A mean-field excitatory network for risk modeling

Séminaire FIME

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23 Janvier 2015

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Motivation

- Mathematical model
  - interacting companies
  - propagation of defaults?
- Interactions between N companies
  - company No. i defaults $\Rightarrow$ other companies $j \neq i$ feel it
  - interaction may be

$$j \text{ more } \underbrace{\text{likely to}}_{\text{excitatory}} \text{ default} \quad / \quad j \text{ less } \underbrace{\text{likely to}}_{\text{inhibitory}} \text{ default}$$

- Model of interactions
  - mean-field interactions with instantaneous effect

Dynamics of one single company

- State of one single company $\rightsquigarrow$ wealth
- wealth dynamics of the company

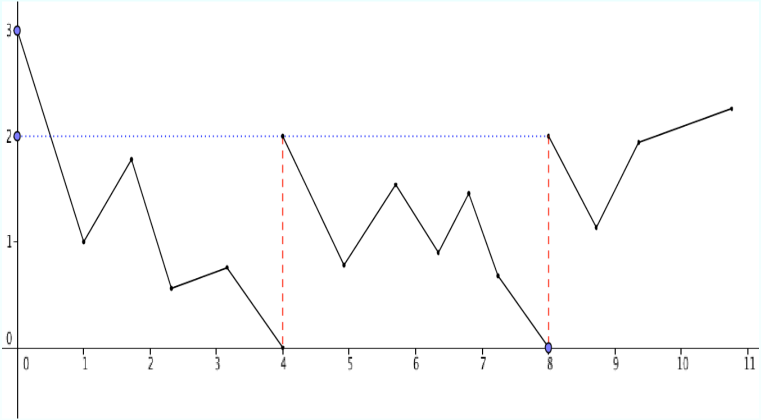
$$V_t = V_0 + \int_0^t b(V_s)ds + I_t + W_t$$

  - $(I_t)_{t \geq 0}$ $\rightsquigarrow$ environment, $(W_t)_{t \geq 0}$ $\rightsquigarrow$ Brownian noise
  - numerical example: $b(x) = -\lambda(x - x_0)$
  - company defaults when V_t hits threshold V_D

$$\tau = \inf\{t \geq 0 : V_t \leq V_D\}$$

  - after bankruptcy $\rightsquigarrow$ stop the modeling
or
wealth is reset to V_R and restart...
  - if reset $\rightsquigarrow$ reset is instantaneous

Example



Model inspired from Neurosciences

- State of one single neuron $\rightsquigarrow$ potential
  - different concentration of ions inside and outside neuron
- integrate and fire dynamics dynamics of the potential

$$V_t = V_0 + \int_0^t b(V_s)ds + I_t + W_t$$

  - $(I_t)_{t\geq 0} \rightsquigarrow$ signal, $(W_t)_{t\geq 0} \rightsquigarrow$ Brownian noise
  - neuron spikes when V_t hits threshold V_F

$$\tau = \inf\{t \geq 0 : V_t \geq V_F\}$$

  - after spike $\rightsquigarrow$ potential is reset to V_R and restart...

Mean-field interaction

- Model with interactions: N companies $V_t^1, \dots, V_t^N$
  - $I_t^i \rightsquigarrow I_t^i(V^j, j \neq i)$ (signal depending on other wealths)
  - $I_t^i(V^j, j \neq i)$ depending on the empirical distribution

$$I_t^i(V^j, j \neq i) = I_t^i\left(N^{-1} \sum_{j \neq i} \delta_{V^j}\right)$$

  - $(W_t^i)_{t \geq 0} \rightsquigarrow$ independent noises on each neuron/company
- Example: $I_t^i(V^j, j \neq i) = -\frac{\alpha}{N} \sum_{j \neq i} \sharp\{\text{defaults}(j) \leq t\}$

$$\begin{aligned} I_t^i(V^j, j \neq i) &- I_{t-}^i(V^j, j \neq i) \\ &= -\frac{\alpha}{N} \sum_{j \neq i} \sharp\{\text{defaults}(j) = t\} \end{aligned}$$

