{if } 1 - p_i + \frac{\alpha}{d_i} \sum_j g_{ij} p_j < 0, \\ 1 & \text{if } 1 - p_i + \frac{\alpha}{d_i} \sum_j g_{ij} p_j > 1, \\ 1 - p_i + \frac{\alpha}{d_i} \sum_j g_{ij} p_j & \text{otherwise} \end{cases}$$

We denote by \mathbf{H} the row-stochastic matrix with typical element $h_{ij} = \frac{g_{ij}}{d_i}$.

4.2 Monopoly pricing

We compute the optimal prices \mathbf{p} charged by a monopolist. Notice that in the price externalities model, the monopolist does not necessarily have an incentive to serve all nodes. It could instead choose to set very high prices at some nodes, resulting in no sale at these nodes but increasing the profit made at neighboring nodes. We do not believe that this phenomenon is likely to happen and assume instead that the monopolist always has an incentive to sell the good and that the following sufficient condition holds:

$$\frac{(1-c)^2}{2} \geq \alpha \sum_j g_{ij} \frac{1}{\deg_j} \bar{p}$$

To understand this formula, notice that the profit at any node is at least equal to the profit a monopolist would make if that node were isolated, $\frac{(1-c)^2}{2}$. On the other hand, the increase in profit that a price \bar{p} can generate is bounded above by the increase in profit on neighboring nodes when the quantity sold is 1, $\alpha \sum_j g_{ij} \frac{1}{\deg_j} \bar{p}$. If externalities are low enough so that the sufficient condition holds, any firm will have an incentive to serve all nodes, and demand will be given by

$$\mathbf{x} = (\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H})\mathbf{p}).$$

The maximization problem of the monopoly is given by

$$\max_{\mathbf{p}} (\mathbf{p} - c\mathbf{1})^T (\mathbf{I} - (\mathbf{I} - \alpha\mathbf{H})\mathbf{p}),$$

resulting in a gradient:

$$\nabla \Pi = -(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T))\mathbf{p} + \mathbf{1} + c(\mathbf{I} - \alpha\mathbf{H}^T)\mathbf{1}.$$

and a Hessian:

$$\nabla^2 \Pi = -(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T)).$$

As long as $\alpha\rho(\mathbf{H} + \mathbf{H}^T) < 2$, the matrix $(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T))$ is positive definite, so that the optimal price is given by the first order condition

$$\mathbf{p}^* = (2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T))^{-1}\mathbf{1} + c(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T))^{-1}(\mathbf{I} - \alpha\mathbf{H}^T)\mathbf{1}.$$

Summarizing,

Proposition 4.1 *Suppose that $\alpha\rho(\mathbf{H} + \mathbf{H}^T) < 2$. The optimal pricing strategy of the monopoly is to charge prices*

$$\mathbf{p}^* = (2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T))^{-1}\mathbf{1} + c(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T))^{-1}(\mathbf{I} - \alpha\mathbf{H}^T)\mathbf{1}.$$

Given this pricing strategy, the equilibrium demand of consumers at each node is given by

$$\mathbf{x}^* = \mathbf{1} - (\mathbf{I} - \alpha\mathbf{H})\mathbf{p}^*.$$

Proposition 4.1 characterizes the equilibrium prices as a function of the Katz-Bonacich centrality with respect to the matrix $\frac{\mathbf{H} + \mathbf{H}^T}{2}$. When $c = 0$, prices are proportional to this Katz-Bonacich centrality measure. Using the power series formula for $(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{H}^T))^{-1}$ and taking α small, we compute the approximation:

$$p_i = \frac{1+c}{2} + \frac{1+c}{4}\alpha\left(1 + \sum_j g_{ij} \frac{1}{\text{deg}_j}\right) + \mathcal{O}(\alpha^2)$$

showing that prices are higher for nodes which have a large number of neighbors with small degrees. In particular, when the social network is a star, the monopolist has an incentive to charge a very high price to the hub of the star in order to influence the valuations of the peripheral agents. This ranking of prices in the star is very intuitive: by raising the price in the hub, the monopoly is able to increase demand at *all* peripheral nodes, whereas an increase in the price of the peripheral node only increases demand at the hub. Hence the indirect positive effect of a price increase is higher for the hub than for a peripheral agent, implying that the optimal price will be higher at the hub.

4.3 Oligopoly pricing

We now suppose that K firms compete in the network, and that prices result from a Nash equilibrium of the pricing game played by oligopolists in the network. The maximization problem of firm k is given by:

$$\max_{\mathbf{p}_{N(k)}} \Pi_k = ((\mathbf{p} - c\mathbf{1})^{N(k)})^T (\mathbf{1} - (\mathbf{I} - \alpha\mathbf{H})\mathbf{p}).$$

Computing the gradient of π_k with respect to $\mathbf{p}_{N(k)}$,

$$\nabla \Pi_k = (\mathbf{1} - (\mathbf{I} - \alpha\mathbf{H})\mathbf{p})_{N(k)} - ((\mathbf{I} - \alpha\mathbf{D} \circ \mathbf{H}^T)(\mathbf{p} - c\mathbf{1}))_{N(k)},$$

And the Hessian is given by:

$$\nabla^2 \Pi_k = -(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{D} \circ \mathbf{H}^T)).$$

If $\alpha\rho(\mathbf{H} + \mathbf{D} \circ \mathbf{H}^T) < 2$, the Hessian is negative definite, and we can characterize the Nash equilibrium of the pricing game by the system of equations:

$$\mathbf{p}^* = (2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{D} \circ \mathbf{H}^T))^{-1} \mathbf{1} + c(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{D} \circ \mathbf{H}^T))^{-1} (\mathbf{I} - \alpha\mathbf{D} \circ \mathbf{H}^T) \mathbf{1}.$$

