Almost sure optimal hedging strategy

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Joint work with N. Landon.

Problem 1: hedging errors due to discrete rebalancing

- Payoff: $g(S_T)$ with **d-dimensional Itô diffusion** $(S_t)_{t\geq 0}$
- Price function: $u(t,x) = \mathbb{E}(g(S_T)|S_t = x)$ (zero interest-rate)
- Price process: $\mathbf{V_t = u(t, S_t) = u(0, S_0) + \int_0^t D_x u(\theta, S_\theta) \cdot dS_\theta}$.
- Rebalancing strategy along time grid: $\pi = \{\mathbf{0 = t_0 < \dots < t_i < \dots < t_N = T}\}$
- Hedging portfolio based on π:

$$\mathbf{V_t^N = u(0, S_0) + \int_0^t D_x u_{\varphi(\theta)} \cdot dS_\theta},$$

where $\varphi(t) = \max\{t_j \in \pi : t_j \leq t\}$.

- Hedging error: $\mathbf{Z_t^N = V_t - V_t^N = \int_0^t \left(D_x u_\theta - D_x u_{\varphi(\theta)}\right) \cdot dS_\theta}$.

Purpose: compute the optimal grids π **minimizing *a.s.***

$$N \langle Z_\cdot^N \rangle_T$$

as $N \to \infty$, over the set of **deterministic and stopping times strategies** π.

Problem 2: optimal timing for portfolio utility maximization

- Utility function U: concave, C^1, $\uparrow\uparrow$, $U'(0^+) = +\infty$ et $U'(+\infty) = 0$;
- Initial wealth V^0 is given;
- Given an allocation strategy $(\delta_t)_t$, terminal wealth process V_T^δ;
- **Problem:** $\max_{(\delta_t)_t} \mathbb{E}(U(V_T^\delta)) =?$ $\delta^* =$**?**
- **Solution in complete market:** we recover a pricing problem. Set $I := (U')^{-1}$ and $L_T := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T}$, then $\exists y_0 > 0$

$$\mathbf{V_t} = \mathbb{E}_{\mathbb{Q}}(\mathbf{e}^{-\int_t^T r_s ds}\mathbf{I}(\mathbf{y_0 e}^{-\int_0^T r_s ds}\mathbf{L_T})|\mathcal{F}_\mathbf{t}).$$

BUT

- in practice, the portfolio reallocation is done at discrete times (once a month...).
- **What is the best timing for this portfolio utility maximization?**

$\rightsquigarrow$ **Similar issue to optimal hedging error**

Organization of the talk

1. Literature background
2. Lower bound and minimizing strategy π (of hitting time type)
3. Results for almost sure convergence of Brownian stochastic integrals, related increments...
4. Numerical experiments
5. Extensions

1. Literature background

$$\pi = \{0 = t_0 < ... < t_i < ... < t_N = T\}, \qquad Z_t^N = \int_0^t \left(D_x u_\theta - D_x u_{\varphi(\theta)}\right) \cdot \mathrm{d}S_\theta.$$

- **Weak convergence:** $\sqrt{\mathbf{N}}\mathbf{Z}_\mathbf{T}^\mathbf{N}$ weakly converges to a Gaussian mixture
  - when π is deterministic **[Bertsimas, Kogan, Lo '01; Hayashi, Mykland '05]** (under rather weak assumptions)
  - when π consists of stopping times **[Fukasawa '11]** (under conditions easy to check in dim 1, and hardly tractable in higher dimension)
- L_2 **norm:** $\mathbb{E}(\mathbf{Z}_\mathbf{T}^\mathbf{N})^\mathbf{2} = \mathbb{E}\langle \mathbf{Z}^\mathbf{N}\rangle_\mathbf{T}$ (under the RN measure)
  - for uniform grids: $\mathbb{E}\langle \mathbf{Z}^\mathbf{N}\rangle_\mathbf{T} \sim \mathbf{C}\mathbf{N}^{-\alpha}$ where $\alpha \in (0,1]$ is the fractional regularity index of $g(S_T)$; **[Zhang' 99, G'-Temam ' 01, Geiss-Geiss ' 04, Geiss-Hujo' 07, G'-Makhlouf '10]**
  - appropriate deterministic non uniform grids give $\mathbb{E}\langle \mathbf{Z}^\mathbf{N}\rangle_\mathbf{T} \sim \mathbf{C}\mathbf{N}^{-\mathbf{1}}$;
  - best n-stopping times **[Martini-Patry '99]** (optimal multi-stopping pb);
  - Asymptotic minimization over stopping times: $\liminf \mathbb{E}(\mathbf{N})\mathbb{E}\langle \mathbf{Z}^\mathbf{N}\rangle_\mathbf{T}$. Convex payoff in dimension 1, mainly within BS model **[Fukasawa 10]**.

