Mathematical Challenges of the Emission Markets

René Carmona

ORFE, Bendheim Center for Finance
Princeton University

Laboratoire FiME,
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Schematic of the Talk

- Descriptive Introduction
  - Zoology of the Carbon Markets: **EU ETS**, and those soon to exist in the **US** (*wishful thinking*)
  - **Lessons** learned from the EU Experience
- First **Mathematical (Equilibrium) Models**
  - Joint Price Formation for Production Goods and Emission Allowances
  - **Costs** associated to a Cap-and-Trade scheme
  - Design of Cap-and-Trade Schemes: the Allocation Mechanism
  - Multi-periods, Multi-markets Models & the **CDM**
- Reduced Form Models
  - **Information Flows** and **jumps**
  - First Models for **EUA Option Prices**
- Partial Equilibrium Models & **BSDEs**
  - **BSDE** Formulaiton
  - Mathematical Pathologies of **Singular BSDEs**
  - More **Option Pricing**

First Emission Trading Market

- Established in the United States **Clean Air Act of 1990**
- **Acid Rain Program**
- **Program to reduce the primary causes of acid rain**
  - sulfur dioxide (SO_2)
  - nitrogen oxides (NO_x)
- **Program based on BOTH**
  - **regulatory approach**
  - **market mechanisms**
- To achieve this goal **at the lowest cost to society**
- **SOx and NOx Trading**: Great learning experience!
  - Liquidity and Price Collapse Issues
  - They did not create Pollution Hot Spots?
- **TOO SMALL** a scale (Montgomery *flip-flop*)

Kyoto Protocol

- Kyoto Conference **1997**
- Assign **MANDATORY** Green House Gas (GHG) emission limits to signatory nations
  - Reduce emissions of CO_2 and 5 other gases in 2008 - 2012
  - Target level: 95% of 1990 levels
- Set up **Cap & Trade for Green House Gases**
- Clean Development Mechanism (CDM) and Joint Initiative (JI)
- **ENFORCEMENT?** (theory of self-enforced treaties)

Flexible Mechanisms of Kyoto Protocol

- Stimulate sustainable development and emission reductions, when and where it is cheapest to do
- Projects must qualify through a rigorous and public registration and issuance process
  - Ensure **real**, **measurable** and **verifiable** emission reductions
  - **Additional** to what would have occurred without the projects
- **Clean Development Mechanism** (CDM)
  - Projects located in **developing countries**
- **Joint Initiative** (JI)
  - Projects located in economies **in transition**
- We'll use **same mathematical models!**
- Approved projects earn **Certified Emission Reduction** (CER)

CERs

- Using CERs to meet emission reduction targets
  - 1 CER = 1 ton of CO_2 equivalent to meet emission reduction
  - traded and sold on **ANY** market, **NO** date limitation
  - discount due to moral hazard, political, project completion, . . . **RISK**
- **Trading**
  - Program started in 2006
  - More than 1,000 projects already registered
  - Anticipated to produce CERs amounting to more than 2.7 billion tons of CO_2 equivalent for 2008 – 2012
- **Speculative Trading of Spread** between EUAs and CERs
- Role of **CERs** in EUA option prices still a **mystery**

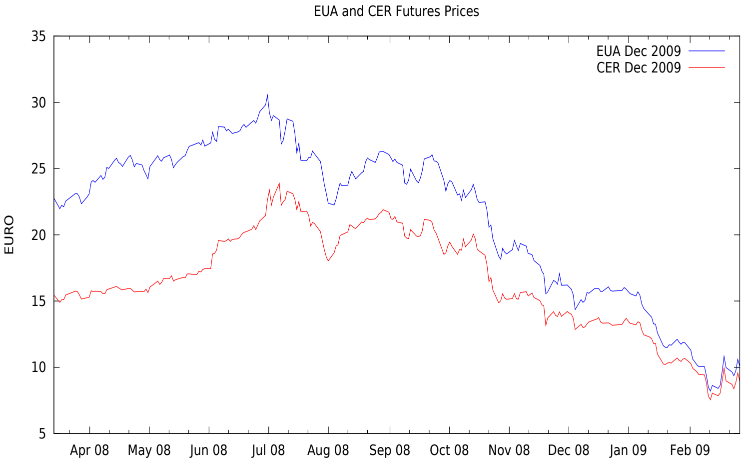


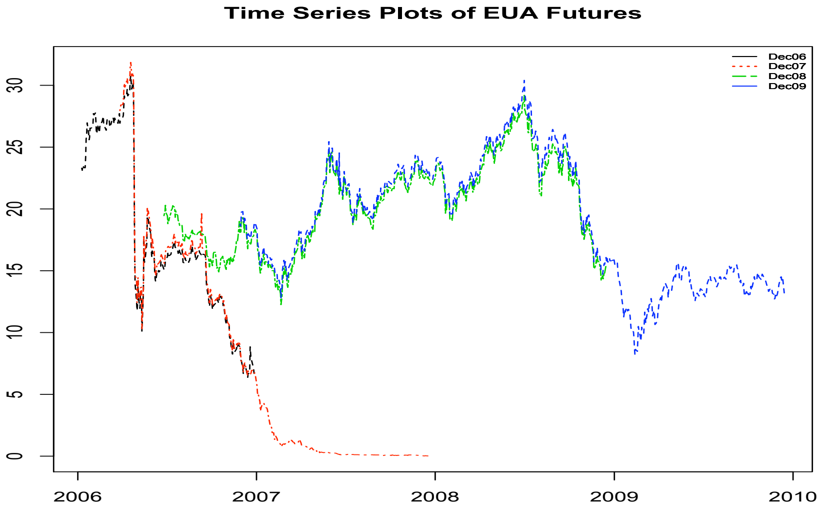
Figure: Prices of the December 2012 EUA futures contract (EU-ETS second phase), together with the price of the corresponding CER futures contract.

EU and the Kyoto Protocol

- **European Climate Change Programme (ECCP)** June 2000
- **All** 25 EU countries ratify Kyoto Protocol on 31 May 2002
- **Directive 2003/87/ec** of the European Parliament of October 13, 2003: *establishment of a scheme for greenhouse gas emission allowance trading.*
- **Each** EU member state proposes a **National Allocation Plan** (NAP) with a **cap**
- **Permit Allocation:**
  Installations covered by ETS are given **allowances for FREE**
  - power plants (capacity $>$ 20MW)
  - steel manufacturers
  - cement factories
  - . . . . . . (1200 installations in EU during first phase)

EU Emission Trading Scheme (ETS)

- Actual trading in EU ETS started **January 2005**
- **400 million** tons of CO_2 equivalent traded the first year
- EU ETS structured in **Three Phases**
  - **Phase I**: January 2005 - December 2007 (trial)
  - **Phase II**: January 2008 - December 2012 (current)
  - **Phase III**: January 2013 - December 2018 (unclear – Copenhagen)
- Exchange traded (standardized & cleared) futures contracts (Dec-05, Dec-06, Dec-07, Dec-08, . . ., Dec-12)
- 1 **contract** = 1 **lot** = 1000 EUAs of 1 ton CO_2 equivalent each
- **Liquid** Front End contract
- Vibrant **option** market on these futures contracts



Prices of the EUA futures contracts.

How Do Things Work?

- Each year, installation receive allowances according to NAP
- Each year, cumulative emissions are tallied up to Dec. 31
- Each installation has up to Apr. 30 to cover its emissions
  - by **surrendering** allowances
  - paying a **penalty** of λ euros per ton not covered by an allowance
  - $\lambda = 40$ euros in **Phase I**; $\lambda = 100$ euros in **Phase II**
  - Paying the penalty is not enough: the corresponding amount of allowances is **withdrawn** from the next allocation
- Phase I was a *trial balloon*
- Phase I allowances **COULD NOT BE USED** beyond their maturities
- Phase II allowances **CAN BE BANKED** for later use

Goal of the Study

- Putting a Price on
  - CO_2 by **internalizing** its Social Cost
  - Goods whose Productions lead to **Emissions**
- Regulatory Economic Instruments
  - Carbon **TAX**
  - Permits Allocation & Trading (**Cap-and-Trade**)
- Calibrate the Different Schemes for
  - **MEANINGFUL** & **FAIR** comparisons

Goal of the Study: Equilibrium Analysis

- **Dynamic Stochastic General Equilibrium**
- **Inelastic** Demand
  - Electricity Production for the purpose of **illustration**
  - **Same** results in **multi-good** Markets
- **Random** Factors
  - Demands for goods $\{D_t^k\}_{t\geq 0}$
  - **Costs** of Production $\{C_t^{i,j,k}\}_{t\geq 0}$
    - Spot Price of Coal
    - Spot Price of Natural Gas

Goal of the Study: **Japan** Case Study

TOKYO unveiled a **Carbon Scheme**

Japanese Electricity Market:

- Eastern & Western Regions (1GW Interconnection)
- Electricity Production: Nuclear, **Coal, Natural Gas**, Oil
  - Coal is **expensive**
  - Visible Impact of Regulation (**fuel switch**)
- **Regulation** Gory Details
  - **Cap** (Emission Target) 300 Mega-ton CO_2 = 20% w.r.t. 2012 BAU
  - Calibration for Fair Comparisons: **Meet Cap** 95% **of time**
    - Penalty 100 USD
    - Tax Level 40 USD
  - Numerical Solution of a **Stochastic Control** Problem (**HJB**) in 4-D

Goal of the Study: Comparisons

Economic Statics to be Compared

- **Actual Emissions**
- **Reduction (Abatment) Costs**
- **Social Costs**
- **Windfall Profits**

Controls to be Varied

- **Penalty**
- **Tax**
- **Allocation Mechanisms**
  - Free Initial Allocation
  - Auctions
  - Dynamic Proportional Allocation
  - Hybrid Allocation Schemes
  - . . . .