Summarizing the analysis,

Proposition 4.2 *Let $\alpha\rho(\mathbf{H} + \mathbf{D} \circ \mathbf{H}^T) < \frac{1}{2}$. In the model of aspiration based reference price, there exists a unique equilibrium of the oligopoly pricing game characterized by*

$$\mathbf{p}^* = (2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{D} \circ \mathbf{H}^T))^{-1} \mathbf{1} + c(2\mathbf{I} - \alpha(\mathbf{H} + \mathbf{D} \circ \mathbf{H}^T))^{-1} (\mathbf{I} - \alpha\mathbf{D} \circ \mathbf{H}^T) \mathbf{1}.$$

Proposition 4.2 characterizes the equilibrium prices for all market structures and admits as special cases the single monopolist (when \mathbf{D} is the unit matrix of the Hadamard product) and the situation where every node is served by a different firm (when $\mathbf{D} = \mathbf{I}$). In the latter case, $\mathbf{D} \circ \mathbf{H}^T = \mathbf{0}$ and, as \mathbf{H} is row-stochastic, $\mathbf{H}^k \mathbf{1} = \mathbf{1}$ for all k , so that

$$\mathbf{p}^* = \frac{1+c}{2-\alpha} \mathbf{1},$$

uniformly across nodes. Hence, even though all firms charge the same price, social comparisons enable firms to extract a surplus above the monopoly surplus, as the equilibrium price \mathbf{p}^* is above the monopoly price $\frac{1+c}{2}$. As opposed to Proposition 3.2, this network irrelevance result is robust to changes in the model. It is easy to check that, when all nodes are served by different firms, given any demand function $\mathbf{x}(\mathbf{p})$, there exists an equilibrium where all firms charge the uniform price

$$p^* = \operatorname{argmax}_p (p - c)x(p(1 - \alpha)).$$

Hence, when every node is served by a different local monopolist, there exists generally an equilibrium where all firms charge the same price so that consumers do not experience any price externality.

5 Extensions

In this Section, we discuss two extensions of the model of monopoly pricing with consumption externalities: optimal pricing with general distributions (sub-section 5.1) and bargaining between the monopolist and consumers on the division of the surplus (sub-section 5.2).

5.1 General Distribution Functions

In the analysis so far, we have restricted attention to linear demands generated by a uniform distribution of valuations. This restriction is motivated by tractability considerations. With linear demands, optimal prices are characterized as the solution to a system of linear equations, allowing for a study of the relation between prices and node centrality in arbitrary network topologies. Furthermore, with uniform distributions, the marginal effect of a change in prices on demand is independent of the price level, eliminating the complexity which would result from the curvature of the demand function. However, we are aware of the fact that the assumption of uniform distribution is restrictive, and we discuss in this extension partial results on monopoly pricing obtained under general distribution functions.

We consider a general distribution of valuations F over a compact interval $[\underline{\theta}, \bar{\theta}] \subset \mathfrak{R}_+$ with continuous and bounded density f . We assume that F satisfies the familiar *monotone hazard rate condition*: $\frac{f(\theta)}{1-F(\theta)}$ is monotonically increasing in θ . As a first step, we analyze conditions under which a unique demand vector can be computed for each \mathbf{p} . Consider the general mixed complementarity problem

$$\begin{aligned} x_i &= \underline{\theta}, & \Psi_i(\theta) &\leq p_i, \\ 0 < x_i &< 1, & \Psi_i(\theta) &= p_i, \\ x_i &= \bar{\theta}, & \Psi_i(\theta) &\geq p_i. \end{aligned}$$

where $\Psi_i(\theta) \equiv \theta_i + \alpha \sum_j g_{ij}(1 - F(\theta_j))$. Following Simsek, Ozdaglar and Acemoglu (2005), a sufficient condition for the existence and uniqueness of a solution to the mixed complementarity problem is that the Jacobian matrix J_Ψ be a P-matrix. A simple computation shows that

$$J_\Psi = \mathbf{I} - \alpha \mathbf{E} \circ \mathbf{G},$$

where \mathbf{E} is a square matrix with entry $e_{ij} = f(\theta_j)$. Hence, a sufficient condition for the invertibility of the demand system is that $\alpha \rho(\mathbf{E} \circ \mathbf{G}) < 1$ for all θ . Using the same arguments as in the proof of Proposition 3.2, we can show that the monopolist chooses prices which generate interior demands so that we restrict attention to $\Psi_i(\theta) = p_i$ for all θ , and write the optimization problem of the monopolist as:

$$\max_{\theta} \Pi(\theta) = [1 - F(\theta)]^T (\Psi(\theta) - c\mathbf{1}).$$

Define the mapping $\Phi : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ by:

$$\Phi_i(\theta) \equiv (1 - F(\theta_i)) - f(\theta_i)[\theta_i + 2\alpha \sum g_{ij}(1 - F(\theta_j)) - c].$$

Then

$$\nabla \Pi = \Phi,$$

and the second order conditions are satisfied if the Jacobian matrix J_Φ is negative semi-definite.⁷ The first order conditions $\Phi(\theta) = 0$ enable us to study the robustness of Proposition 3.2 with respect to changes in the distribution. Define the auxiliary function:

$$G(\theta) \equiv \frac{1 - F(\theta) + f(\theta)\theta}{f(\theta)} = \frac{1 - F(\theta)}{f(\theta)} + \theta. \quad (12)$$

When the distribution $F(\theta)$ is uniform, the function $G(\theta)$ is identically equal to zero. In general, the discussion of the optimal pricing rules differs when $G(\theta)$ is increasing and decreasing. We provide a partial comparison of optimal prices, considering two nodes i and j such that the neighborhood of i is contained in the neighborhood of j .