2. Lower bounds and minimizing stopping times

Asymptotic framework

- Positive deterministic real numbers $(\varepsilon_n)_{n\geq 0}$ such that $\sum_{\mathbf{n}\geq\mathbf{0}} \varepsilon_{\mathbf{n}}^{\mathbf{2}} < +\infty$
- Strategy (indexed by $n = 0, 1, \dots$) = sequence of stopping times

$$\mathcal{T}^n := \{\tau_0^n = 0 < \tau_1^n < \dots < \tau_i^n < \dots \leq \tau_{N_T^n}^n = T\} \qquad (N_T^n \text{ may be random}).$$

- Let $\rho_{\mathbf{N}} \in [\mathbf{1}, \frac{\mathbf{4}}{\mathbf{3}})$. A sequence of strategies $(\mathcal{T}^n)_{n\geq 0}$ is admissible if *a.s.*

$$\sup_{\mathbf{n}\geq\mathbf{0}} \Big(\varepsilon_{\mathbf{n}}^{-\mathbf{1}} \sup_{\mathbf{1}\leq\mathbf{i}\leq\mathbf{N_T^n}} \sup_{\mathbf{t}\in(\tau_{\mathbf{i-1}}^{\mathbf{n}}, \tau_{\mathbf{i}}^{\mathbf{n}}]} |\mathbf{S_t} - \mathbf{S}_{\tau_{\mathbf{i-1}}^{\mathbf{n}}}|\Big) < +\infty, \qquad \sup_{\mathbf{n}\geq\mathbf{0}} \Big(\varepsilon_{\mathbf{n}}^{\mathbf{2}\rho_{\mathbf{N}}} \mathbf{N_T^n}\Big) < +\infty.$$

- **Deterministic times:** if $\rho_N > 1$, a sequence of strategy with $N_T^n \sim C\varepsilon_n^{-2\rho_N}$ deterministic times and mesh size $\sup_{1\leq i\leq N_T^n} \Delta\tau_i^n \leq C\varepsilon_n^{2\rho_N}$ is admissible.
- **Hitting times of random "ellipsoids":** the strategy given by

$$\tau_0^n := 0, \qquad \tau_i^n := \inf\left\{t \geq \tau_{i-1}^n : (S_t - S_{\tau_{i-1}^n})^* H_{\tau_{i-1}^n} (S_t - S_{\tau_{i-1}^n}) > \varepsilon_n^2\right\} \wedge T,$$

defines an admissible sequence if H is a continuous adapted positive-definite $d \times d$-matrix process.

Purpose:

1. Compute the ***a.s.*** $\liminf_{\mathbf{n}\to\infty} \mathbf{N^n_T}\langle \mathbf{Z^n}\rangle_\mathbf{T}$ over the set of admissible sequence of strategies.
2. Provide a minimizing sequence.