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  - j defaults, $j \neq i \rightsquigarrow$ instantaneous jump of $-\frac{\alpha}{N}$ in V^i
  - excitatory (>0) or inhibitory (<0) interaction
  - α independent of $i \rightsquigarrow$ exchangeable companies

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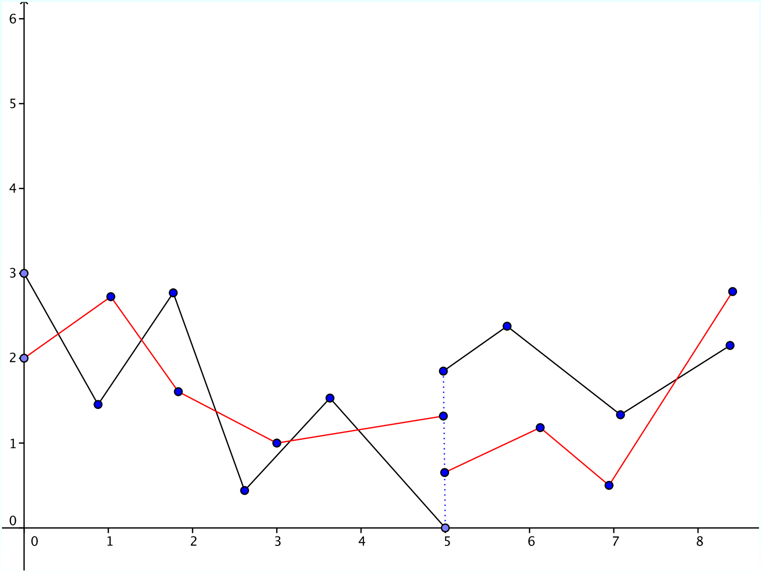
  - dynamics before default (restarts from V_R after default)

$$V_t^i = V_0^i + \int_0^t b(V_s^i)ds - \frac{\alpha}{N} \sum_{j \neq i} \sharp\{\text{defaults}(j) \leq t\} + W_t^i$$

Example



Averaging principle

- Asymptotic model when $N \to +\infty$? McKean-Vlasov version?
  - N particles $\rightsquigarrow$ one typical particle interacting with its law?
- Usual setting

$$V_t^i = V_0^i + \int_0^t b(V_s^i)ds + \int_0^t I\Big(\frac{1}{N}\sum_{j\neq i}\delta_{V_s^j}\Big)ds + dW_t^i$$

  - expect decorrelation between companies as $N \to +\infty$
  - exchangeability + decorrelation $\Rightarrow$ expect LLN

$$\int_0^t I\Big(\frac{1}{N}\sum_{j\neq i}\delta_{V_s^j}\Big)ds \sim \int_0^t I(\mathcal{L}(V_s))ds$$

- Typical company as $N \to \infty$

$$dV_t = b(V_t)dt + I(\mathcal{L}(V_t))dt + dW_t$$

Averaging principle

- Asymptotic model when $N \to +\infty$? McKean-Vlasov version?
  - N particles $\rightsquigarrow$ one typical particle interacting with its law?
- Instantaneous interaction is highly-singular
  - does not satisfy usual McKean-Vlasov requirements
- Heuristics

$$I_t^i(V^j, j \neq i) \underset{N \to +\infty}{\sim} -\alpha \mathbb{E}(M_t)$$

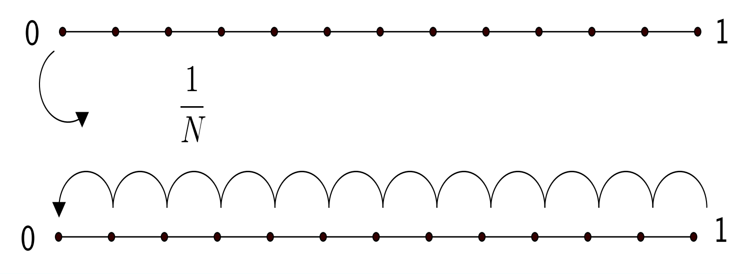
  - M_t = number of defaults for typical company up to t
- Typical company for $N \to \infty$ (before default)

$$V_t = V_0 + \lambda \int_0^t b(V_s) ds - \alpha \mathbb{E}(M_t) + W_t$$

  - $M_t = \sharp\{t \geq 0 : V_{t-} = V_D\}$ depends on V!
  - if no reset $\Rightarrow \mathbb{E}(M_t) \rightsquigarrow \mathbb{P}(\text{default} \leq t)$