- **Finite set** $\mathcal{I}$ of **risk neutral firms**
- **Producing a finite set** $\mathcal{K}$ of **goods**
- Firm $i \in \mathcal{I}$ can use **technology** $j \in \mathcal{J}^{i,k}$ to produce good $k \in \mathcal{K}$
- **Discrete time** $\{0, 1, \cdots, T\}$
- **No Discounting** Work with T-Forward Prices
- **Inelastic Demand**

$$\{D^k(t);\ t = 0, 1, \cdots, T-1,\ k \in \mathcal{K}\}.$$

- . . . .

Regulator Input (EU ETS)

At inception of program (i.e. time $t = 0$)

- **INITIAL DISTRIBUTION** of $\Lambda = \Lambda_0$ **allowance certificates**

$$\Lambda_0 = \sum_{i \in \mathcal{I}} \Lambda_0^i, \qquad \Lambda_0^i \quad \text{to firm } i \in \mathcal{I}.$$

- Set **PENALTY** λ for emission unit **NOT** offset by allowance certificate at end of **compliance period**

Extensions discussed later on.

- **Multi-period**, multi-market extensions
- **Alternative allocation** mechanisms
- **Risk aversion** and agent preferences
- **Elastic** demand (e.g. smart meters for electricity)
- **Investments in new technologies** (wind, solar, CCS,...)
- . . . .

Goal of Equilibrium Analysis

Find **stochastic processes**

- **Price of one allowance**

$$A = \{A_t\}_{t \geq 0}$$

- **Prices of goods**

$$S = \{S_t^k\}_{k \in K,\, t \geq 0}$$

satisfying the usual conditions for the existence of a

competitive equilibrium

(to be spelled out below) and study the fine properties of these processes.

Individual Firm Problem

During each time period $[t, t+1)$

- Firm $i \in \mathcal{I}$ **produces** $\xi_t^{i,j,k}$ of good $k \in \mathcal{K}$ with technology $j \in \mathcal{J}^{i,k}$
- Firm $i \in \mathcal{I}$ **holds** a position θ_t^i in emission credits

$$
\begin{aligned}
L^{A,S,i}(\theta^i, \xi^i) := \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}^{i,k}} \sum_{t=0}^{T-1} (S_t^k - C_t^{i,j,k}) \xi_t^{i,j,k} \\
+ \theta_0^i A_0 + \sum_{t=0}^{T-1} \theta_{t+1}^i (A_{t+1} - A_t) - \theta_{T+1}^i A_T \\
- \lambda(\Gamma^i + \Pi^i(\xi^i) - \theta_{T+1}^i)^+
\end{aligned}
$$

where

$$
\Gamma^i \text{ random}, \qquad \Pi^i(\xi^i) := \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}^{i,k}} \sum_{t=0}^{T-1} e^{i,j,k} \xi_t^{i,j,k}
$$

Random Inputs

- Γ^i uncontrolled emissions
- $C_t^{i,j,k}$ costs of productions (e.g. fuel prices)

Individual Firm Problem (cont.)

Problem for (risk neutral) firm $i \in I$

$$\max_{(\theta^i,\xi^i)} \mathbb{E}\{L^{A,S,i}(\theta^i,\xi^i)\}$$

Choose

- Production strategy ξ^i
- Trading strategy θ^i

in order to

- Maximize its own **expected P&L**
- Satisfy the demand

Equilibrium Definition for Emissions Market

The processes $A^* = \{A^*_t\}_{t=0,1,\cdots,T}$ and $S^* = \{S^*_t\}_{t=0,1,\cdots,T}$ form an equilibrium if for each agent $i \in \mathcal{I}$ there exist strategies $\theta^{*i} = \{\theta^{*i}_t\}_{t=0,1,\cdots,T}$ (**trading**) and $\xi^{*i} = \{\xi^{*i}_t\}_{t=0,1,\cdots,T}$ (**production**)

- **(i) All financial positions are in constant net supply**

$$\sum_{i \in I} \theta^{*i}_t = \sum_{i \in I} \theta^i_0, \qquad \forall\, t = 0, \ldots, T+1$$

- **(ii) Supply meets Demand**

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}^{i,k}} \xi^{*i,j,k}_t = D^k_t, \qquad \forall\, k \in \mathcal{K}, \;\; t = 0, \ldots, T-1$$

- **(iii) Each agent $i \in I$ is satisfied by its own strategy**

$$\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i}, \xi^{*i})] \geq \mathbb{E}[L^{A^*,S^*,i}(\theta^i, \xi^i)] \qquad \text{for all } (\theta^i, \xi^i)$$

Business As Usual (i.e. $\lambda = 0$)

The corresponding prices of the goods are

$$S_t^{*k} = \max_{i \in \mathcal{I},\, j \in \mathcal{J}^{i,k}} C_t^{i,j,k} \mathbf{1}_{\{\xi_t^{*i,j,k} > 0\}},$$

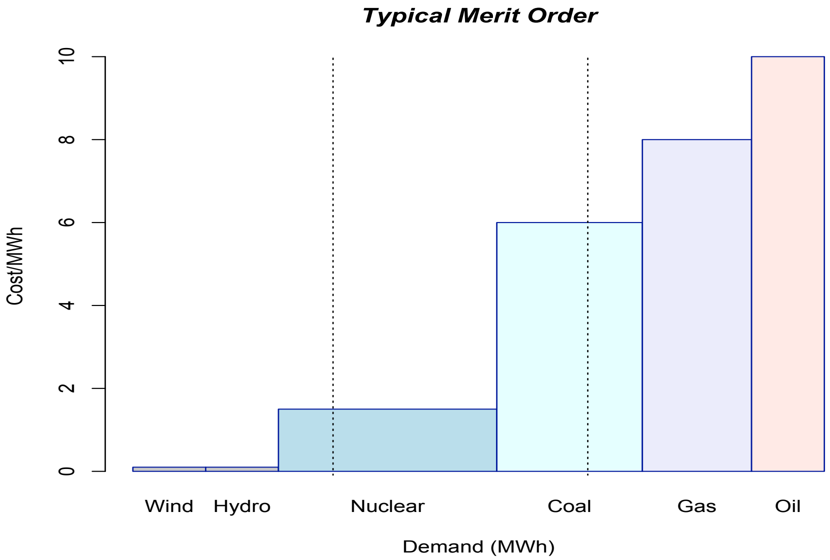
Classical **MERIT ORDER**

- At each time t and for each good k
- Production technologies ranked by increasing production costs $C_t^{i,j,k}$
- Demand D_t^k met by producing from the cheapest technology first
- Equilibrium spot price is the marginal cost of production of the most expansive production technoligy used to meet demand

Business As Usual

(typical scenario in Deregulated **electricity markets**)

