A MATHEMATICAL TREATMENT OF BANK MONITORING INCENTIVES

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Outline

The model

We consider a model with universal risk neutrality in continuous time with infinite maturity. The risk-free interest rate is supposed, without loss of generality, to be equal to 0. A bank has the opportunity to set up a pool of I unit loans indexed by $i = 1, \ldots, I$

- The loans are ex-ante identical.
- Each loan is a defaultable perpetuity yielding cash flow μ per unit time until it defaults.
- Once a loan defaults it gives no further payments.

The model

Denote

$$N_t = \sum_{i=1}^{I} 1_{\{\tau^i \leq t\}},$$

the sum of individual loan default indicators, where τ^i denotes the default time of loan i. The current size of the pool is $I - N_t$.

The action of the bank consists on deciding at each time t whether it monitors the different loans. These actions are summarized by the functions e_t^i such that for $1 \leq i \leq I - N_t$, $e_t^i = 1$ if loan i is monitored at time t, and $e_t^i = 0$ otherwise.

The model

The rate at which loan i defaults is controlled by the hazard rate α_t^i specifying its instantaneous probability of default conditional on history up to time t. Individual hazard rates are assumed to depend both on the monitoring choice of the bank and on the size of the pool.

$$\alpha_t^i = \alpha_{I-N_t}\left(1 + (1 - e_t^i)\epsilon\right), \tag{1}$$

where the parameters $\{\alpha_j\}_{1 \leq j \leq I}$ represent individual risk under monitoring when the number of loans is j and ϵ is the proportional impact of shirking on default risk.

The model

We define a shirking process k by

$$k_t = \sum_{i=1}^{I-N_t} \left(1 - e_t^i\right),$$

which represents the number of loans that the bank fails to monitor at time t.

Then we define an aggregate default intensity by

$$\lambda_t^k = \alpha_{I-N_t} \left(I - N_t + \epsilon k_t\right). \tag{2}$$

The bank can fund the pool internally at a cost $r \geq 0$. The bank can also raise funds from competitive investors who value income streams at the prevailing riskless interest rate of zero. We assume that both the bank and investors observe the history of defaults and liquidations.

The contracts

The contracts are agreed upon at time 0 and determine how cash flows are shared and how loans are liquidated, conditionally on past defaults and liquidations. We denote by $D = \{D_t\}_{t\geq 0}$ the càdlàg, positive and increasing process describing cumulative transfers from the investors to the bank, such that

$$\mathbb{E}^{\mathbb{P}}\left[D_\tau\right] < +\infty, \tag{3}$$

where τ is the liquidation time of the pool and where we assumed that $D_0 = 0$.

The contracts

Let then $H_t := 1_{\{t \geq \tau\}}$ be the liquidation indicator of the whole pool. The contract specifies the probability θ_t with which the pool is maintained given default ($dN_t = 1$), so that at each point in time

$$dH_t = \begin{cases} 0 & \text{with probability } \theta_t, \\ dN_t & \text{with probability } 1 - \theta_t. \end{cases}$$

With our notations,the hazard rates associated with the default and liquidation processes N_t and H_t are λ_t^k and $(1 - \theta_t)\, \lambda_t^k$, respectively.

The contract also specifies when liquidation occurs. We assume that liquidations can only take the form of the stochastic liquidation of all loans following immediately default. The above properties translate into

$$\mathbb{P}\left(\tau \in \left\{\tau^1, ..., \tau^I\right\}\right) = 1, \text{ and } \mathbb{P}(\tau = \tau^i | \mathcal{F}_{\tau^i}, \tau > \tau^{i-1}) = 1 - \theta_{\tau^i}.$$

The contracts

We summarize the above details of the contracts, which are completely specified by the choice of (D, θ). Each infinitesimal time interval $(t, t + dt)$ unfolds as follows :

- $I - N_t$ loans are performing at time t.
- The bank chooses to leave $k_t \leq I - N_t$ loans unmonitored and monitors the $I - N_t - k_t$ other loans, enjoying private benefits $k_t B \, dt$.
- The investor receives $(I - N_t) \, \mu \, dt$ from the cash flows generated by the pool and pays $dD_t \geq 0$ as fees to the bank.
- With probability $\lambda_t^k \, dt$ defined by (2) there is a default $(dN_t = 1)$.
- Given default the pool is maintained $(dH_t = 0)$ with probability θ_t or liquidated $(dH_t = 1)$ with probability $1 - \theta_t$.

Outline

Utilities

We consider that the monitoring choices of the bank affect only the distribution of the size of the pool. Formally, we can define a probability measure $\mathbb{P}^k$ equivalent to $\mathbb{P}$ such that

$$N_t - \int_0^t \lambda_t^k ds,$$

is a $\mathbb{P}^k$-martingale.