Proposition 5.1 *Consider two nodes i and j such that $g_{ik} = 1 \Rightarrow g_{jk} = 1$. Then, for any distribution, $\theta_i \geq \theta_j$. If $G(\theta)$ is decreasing, $p_j \geq p_i$; if $G(\theta)$ is increasing, $p_i \geq p_j$.*

We illustrate Proposition 5.1 by considering a parameterized family of distributions covering both increasing and decreasing functions $G(\theta)$.

Example 5.2 *Suppose that $n = 3$ and $g_{12} = g_{23} = 1$, $g_{13} = 0$. Let $F(\theta) = \theta^2(\frac{3\beta}{2} - \frac{1}{2}) + \frac{3}{2}(1 - \beta)$ for $\beta, \theta \in [0, 1]$*

Notice that, for $\beta < \frac{1}{3}$, the function $G(\theta)$ is increasing, for $\beta > \frac{1}{3}$, the function $G(\theta)$ is decreasing, and for $\beta = \frac{1}{3}$, the distribution F is uniform. Let p be the optimal price charged at the peripheral nodes 1 and 3 and q the price charged at the central node 2. The following table lists the optimal prices for different values of β with $\alpha = 0.1$:

⁷There is no simple sufficient condition on the distribution function $F(\cdot)$ to guarantee that the matrix J_Φ is negative semi-definite. However, the condition is verified when $F(\cdot)$ is a uniform distribution and the matrix of external effects satisfies $\alpha\rho(\mathbf{G}) < 1$ and, by continuity, the condition is also satisfied for any distribution which is close to the uniform distribution.

β	0	0.2	0.4	0.6	0.8	1.0
p	0.408	0.457	0.517	0.555	0.580	0.598
q	0.397	0.448	0.522	0.572	0.608	0.636

Table 2: Optimal prices for different distributions

Table 2 shows that, in accordance with proposition 5.1, the ranking of prices at the central and peripheral node varies with β : when $\beta < \frac{1}{3}$ and $G(\theta)$ is increasing, the price charged at the peripheral nodes is higher than at the central nodes. When $\beta > \frac{1}{3}$ and $G(\theta)$ is decreasing, the price charged at the central node is higher than at the peripheral nodes.

5.2 Bargaining

In the previous sections, we assumed that the monopoly supplier had all the bargaining power. We now allow consumers at each node to exercise bargaining power, and briefly discuss the outcome of a bargaining process between the monopoly and the suppliers. In order to avoid complications due to asymmetric information, we assume that bargaining takes place *before* consumers learn their valuations. At that point, at every node, the consumer and the monopoly bargain over the expected surplus, and agree on an unconditional transfer of the good to the customer. The main difficulty in the bargaining process arises from the externalities created by trade at neighboring nodes.

5.2.1 Nash bargaining

We first discuss a simple Nash bargaining problem. Suppose that the monopolist first selects a set of nodes M , for which bargaining surplus is positive, and then bargains simultaneously with every consumer at every node in M . If bargaining fails at some node i , the surplus is reduced at every node j in M such that $g_{ij} = 1$. Assuming equal bargaining power between the monopoly and the consumer, we thus compute disagreement points at node i as:

$$d_m = \sum_{j|j \in M, j_n \in q_i} u_{mj} - \alpha \frac{\sum_{j|j \in M} g_{ij}}{2} \quad \begin{array}{l} d_i = 0 \quad \text{for consumer } i, \\ \text{for the monopoly.} \end{array}$$

where u_{mj} denotes the utility of the monopolist in the transaction with consumer j , and the surplus to be divided between the two agents is⁸:

$$\begin{aligned} S_i &= E(\theta) - c + \alpha \sum_{j \in M} g_{ij}, \\ &= \frac{1}{2} - c + \alpha \sum_{j \in M} g_{ij}. \end{aligned}$$

Proposition 5.3 *In the Nash bargaining problem with equal bargaining power, disagreement points (d_i, d_m) and surplus S_i , the utility of the consumer at node i is given by:*

$$u_i = \frac{1}{4} - \frac{c}{2} + \frac{3}{4}\alpha \sum_{j \in M} g_{ij}$$

and the profit of the monopolist as

$$\Pi = \sum_{i \in M} \frac{1}{4} - \frac{c}{2} + \frac{3}{4}\alpha \sum_{j \in M} g_{ij}.$$

Proposition 5.3 characterizes the bargaining shares obtained by the monopolist and consumers at every node. As $u_i = E(\theta) + \alpha \sum_{j \in M} g_{ij} - p_i$, the implicit price paid by a consumer i is given by

$$p_i = \frac{1}{4} + \frac{c}{2} + \frac{1}{4}\alpha \sum_{j \in M} g_{ij}.$$

Hence, consumers with higher degree enjoy a larger surplus, and face a higher implicit price. The intuition for this result is easy: consumers with a more central position generate a higher surplus, which is shared among consumers and the monopolist. The share accruing to the monopolist is directly related to the price.