Example of a Classical Merit Order Plot



Necessary Conditions

Assume

- (A^*, S^*) is an equilibrium
- (θ^{*i}, ξ^{*i}) optimal strategy of agent $i \in I$

then

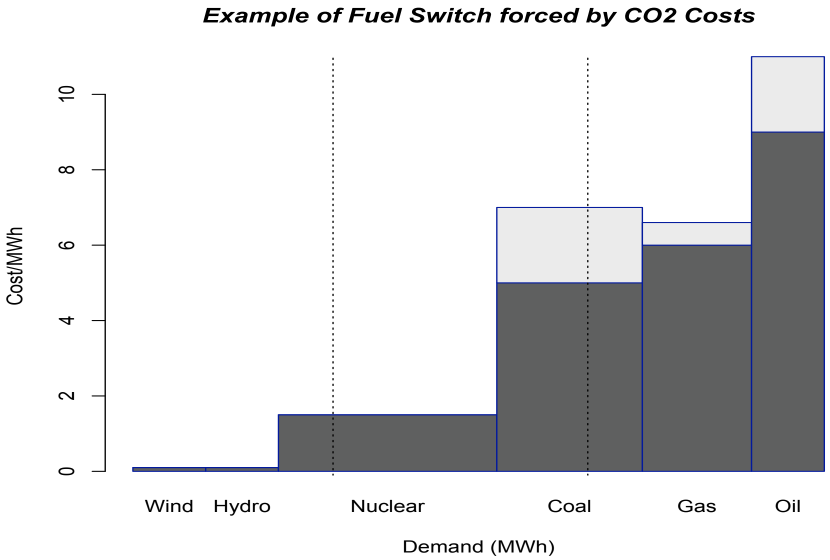
- The allowance price A^* is a **bounded martingale** in $[0, \lambda]$
- Its terminal value is given by

$$A^*_T = \lambda \mathbf{1}_{\{\Gamma^i + \Pi(\xi^{*i}) - \theta^{*i}_{T+1} \geq 0\}} = \lambda \mathbf{1}_{\{\sum_{i \in \mathcal{I}} (\Gamma^i + \Pi(\xi^{*i}) - \theta^{*i}_0) \geq 0\}}$$

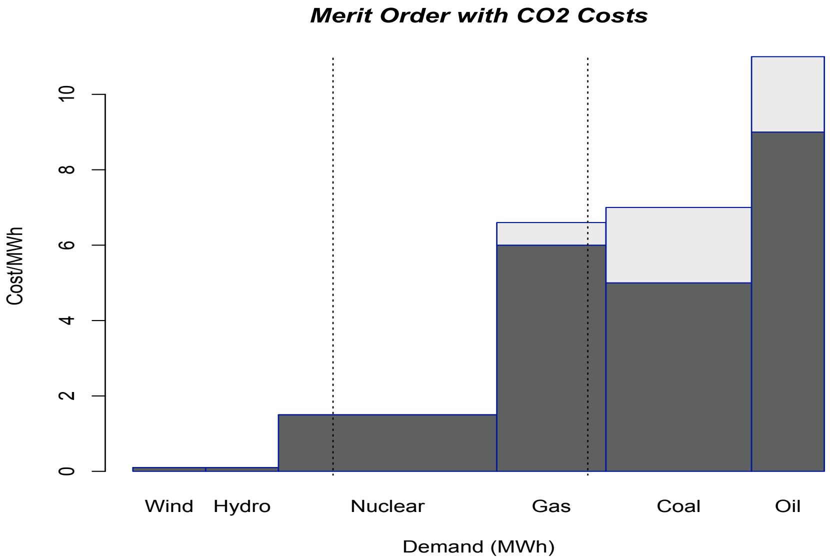
- The **spot prices** S^{*k} of the goods and the **optimal production strategies** ξ^{*i} are given by the **merit order** for the equilibrium with **adjusted costs**

$$\tilde{C}^{i,j,k}_t = C^{i,j,k}_t + e^{i,j,k} A^*_t$$

Example of a Fuel Switch forced by Regulation



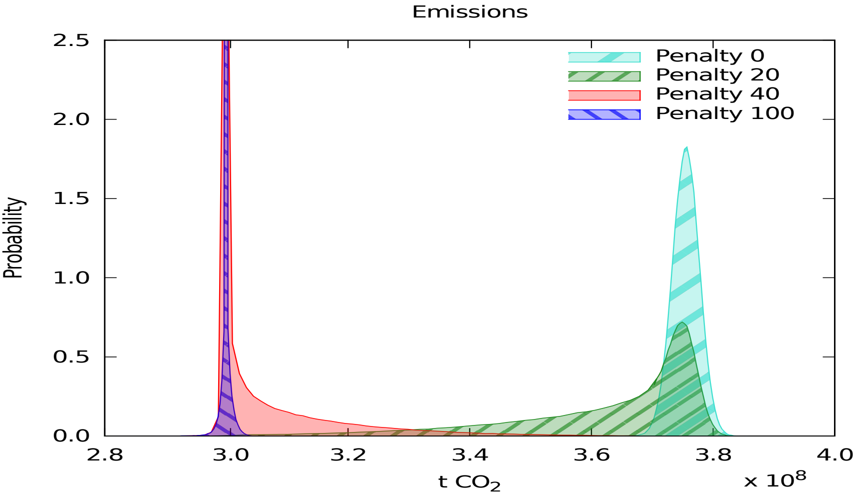
Example of a Merit Order Plot Including CO_2



Impact of the Penalty

- Trial Phase of EU ETS (2005 - 2007): 40 **Euros**
- First Phase of EU ETS (2008 - 2012): 100 **Euros**
- RGGI: Market Participants ***do not really pay attention***
- Option Data show Market Participants **DO NOT BELIEVE** the market will **EVER BE SHORT**
  - Influx of CERs
  - Hot Air (Russia, Poland .... excess allocation)
  - Lobbying & Political Pressure to put FLOORs and CIELINGs

Effect of the Penalty on Emissions



Costs in a Cap-and-Trade

- **Consumer Burden**

$$\mathrm{SC} = \sum_t \sum_k (S_t^{k,*} - S_t^{k,BAU*}) D_t^k .$$

- **Reduction Costs** (producers' burden)

$$\sum_t \sum_{i,j,k} (\xi_t^{i,j,k*} - \xi_t^{BAU,i,j,k*}) C_t^{i,j,k}$$

- **Excess Profit**

$$\sum_t \sum_k (S_t^{k,*} - S_t^{k,BAU*}) D_t^k - \sum_t \sum_{i,j,k} (\xi_t^{i,j,k*} - \xi_t^{BAU,i,j,k*}) C_t^{i,j,k} - \lambda (\sum_t \sum_{ijk} \xi_t^{ijk} e_t^{ijk} - \theta_0)^+$$

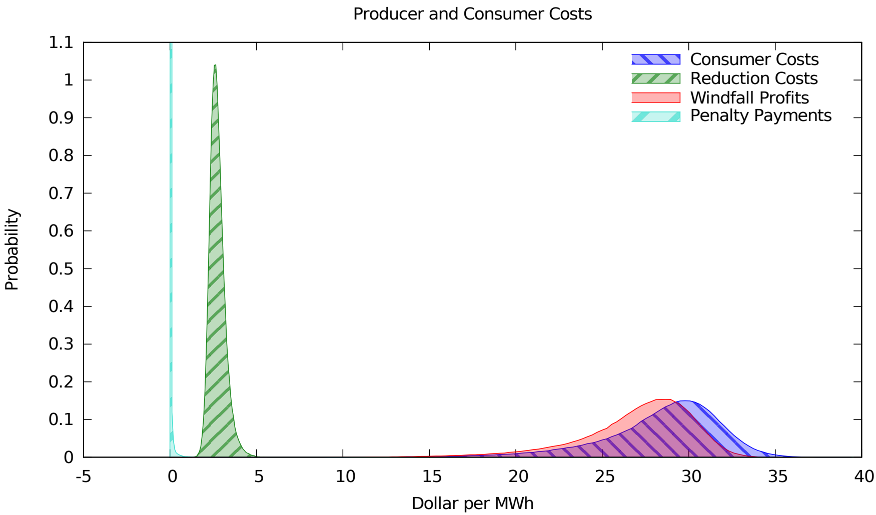
- **Windfall Profits**

$$\mathrm{WP} = \sum_{t=0}^{T-1} \sum_{k \in K} (S_t^{*k} - \hat{S}_t^k) D_t^k$$

where

$$\hat{S}_t^k := \max_{i \in I, j \in J^{i,k}} C_t^{i,j,k} \mathbf{1}_{\{\xi_t^{*i,j,k} > 0\}} .$$

Costs in a Cap-and-Trade Scheme



Histograms of consumer costs, social costs, windfall profits and penalty payments of a standard cap-and-trade scheme calibrated to reach the emissions target with 95% probability and BAU.