Then, given a contract (D, θ) and a shirking process k, the bank's expected utility at $t = 0$ is given by

$$u_0^k(D,\theta) := \mathbb{E}^{\mathbb{P}^k}\left[\int_0^T e^{-rt}(dD_t + Bk_t\, dt)\right], \quad (4)$$

while that of the investor is

$$v_0^k(D,\theta) := \mathbb{E}^{\mathbb{P}^k}\left[\int_0^T (I - N_t)\,\mu\, dt - dD_t\right]. \quad (5)$$

Incentive compatibility

Definition

A shirking decision k is incentive-compatible with respect to the contract (D, θ) if it maximizes (4).

Then, the problem faced by the investors is to design a contract (D, θ) and an incentive-compatible advice on the monitoring k that maximize their expected discounted payoff, subject to a given reservation utility for the bank

$$v_I(u) := \sup_{(D,\theta,k)} \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau (I - N_t)\,\mu dt - dD_t\right] \qquad (6)$$

subject to $\mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau e^{-rt}(dD_t + Bk_t\,dt)\right] \geq u_0$

k incentive-compatible with respect to (D, θ).

Incentive compatibility

Define u_t^k the dynamic version of the bank's continuation utility

$$u_t^k(D,\theta) := 1_{\{t\leq\tau\}}\mathbb{E}^{\mathbb{P}^k}\left[\int_t^\tau e^{-r(s-t)}\left(dD_s + Bk_s ds\right)\,\middle|\,\mathcal{G}_t\right]. \qquad (7)$$

By the martingale representation property, there exists processes h^1 and h^2 such that

$$du_t^k + (dD_t + Bk_t dt) = ru_t^k\,dt - h_t^1\left(dN_t - \lambda_t^k\,dt\right) \\ -h_t^2\left(dH_t - (1-\theta_t)\lambda_t^k\,dt\right).$$

Incentive compatibility

Theorem

Given a contract (D,θ), $k=0$ *is incentive-compatible if and only if*

$$h_t^1 + (1-\theta_t)h_t^2 \geq \frac{B}{\varepsilon \alpha_{I-N_t}}, \;\; t \in [0,\tau], \;\; \mathbb{P}-a.s. \qquad (8)$$

Limited liability

Define

$$b_i := \frac{B}{\varepsilon \alpha_i},$$

We assume that the bank has limited liability. This means that the bank's continuation utility must exceed the lower bound b_{i-1}, because otherwise it would not be possible for the investors to apply the required penalties following default. This implies that the pool can be maintained only if the following condition is not violated

$$\text{For all } 1 \leq i \leq I,\ u^0_{t^-} - h^1_t \geq b_{i-1},\ \text{ on } \{N_t = I - i\}. \tag{9}$$

Limited liability

We assume that the bank forfeits any rights to cash flows once the pool is liquidated. The constraint $u^0_\tau = 0$ implies in turn that at all times

$$u^0_t = h^1_t + h^2_t. \tag{10}$$

Indeed, the utility of the bank must jump to 0 just after the liquidation of the pool. Since the penalty after liquidation is exactly $h^1 + h^2$, (10) must hold at each time.

Admissible contracts

Our set of admissible contracts is therefore

$$\widetilde{\mathcal{A}}^0(x) :=$$
$$\{(D,\theta,h^1,h^2),\ \theta \text{ is a predictable process with values in } [0,1],$$
$$D \text{ is a positive càdlàg non-decreasing process which satisfies } (3),$$
$$h^1 \text{ and } h^2 \text{ are predictable processes, integrable, and satisfy}$$
$$u^0_{t^-} - h^1_t \geq b_{I-N_t-1},\ u^0_t = h^1_t + h^2_t,\ x \leq u^0_0(D,\theta).\}$$

Outline

The HJB equation

Let $v_j(u)$ denote the investor's value function, i.e., the maximum expected utility an investor can achieve given a pool of size j and a reservation utility for the bank u. Then, we expect the investor's value function to solve the following system of Hamilton-Jacobi-Bellman equations with initial condition $v_0(u) = 0$

$$\sup_{(\delta,\theta,h^1,h^2)\in\mathcal{C}^j} \left\{ \left(ru + \lambda_j \left(h^1 + (1-\theta)h^2 \right) - \delta \right) v_j'(u) + j\mu - \delta \right.$$
$$\left. -\theta\lambda_j \left(v_j(u) - v_{j-1}(u-h^1) \right) - (1-\theta)\lambda_j v_j(u) \right\} = 0, \;\; u > b_j, \tag{11}$$

where the $\mathcal{C}^j$ are our admissible strategies sets defined by

$$\mathcal{C}^j := \left\{ \delta \geq 0, \;\; \theta \in [0,1], \;\; h^1 + (1-\theta)h^2 \geq b_j, \;\; u - h^1 \geq b_{j-1}, \;\; u = h^1 +$$