Mathematical question

- Well-posedness and influence of the excitation parameter α?
- Example: runaway behavior if reset ($V_R = 1$, $V_D = 0$)
  - choose $\alpha = 1$ and $V_0^i = i/N$, $\quad i = 0, \ldots, N-1$,



  - particles keep jumping!
  - $\alpha < 1 \Rightarrow$ no way for defaulting twice at same time
- Behavior of the mean-field model when $\alpha < 1$?

Mean-field model

- Dynamics (with reset)

$$V_t = V_R + \int_0^t b(V_s)ds - \alpha\mathbb{E}(M_t) + W_t$$

  - default value $V_D = 0$, reset (after default) $V_R = 1$

- Crucial question: what class of admissible solutions?

  - class of solutions dictates regularity for $\mathbb{E}(M_t)$

$$\mathbb{E}(M_{t+h} - M_t)$$

$\sim_{N=\infty}$ probability of default in $[t, t+h]$

$\sim_{N<\infty}$ proportion of companies default in $[t, t+h]$

  - $\mathbb{E}(M_t)$ is allowed to jump $\leftrightarrow$ large proportion of companies may default at the same time

  - may stand for a massive default in the system

Instantaneous default rate

- Meaning for requiring $e : t \mapsto \mathbb{E}(M_t)$ to be differentiable?

$$\text{probability of default in } [t, t+h] \sim e'(t)h$$

- Dynamics of V (before default) if differentiability

$$dV_t = b(V_t)dt - \alpha e'(t)dt + dW_t$$

  - SDE $\rightsquigarrow$ stochastic calculus and regularizing effect
  - $\mathbb{P}(V_t \in dy) = p(t,y)dy, \quad t > 0, \quad y > 0$

- Fokker Planck equation

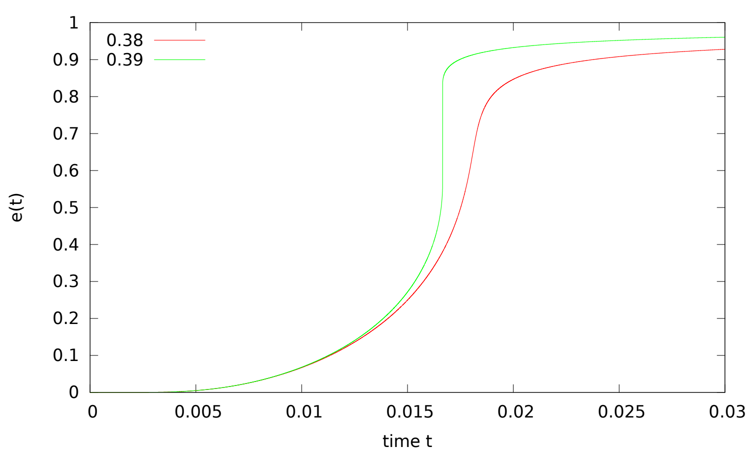
$$\partial_t p(t,y) + \partial_y \big[\big(b(y) - \alpha e'(t) \big) p(t,y) \big] - \frac{1}{2}\partial^2_{yy} p(t,y) = e'(t)\delta_1$$

  - $p(t,0) = 0$ and $\partial_y p(t,0) = \frac{1}{2} e'(t)$
  - control of e' $\Leftrightarrow$ control of the mass near 0

