Management of a LNG contract portfolio

Zhihao CEN[1,2] J. Frédéric BONNANS[1] Thibault CHRISTEL[2]

[1]INRIA-Saclay and CMAP, Ecole Polytechnique

[2]Total

Conférence du Laboratoire FIME, HEC, June 28-29, 2010

Outline

Original formulation

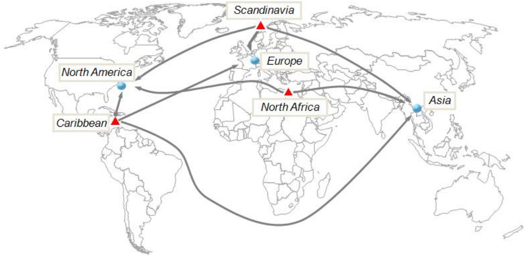
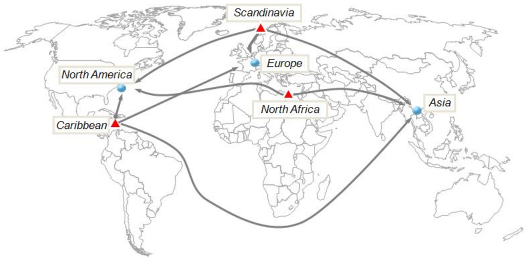


Figure: A supply and demand portfolio, as well as the possible route. ▲: producing country; •: consuming country.

Basic rules

- Long term buying and selling contracts (min-max amounts per month and per year);
- Route: between two ports able to receive ships of a format (in discrete number);
- Seller-buyer price formulas based on various commodities indexes;
- Income (uncertain): function of route and indexes;
- Discrete decisions: how many ships on each route, each month.

Original formulation



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- Income (uncertain): function of route and indexes;
- ~~Discrete decisions: how many ships on each route, each month.~~ (relaxation)

Mathematical model

Mathematical formulation

- Time $t = 1, \dots, T = 12$;
- State variables I: traded amounts (starting from 0) $x_t \geq 0$;
- State variables II: (random) commodities indexes ξ_t;
- Decision variables: number of ships per route u_t;
- State equation I: $x_{t+1} = x_t + Au_t$;
- Running cost $c_t(\xi_t)$ per ship per route;
- Final cost $g(\xi_T, x_{T+1})$;
- Filtration $\mathcal{F}_t$ generated by ξ_t;

Problem

Problem

$$\begin{array}{rl} \inf\limits_{(u_t)_{t=1,\dots,T-1}} & \mathbb{E}\left[\sum_{t=1}^{T} c_t(\xi_t)\cdot u_t + g(\xi_T, x_{T+1})\right] \\ s.t. & u_t \in \mathfrak{U}_t \text{ and } \mathcal{F}_t - \text{measurable} \\ \text{(dynamic)} & x_{t+1} = x_t + A_t u_t \\ \text{(final condition)} & x_{T+1} \in \mathfrak{X}_{T+1} \end{array} \quad (1)$$

where $u_t \in \mathbb{R}^n$, $x_t \in \mathbb{R}^m$, $x_1 = 0$;
$(\xi_t) \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathbb{R}^d)$ Markov chain :

$$\xi_{t+1} = f(W_t, \xi_t, \alpha_t) \quad t = 1, \dots, T-1 \quad (2)$$

$\xi_1 = \xi_1$, (W_t) i.i.d. r.v. on $\mathbb{R}^d$, $W_t \perp\!\!\!\perp \xi_t$;
$\mathcal{F}_t = \sigma(\xi_s, 1 \leq s \leq t)$;
$g(\xi, x)$ final cost function, convex w.r.t. x, Lipschitz;
$c_t(\xi)$ Lipschitz.

Dynamic programming (DP.) formulation

At stage t ($t = 1 \dots T$):

$$
\begin{aligned}
Q(t, x_t, \xi_t) := \quad & \inf_{u_t} \quad c_t(\xi_t) \cdot u_t + \mathcal{Q}(t+1, x_{t+1}, \xi_t) \\
& \text{s.t.} \quad u_t \in \mathfrak{U}_t \\
& \qquad\; x_{t+1} = x_t + A_t u_t
\end{aligned}
\tag{3}
$$

and final cost ($T + 1$):

$$
\mathcal{Q}(T+1, x_{T+1}, \xi_T) = \begin{cases} g(\xi_T, x_{T+1}) & \text{if } x_{T+1} \in \mathfrak{X}_{T+1} \\ +\infty & \text{otherwise} \end{cases}
\tag{4}
$$

where $\mathcal{Q}(t+1, x_{t+1}, \xi_t) = \mathbb{E}\Big[Q(t+1, x_{t+1}, \xi_{t+1}) \mid \mathcal{F}_t\Big]$.