One of many Alternative Designs

Introduction of **Taxes / Subsidies**

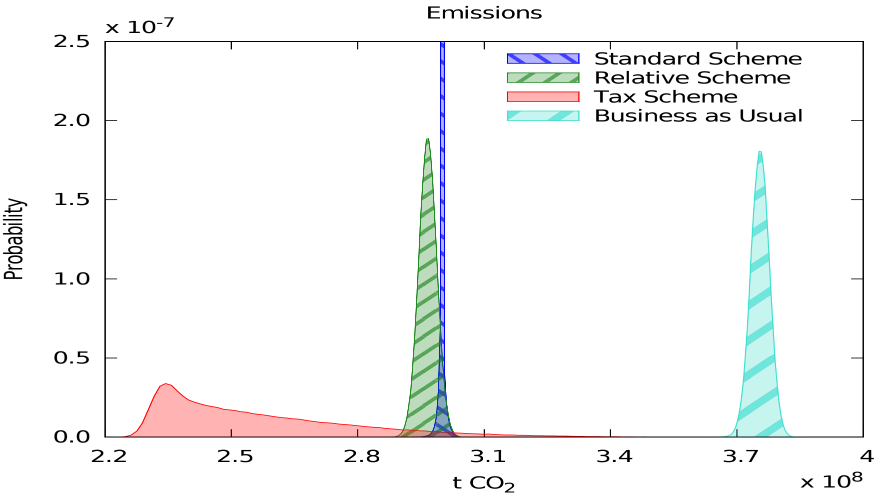
$$H^{A,S,i}(\theta^i,\xi^i) := -\sum_{t=0}^{T-1} V_t^i + \sum_{t=0}^{T-1}\sum_{(j,k)\in M_i}(S_t^k - C_t^{i,j,k} - Z_t^k)\xi_t^{i,j,k}$$
$$+\sum_{t=0}^{T-1}\theta_t^i(A_{t+1}-A_t) - \theta_T^i A_T$$
$$-\lambda\left(\Delta^i + \Pi^i(\xi^i) - \sum_{t=0}^{T-1}\left(X_t^i + \sum_{(j,k)\in M_i} Y_t^k\xi_t^{i,j,k}\right) - \theta_T^i\right)^+ . \qquad (1)$$

then in equilibrium **allowance price does not change** but

$$S_t^{\dagger k} = S_t^{*k} + Z_t^k - Y_t^k A_t^* \quad \text{for all } k\in\mathcal{K},\ t=0,\ldots,T-1 \qquad (2)$$

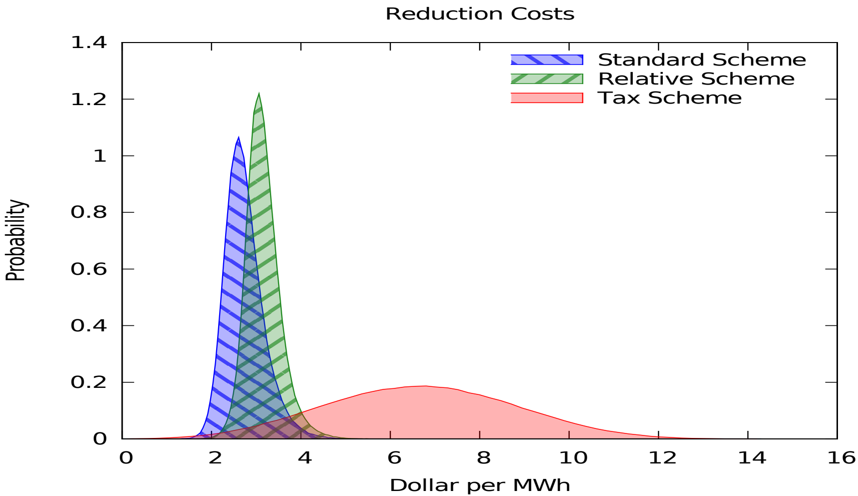
- Cost of the **tax passed along** to the end consumer
- **Proportional allocation reduces the prices** of the goods

Yearly Emissions Equilibrium Distributions



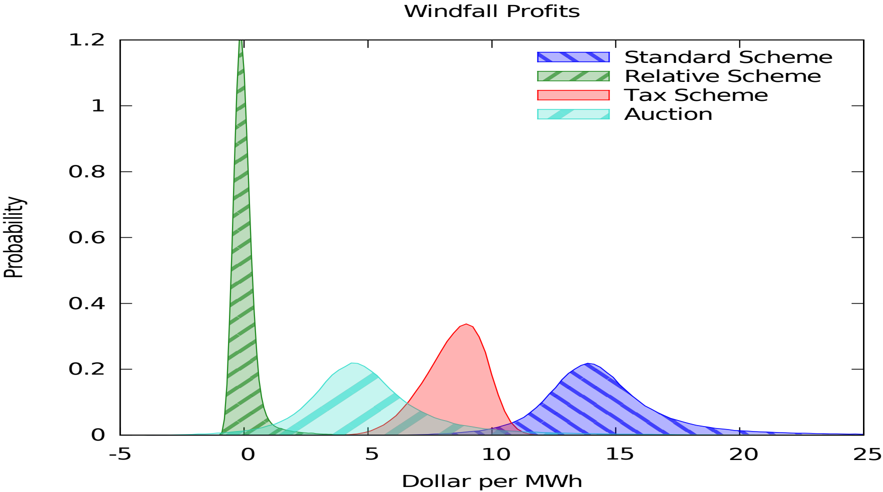
Yearly emissions from electricity production for the Standard Scheme, the Relative Scheme, a Tax Scheme and BAU.

Abatement Costs



Yearly abatement costs for the Standard Scheme, the Relative Scheme and a Tax Scheme.

Windfall Profits



Histograms of the yearly distribution of windfall profits for the Standard Scheme, a Relative Scheme, a Standard Scheme with 100% Auction and a Tax Scheme

What is Next?

- Why would we want to reduce **Windfall Profits**?
- Can one **Design** a cap-and-trade scheme to reach **Prescribed Distributions** for profits and costs?
- Optimizing **irreversible investment** decisions (installing **scrubbers, ...**)
- Need for **Partial Equilibrium** and/or **Reduced Form** Models
  - Require early **active trading**
  - Illustrate **Leakage** and/or **Market Exits**
  - Illustrate and identify **Market Impact** and/or **Manipulations**

Multi-Compliance Periods Markets

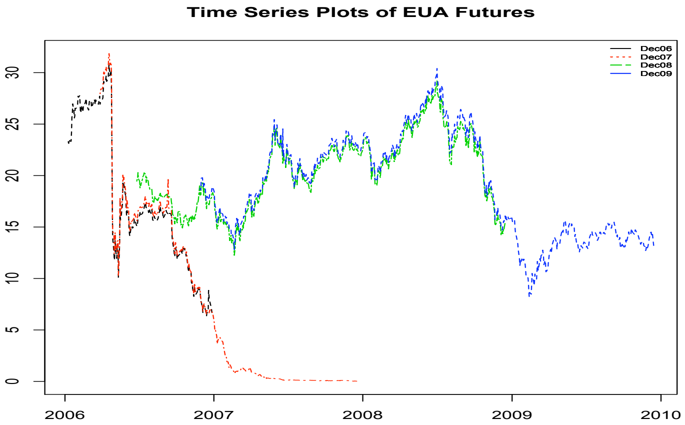


Figure: Price drop before the end of the first phase of the EU-ETS.

Rules Governing Successive Compliance Periods

- **Borrowing** allows for the transfer of a (limited) number of allowances from the next period into the present one;
- **Banking** allows for the transfer of a (limited) number of (unused) allowances from the present period into the next;
- **Withdrawal** penalizes firms which fail to comply in two ways:
  1. Penalty payment for each unit of pollutant not covered by credits
  2. Withdrawal of the missing allowances from next period allocation.

Existing markets

unlimited banking, no borrowing, withdrawal

Two-period Market Model

- Periods $[0, T]$ and $[T, T']$
- $(A_t)_{t\in[0,T]}$ futures contract with compliance at T
- $(A'_t)_{t\in[0,T']}$ futures contract with compliance at T
- $N \in \mathcal{F}_T$ non-compliance at the end of the first period
- $N' \in \mathcal{F}_{T'}$ non-compliance at the end of the first period

No arbitrage implies

$$A_T \mathbf{1}_{\Omega\setminus N} = \kappa A'_T \mathbf{1}_{\Omega\setminus N},$$

$\kappa \in (0, \infty)$ discount factor
and **withdrawal rule** implies

$$A_T \mathbf{1}_N = \kappa A'_T \mathbf{1}_N + \lambda \mathbf{1}_N.$$

$$A_t - \kappa A'_t = \mathbb{E}^{\mathbb{Q}}(A_T - \kappa A'_T \,|\, \mathcal{F}_t) = \lambda \mathbb{E}^{\mathbb{Q}}(\mathbf{1}_N \,|\, \mathcal{F}_t) \qquad t \in [0, T]$$

is a $[0, \lambda]$-valued martingale with binary terminal value!