Solvability of the regular model

- Existence of regular solutions in arbitrary time?
  - avoid blow-up of e' in finite time?
  - $\Leftrightarrow$ avoid massive defaults?
- Caceres, Carrillo, Perthame (2011)
  - for any $\alpha > 0$, $\exists V_0 > 0$ such that blow-up in finite time!
- D., Inglis, Rubenthaler and Tanré (2014)
  - for $V_0 > 0$, $\exists!$ solution without blow-up for α small enough
  - explicit (but non-optimal) bounds on critical values α
- Brownian example: $b = 0$ and $V_0 = .2$ ($V_D = 0$, $V_R = 1$)
  - existence of regular solutions if $\alpha \leq 0.10$
  - no regular solutions if $\alpha \geq 0.54$
  - numerically, critical value $\sim 0.38\ldots$
- Exemple O-U $\lambda \to \infty \Rightarrow$ critical $\alpha \to 1$ ($\Leftrightarrow \lambda$ fixed and $\sigma \to 0$)

Illustration



Model with a common noise

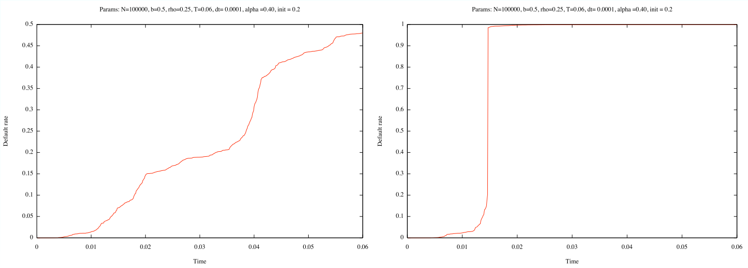
- Common source of noise in dynamics companies

$$V_t^i = V_0^i + \int_0^t b(V_s^i)ds + I_t^i + W_t^i + W_t^0$$

- Mean-field modeling

$$V_t = V_R + \int_0^t b(V_s)ds - \alpha\mathbb{E}(M_t | W^0) + W_t + W_t^0$$

  - same $\alpha \rightsquigarrow$ competition with common noise

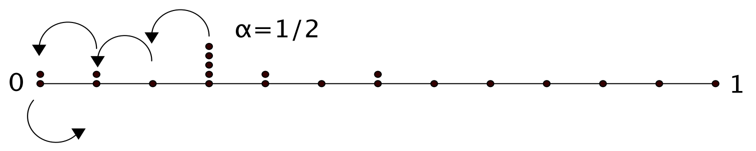


Sketch of the proof

- Competition between noise and mean-field

  ◦ Control regularity of $e \Leftrightarrow$ the mass near the boundary along the construction

- Condition for continuity of e?



$$\Delta e(t) = e(t) - e(t-) = 0$$

$$\Leftrightarrow \exists \delta_n \downarrow 0 : \underbrace{\text{kick due to particles in } [0,\delta_n)}_{\alpha \int_0^{\delta_n} p(t-,y)dy} < \delta_n$$

◦ if $p(t,y) < 1/\alpha$ for $y \in [0,\varepsilon)$ then $e(t) = e(t-)$

Sketch of the proof

- Typical scheme for nonlinear models
  - ∘ Existence and uniqueness in short time on $[0, T^\star]$

- Short time result
  - ∘ if $\frac{1}{dy}\mathbb{P}(V_0 \in dy) \leq \beta y$ for $y \in (0, \varepsilon)$

    $\Rightarrow$ existence and uniqueness on $[0, T^\star(\alpha, \beta, \varepsilon)]$
  - ∘ Picard's fixed point argument

$$e \in \mathcal{C}^1([0,T]) \mapsto \left(\Gamma(e)(t) = \mathbb{E}\left(\sum_{s \leq t} \mathbf{1}_{\{V_{s-}=1\}}\right)\right)_{0 \leq t \leq T}$$

  - ∘ where $dV_t = b(V_t)dt + \alpha e'(t)dt + dW_t$ before default

Sketch of the proof

- Typical scheme for nonlinear models
  - Existence and uniqueness in short time on $[0, T^\star]$
  - Estimate of $\frac{1}{dy}\mathbb{P}(V_{T^\star} \in dy)$ and iteration
- Short time result
  - if $\frac{1}{dy}\mathbb{P}(V_0 \in dy) \leq \beta y$ for $y \in (0, \varepsilon)$