Difficulties

- high dimension (u_t and x_t);
- many conditional expectation calculations.

Feasibility

Since (ξ_t) only present in objective function, we note:

$$
\begin{aligned}
\mathfrak{U}^{ad} &:= \left\{ (u_t)_{t\in[1,T]} \,\middle|\, \sum_{t=1}^{T} A_t u_t \in \mathfrak{X}_{T+1}, u_t \in \mathfrak{U}_t \right\} \\
\mathfrak{U}_t^{ad}(x_t) &:= \left\{ u_t \in \mathfrak{U}_t \,\middle|\, x_t + \sum_{s=t}^{T} A_s u_s \in \mathfrak{X}_{T+1}, u_s \in \mathfrak{U}_s \right\} \\
\mathfrak{X}_t &:= \mathfrak{X}_{T+1} - \sum_{s=t}^{T} A_s \mathfrak{U}_s
\end{aligned}
$$

$\Rightarrow$ relatively complete recourse problem.

$\mathfrak{U}_t^{ad}(x_t)$ takes into account feasibility cuts.

Idea

- Bender's decomposition for high dimension problem (x_t);
- a simple way to calculate conditional expectation (??);

Conditional expectation

- space discretization (tree method);
- regression (Longstaff and Schwartz);
- Malliavin calculus + Monte Carlo.

Optimal quantization

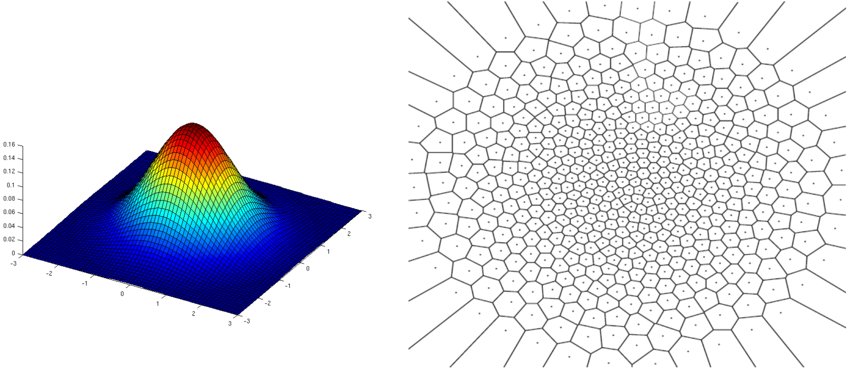


Figure: 2d Normal distribution and its optimal quantization

Optimal quantization : $\inf\left\{\|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_2 \;|\; \hat{\boldsymbol{\xi}} \in \Gamma = \left\{\xi^1, \ldots, \xi^N\right\}\right\}$

$$\Rightarrow \hat{\boldsymbol{\xi}} = \mathbf{Proj}_\Gamma(\boldsymbol{\xi}) \qquad p^i = \int \mathbf{1}_{\{\mathbf{Proj}_\Gamma(\xi) = \xi^i\}} \mu(d\boldsymbol{\xi})$$

Quantization tree

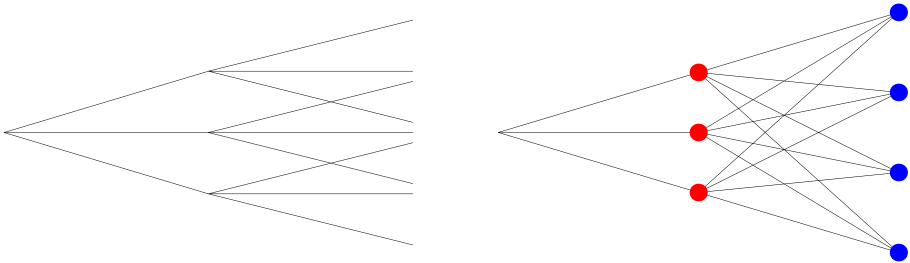


Figure: transitional tree v.s. quantization tree

Building method

- stochastic gradient method : Competitive Learning Vector Quantization(CLQV) (see Bally and Pagès);
- Gaussian case (ex. Brownian motion) : `http://quantize.math-fi.com`.