Sample Result for the CDM and CER Prices

For an emission market $m \in M$ and a compliance period $[T_q^m, T_{q+1}^m]$

$$A^{q,m}_{T^m_q} = (\lambda^{q,m} + \mathbb{E}[A^{q+1,m}_{T^m_{q+1}} | \mathcal{F}_{T^m_q})]\mathbf{1}_{\{\beta_{T^m_q} > 0\}} \tag{3}$$
$$+ \left(\mathbb{E}[A^{q+1,m}_{T^m_{q+1}} | \mathcal{F}_{T^m_q}]\mathbf{1}_{\{\gamma_{T^m_q} > 0\}} + \mathbb{E}[C^{p+1}_{T_{q+1}} | \mathcal{F}_{T^m_q})\mathbf{1}_{\{\gamma_{T^m_q} = 0\}} \right) \mathbf{1}_{\{\beta_{T^m_q} = 0\}}$$

R.C. - M. Fehr

- When $\{\beta_{T^m_q} > 0\}$ market m is short of allowances despite the usage of CERs, the allowance price is given by the penalty $\lambda^{q,m}$ plus the cost of the allowances from the next period
- When $\{\beta_{T^m_q} = 0\}$ (not short of allowances at time of compliance) the allowance price is either the expected value of an allowance for the next period on the event $\{\gamma_{T^m_q} > 0\}$ that the allowances are banked for use in the next period, or the expected value of a CER in the next period on the event $\{\gamma_{T^m_q} = 0\}$ that the allowances are not banked.

Modeling Partial Information

Cetin-Verschuere (T=Dec-07 & T'=Dec-08 futures contracts)

- A'_t value at time t of Dec-08 EUA futures contract

$$dA'_t = A'_t[\mu + \alpha\theta_t]dt + A'_t\sigma dW_t$$

- σ, μ, α constants, $A'_0 = x$
- θ_t two-state continuous-time Markov chain **independent** of Wiener process W_t
  - $\theta_t = 1$ market is *long allowances* at time t
  - $\theta_t = -1$ market is *short allowances* at time t
- $T = Dec - 07$ end of Phase I
- A_t value at time $t \leq T$ of Dec-07 EUA futures contract

$$A_T = \begin{cases} A'_T + \lambda & \text{if } \theta_T \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

- Pricing & Hedging in **Incomplete Market** (**two** sources of randomness, **one** underlier)

Filtering Techniques

- Observe $\mathcal{F}^{A'} = \{\mathcal{F}^{A'}_t\}_t$ filtration of S_t
- One time announcement of true value of θ at time T

$$\mathcal{G}_t = \begin{cases} \mathcal{F}^{A'}_t & \text{for } t < T \\ \mathcal{F}^{A'}_t \vee \sigma(\theta_T) & \text{for } t = T \end{cases}$$

- Optional projection $\overline{\theta}_t = \mathbb{E}\{\theta_t | \mathcal{F}^{A'}_t\}$
- $\overline{W}_t = \int_0^t \frac{1}{\sigma A'_s}[dA'_s - (\mu - \alpha\overline{\theta}_s)A'_s ds]$ is a $\mathcal{G}$ Brownian motion
- $d\overline{\theta}_t = -2\lambda\overline{\theta}_t dt + \frac{\alpha}{\sigma}(1 - \overline{\theta}_t^2)d\overline{W}_t$ with $\overline{\Lambda} = 2p - 1$, and $p = \mathbb{P}\{\Lambda = 1\}$.
- $Z_t = \mathbf{1}_{\{t=T\}}(\theta_T - \overline{\theta}_T)$ is a $\mathcal{G}$ martingale orthogonal to $\overline{W}$
- $\overline{A}_t$ fair price of A_t

$$\overline{A}_t = \mathbb{E}^*\{\frac{1-\theta_T}{2}(A'_T + \lambda)|\mathcal{G}_t\}$$

where $\mathbb{E}^*$ is expectation w.r.t. **minimal martingale measure** $\mathbb{P}^*$ (**Foellmer-Schweizer**)

What Happened in April 06? Special Announcement

- **TRUE** value θ_{t_0} of θ_t revealed at time t_0
- Replace $\overline{\theta}_t$ by $\tilde{\theta}_t = \mathbb{E}\{\theta_t | \mathcal{F}_t^{A'}, \theta_{t_0}\}$ for $t > t_0$
- Fair price of T=Dec-07 contract now given by

$$A_t = \begin{cases} Z_t^h + h(t, S_t, \overline{\theta}_t) & \text{for } t < t_0 \\ h(t, S_t, \overline{\theta}_t) - Z_t(S_t + \lambda)/2 & \text{for } t > t_0 \end{cases}$$

and

$$\Delta A_{t_0} = h(t_0, A'_{t_0}, \theta_{t_0}) - h(t_0, A'_{t_0}, \overline{\theta}_{t_0})$$

where h is the solution of a specific PDE (full observation model) and

$$Z_t^h = \mathbb{E}^*\{h(t_0, A'_{t_0}, \theta_{t_0}) - h(t_0, A'_{t_0} | \mathcal{G}_t\}$$

Explicit formula for the size of the jump in price!

Reduced Form Models & Option Pricing

(**Uhrig-Homburg-Wagner, R.C - Hinz**)

- Emissions Cap-and-Trade Markets **SOON** to exist in the US (and Canada, Australia, Japan, ...)
- Liquid **Option** Market **ALREADY** exists in Europe
    - Underlying $\{A_t\}_t$ non-negative martingale with **binary terminal value**
    - Think of A_t as of a binary option
    - Underlying of binary option should be *Emissions*
- Need for **Formulae** (closed or computable)
    - Prices and Hedges difficult to compute (only numerically)
    - Jumps due to announcements (**Cetin** et al.)
- **Reduced Form Models**

Option quotes on Jan. 3, 2008

Option Maturity	Option Type	Volume	Strike	Allowance Price	Implied Vol	Settlement Price
Dec-08	Call	150,000	24.00	23.54	50.50%	4.19
Dec-08	Call	500,000	26.00	23.54	50.50%	3.50
Dec-08	Call	25,000	27.00	23.54	50.50%	3.20
Dec-08	Call	300,000	35.00	23.54	50.50%	1.56
Dec-08	Call	1,000,000	40.00	23.54	50.50%	1.00
Dec-08	Put	200,000	15.00	23.54	50.50%	0.83

Option quotes on Jan. 4, 2008

Option Maturity	Option Type	Volume	Strike	Allowance Price	Implied Vol	Settlement Price
Dec-08	Cal	200,000	22.00	23.55	51.25%	5.06
Dec-08	Call	150,000	26.00	23.55	51.25%	3.57
Dec-08	Call	450,000	27.00	23.55	51.25%	3.27
Dec-08	Call	100,000	28.00	23.55	51.25%	2.99
Dec-08	Call	125,000	29.00	23.55	51.25%	2.74
Dec-08	Call	525,000	30.00	23.55	51.25%	2.51
Dec-08	Call	250,000	40.00	23.55	51.25%	1.04
Dec-08	Call	700,000	50.00	23.55	51.25%	0.45
Dec-08	Put	1,000,000	14.00	23.55	51.25%	0.64
Dec-08	Put	200,000	15.00	23.55	51.25%	0.86
Dec-08	Put	200,000	15.00	23.55	51.25%	0.86
Dec-08	Put	400,000	16.00	23.55	51.25%	1.13
Dec-08	Put	100,000	17.00	23.55	51.25%	1.43
Dec-08	Put	1,000,000	18.00	23.55	51.25%	1.78
Dec-08	Put	500,000	20.00	23.55	51.25%	2.60
Dec-08	Put	200,000	21.00	23.55	51.25%	3.07
Dec-08	Put	200,000	22.00	23.55	51.25%	3.57

Reduced Form Models and Calibration

Allowance price should be of the form

$$A_t = \lambda \mathbb{E}\{\mathbf{1}_N \mid \mathcal{F}_t\}$$

for a non-compliance set $N \in \mathcal{F}_t$. Choose

$$N = \{\Gamma_T \geq 1\}$$

for a random variable Γ_T representing the normalized emissions at compliance time. So

$$A_t = \lambda \mathbb{E}\{\mathbf{1}_{\{\Gamma_T \geq 1\}} \mid \mathcal{F}_t\}, \qquad t \in [0, T]$$

We choose Γ_T in a parametric family

$$\Gamma_T = \Gamma_0 \exp\left[\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2\, ds\right]$$

for some square integrable deterministic function

$$(0, T) \ni t \hookrightarrow \sigma_t$$

Dynamic Price Model for $a_t = \frac{1}{\lambda} A_t$

- a_t is given by

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \qquad t \in [0, T)$$

  where Φ is standard normal c.d.f.