5.2.2 Alternating offer bargaining

Finally, we consider the outcome of a non-cooperative bargaining process based on Rubinstein (1982)'s alternating-offers game. Suppose that the monopolist simultaneously bargains with all consumers according to the following procedure. At time $t = 1$, the monopolist makes offers (x_1^1, \dots, x_n^1) to all

⁸Notice that this surplus is not necessarily positive. Hence, the monopolist first chooses a set M of nodes for which this surplus is positive

consumers. The consumers simultaneously decide whether to accept or reject. If a subset S of consumers reject, one time period elapses, and consumers in S simultaneously and independently make offers to the monopolist, y_j^2 for all $i \in S$, etc. Consumers observe the outcome of the bargaining but not the offers made to other consumers. All players discount the future at a constant rate δ . A *state* s in the game at period t is a list of the consumers who are still active at period t , and of the current offers. If it is the consumer's time to respond, (t is odd), they only know their individual offer x_i^t . At even periods, the monopolist observes all offers y_j^t . A *stationary perfect equilibrium* is a list of stationary strategies for the monopolist and the consumers such that, at any period t , all agents act optimally.

Proposition 5.4 *Let M be the set of consumers who have an expected positive surplus. The non-cooperative bargaining procedure admits a stationary subgame perfect equilibrium where all consumers in M accept the monopoly's offer in the first period. In this equilibrium, at odd periods t , the monopolist makes offers x_i^t to every consumer i in M , and at even periods t , consumer i in M makes an offer y_i^t to the monopolist, where*

$$\begin{aligned} x_i^t &= \delta[\delta^t(\frac{1}{2} - c + \alpha \sum_{j \in M} g_{ij}) - y_i^t], \\ y_i^t &= \delta[\delta^t(\frac{1}{2} - c + \alpha \sum_{j \in M} g_{ij}) - x_i^t], \end{aligned}$$

Proposition 5.4 shows that, in the Rubinstein bargaining game, there exists a stationary perfect equilibrium where all consumers anticipate that other consumers will accept the offer, and consumer i obtains a share of the surplus given by:

$$u_i = \frac{\delta}{1 + \delta}(\frac{1}{2} - c + \alpha \sum_{j \in M} g_{ij}).$$

As δ goes to one, the expected utility of a consumer converges to

$$u_i = \frac{1}{4} - \frac{c}{2} + \frac{1}{2}\alpha \sum_{j \in M} g_{ij}$$

and, as $u_i = \frac{1}{2} + \alpha \sum_{j \in M} g_{ij} - p_i$, the implicit price paid by a consumer i is given by

$$p_i = \frac{1}{4} + \frac{c}{2} + \frac{1}{2}\alpha \sum_{j \in M} g_{ij}.$$

so that consumers who are more central are again charged a higher price. This result simply stems from the fact that the surplus generated by consumers who are more central is higher, resulting in larger bargaining shares for the monopolist and the consumer.

6 Conclusions

This paper contributes to an emerging literature which tries to understand how a monopolist optimally discriminates in a social network according to consumer's centrality. As opposed to some recent contributions in computer science, which focus on sequential consumption decisions among myopic consumers, we consider simultaneous consumption choices among perfectly rational agents. We show that in a model of local network externalities where consumers are positively affected by the consumption of their neighbors, a single monopolist does not discriminate across the network. However, in variants of the model with quadratic costs and directed influence, the network irrelevance result disappears, and optimal prices depend on the centrality of consumers in the social network. With quadratic costs, more central agents are charged a higher price. In directed networks, agents with more influence are charged a lower price. When the network is served by different firms, equilibrium prices also differ at different nodes. When consumers compare the price they receive with the average price in their social neighborhood, a single monopolist has an incentive to charge a higher price to a node which has many neighbors of small degree, like the hub of a star. Local monopolies do not internalize the price externalities and in equilibrium charge a uniform price across the network.

The paper thus shows that an unregulated monopolist chooses to price discriminate by trading off "influence" and "exploitation". According to the specific model, this trade-off either leads the monopolist to charge lower prices at more central nodes in order to maximize influence over neighboring nodes, or to charge higher prices at more central nodes in order to exploit the higher valuation of more central consumers. Our study of bargaining also suggests that, because more central consumers enjoy a larger valuation, they should face higher prices in equilibrium. It is difficult to draw precise recommendations for public intervention from the analysis. One immediate conclusion from our model is that price discrimination in a model of local network externalities is desirable, as it allows to increase the total surplus. The exact shape of the socially optimal price discrimination strategy is not easy to characterize and remains an open question for future research.

Finally, we would like to mention two other open problems that deserve further study. First, the study of endogenous formation of the social network requires a detailed analysis of the marginal value of additional links that we would like to undertake in future research. Second, as in any model of price discrimination, consumers located at different nodes in the social network end up paying different prices for the good, and could resell the good to one another. The study of models of resale along social networks is obviously an important area for future research.

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8 Proofs

Proof of Proposition 3.1: Simsek, Ozdaglar and Acemoglu (2005) show that the linear instance of a mixed complementarity problem admits a unique solution if the matrix $(\mathbf{I} - \alpha\mathbf{G})$ is a P-matrix (Simsek, Ozdaglar and Acemoglu (2005), Theorem 4).

By the definition in Berman and Plemmons (1984) (Chapter 6), if $\alpha\rho(\mathbf{G}) < 1$, $(\mathbf{I} - \alpha\mathbf{G})$ is a nonsingular M-matrix. Any non-singular M-matrix is a P-matrix (Berman and Plemmons (1984), Theorem 6.2.3), so that the sufficient condition of Simsek, Ozdaglar and Acemoglu (2005) is satisfied by $\mathbf{I} - \alpha\mathbf{G}$.