Transition probability

Denote p_t^{ij} the transition probability from ξ_t^i to ξ_{t+1}^j:

$$p_t^{ij} = \mathbb{P}\left(\mathbf{Proj}_{\Gamma_{t+1}}(\boldsymbol{\xi}_{t+1}) = \xi_{t+1}^j \;\middle|\; \mathbf{Proj}_{\Gamma_t}(\boldsymbol{\xi}_t) = \xi_t^i\right)$$

in practice, computed by Monte-Carlo method

$$\hat{p}_t^{ij} = \frac{\sum_{m=1}^M \mathbf{1}_{\{\mathbf{Proj}_{\Gamma_t}(\xi_t^m)=\xi_t^i\}} \mathbf{1}_{\{\mathbf{Proj}_{\Gamma_{t+1}}(\xi_{t+1}^m)=\xi_{t+1}^j\}}}{\sum_{m=1}^M \mathbf{1}_{\{\mathbf{Proj}_{\Gamma_t}(\xi_t^m)=\xi_t^i\}}}$$

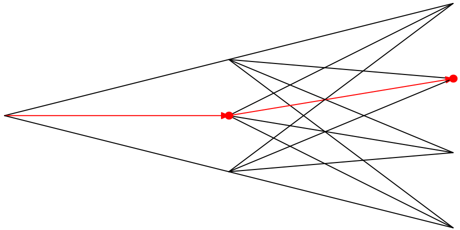
where $M(\approx 10^6)$ is the number of sample in Monte-Carlo simulation.

Conditional expectation calculation

$$\begin{aligned}
\mathcal{Q}(t+1, x_{t+1}, \boldsymbol{\xi}_t) &= \mathbb{E}\left[Q(t+1, x_{t+1}, \boldsymbol{\xi}_{t+1}) \mid \boldsymbol{\xi}_t\right] \\
\text{(scheme 0)} &\approx \mathbb{E}\left[Q(t+1, x_{t+1}, \boldsymbol{\xi}_{t+1}) \mid \xi_t^i = \mathbf{Proj}_{\Gamma_t}(\boldsymbol{\xi}_t)\right] \\
\text{(space discretization)} &\approx \sum_{\xi_{t+1}^j \in \Gamma_{t+1}} \hat{p}_t^{ij} Q(t+1, x_{t+1}, \xi_{t+1}^j)
\end{aligned}$$

Dual dynamic programming

Forward pass (in sample)



$$Q(t,x_t,\xi_t) \approx \inf_{u_t} \quad c_t(\xi_t)\cdot u_t + Q(t+1,x_{t+1},\hat{\xi}_t=\xi_t^i) \tag{5}$$

$$\approx \inf_{u_t} \quad c_t(\xi_t)\cdot u_t + \vartheta(t+1,x_{t+1},\xi_t^i) \tag{6}$$

$$s.t. \quad x_{t+1} = x_t + A_t u_t$$

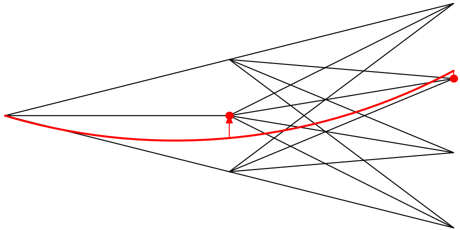
$$\text{(feasibility)} \quad u_t \in \mathfrak{U}_t^{ad}(x_t)$$

$$\text{(optimality)} \quad \vartheta(t+1,x_{t+1},\xi_t^i) \geq \left[\sum\nolimits_{\xi_{t+1}^j\in\Gamma_{t+1}} \hat{p}_t^{ij}(x_k^*\cdot x_{t+1}+e_k)\right] \quad k\in K(t+1,\xi_{t+1}^j)$$

where $\hat{\xi}_t = \mathbf{Proj}_{\Gamma_t}(\xi_t)$.

Dual dynamic programming

Forward pass (out of sample)



$$
\begin{aligned}
Q(t, x_t, \xi_t) &\approx \inf_{u_t} \quad c_t(\xi_t) \cdot u_t + Q(t+1, x_{t+1}, \hat{\xi}_t = \xi_t^i) && (5)\\
&\approx \inf_{u_t} \quad c_t(\xi_t) \cdot u_t + \vartheta(t+1, x_{t+1}, \xi_t^i) && (6)\\
& \quad s.t. \quad x_{t+1} = x_t + A_t u_t \\
\text{(feasibility)} & \quad u_t \in \mathfrak{U}_t^{ad}(x_t) \\
\text{(optimality)} & \quad \vartheta(t+1, x_{t+1}, \xi_t^i) \geq \left[\sum_{\xi_{t+1}^j \in \Gamma_{t+1}} \hat{p}_t^{ij} (x_k^* \cdot x_{t+1} + e_k) \right] \quad k \in K(t+1, \xi_{t+1}^j)
\end{aligned}
$$

where $\hat{\xi}_t = \mathbf{Proj}_{\Gamma_t}(\xi_t)$.