- a_t solves the SDE

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{z_t}dW_t$$

  where the positive-valued function $(0, T) \ni t \hookrightarrow z_t$ is given by

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \qquad t \in (0, T)$$

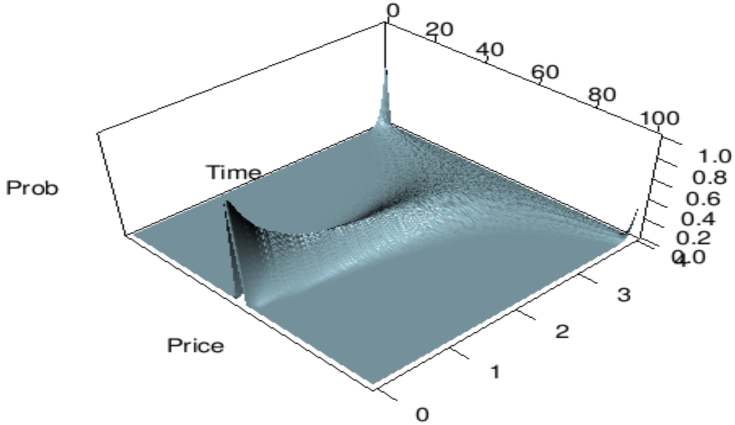


Figure: Histograms for each day of a 4 yr compliance period of 10^5 simulated risk neutral allowance price paths.

Aside: Binary Martingales as Underliers

Allowance prices are given by $A_t = \lambda a_t$ where $\{a_t\}_{0 \le t \le T}$ satisfies

- $\{a_t\}_t$ is a martingale
- $0 \le a_t \le 1$
- $\mathbb{P}\{\lim_{t \to T} a_t = 1\} = 1 - \mathbb{P}\{\lim_{t \to T} a_t = 0\} = p$ for some $p \in (0, 1)$

The model

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{z_t} dW_t$$

suggests looking for martingales $\{Y_t\}_{0 \le t < \infty}$ satisfying

- $0 \le Y_t \le 1$
- $\mathbb{P}\{\lim_{t \to \infty} Y_t = 1\} = 1 - \mathbb{P}\{\lim_{t \to \infty} T_t = 0\} = p$ for some $p \in (0, 1)$

and do a **time change** to get back to the (compliance) interval $[0, T)$

Gives conditions for the SDE

$$da_t = \Theta(a_t)dW_t$$

for $x \hookrightarrow \Theta(x)$ satisfying

- $\Theta(x) > 0$ for $0 < x < 1$
- $\Theta(0) = \Theta(1) = 0$

to

- Converge to the boundaries 0 and 1
- NOT explode (i.e. NOT reach the boundaries in finite time)

Interestingly enough the solution of

$$dY_t = \Phi'(\Phi^{-1}(Y_t))dW_t$$

IS ONE OF THEM !

Explicit Examples

The SDE

$$dX_t = \sqrt{2}dW_t + X_t dt$$

has the solution

$$X_t = e^t\left(x_0 + \int_0^t e^{-s}dW_s\right)$$

and

$$\lim_{t\to\infty} X_t = +\infty \qquad \text{on the set } \left\{\int_0^\infty e^{-s}dW_s > -x_0\right\}$$

$$\lim_{t\to\infty} X_t = -\infty \qquad \text{on the set } \left\{\int_0^\infty e^{-s}dW_s < -x_0\right\}$$

Moreover Φ is **harmonic** so if we choose

$$Y_t = \Phi(X_t)$$

we have a martingale with the desired properties.

Another (explicit) example can be constructed from Ph. Carmona, Petit and Yor on Dufresne formula.

Calibration

Has to Be Historical !!!!

- Choose **Constant** Market Price of Risk
- **Two-parameter** Family for Time-change

$$\{z_t(\alpha,\beta) = \beta(T-t)^{-\alpha}\}_{t\in[0,T]}, \qquad \beta > 0, \alpha \geq 1.$$

Volatility function $\{\sigma_t(\alpha,\beta)\}_{t\in(0,T)}$ given by

$$\begin{aligned}
\sigma_t(\alpha,\beta)^2 &= z_t(\alpha,\beta)e^{-\int_0^t z_u(\alpha,\beta)du} \\
&= \begin{cases} \beta(T-t)^{-\alpha}e^{\beta\frac{T^{-\alpha+1}-(T-t)^{-\alpha+1}}{-\alpha+1}} & \text{for } \beta > 0, \alpha > 1 \\ \beta(T-t)^{\beta-1}T^{-\beta} & \text{for } \beta > 0, \alpha = 1 \end{cases}
\end{aligned}$$

Maximum Likelihood

Call Option Price in One Period Model

for $\alpha = 1$, $\beta > 0$, the price of an European call with strike price $K \geq 0$ written on a one-period allowance futures price at time $\tau \in [0, T]$ is given at time $t \in [0, \tau]$ by

$$
\begin{aligned}
C_t &= e^{-\int_t^\tau r_s ds} \mathbb{E}\{(A_\tau - K)^+ \mid \mathcal{F}_t\} \\
&= \int (\lambda \Phi(x) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{t,\tau} &= \Phi^{-1}(A_t/\lambda) \sqrt{\left(\frac{T-t}{T-\tau}\right)^\beta} \\
\nu_{t,\tau} &= \left(\frac{T-t}{T-\tau}\right)^\beta - 1.
\end{aligned}
$$

Easily extended to several periods

Price Dependence on T and Sensitivity to β

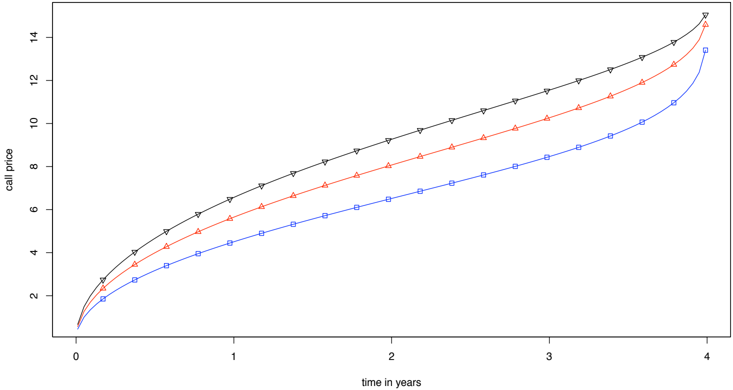
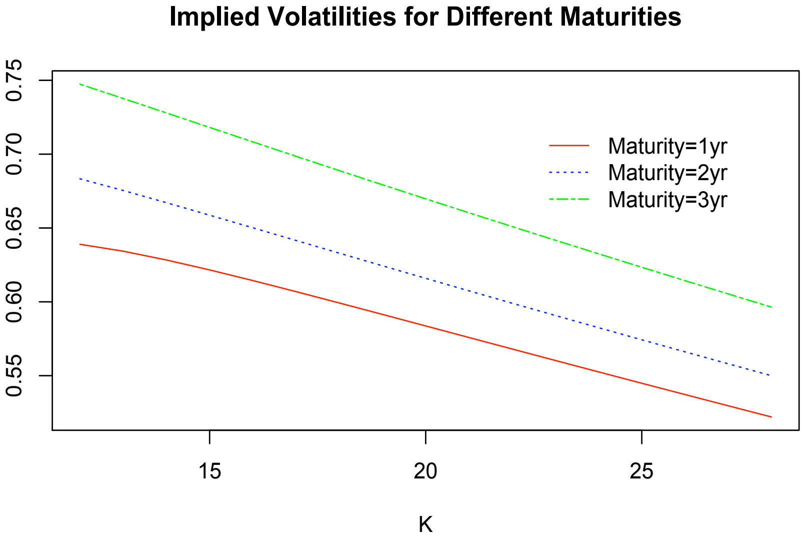
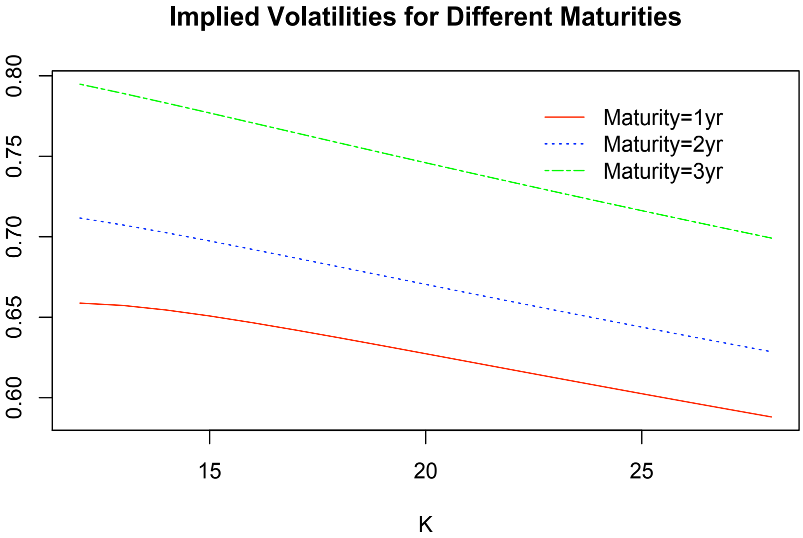


Figure: Dependence $\tau \mapsto C_0(\tau)$ of Call prices on maturity τ. Graphs $\square$, $\triangle$, and $\triangledown$ correspond to $\beta = 0.5$, $\beta = 0.8$, $\beta = 1.1$.