Once existence and uniqueness are obtained, the characterization of equilibrium is easy. We partition the set of agents into agents for whom $x_i = 0$, for whom $x_i = 1$ and for whom $0 < x_i < 1$. To characterize the interior values x_i we focus attention on the agents in the index set S and compute the S -dimensional square matrix $(\mathbf{I} - \alpha\mathbf{G})_S$ which is the restriction of $(\mathbf{I} - \alpha\mathbf{G})$ to the rows and columns in S . We also take into account the fact that all agents in S_1 have a demand of one, by writing the linear complementarity problem of agents in S as:

$$(\mathbf{I} - \alpha\mathbf{G})_S \mathbf{x}_S = \mathbf{u}_S,$$

where

$$\mathbf{u} = \mathbf{1} - p + \alpha\mathbf{G}\mathbf{1}^{S_1}.$$

The solution follows from standard arguments, as in Ballester and Calvó-Armengol (2010), Theorem 2, p. 401.

Proof of Proposition 3.2:

Step 1: At the optimal price \mathbf{p}^* , $\mathbf{F}(\mathbf{x}^*) \equiv (\mathbf{1} - \mathbf{p}^*) - (\mathbf{I} - \alpha\mathbf{G})\mathbf{x}^* = 0$. Suppose first that there exists a node i such that $\mathbf{F}_i(\mathbf{x}^*) > 0$ and $x_i^* = 1$. As \mathbf{F}_i is continuous in p_i , there exists $\epsilon > 0$ such that $\mathbf{F}_i(\mathbf{x}^*, p_i^* + \epsilon) > 0$, so that (\mathbf{x}^*) remains a solution to the bounded linear complementarity problem under the new price vector $\bar{p} = (p_i^* + \epsilon, p_j^*, j \neq i)$. The profit of the monopolist increases from $\Pi = (\mathbf{p}^* - c\mathbf{1})\mathbf{x}^*$ to $\Pi' = \sum_{j \neq i} (p_j^* - c)x_j^* + (p_i^* + \epsilon - c) = \Pi + \epsilon > \Pi$.

Next, suppose that there exists a node i such that $\mathbf{F}_i(\mathbf{x}^*) < 0$ and $x_i^* = 0$. We construct a new vector of prices \bar{p} and show that the profit of the monopolist is strictly higher under that new price vector. Consider first agent i . Because \mathbf{F}_i is a continuous strictly decreasing function of p_i , and, at $p_i = c$, $\mathbf{F}_i = (1 - c) + \alpha \sum_j g_{ij}x_j^* > 0$, there exists $\bar{p}_i > c$ such that $1 > \bar{x}_i = 1 - \bar{p}_i + \alpha \sum_j g_{ij}x_j^* > 0$. Pick this price \bar{p}_i for agent i , and we will construct prices \bar{p}_j for all other agents $j \neq i$, in such a way that x_j^* remains a solution to the bounded linear complementarity problem under the new prices \bar{p} .

Consider first a node $j \in S_0$. Pick a price $\bar{p}_j \gg 1$ so that $1 - \bar{p}_j + \alpha(\sum_{k \neq i, j} g_{jk}x_k^* + g_{ki}\bar{x}_i) < 0$.

Second, consider a node $j \in S_1$, keep the price $\bar{p}_j = p_j^*$. Then,

$$\begin{aligned} 1 - \bar{p}_j + \alpha\left(\sum_{k \neq i, j} g_{jk}x_k^* + g_{ki}\bar{x}_i\right) &\geq 1 - \bar{p}_j + \alpha \sum_{k \neq i, j} g_{jk}x_k^*, \\ &> 0. \end{aligned}$$

Finally, consider a node $j \in S$, define:

$$\bar{p}_j \equiv p_j^* + \alpha g_{ji} \bar{x}_i \geq p_j^*.$$

By construction,

$$\begin{aligned} 1 - \bar{p}_j + \alpha \left(\sum_{k \neq i, j} g_{jk} x_k^* + g_{ki} \bar{x}_i \right) &= 1 - p_j^* + \alpha \sum_{k \neq i, j} g_{jk} x_k^*, \\ &= 0. \end{aligned}$$

Hence, under the new price vector \bar{p} , (\bar{x}_i, x_j^*) is a solution to the mixed complementarity problem. As the problem admits a unique solution, this is in fact the unique solution to the problem. The profit of the monopolist under the new price vector \bar{p} is:

$$\begin{aligned} \bar{\Pi} &= (\bar{p}_i - c) \bar{x}_i + \sum_{j \in S} (\bar{p}_j - c) x_j^* + \sum_{j \in S_1} (\bar{p}_j - c), \\ &> \sum_{j \in S} (p_j^* - c) x_j^* + \sum_{j \in S_1} (p_j^* - c), \\ &= \Pi. \end{aligned}$$

Step 2: The optimal price chosen by the monopolist is given by: $\mathbf{p}^* = \frac{1+c}{2} \mathbf{1}$. The proof of Step 2 can be found in the text.

Proof of Proposition 3.3: As a first step, we show that prices are chosen so that demand are always interior. The only difference with the proof of Proposition 3.2 stems from the case where $\mathbf{F}_i(\mathbf{x}^*) < 0$ and $x_i^* = 0$. Because \mathbf{F}_i is a continuous strictly decreasing function of p_i , and, at $p_i = 0$, $\mathbf{F}_i = 1 + \alpha \sum_j g_{ij} x_j^* > 0$, there exists $\bar{p}_i > 0, \bar{x}_i > 0$ such that $1 > \bar{x}_i = 1 - \bar{p}_i + \alpha \sum_j g_{ij} x_j^* > 0$. Furthermore, for all $\epsilon > 0$, $\bar{x}_i < \epsilon \Rightarrow \bar{p}_i > 1 - \epsilon$. Hence, we can pick a price \bar{p}_i and a quantity \bar{x}_i so that $\bar{p}_i \bar{x}_i > c \bar{x}_i^2$. Next define, as in the proof of Proposition 3.2, prices at other nodes, $\bar{p}_j, j \neq i$ so that the quantities demanded remain constant, $\bar{x}_j = x_j^*$. The rest of the argument follows.