Control variate for forward pass

Take the following variable as control variate for forward pass:

$$v_{cv}(\xi) = \sum_{t=1}^{T} c_t(\xi_t) \cdot u_t^* \tag{7}$$

where u_t^* is one optimal solution for problem (v_{cv}^* the optimal value):

$$\begin{aligned} \inf_{(u_t)} \quad & \sum_{t=1}^{T} \mathbb{E}\left[c_t(\xi_t)\right] \cdot u_t \\ \text{s.t.} \quad & \sum_{t=1}^{T} A_t u_t \in \mathfrak{X}_{T+1} \\ & u_t \in \mathfrak{U}_t \qquad t = 1, \ldots, T \end{aligned} \tag{8}$$

Evaluate $\mathbb{E}[c_t(\xi_t)]$ by quantization: $\mathbb{E}[c_t(\xi_t)] \approx \sum_{\xi_t^i \in \Gamma_t} p_t^i c_t(\xi_t^i)$

$$\mathbb{E}[v_{cv}(\xi)] = v_{cv}^* \tag{9}$$

Algorithm II

Forward pass (upper bound)

Step 1 **Initialization**

Let $u_1 \in \mathfrak{U}_1^{ad}$ current control and calculate state variable x_2.

Let $\underline{v}^0 = -\infty$.

Go to Step 2.

Step 2 **Forward pass**

Simulate M_f Markov chain following (2);

At $t = 2, \ldots, T$ do

for $m = 1, \ldots, M_f$ do

calculate $\hat{\xi}_t^m = \mathbf{Proj}_{\Gamma_t}(\xi_t^m)$;

solve the subproblem (6).

v^m optimal value of scenario ξ^m.

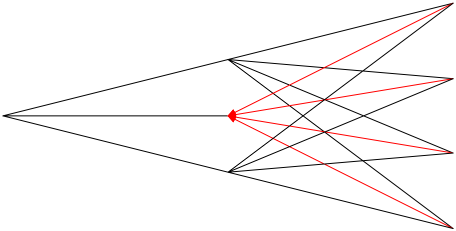
Compute the forward statistic $\overline{v} = \frac{1}{M_f} \sum_{m=1}^{M_f} v^m$

and $s = \frac{1}{M_f} \sqrt{\sum_{m=1}^{M_f} (v^m - \overline{v})^2}$.

Go to Step 3.

Dual dynamic programming

Backward pass



$$
\begin{aligned}
Q(t, x_t, \xi_t^i) \quad &= \inf_{u_t} \quad c_t(\xi_t^i) \cdot u_t + Q(t+1, x_{t+1}, \xi_t^i) && (10)\\
&\approx \inf_{u_t} \quad c_t(\xi_t^i) \cdot u_t + \vartheta(t+1, x_{t+1}, \xi_t^i) && (11)\\
&\quad s.t. \quad x_{t+1} = x_t + A_t u_t \\
\text{(feasibility)} \quad & \quad u_t \in \mathfrak{U}_t^{ad}(x_t) \\
\text{(optimality)} \quad & \quad \vartheta(t+1, x_{t+1}, \xi_t^i) \geq \left[\sum_{\xi_{t+1}^j \in \Gamma_{t+1}} \hat{p}_t^{ij} (x_k^* \cdot x_{t+1} + e_k) \right] \quad k \in K(t+1, \xi_{t+1}^j)
\end{aligned}
$$

where $\hat{\xi}_t = \mathbf{Proj}_{\Gamma_t}(\xi_t)$.

Algorithm III

Backward pass (lower bound)

Step 3 **Backward pass**

At $t = T, \ldots, 2$ do

for $m = 1, \ldots, M_b$ do

solve the subproblem (11) on all vertices in Γ_t;

calculate new optimality cuts and add them to vertex $(t-1, \xi_{t-1})$.

At $t = 1$

solve the subproblem;

calculate the backward value $\underline{v}^{it}$, *it* the iteration number.

Go to Step 4.

Step 4 **Check stop condition**

If $\underline{v}^{it} \in [\overline{v} - 1.96s, \overline{v} + 1.96s]$ and $|\underline{v}^{it} - \underline{v}^{it-1}| \leq \epsilon |\underline{v}^{it}|$

STOP;

else

go to Step 2.