Assumptions

- **Model of d risky assets:** $\mathbf{S_t} = \mathbf{S_0} + \int_0^\mathbf{t} \mathbf{b_s}\mathrm{d}\mathbf{s} + \int_0^\mathbf{t} \sigma_\mathbf{s}\mathrm{d}\mathbf{B_s}$. W.l.o.g. $b \equiv 0$. To simplify $\sigma_t = \sigma(t, S_t)$ with $\sigma(.)$ Lipschitz.
- **Pathwise ellipticity:** $\mathbf{0} < \lambda_{\min}(\sigma_\mathbf{t}\sigma^*_\mathbf{t}), \quad \forall\, \mathbf{0} \le \mathbf{t} \le \mathbf{T}$.
- **First Greeks are *a.s.* finite** in a small tube around the (t, S_t):

$$\mathbb{P}\left(\lim_{\delta\to 0}\sup_{0\le t<T}\sup_{|x-S_t|\le\delta}\left(\left|D^2_{xx}u(t,x)\right| + \left|D^2_{tx}u(t,x)\right| + \left|D^3_{xxx}u(t,x)\right|\right) < +\infty\right) = 1.$$

Main results

LEMMA (MATRIX EQUATION) Let $c \in \mathcal{S}^d(\mathbb{R})$. Then, there is a unique solution $x(c) \in \mathcal{S}^d_+(\mathbb{R})$ to the equation $\mathbf{2}\mathrm{Tr}(\mathbf{x})\mathbf{x} + \mathbf{4x^2} = \mathbf{c^2}$ and $c \mapsto x(c)$ is continuous.

THEOREM (LOWER BOUND) Let X be the solution of the matrix equation with $c := \sigma^* D^2_{xx} u \sigma$. Then, for any admissible sequence of strategies,

$$\liminf_{\mathbf{n}\to+\infty} \mathbf{N^n_T}\langle \mathbf{Z^n}\rangle_\mathbf{T} \geq \left(\int_\mathbf{0}^\mathbf{T} \mathrm{Tr}\,(\mathbf{X_t})\,\mathrm{d}\mathbf{t} \right)^\mathbf{2}, \quad a.s..$$

THEOREM (MINIMIZING SEQUENCE) For any $\mu > 0$, we can exhibit an admissible sequence of strategies such that

$$\limsup_{\mathbf{n}\to+\infty} \left| \mathbf{N^n_T}\langle \mathbf{Z^n}\rangle_\mathbf{T} - (\int_\mathbf{0}^\mathbf{T} \mathrm{Tr}(\mathbf{X_t})\mathrm{d}\mathbf{t})^\mathbf{2} \right| \leq \mu \mathbf{C}_\mu \quad a.s.,$$

where the random variable C_μ is $a.s.$ finite (locally uniformly in μ).

- If the Gamma matrix is positive-definite (unif. in (t, ω)), we can take $\mu = 0$.
- The μ-optimal strategies are of the hitting time form $\rightsquigarrow$ deterministic times are suboptimal.

Explicit representation of the optimal strategies

Let $\chi(.) \in C^\infty(\mathbb{R})$ with $\mathbf{1}_{]-\infty,1/2]} \leq \chi(.) \leq \mathbf{1}_{]-\infty,1]}$. Set $\chi_\mu(x) = \chi(x/\mu)$.

- In the one dimensional case, the μ-optimal stopping times read $\tau_0^n := 0$ and

$$\boldsymbol{\tau_i^n = \inf \left\{ t \geq \tau_{i-1}^n : |S_t - S_{\tau_{i-1}^n}| > \frac{\varepsilon_n}{\sqrt{|D^2_{xx} u_{\tau_{i-1}^n}|/\sqrt{6} + \mu \chi_\mu(|D^2_{xx} u_{\tau_{i-1}^n}|/\sqrt{6})}} \right\} \wedge T.}$$

  Rebalancing frequency depends on the Gamma of the option.

- In the general case, we have to set

$$\Lambda_t := (\sigma_t^{-1})^* X_t \sigma_t^{-1} \quad \text{and} \quad \Lambda_t^\mu := \Lambda_t + \mu \chi_\mu(\lambda_{\min}(\Lambda_t)) I_d.$$

  Then the μ-optimal strategy is defined by

$$\boldsymbol{\tau_i^n = \inf \left\{ t \geq \tau_{i-1}^n : (S_t - S_{\tau_{i-1}^n})^* \Lambda^\mu_{\tau_{i-1}^n} (S_t - S_{\tau_{i-1}^n}) > \varepsilon_n^2 \right\} \wedge T.}$$

  $\rightsquigarrow$ Hitting times of ellipsoids.