The computations leading to the formula

$$\mathbf{p}^* = \mathbf{1} - \frac{1}{2} (\mathbf{I} - \alpha \mathbf{G}) ((1+c)\mathbf{I} - \alpha \mathbf{G})^{-1} \mathbf{1}.$$

are given in the text. For the approximation formula, expand the power series

$$((1+c)\mathbf{I} - \alpha\mathbf{G})^{-1} = \frac{1}{1+c} \sum_{k=0}^{\infty} \frac{\alpha^k}{(1+c)^k} \mathbf{G}^k,$$

so that

$$\frac{1}{2}(\mathbf{I} - \alpha\mathbf{G})((1+c)\mathbf{I} - \alpha\mathbf{G})^{-1} = \frac{1}{2(1+c)} \left(\mathbf{I} - \sum_{k=1}^{\infty} \frac{c\alpha^k}{(1+c)^k} \mathbf{G}^k \right).$$

yielding the desired formula.

Proof of Proposition 3.4: The proof is identical to the proof of Proposition 3.2 with the exception of the following step, which yields the power series formula:

$$\begin{aligned} (\mathbf{I} - \alpha\mathbf{G})(2\mathbf{I} - \alpha(\mathbf{G} + \mathbf{G}^T))^{-1} &= \frac{1}{2}(\mathbf{I} - \alpha\mathbf{G}) \sum_{k=0}^{\infty} \frac{\alpha^k}{2^k} (\mathbf{G} + \mathbf{G}^T)^k, \\ &= \frac{1}{2}\mathbf{I} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\alpha^k}{2^k} ((\mathbf{G} + \mathbf{G}^T)^k - \mathbf{G}(\mathbf{G} + \mathbf{G}^T)^{k-1}) \\ &= \frac{1}{2}\mathbf{I} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\alpha^k}{2^k} ((\mathbf{G}^T - \mathbf{G})(\mathbf{G} + \mathbf{G}^T)^{k-1}) \end{aligned}$$

yielding the desired formula.

Proof of Lemma 3.5 By Proposition 3.1, we know that the system of equations (3) admits a unique solution under both prices \mathbf{p} and \mathbf{p}' . Now partition the set of agents into $S = \{i | \mathbf{x}_i^* < \mathbf{x}_i^{*'}\}$ and $T = \{i | \mathbf{x}_i^* \geq \mathbf{x}_i^{*'}\}$. For any $i \in S$, by definition of the bounded linear complementarity problem, we must have:

$$(1 - p'_i) - x_i^{*'} + \alpha \sum_{j \neq i} g_{ij} x_j^{*'} \geq (1 - p_i) - x_i^* + \alpha \sum_{j \neq i} g_{ij} x_j^*.$$

or

$$(p_i - p'_i) - (x_i^{*'} - x_i^*) + \alpha \sum_{j \neq i} g_{ij} (x_j^{*'} - x_j^*) \geq 0.$$

Now, for any $j \in T$, $(x_j^{*'} - x_j^*) \leq 0$, so that

$$\begin{aligned}
(p_i - p'_i) - (x_i^{*'} - x_i^*) + \alpha \sum_{j \neq i, j \in S} g_{ij}(x_j^{*'} - x_j^*) &\geq (p_i - p'_i) - (x_i^{*'} - x_i^*) + \alpha \sum_{j \neq i} g_{ij}(x_j^{*'} - x_j^*), \\
&\geq 0.
\end{aligned}$$

Summing up over all agents in S , we obtain:

$$(\mathbf{p} - \mathbf{p}')_S^T \mathbf{1}_S + ((\mathbf{I} - \alpha \mathbf{G})_S (\mathbf{x}^* - \mathbf{x}^{*'}))_S^T \mathbf{1}_S \geq 0, \quad (13)$$

As $(\mathbf{I} - \alpha \mathbf{G})$ satisfies strict row diagonal dominance, so do all principal submatrices of $(\mathbf{I} - \alpha \mathbf{G})$ and in particular the $S \times S$ submatrix of $(\mathbf{I} - \alpha \mathbf{G})$ restricted to the set S . Hence, for all positive vector \mathbf{x}_S , $(\mathbf{I} - \alpha \mathbf{G})_S \mathbf{x}_S \geq 0$ and $(\mathbf{I} - \alpha \mathbf{G})_S \mathbf{x}_S \neq 0$. Hence,

$$((\mathbf{I} - \alpha \mathbf{G})_S (\mathbf{x}^* - \mathbf{x}^{*'}))_S^T \mathbf{1}_S < 0.$$

Furthermore, by assumption, $(\mathbf{p} - \mathbf{p}')_S \mathbf{1}_S \leq 0$, so that

$$(\mathbf{p} - \mathbf{p}')_S^T \mathbf{1}_S + ((\mathbf{I} - \alpha \mathbf{G})_S (\mathbf{x}^* - \mathbf{x}^{*'}))_S^T \mathbf{1}_S < 0,$$

contradicting equation (13).

Proof of Proposition 3.6: Fix the prices p_i for $i \in N \setminus N(k)$. We first show that firm k optimally chooses prices such that $\mathbf{F}_i = 0$ for all $i \in N(k)$. If $\mathbf{F}_i > 0$ and $x_i^* = 1$, as in the proof of Proposition 3.2, firm k can increase its profit by raising the price of node i to $p_i^* + \epsilon$ without changing the demands.