Price dynamic

$$\ln F^i(s+1,t) - \ln F^i(s,t) = \sigma_i W_s^i - \frac{\sigma_i^2}{2} \quad i = 1,\dots,d \tag{12}$$

$$\xi_t^i = F^i(t,t) = F^i(0,t)\exp\left(\sigma_i \sum_{s=1}^{t-1} W_s - \frac{\sigma_i^2}{2}(t-1)\right) \tag{13}$$

where $F(s,t)$ the forward contract price at time s with maturity t;
W_s a $d-$dimension i.i.d. r.v. $N(0,\Sigma)$;

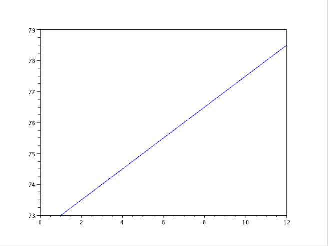


Figure: $F^1(0,t)$ OIL price (US$ /barrel)

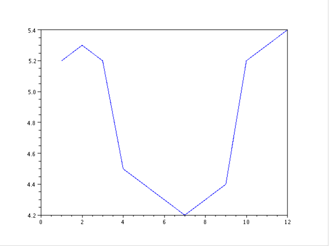


Figure: $F^2(0,t)$ NA NG price (US$ / MMBtu)

Portfolio

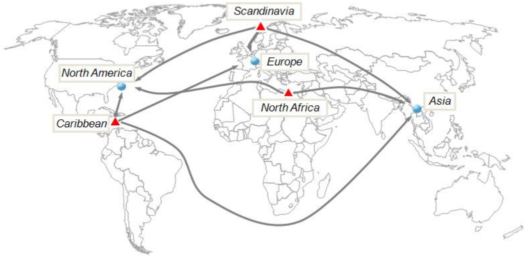
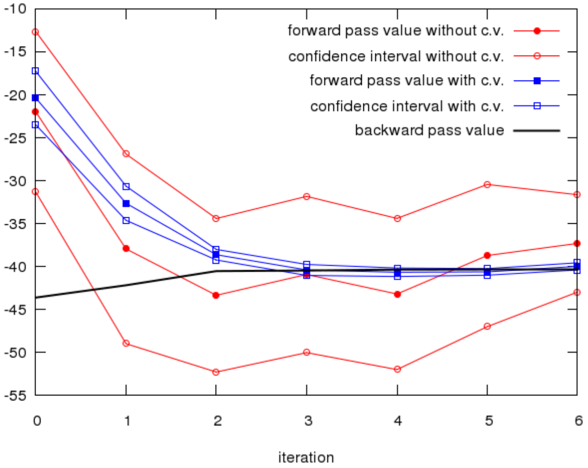


Figure: A supply and demand portfolio, as well as the possible route. ▲: producing country; ●: consuming country.

Port	Annual QC.	Monthly QC.	Price formula (in $/MMBtu)
Caribbean	50.0	[0, 6.0]	NA NG – 0.1
Scandinavia	25.0	[0, 3.0]	$\begin{cases} 0.05\text{OIL} + 2.5 & \text{if OIL} \le 75 \\ 0.07\text{OIL} + 1.0 & \text{elsewise} \end{cases}$
North Africa	100.0	[0, 12.0]	$\begin{cases} 0.9\text{NA NG} + 0.4 & \text{if NA NG} \le 5 \\ 0.8\text{NA NG} + 0.9 & \text{otherwise} \end{cases}$
North American	85.0	[0, 8.0]	NA NG
Europe	70.0	[0, 7.0]	$\begin{cases} \text{EU NG} & \text{(3 random)} \\ 0.055\text{OIL} + 0.8 & \text{(2 random)} \end{cases}$
Asia	20.0	[0, 4.0]	0.08OIL – 0.8

Result (2d)

Quantization size: $N = 8000$;
Algorithm parameter: $M_f = 3000$ to 6000, $M_b = 8$ to 12;
Process parameters $\sigma_1 = \sigma_2 = 66\%$, $\rho_{12} = 0.3$.