About the assumption

$$(\bigstar) \quad \mathbb{P}\left(\lim_{\delta\to\mathbf{0}} \sup_{\mathbf{0}\le\mathbf{t}<\mathbf{T}} \sup_{|\mathbf{x}-\mathbf{S_t}|\le\delta} \left(\left|\mathbf{D^2_{xx}u(t,x)}\right| + \left|\mathbf{D^2_{tx}u(t,x)}\right| + \left|\mathbf{D^3_{xxx}u(t,x)}\right|\right) < +\infty\right) = \mathbf{1}.$$

- In the BS model in dimension 1, for Call option $\mathbf{g(S) = (S - K)_+}$: **OK** because the strike K is negligible for the law of S_T.
- Digital payoff $\mathbf{g(S) = 1_{S\ge K}}$: **OK**
- General diffusion with smooth coefficients:
  - $\mathbf{g(S) = \Phi(S)}$ where Φ is smooth: **OK**
  - $\mathbf{g(S) = 1_{S\in F}\Phi(S)}$ and F is a closed set which boundary has zero Lebesgue-measure: **OK** under ellipticity or hypo-ellipticity assumption.

  $\rightsquigarrow$ includes all the "usual" continuous and discontinuous payoffs.
- **Open issue**: find a payoff g violating the $(\bigstar)$-condition.

3. Almost sure convergence results

A new general result inspired by Borel-Cantelli, Lenglart, Karandikar, Bichteler works.

LEMMA. Let $\mathcal{M}_0^+$ be the set of non-negative measurable processes vanishing at $t = 0$.

Let $(U^n)_{n\geq 0}$ and $(V^n)_{n\geq 0}$ be two sequences of processes in $\mathcal{M}_0^+$.

Assume that

1. the series $\sum_{n\geq 0} V_t^n$ converges for all $t \in [0, T]$, almost surely;
2. the above limit is upper bounded by a process $\bar{V} \in \mathcal{M}_0^+$ and that $\bar{V}$ is continuous *a.s.* ;
3. there is a constant $c \geq 0$ such that, for every $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $t \in [0, T]$, we have

$$\mathbb{E}[U^n_{t\wedge\theta_k}] \leq c\mathbb{E}[V^n_{t\wedge\theta_k}]$$

   with the random time $\theta_k := \inf\{s \in [0, T] : \bar{V}_s \geq k\}$

Then for any $t \in [0, T]$, **the series $\sum_{n\geq 0} U_t^n$ converges almost surely**. As a consequence, $U_t^n \xrightarrow{a.s.} 0$.

A direct application

COROLLARY. Let $p > 0$ and let $\{(M_t^n)_{0 \le t \le T} : n \ge 0\}$ be a sequence of scalar continuous local martingales vanishing at zero. Then,

$$\sum_{\mathbf{n} \ge \mathbf{0}} \langle \mathbf{M^n} \rangle_{\mathbf{T}}^{\mathbf{p/2}} < +\infty \quad a.s. \qquad \Longleftrightarrow \qquad \sum_{\mathbf{n} \ge \mathbf{0}} \sup_{\mathbf{0} \le \mathbf{t} \le \mathbf{T}} |\mathbf{M_t^n}|^{\mathbf{p}} < +\infty \quad a.s..$$

A non-trivial application

Directly estimating $\Delta\tau_i^n = \tau_i^n - \tau_{i-1}^n$ is very difficult since the distribution is not explicit. But the key lemma gives

PROPOSITION. Consider an admissible sequence of strategies and let $p \ge 0$. Then

$$\sum_{\mathbf{n} \ge \mathbf{0}} \varepsilon_{\mathbf{n}}^{-\mathbf{2(p-1)+2}\rho_{\mathbf{N}}} \sum_{\tau_{\mathbf{i-1}}^{\mathbf{n}} < \mathbf{T}} (\boldsymbol{\Delta}\tau_{\mathbf{i}}^{\mathbf{n}})^{\mathbf{p}} < +\infty \quad a.s..$$