Next, suppose that there exists a node i such that $\mathbf{F}_i(\mathbf{x}^*) > 0$. Then, $x_i^* = 0$. Because \mathbf{F}_i is a continuous strictly decreasing function of p_i , and, at $p_i = c$, $\mathbf{F}_i = (1 - c) + \alpha \sum_j g_{ij} x_j^* > 0$, there exists $\bar{p}_i > c$ such that $1 > \bar{x}_i = 1 - \bar{p}_i + \alpha \sum_j g_{ij} x_j^* > 0$. Pick this price \bar{p}_i for node i , and let all other prices remain fixed. By Lemma 3.5, when p_i decreases, all solutions x_1^*, \dots, x_n^* of the bounded linear complementarity problem weakly increase to $\bar{x}_1^*, \dots, \bar{x}_n^*$.

The profit of firm k at this new price vector is given by

$$\begin{aligned}
\bar{\Pi}_k &= (\bar{p}_i - c)\bar{x}_i^* + \sum_{j \in N(k), j \neq i} p_j \bar{x}_j^*, \\
&> \sum_{j \in N(k), j \neq i} p_j \bar{x}_j^*, \\
&\geq \sum_{j \in N(k), j \neq i} p_j x_j^*, \\
&= \Pi_k,
\end{aligned}$$

so that firm k has an incentive to deviate.

Next, define $\mathbf{A} \equiv (\mathbf{I} - \alpha \mathbf{G})^{-1}$. We first check that the Hessian $\nabla^2 \Pi_k = -2(\mathbf{A} + \mathbf{D} \circ \mathbf{A})_{N(k)}$ is negative definite. Recall that if $(\mathbf{I} - \alpha \mathbf{G})$ satisfies strict row diagonal dominance, it is a symmetric P-matrix and hence is positive definite. As the inverse of a positive definite matrix is also positive definite, \mathbf{A} is positive definite. To show that the matrix \mathbf{D} is positive semi-definite, compute the leading principal minors $|D_k|$. Either there exist two columns in D_k corresponding to two nodes controlled by the same firm, and then the column vectors are identical and the determinant $|D_k|$ is zero, or all nodes are controlled by different firms in which case D_k is a diagonal matrix, and the determinant is $|D_k| = 1$. By the Schur product theorem, $\mathbf{D} \circ \mathbf{A}$ is positive semi-definite, and $(\mathbf{A} + \mathbf{D} \circ \mathbf{A})$ is positive definite, as the sum of one positive definite and one positive semi-definite matrix. The principal submatrix of a symmetric positive definite matrix is also positive definite, so that $(\mathbf{A} + \mathbf{D} \circ \mathbf{A})_{N(k)}$ is positive definite, establishing that the Hessian is negative definite.

We now compute the approximation formula. Recall that equilibrium prices are characterized by the first order condition

$$[(\mathbf{A}\mathbf{1} + c(\mathbf{D} \circ \mathbf{A})\mathbf{1}) - (\mathbf{A} + \mathbf{D} \circ \mathbf{A})\mathbf{p}] = 0.$$

To compute an approximation formula for this system of equations, we go through the following algebraic steps:

$$\begin{aligned}
(\mathbf{A} + \mathbf{D} \circ \mathbf{A})\mathbf{p} &= \mathbf{A}\mathbf{1} + c(\mathbf{D} \circ \mathbf{A})\mathbf{1}, \\
(\mathbf{A} + \mathbf{D} \circ \mathbf{A})(\mathbf{p} - c\mathbf{1}) &= (1 - c)\mathbf{A}\mathbf{1}, \\
(\mathbf{I} + \mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A}))(\mathbf{p} - c\mathbf{1}) &= (1 - c)\mathbf{1}, \\
\frac{1}{2}(\mathbf{I} + \mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A}))(\mathbf{p} - c\mathbf{1}) &= \frac{1}{2}(1 - c)\mathbf{1}, \\
(\mathbf{I} - \frac{1}{2}(\mathbf{I} - \mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A}))) (\mathbf{p} - c\mathbf{1}) &= \frac{1}{2}(1 - c)\mathbf{1}.
\end{aligned}$$

Assume that α is small enough so that

$$\frac{1}{2}\rho(\mathbf{I} - (\mathbf{I} - \alpha\mathbf{G})(\mathbf{D} \circ (\mathbf{I} - \alpha\mathbf{G})^{-1})) < 1.$$

Then we can invert $(\mathbf{I} - \frac{1}{2}(\mathbf{I} - \mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A})))$ to obtain:

$$(\mathbf{p} - c\mathbf{1}) = \frac{1}{2}(1 - c)(\mathbf{I} - \frac{1}{2}(\mathbf{I} - \mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A})))^{-1}\mathbf{1}.$$

Next recall that

$$\mathbf{A} = (\mathbf{I} - \alpha\mathbf{G})^{-1} = \sum_{k=0}^{\infty} \alpha^k \mathbf{G}^k.$$