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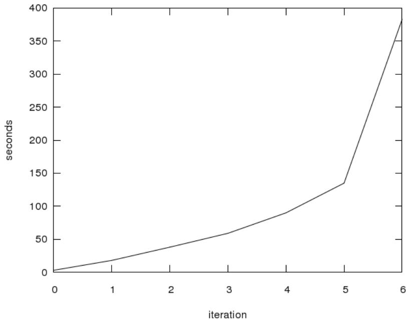


Figure: forward time-consuming

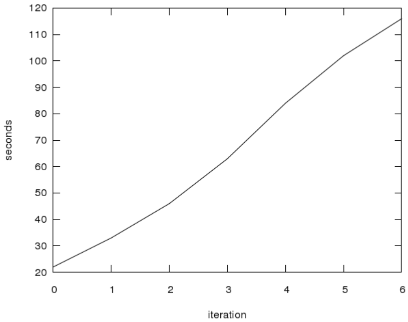
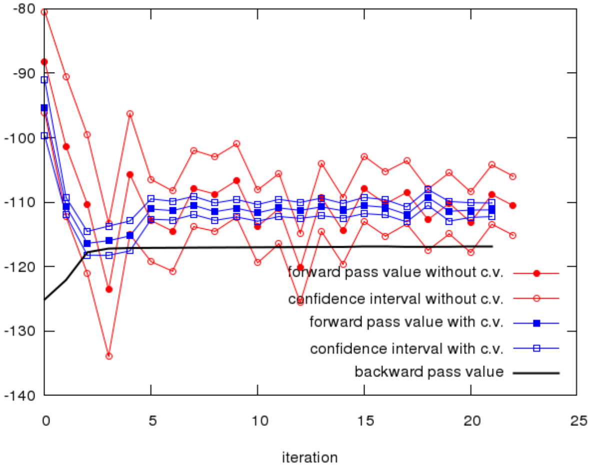


Figure: backward time-consuming

Result (3d)

Quantization size: $N = 30000$;
Algorithm parameter: $M_f = 3000$ to 6000, $M_b = 8$ to 12;
Process parameters $\sigma_1 = \sigma_2 = \sigma_3 = 66\%$; $\rho_{12} = 0.7, \rho_{13} = 0.2, \rho_{23} = 0.4$.



Result (3d)

Quantization size: $N = 30000$;
Algorithm parameter: $M_f = 3000$ to 6000, $M_b = 8$ to 12;
Process parameters $\sigma_1 = \sigma_2 = \sigma_3 = 66\%$; $\rho_{12} = 0.7, \rho_{13} = 0.2, \rho_{23} = 0.4$.

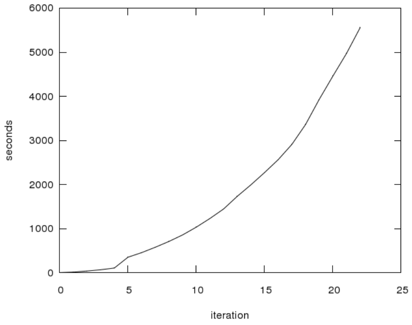


Figure: forward time-consuming

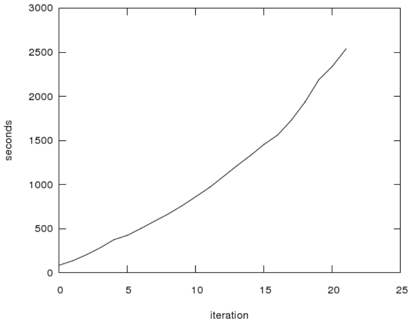


Figure: backward time-consuming

Heuristic method for integer solution

Idea

Using the Bellman function obtained in continuous relaxation to simulate the integer strategy. After converge in continuous problem, solve integer strategy in forward pass.

$$
\begin{aligned}
Q(t, x_t, \xi_t) \quad & \approx \inf_{u_t} \quad c_t(\xi_t) \cdot u_t + Q(t+1, x_{t+1}, \hat{\xi}_t = \xi_t^i) && (14)\\
& \approx \inf_{u_t} \quad c_t(\xi_t) \cdot u_t + \vartheta(t+1, x_{t+1}, \xi_t^i) && (15)\\
& \quad s.t. \quad x_{t+1} = x_t + A_t u_t \\
\text{(feasibility)} \quad & \quad u_t \in \mathbb{Z}^n \cap \mathfrak{U}_t^{ad}(x_t) \\
\text{(optimality)} \quad & \quad \vartheta(t+1, x_{t+1}, \xi_t^i) \geq \left[\sum_{\xi_{t+1}^j \in \Gamma_{t+1}} \hat{p}_t^{ij} (x_k^* \cdot x_{t+1} + e_k) \right] \quad k \in K(t+1, \xi_{t+1}^j)
\end{aligned}
$$