Since p is arbitrary, we get

COROLLARY. For any $\rho > 0$, we have $\sup_{\mathbf{n} \ge \mathbf{0}} \left(\varepsilon_{\mathbf{n}}^{\rho - \mathbf{2}} \sup_{\mathbf{1} \le \mathbf{i} \le \mathbf{N_T^n}} \boldsymbol{\Delta}\tau_{\mathbf{i}}^{\mathbf{n}} \right) < +\infty \quad a.s..$

Another non-trivial application

PROPOSITION. Let

1. $\mathcal{T} = (\mathcal{T}^n)_{n\geq 0}$ be an admissible sequence of strategies,
2. $((M_t^n)_{0\leq t\leq T})_{n\geq 0}$ be a sequence of $\mathbb{R}$-valued continuous local martingales s.t.
   (a) $\langle M^n\rangle_t = \int_0^t \alpha_r^n \mathrm{d}r$ for a non-negative measurable adapted α^n
   (b) there exists a non-negative *a.s.* finite random variable C_α and a parameter $\theta \geq 0$ such that

$$0 \leq \alpha_r^n \leq C_\alpha(|S_r - S_{\varphi(r)}|^{2\theta} + |r-\varphi(r)|^{\theta}), \quad \forall\, 0 \leq r < T, \forall n \geq 0, \quad a.s..$$

Then,

1. for any $p>0$: $\sum_{\mathbf{n}\geq\mathbf{0}} \left(\varepsilon_{\mathbf{n}}^{\mathbf{2}-(\mathbf{1}+\theta)\mathbf{p}+\mathbf{2}\rho_{\mathbf{N}}} \sum_{\tau_{\mathbf{i}-\mathbf{1}}^{\mathbf{n}}<\mathbf{T}} \sup_{\tau_{\mathbf{i}-\mathbf{1}}^{\mathbf{n}}\leq\mathbf{t}\leq\tau_{\mathbf{i}}^{\mathbf{n}}} |\mathbf{M}_{\mathbf{t}}^{\mathbf{n}} - \mathbf{M}_{\varphi(\mathbf{t})}^{\mathbf{n}}|^{\mathbf{p}}\right) < +\infty, \quad a.s..$
2. for any $\rho > 0$: $\sup_{\mathbf{n}\geq\mathbf{0}} \left(\varepsilon_{\mathbf{n}}^{\rho-(\mathbf{1}+\theta)} \sup_{\mathbf{1}\leq\mathbf{i}\leq\mathbf{N}_{\mathbf{T}}^{\mathbf{n}}} \sup_{\tau_{\mathbf{i}-\mathbf{1}}^{\mathbf{n}}\leq\mathbf{t}\leq\tau_{\mathbf{i}}^{\mathbf{n}}} |\mathbf{M}_{\mathbf{t}} - \mathbf{M}_{\varphi(\mathbf{t})}^{\mathbf{n}}|\right) < +\infty, \quad a.s..$