Implied Volatilities $\beta = 1.2$



Implied Volatilities $\beta = 0.6$, $\lambda = 100$



Option quotes on April 9, 2010

With a Smile Now!

Option Maturity	Option Type	Volume	Strike	Allowance Price	Implied Vol	Settlement Price
Dec-10	Call	750,000	14.00	13.70	29.69	1.20
Dec-10	Call	150,000	15.00	13.70	29.89	0.85
Dec-10	Call	250,000	16.00	13.70	30.64	0.61
Dec-10	Call	250,000	18.00	13.70	32.52	0.34
Dec-10	Call	1,000,000	20.00	13.70	33.07	0.17
Dec-10	Put	1,000,000	10.00	13.70	37.42	0.29
Dec-10	Put	500,000	12.00	13.70	32.12	0.67
Dec-10	Put	500,000	13.00	13.70	30.37	1.01

- Relax **demand inelasticity**
- Include preferences to relax **risk neutrality** (**Touzi et al., RC-Espinosa-Touzi**)
- "Representative Agent" form already considered in **Seifert-Uhrig-Homburg-Wagner**, **RC-Fehr-Hinz**

Mathematical Set-Up (continuous time)

- $(\Omega, \mathcal{F}, \mathbb{P})$ **historical** probability structure
- W D-dimensional Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$
- $T > 0$ finite horizon (end of the **single** compliance period)
- $\mathbb{F} = \{\mathcal{F}_t;\ 0 \leq t \leq T\}$ filtration of W

Goal of equilibrium analysis is to derive pollution permit price $\{A_t;\ 0 \leq t \leq T\}$ allowing firms to **maximize their expected utilities simultaneously**

Emissions Dynamics

Assume allowance price $A = \{A_t;\ 0 \le t \le T\}$ exists.

- A is a $\mathbb{F}$-martingale under $\mathbb{Q}$
- $dA_t = Z_t\, dB_t$ for some adapted process Z s.t. $Z_t \neq 0$ a.s. and B D-dim Wiener process for spot martingale measure $\mathbb{Q}$
- $A_T = \lambda \mathbf{1}_{[\Lambda,\infty)}(E_T)$ where
    - λ is the penalty
    - $E_t = \sum_{i \in \mathcal{I}} E_t^i$ is the aggregate of the E_t^i representing the **cumulative emission** up to time t of firm i
    - Λ is the cap imposed by the regulator

Assume the following dynamics **under** $\mathbb{P}$

$$dE_t^i = (b_t^i - \xi_t^i)dt + \sigma_t^i dW_t, \qquad E_0^i = 0.$$

- $\{E_t^i(\xi_t^i \equiv 0)\}_{0 \le t \le T}$ cumulative emissions of firm i in BAU
- $\{\xi_t^i)\}_{0 \le t \le T}$ abatement rate of firm i
- Assumptions on emission rates b_t^i and volatilities σ_t^i to be articulated later

Individual Firm Optimization Problems

Abatement costs for firm i given by cost function $c_t^i : \mathbb{R} \to \mathbb{R}$

- c^i is C^1 and strictly convex
- c^i satisfies Inada-like conditions for each $t \in [0, T]$

$$(c^i)'(-\infty) = -\infty \qquad \text{and} \qquad (c^i)'(+\infty) = +\infty.$$

- $c^i(0) = \min c_t^i$ ($\xi^i \equiv 0$ corresponds to BAU)

Typical example for c^i

$$\lambda |x|^{1+\alpha},$$

for some $\lambda > 0$ and $\alpha > 0$.

Each firm chooses its **abatement strategy** ξ^i and its **investment** θ^i in allowances. Its **wealth** is given by

$$X_t^i = X_t^{i,\xi,\theta} = x^i + \int_0^T \theta_t^i dA_t - \int_0^T c^i(\xi_t^i)dt - E_T^i A_T.$$

Solving the Individual Firm Optimization Problems

Preferences of firm i given by a C^1, increasing, strictly concave **utility function** $U^i : \mathbb{R} \to \mathbb{R}$ satisfying Inada conditions:

$$(U^i)'(-\infty) = +\infty \qquad \text{and} \qquad (U^i)'(+\infty) = 0.$$

The optimization problem for firm i is:

$$V(x^i) := \sup_{(\xi^i,\theta^i)\in\mathcal{A}^i} \mathbb{E}^{\mathbb{P}}\{U^i(X_T^{i,\xi^i\theta^i})\}$$

If no non-standard restriction on $\mathcal{A}^i$ set of admissible strategies for firm i

Proposition

If an equilibrium allowance price $\{A_t\}_{0\le t\le T}$ exists, then the optimal abatement strategy $\hat{\xi}^i$ is given by

$$\hat{\xi}_t^i = [(c^i)']^{-1}(A_t).$$

NB: The optimal abatement strategy $\hat{\xi}^i$ is independent of the utility function U^i!

Finding the Equilibrium Allowance Price

Complete Market Intuition $\Longrightarrow$ **Representative Agent** (Informed Central Planner) approach

- Recall

$$dE_t^i = \left[\tilde{b}_t^i - [(c^i)']^{-1}(A_t)\right] dt + \sigma_t^i dB_t, \qquad E_0^i = 0, \text{ for each } i$$

- Assume
  - $\forall i, \tilde{b}_t^i = \tilde{b}^i(t)E_t^i$ or $\forall i, \tilde{b}_t^i = \tilde{b}^i(t)$
  - $\forall i, \sigma_t^i = \sigma^i(t)$.
- Set

$$b := \sum_{i\in\mathcal{I}} \tilde{b}^i, \ \sigma := \sum_{i\in\mathcal{I}} \sigma^i, \text{ and } f := \sum_{i\in\mathcal{I}} [(c^i)']^{-1}.$$

Therefore we have the following FBSDE

$$dE_t = \{b(t, E_t) - f_t(A_t)\}dt + \sigma(t)dB_t, \qquad E_0 = 0 \tag{4}$$

$$dA_t = Z_t dB_t, \qquad A_T = \lambda \mathbf{1}_{[\kappa,+\infty)}(E_T), \tag{5}$$

with $b(t, E_t) = b(t)E_t^\beta$ with $\beta \in \{0, 1\}$ and f increasing.

Theoretical Existence and Uniqueness

Theorem

If $\sigma(t) \geq \underline{\sigma} > 0$ then for any $\lambda > 0$ and $\kappa \in \mathbb{R}$, FBSDE (4)-(5) admits a unique solution $(E, A, Z) \in M^2$. Moreover, A_t is nondecreasing w.r.t λ and nonincreasing w.r.t κ.