Problem and result

Relaxed problem

Port	Cargo Size	Annual QC.	Monthly QC.	Price formula (in $/MMBtu)
Caribbean	3	[46, 54]	[0, 6.0]	NA NG − 0.1
Scandinavia	3	[21, 29]	[0, 3.0]	$\begin{cases} 0.05\text{OIL} + 2.5 & \text{if OIL} \leq 75 \\ 0.07\text{OIL} + 1.0 & \text{elsewise} \end{cases}$
North Africa	4	[96, 104]	[0, 12.0]	$\begin{cases} 0.9\text{NA NG} + 0.4 & \text{if NA NG} \leq 5 \\ 0.8\text{NA NG} + 0.9 & \text{otherwise} \end{cases}$
North American	3, 4	[81, 89]	[0, 8.0]	NA NG
Europe	3, 4	[66, 74]	[0, 7.0]	0.055OIL + 0.8
Asia	3, 4	[16, 24]	[0, 4.0]	0.08OIL − 0.8

Result

	upper Bound	c.v. of upper bound	lower Bound	$v^*_{cv}(9)$
Continuous	-54.0464	1.79238	-54.7472	-31.5404
Integer	-31.8133	4.29456	nan	-28.0352

Problem with CV@R

Problem with CV@R

$$\begin{aligned} \inf_{(u_t)} \quad & \mathbb{E}\left[Y_{T+1}\right] + \lambda\, \mathrm{CVaR}_\beta(Y_{T+1}) && (16)\\ s.t. \quad & u_t \in \mathfrak{U}_t \text{ and } \mathcal{F}_t - \text{measurable} \\ \text{(dynamic)} \quad & x_{t+1} = x_t + A_t u_t \\ \text{(final condition)} \quad & x_{T+1} \in \mathfrak{X}_{T+1} \\ & Y_{T+1} = \sum_{t=1}^{T} e^{-rt} c_t(\xi_t) \cdot u_t \end{aligned}$$

where $\lambda > 0$ and $\beta \in (0, 1)$;

CV@R

$$\mathrm{CVaR}_\beta(Y) = \inf_{z \in \mathbb{R}} z + (1-\beta)^{-1}\mathbb{E}\left[(Y-z)_+\right] \tag{17}$$

$$\inf_{(u_t)} \mathbb{E}\left[Y\right] + \lambda \mathrm{CVaR}_\beta(Y) = \inf_{(u_t), z} \mathbb{E}\left[Y + \lambda\left(z + (1-\beta)^{-1}(Y-z)_+\right)\right] \tag{18}$$

DP. formulation with CV@R

DP. formulation

$$\begin{aligned} Q(t, x_t, Y_t, \xi_t, z) &= \inf_{u_t \in \mathfrak{U}_t} c_t(\xi_t) u_t + Q(t+1, x_{t+1}, Y_{t+1}, \xi_t, z) \\ s.t. & \quad x_{t+1} = x_t + A_t u_t \qquad Y_{t+1} = Y_t + e^{-r} c_t(\xi_t) u_t \end{aligned} \tag{19}$$

The first stage :

$$\begin{aligned} Q(t=1, x_1=0, Y_1=0, \xi_1) &= \inf_{u_1 \in \mathfrak{U}_1, z \in \mathbb{R}} c_1(\xi_1) u_1 + Q(2, x_2, Y_2, \xi_1, z) \\ s.t. & \quad x_2 = x_1 + A_1 u_1 \qquad Y_2 = Y_1 + e^{-r} c_1(\xi_1) u_1 \end{aligned} \tag{20}$$

The final stage :

$$Q(T+1, x_{T+1}, Y_{T+1}, \xi_T, z) = \begin{cases} \lambda\Big(z + (1-\beta)^{-1}(Y_{T+1} - z)_+\Big) & \text{if } x_{T+1} \in \mathfrak{X}_{T+1} \\ \infty; & \text{otherwise} \end{cases} \tag{21}$$

Lemma

$Q(t, x_t, Y_t, \xi_t, z)$ is a convex function of (x_t, Y_t, z).