    $\Rightarrow$ existence and uniqueness on $[0, T^\star(\alpha, \beta, \varepsilon)]$
  - Picard's fixed point argument

$$e \in \mathcal{C}^1([0, T]) \mapsto \left(\Gamma(e)(t) = \mathbb{E}\left(\sum_{s \leq t} \mathbf{1}_{\{V_{s-}=1\}}\right)\right)_{0 \leq t \leq T}$$

  - where $dV_t = b(V_t)dt + \alpha e'(t)dt + dW_t$ before default

Scheme for the a priori estimate

- Assume $\exists$ solution with $e \in \mathcal{C}^1$ on $[0, T]$
  - where $dV_t = b(V_t)dt + \alpha e'(t)dt + dW_t$ before default
- Four steps $\left\{ \begin{array}{l} \circ \text{ bound for } p(t,y) = \mathbb{P}(V_t \in dy)/dy \\ \circ \text{ } 1/2 \text{ Hölder bound for } e \\ \circ \text{ Hölder regularity of } p(t,y) \text{ in } y \\ \circ \text{ Lipschitz regularity of } p(t,y) \text{ in } y \end{array} \right.$
- Bound of $p(t,y)$
  - rough bound using (non-killed) Gaussian kernels

$$V_0 > \varepsilon \Rightarrow p(t,y) \leq C(\varepsilon, \alpha), \quad y \in (0, \varepsilon/4)$$

  - very bad (can't see $p(t,0) = 0$) but explicit
  - if $C(\varepsilon, \alpha)\alpha < 1$ then continuity of e (here is α small!)
  - continuity dictated by Brownian: e $1/2$-Hölder

Regularity of p close to the boundary

- Recall Dirichlet condition $p(t,1)=0$
  - p satisfies Fokker-Planck $\rightsquigarrow$ Feynman-Kac

$$p(T,y) = \mathbb{E}\left[p(T-\rho, Y_\rho) \exp\left(-\int_0^\rho b'(Y_s)ds \right) \Big| Y_0 = y \right]$$

  - where $dY_t = -b(Y_t)dt + \alpha e'(T-t)dt + dW_t$
  - $\rho = \inf\{t \geq 0 : Y_t \notin (0,\delta)\} \wedge T$
- Regularity of p at the boundary $\leftrightarrow \mathbb{P}\{Y_\rho = 0\}$
- Probability to hit the boundary
  - competition between B and e
  - e $1/2$ Hölder $\Rightarrow$ B wins with >0 probability
  - get Hölder decay and then Lipschitz

Solutions with blow-up

- Limit of particle system $\Rightarrow \exists$ solutions with blow-up
  - risk modeling $\rightsquigarrow$ massive/systemic default?
- Description of the jumps of $e(t) = \mathbb{E}(M_t)$ when blow-up?

$$\Delta e(t) = e(t) - e(t-) \geq \delta_0$$
$$\Leftrightarrow \forall \delta \leq \delta_0,\ \delta - \text{kick due to particles in } [0,\delta) \leq 0$$

$$\Delta e(t) = \sup\Big\{\delta_0 : \forall \delta \leq \delta_0, \qquad \underbrace{\alpha \int_0^\delta p(t-,y)dy}_{\text{kick due to particles in } [0,\delta)} \qquad \geq \delta\Big\}$$

  - restart with density $p(t,y) = p\big(t-, y + \Delta e(t)\big)$ for y near 1
- Uniqueness? regularization of e just after default?

Convergence of the particle system

- Main difficulty: singularity of the counter of spikes/defaults

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{s\le t}\mathbf{1}_{\{V^j_{s-}=1\}}$$

  - ∘ requires tightness for a suitable topology and continuity

- Topology : counter is increasing in time $\rightsquigarrow$ Skorohod M1

$$\mathrm{Law}\left(\bar{\mu}^N = \frac{1}{N}\sum_{j=1}^{N}\delta_{V^j}\right) \to \mathrm{Law}(\mu) \quad \text{(up to subsequence)}$$

  - ∘ continuity of counter: μ a.s. hitting $V_F \Leftrightarrow$ crossing V_F
  - ∘ if ! solution to mean-field equation $\Rightarrow \mu$ is Dirac at solution
  - ∘ if no ! $\Rightarrow \mu$ charges solutions $\rightsquigarrow$ way to prove $\exists$