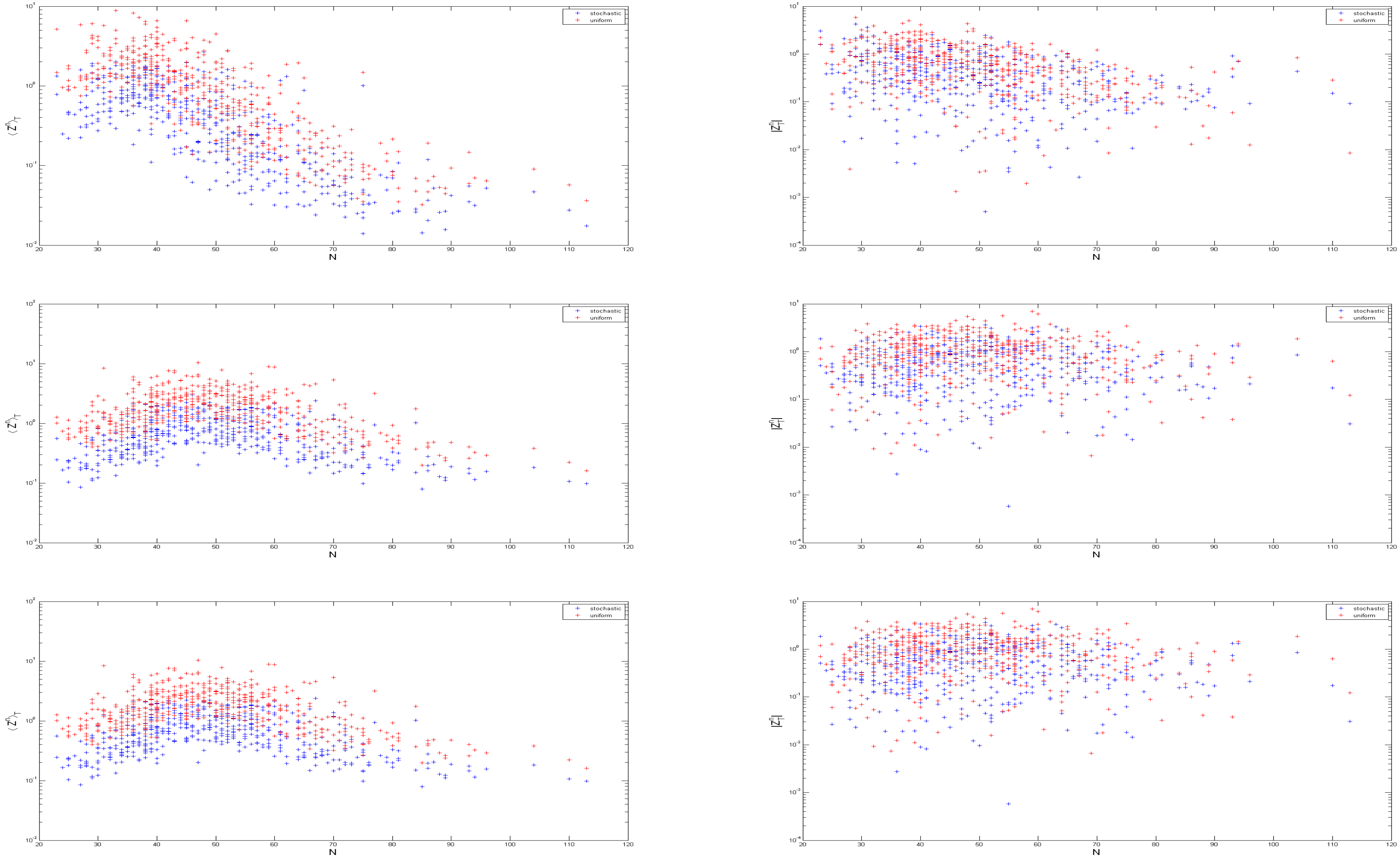
4. Numerical experiments

Model: 1 and 2-dimensional Geometric Brownian Motion with

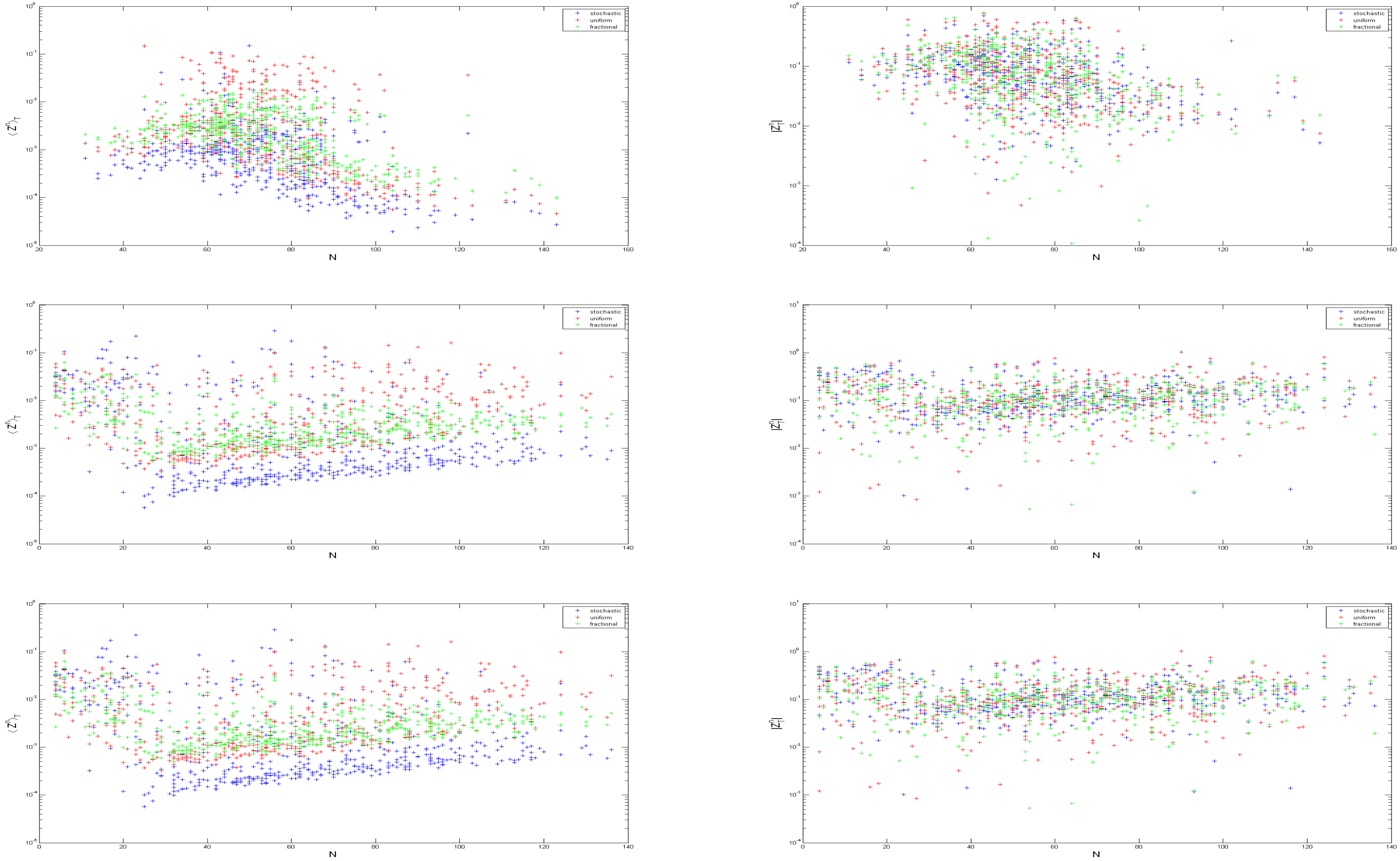
- $S_0^1 = 100$, $S_0^2 = 100$
- $\sigma_1 = 30\%$, $\sigma_2 = 40\%$, $\rho = 50\%$, $T = 1$
- $\varepsilon_n = 0.05$
- $\mu = 0$

We plot the quadratic variation $\langle Z^n \rangle_T$ versus N_T^n for different strategies (optimal, uniform, fractional).