Hence,

$$\begin{aligned}
\mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A}) &= (\mathbf{I} - \alpha\mathbf{G}) \sum_{k=0}^{\infty} \alpha^k (\mathbf{D} \circ \mathbf{G}^k), \\
&= \sum_{k=0}^{\infty} \alpha^k (\mathbf{D} \circ \mathbf{G}^k) - \sum_{k=0}^{\infty} \alpha^{k+1} \mathbf{G} (\mathbf{D} \circ \mathbf{G}^k).
\end{aligned}$$

and

$$\frac{1}{2}(\mathbf{I} - \mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A})) = \frac{1}{2} \sum_{k=1}^{\infty} \alpha^k (\mathbf{G}(\mathbf{D} \circ \mathbf{G}^{k-1}) - (\mathbf{D} \circ \mathbf{G}^k)).$$

so that

$$[\mathbf{I} - \frac{1}{2}(\mathbf{I} - \mathbf{A}^{-1}(\mathbf{D} \circ \mathbf{A}))]^{-1} = \sum_{l=0}^{\infty} [\frac{1}{2} \sum_{k=1}^{\infty} \alpha^k (\mathbf{G}(\mathbf{D} \circ \mathbf{G}^{k-1}) - (\mathbf{D} \circ \mathbf{G}^k))]^l.$$

In order to expand this power series, we look for a sequence of matrices \mathbf{C}_m such that

$$\sum_{l=0}^{\infty} \left[\frac{1}{2} \sum_{k=1}^{\infty} \alpha^k (\mathbf{G}(\mathbf{D} \circ \mathbf{G}^{k-1}) - (\mathbf{D} \circ \mathbf{G}^k)) \right]^l = \sum_{m=0}^{\infty} \alpha^m \mathbf{C}_m.$$

The solution to this problem is given by a combinatorial formula, known as the Faà di Bruno formula on the composition of power series.⁹ To express this formula, consider the composition of two power series:

$$\sum_l \left(\sum_k a_k \alpha_k \right)^l = \sum c_m \alpha^m,$$

For any integer m , let $\mathcal{P}(m)$ denote the set of all partitions of the integer m , i.e., sets of integers k_1, \dots, k_R such that $\sum_r k_r = m$. Then, the Faà di Bruno formula states that:

$$c_m = \sum_{k_1, \dots, k_R \in \mathcal{P}(m)} a_{k_1} a_{k_2} \dots a_{k_R}.$$

Applying the formula, we find:

$$\mathbf{C}_m = \frac{1}{2^m} \sum_{k_1, k_2, \dots, k_R | \sum k_r = m} \prod \mathbf{G}(\mathbf{D} \circ \mathbf{G}^{k_r-1}) - (\mathbf{D} \circ \mathbf{G}^{k_r}). \quad (14)$$

The first terms of the sequence can be computed as

\mathbf{C}_0	\mathbf{I}
\mathbf{C}_1	$\frac{1}{2}(\mathbf{G} - \mathbf{D} \circ \mathbf{G})$
\mathbf{C}_2	$\frac{1}{4}(\mathbf{G}^2 - (\mathbf{D} \circ \mathbf{G})\mathbf{G} + (\mathbf{D} \circ \mathbf{G})^2 - \mathbf{D} \circ \mathbf{G}^2)$

resulting in the approximation formula:

$$\begin{aligned} \mathbf{p} &= \frac{1+c}{2} \mathbf{1} + \frac{\alpha}{4} (\mathbf{G} - \mathbf{D} \circ \mathbf{G}) \mathbf{1} \\ &\quad + \frac{\alpha^2}{8} (\mathbf{G}^2 - (\mathbf{D} \circ \mathbf{G})\mathbf{G} + (\mathbf{D} \circ \mathbf{G})^2 - \mathbf{D} \circ \mathbf{G}^2) \mathbf{1} + \mathcal{O}(\alpha^3). \end{aligned}$$

Proof of Proposition 4.1: In the text.

Proof of Proposition 4.2: In the text.

⁹See Johnson (2002) for an historical account of the formula and its uses and variants.

Proof of Proposition 5.1: Consider two nodes i and j such that $g_{ij} = 0$ and $g_{jk} = 1$ whenever $g_{ik} = 1$. By the first order conditions,

$$\left(\frac{1-F(\theta_j)}{f(\theta_j)}-\theta_j-\alpha g_{ij}F(\theta_j)\right)-\left(\frac{1-F(\theta_i)}{f(\theta_i)}-\theta_i-\alpha g_{ij}F(\theta_i)\right)=2\alpha\sum_{k\neq i,j}(g_{jk}-g_{ik})(1-F(\theta_k))\geq 0.$$

By the monotone likelihood ratio property, $\frac{1-F(\theta)}{f(\theta)}$ is a decreasing function, so that $\theta_i \geq \theta_j$. By the first order condition, at the optimum,

$$p_i = \theta_i + \alpha \sum_j g_{ij}(1 - F(\theta_j)) = \frac{G(\theta_i) + c}{2},$$

establishing the Proposition.

Proof of Proposition 5.3: In the text.

Proof of Proposition 5.4: Consider a period t at which S consumers are active in the game and it is the monopolists's time to make an offer (t is odd). Suppose that all consumers $j, j \in S, j \neq i$ accept the monopolists's offer. Then the bargaining surplus generated by trade with consumer i is given by: $\frac{1}{2} - c + \alpha \sum_{j \in M} g_{ij}$. Following Rubinstein (1982) and Shaked and Sutton (1986), in equilibrium offers of the consumer and the monopolist are given by the stationary formulae x_i^t and y_i^t , and consumer i accepts the monopolists's offer immediately. Hence, there exists an equilibrium where all consumers in S accept the monopolists' offer immediately, and the offers x_i^t and y_i^t are equilibrium offers for all i . If t is even and it is the consumer's time to make an offer, notice that, if the monopolist rejects a subset S' of offers, next period he will make offers x_i^{t+1} to all $i \in S'$. Given that y_i^t is computed so that the monopolist is indifferent between accepting y_i^t today or obtaining x_i^{t+1} tomorrow for all $i \in S$, the monopolist has no incentive to reject any offer.