The HJB equation

Given the constraints in the definition of $\mathcal{C}^i$, we reparametrize the problem in terms of the variable $z := \theta(u - h^1)$. This leads to the simpler system of HJB equations

$$\sup_{(\delta,\theta,z)\in\widetilde{\mathcal{C}}^j} \Big\{ \left(ru + \lambda_j(u - z) - \delta\right) v_j'(u) + j\mu - \delta - \lambda_j \left(v_j(u) - \theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\Big\} = 0, \ \ u > b_j,$$

where the constraints become

$$\widetilde{\mathcal{C}}^j := \left\{ \delta \geq 0, \ \ \theta \in \left[0, 1 \wedge \frac{u - b_j}{b_{j-1}}\right], \text{ and } z \in [b_{j-1}\theta, u - b_j] \right\}.$$

Guess of the value function

Under certain assumptions on the value functions (mainly concavity), we can formally obtain the following system of ODEs

$$(ru + \lambda_j b_j)\, v_j'(u) + j\mu - \lambda_j \left(v_j(u) - v_{j-1}(u - b_j)\right) = 0, \qquad u \in (b_j, \gamma_j]$$
$$v_j'(u) = -1, \;\; u \geq \gamma_j.$$

where γ_j is the first level for which $v_j' = -1$.

Under certain assumptions on the parameters, the above system admits a unique maximal solution which verifies all our guesses.

Guess of the optimal contract

Starting from a reservation utility $x \leq \gamma_I$ for the bank, the following contract unfolds.

(i) Given size j, the pool remains in operation (i.e. there is no liquidation) with one less unit at any time there is a default in the range $[b_j + b_{j-1}, \gamma_j]$.

(ii) The flow of fees paid to the bank given j is $\delta_t^j = \lambda_j b_j + r\gamma_j$ as long as $u_t = \gamma_j$ and no default occurs, where δ^j is the density of D with respect to the Lebesgue measure. Otherwise $\delta_t = 0$.

(iii) Liquidation of the whole pool occurs with probability $\theta_t^j = (u_t - b_j) / b_{j-1}$ in the range $[b_j, b_j + b_{j-1})$.

The verification result

Theorem

For any starting condition $u_0 > b_I$, let u_t be the solution of

$$du_t = (ru_t - \delta^{I-N_t}(u_t))dt - h^{1,I-N_t}(u_t)(dN_t - \lambda_{I-N_t}dt) - h^{2,I-N_t}(u_t)(dH_t - \lambda_{I-N_t}(1 - \theta^{I-N_t}(u_t))dt), \ t < \tau.$$

Then, the contract defined by $\left(\delta^{I-N_t}(u_t), \theta^{I-N_t}(u_t)\right)$ is incentive compatible, has value u_0 and $v_I(u_0)$ for the bank and the investors.

Theorem

For any contract $(D, \theta) \in \widetilde{\mathcal{A}}^0(u_0)$, the utility the investors can obtain is bounded from above by $v_I(u_0)$, where u_0 is the utility obtained by the bank.

Numerical results

μ	0.06
r	0.02
B	0.002
ε	0.25
$(\alpha_j)_{1\leq j\leq 14}$	0.055
$(\alpha_j)_{15\leq j\leq 18}$	0.05
$(\alpha_j)_{19\leq j\leq 20}$	0.044

Numerical results

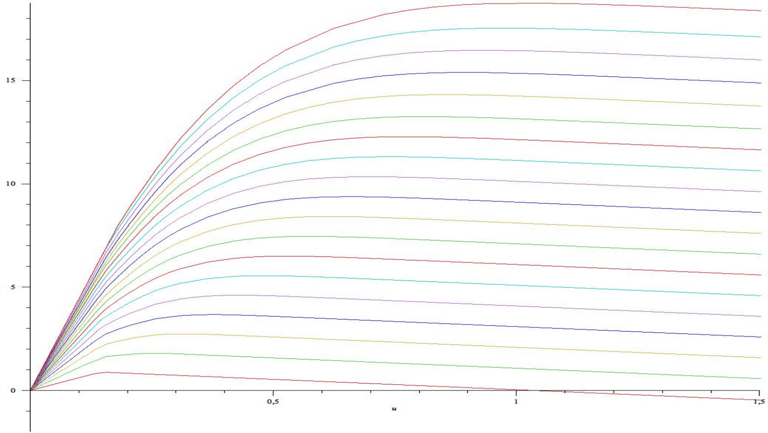

Figure: Functions $v_j(u)$ for $j = 1..20$.

THANK YOU FOR YOUR ATTENTION