Stability of optimal value

Theorem (W. Römisch 06)

Under some assumptions, there exist positive constants L and δ such that:

$$|\mathrm{val}(\xi) - \mathrm{val}(\hat{\xi})| \leq L\left(\|\xi - \hat{\xi}\|_p + D_f(\xi, \hat{\xi})\right)$$

hold for all random element $\hat{\xi} \in l^p$ (same as ξ) with $\|\xi - \hat{\xi}\|_p \leq \delta$, where

$$D_f(\xi, \hat{\xi}) := \inf_{u \in S(\xi), \hat{u} \in S(\hat{\xi})} \sum_{t=2}^{T-1} \|u_t - \mathbb{E}[u_t|\hat{\mathcal{F}}_t]\|_{p'} \vee \|\hat{u}_t - \mathbb{E}[\hat{u}_t|\mathcal{F}_t]\|_{p'}$$

$$\hat{\mathcal{F}}_t = \sigma(\hat{\xi}_s : 0 \leq s \leq t)$$

Stability of optimal solution

Theorem (W. Römisch 06)

Under previous assumptions, and that the solution sets $S(\xi)$ and $S(\hat{\xi})$ are nonempty, then there exist positive constants $\bar{L}$ and $\bar{\epsilon}$ such that

$$dl_\infty(S_\epsilon(\xi), S_\epsilon(\hat{\xi})) \leq \frac{\bar{L}}{\epsilon}(\|\xi - \hat{\xi}\|_p + D_f^*(\xi, \hat{\xi})) \tag{22}$$

hold for any $\epsilon \in (0, \bar{\epsilon})$, where

$$D_f^*(\xi, \hat{\xi}) := \sup_{\|u\|_{p'} \leq 1} \sum_{t=2}^{T} \|\mathbb{E}[u_t|\hat{\mathcal{F}}_t] - \mathbb{E}[u_t|\mathcal{F}_t]\|_{p'}$$

$$dl_\infty(C, D) := \lim_{\rho \to \infty} \inf\{\eta \geq 0 \mid C \cup \rho\mathbb{B} \subset D + \eta\mathbb{B}, D \cup \rho\mathbb{B} \subset C + \eta\mathbb{B}\}$$

Convergence rate of distortion

$$\text{(distrotion of quantization)} \qquad D_N^{\mu,p} = \|\xi - \hat{\xi}\|_p^p$$
$$\text{(optimal distrotion)} \qquad \underline{D}_N^{\mu,p} = \inf_{|\Gamma|\leq N} D_N^{\mu,p}$$

Theorem (Zador 82)

Assume that $\mathbb{E}\|\xi\|^{p+\eta} = \int_{\Xi} |\xi|^{p+\eta}\mu(d\xi) < \infty$ *for some* $\eta > 0$. *Then*

$$\lim_{N\to\infty} (N^{\frac{p}{d}} \underline{D}_N^{\mu,p}) = J_{p,d}\|\varphi\|_{\frac{d}{d+p}} \tag{23}$$

where $\mu(d\xi) = \varphi(\mu)\cdot\lambda_d(d\xi) + \upsilon$ *(*λ_d *Lebesgue measure on* $\mathbb{R}^d$*), and* $\upsilon \perp \lambda_d$*. The constant* $J_{p,d}$ *corresponds to the case of the uniform distribution on* $[0,1]^d$ *(or any Borel set of Lebesgue measure* 1*).*

quantization tree: $\inf_{|\Gamma|\leq N} \|\xi - \hat{\xi}\|_p = \left(\sum_{t=1}^{T} \underline{D}_{N_t}^{\mu_t,p}\right)^{1/p} = O(N^{-1/d})$ where $N = \sum_{t=1}^{T} N_t$.

Bibliography I

V. Bally and P. Pagès.
Error analysis of the quantization algorithm for obstacle problems.
Stochastic Processes & Their Applications, 106(1):1–40, 2003.

V. Bally and P. Pagès.
A quantization algorithm for solving discrete time multi-dimensional optimal stopping problems.
Bernoulli, 9(6):1003–1049, 2003.

V. Bally, P. Pagès, and J. Printemps.
A quantization tree method for pricing and hedging multidimensional american options.
Mathematical Finances, 15(1):119–168, 2005.

O. Bardou, S. Bouthemy, and G. Pagès.
Pricing swing options using optimal quantization, 2007.

Bibliography II

H. Heitsch and W. Römisch.
Stability and scenario trees for multistage stochastic programs, 2006.
preprint 324, DFG Research Center Matheon “Mathematics for key technologies”.

H. Heitsch, W. Römisch, and C. Strugarek.
Stability of multistage stochastic programs.
SIAM J. on Optimization, 17:511–525, 2006.

M.V.F. Pereira and L.M.V.G. Pinto.
Multi-stage stochastic optimization applied to energy planning.
Mathematical Programming, 52:359–375, 1991.

A.B. Philpott and Z. Guan.
On the convergence of stochastic dual dynamic programming and related methods.
Operation Research Letters, 2008.