Proof

- Approximate the singular terminal condition $\lambda \mathbf{1}_{[\kappa,+\infty)}(E_T)$ by increasing and decreasing sequences $\{\varphi_n(E_T)\}_n$ and $\{\psi_n(E_T)\}_n$ of smooth monotone functions of E_T
- Use
    - comparison results for BSDEs
    - the fact that E_T has a density

  to control the limits

PDE Characterization

Assume GBM for BAU emissions (**Chesney-Taschini**, **Seifert-Uhrig-Homburg-Wagner**) i.e. $b(t,e) = be$ and $\sigma(t,e) = \sigma e$

$$\begin{cases} E_t = E_0 + \int_0^t (bE_s - f(Y_s))ds + \int_0^t \sigma E_s d\tilde{W}_s \\ A_t = \lambda \, \mathbf{1}_{[\Lambda,\infty)}(E_T) - \int_t^T Z_t d\tilde{W}_t. \end{cases} \tag{6}$$

Allowance price A_t constructed as $A_t = v(t, E_t)$ for a function v which **MUST** solve

$$\begin{cases} \partial_t v(t,e) + (be - f(v(t,e)))\partial_e v(t,e) + \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} v(t,e) = 0, \\ v(T,.) = \mathbf{1}_{[\Lambda,\infty)} \end{cases} \tag{7}$$

The price at time t of a **call option** with maturity τ and strike K on an allowance forward contract maturing at time $T > \tau$ is given by

$$V(t, E_t) = \mathbb{E}_t\{(Y_\tau - K)^+\} = \mathbb{E}_t\{(v(\tau, E_\tau) - K)^+\}.$$

V solves:

$$\begin{cases} \partial_t V(t,e) + (be - f(v(t,e)))\partial_e V(t,e) + \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} V(t,e) = 0, \\ V(\tau,.) = (v(\tau,.) - K)^+ \end{cases} \tag{8}$$

Black-Scholes Case: $f \equiv 0$.

$$v^0(t,e) = \lambda \mathbb{P}\left[E_T^0 \geq \Lambda | E_t^0 = e\right] = \lambda \Phi\left(\frac{\ln(e/\Lambda e^{-b(T-t)})}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)$$

$$V^0(t,e) = \mathbb{E}\left[(v^0(\tau, E_\tau^0) - K)^+ | E_t^0 = e\right],$$

where E^0 is the geometric Brownian motion:

$$dE_t^0 = E_t^0[bdt + \sigma d\tilde{W}_t].$$

used as proxy estimation of the cumulative emissions in business as usual.

R.C. - Espinosa - Touzi For $\epsilon \geq 0$ small, let v^ϵ and V^ϵ be the prices of the allowances and the option for $f = \epsilon f_0$. We denote by .

$$v^\epsilon(T,.) = \lambda \mathbf{1}_{[\Lambda,\infty)} \quad \text{and} \quad -\partial_t v^\epsilon - (be - \epsilon f_0(v^\epsilon))\partial_e v^\epsilon - \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} v^\epsilon = 0,$$

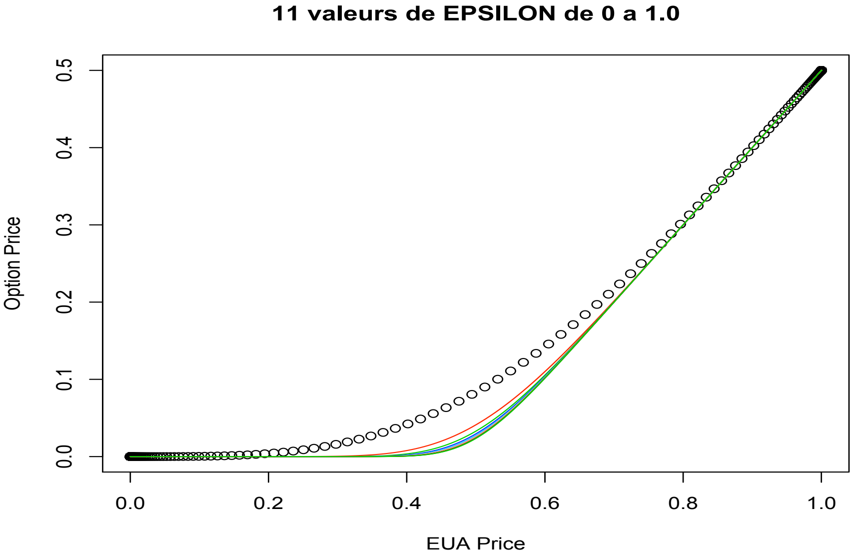
$$V^\epsilon(T,.) = (v^\epsilon(T,.) - K)^+ \quad \text{and} \quad -\partial_t V^\epsilon - (be - \epsilon f_0(v^\epsilon))\partial_e V^\epsilon - \frac{1}{2}\sigma^2 e^2 \partial^2_{ee} V^\epsilon = 0,$$

Proposition

As $\epsilon \to 0$, we have

$$V^\epsilon(t,s) = V^0(t,s)$$
$$+\epsilon\, \mathbb{E}_{t,e}\left[\mathbf{1}_{[\Lambda,\infty)}(v^0)(\tau, E^0_\tau)\int_t^T f_0(v^0)(s, E^0_s)\partial_e v^0(s\vee\tau, E^0_{s\vee\tau})\frac{E^0_{s\vee\tau}}{E^0_s}ds\right]$$
$$+\circ(\epsilon),$$

Implied Volatilities $\beta = 1.2$



A Slightly Different Model

Single good (e.g. **electricity**) regulated economy, with price dynamics given **exogenously**!

$$\frac{dP_t}{P_t} = \mu(t, P_t)dt + \sigma(t, P_t)dW_t$$

Firm i

- Controls its *instataneous rate of production* q_t^i
- **Production** over $[0, t]$

$$Q_t^i := \int_0^t q_t^i dt.$$

- **Costs of production** given by $c_t^i : \mathbb{R}_+ \mapsto \mathbb{R}$ C^1 strictly convex satisfying Inada-like conditions

$$(c_t^i)'(0) = 0, \qquad (c_t^i)'(+\infty) = +\infty$$

- **Cumulative emissions** $E_t^i := e^i Q_t^i$
- **P&L** (wealth)

$$X_t^i = X_t^{i,q^i,\theta^i} = x^i + \int_0^T \theta_t^i dA_t - \int_0^T [P_t q_t^i - c_t^i(q_t^i)]dt - e^i Q_T^i A_T.$$

Individual Firm Optimization Problem

Proposition

If such an equilibrium exits, the optimal production strategy $\hat{q}^i$ is given by:

$$\hat{q}^i_t = [(c^i)']^{-1}(P_t - e^i Y_t).$$

NB: As before the optimal production schedule $\hat{q}^i$ **DOES NOT DEPEND** upon the utility function!

Existence of Allowance Equilibrium Prices

- Set $E_t := \sum_{i \in \mathcal{I}} E_t^i$ for the total aggregate emissions up to time t
- Define $f(p, y) := \sum_{i \in \mathcal{I}} \varepsilon^i [(c^i)']^{-1}(p - \varepsilon^i y)$

Then the corresponding FBSDE under $\mathbb{Q}$ reads

$$\begin{cases} dP_t &= \sigma(t, P_t) dB_t, \quad P_0 = p \\ dE_t &= f(P_t, A_t) dt, \quad E_0 = 0 \\ dA_t &= Z_t dB_t, \quad A_T = \lambda \mathbf{1}_{[\kappa, +\infty)}(E_T). \end{cases}$$

NB: The volatility of the forward equation is **degenerate!**

Still, **Natural Conjecture**: For $\lambda > 0$ and $\kappa \in \mathbb{R}$, the above FBSDE has a unique solution (P, E, A, Z).

An Enlightening Example (R.C. - Delarue)

$$\begin{cases} dP_t = dW_t, & P_0 = p \\ dE_t = (P_t - A_t)\,dt, & E_0 = e \\ dA_t = Z_t dW_t, \quad 0 \le t \le T, & A_T = \mathbf{1}_{[\Lambda,\infty)}(E_T) \end{cases} \tag{9}$$

Theorem

- There exists a unique progressively measurable triple $(P_t, E_t, A_t)_{0 \le t \le T}$ satisfying (9) and

$$\mathbf{1}_{(\Lambda,\infty)}(E_T) \le A_T \le \mathbf{1}_{[\Lambda,\infty)}(E_T).$$

- The marginal distribution of E_t
  - is absolutely continuous for $0 \le t < T$
  - has a Dirac mass at Λ when $t = T$, $\mathbb{P}\{E_T = \Lambda\} > 0$.

The terminal condition $A_T = \mathbf{1}_{[\Lambda,\infty)}(E_T)$ may not be satisfied!

Lectures based on)

1. R.C., M. Fehr and J. Hinz: Mathematical Equilibrium and Market Design for Emissions Markets Trading Schemes. *SIAM J. Control and Optimization* (2009)
2. R.C., M. Fehr, J. Hinz and A. Porchet: Mathematical Equilibrium and Market Design for Emissions Markets Trading Schemes. *SIAM Review* (2010)
3. R.C., M. Fehr and J. Hinz: Properly Designed Emissions Trading Schemes do Work! (working paper)
4. R.C., and J. Hinz: Risk-Neutral Modeling of Emission Allowance Prices and Option Valuation (working paper)
5. R.C. & M. Fehr: Auctions and Relative Allocation Mechanisms for Cap-and-Trade Schemes (working paper)
6. R.C. & M. Fehr: The Clean Development Mechanism: a Mathematical Model. *Proc. 2008 Ascona Conf.*
7. R.C., G.E. Espinosa and N. Touzi: BSDEs and Option Pricing for the Emissions Markets! (working paper)
8. R.C., and F. Delarue: Limiting Behavior of Singular BSDEs (working paper)