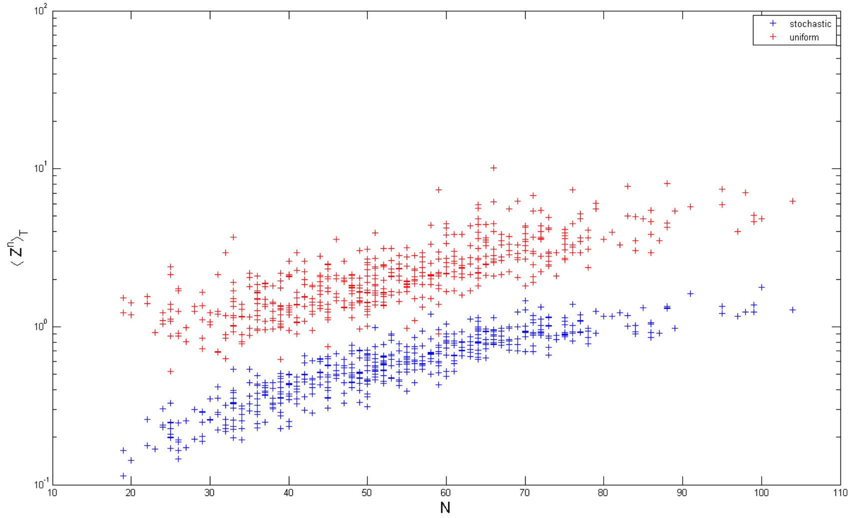
1d - Call option: $K = 80, 100, 120$. Left: $\langle Z^n \rangle_T$. Right: $|Z^n_T|$

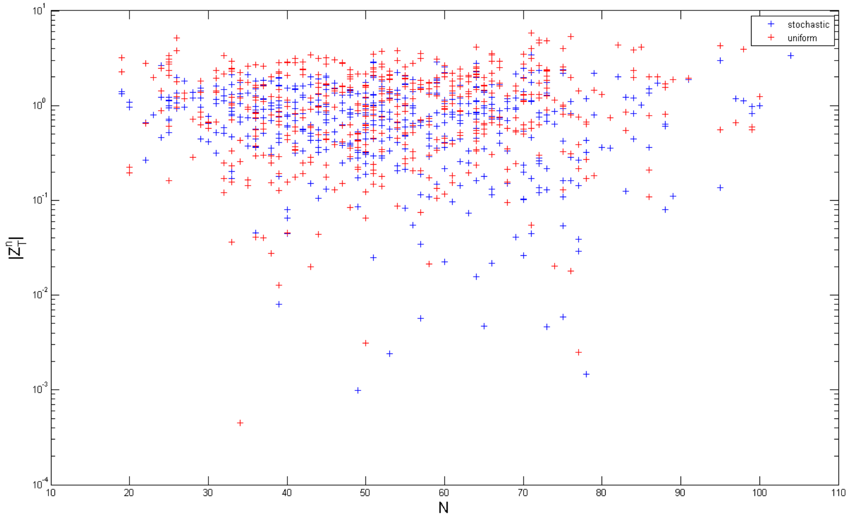


1d - Binary option: $K = 80, 100, 120$

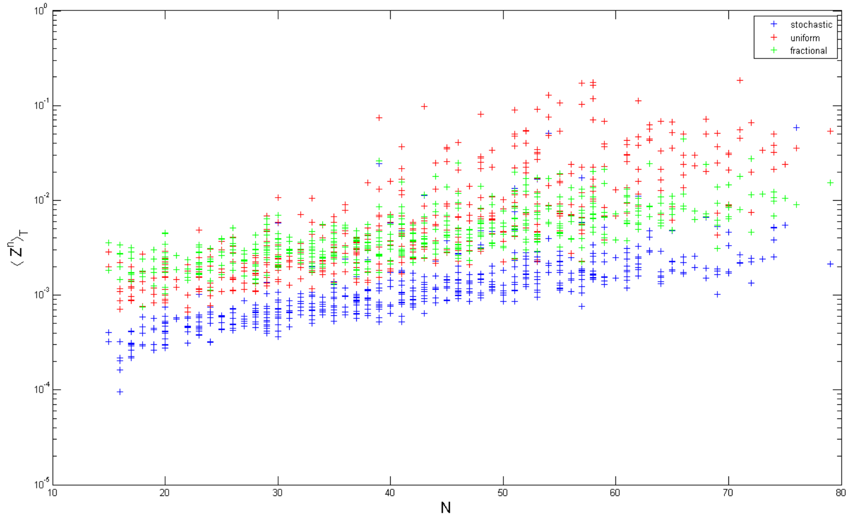


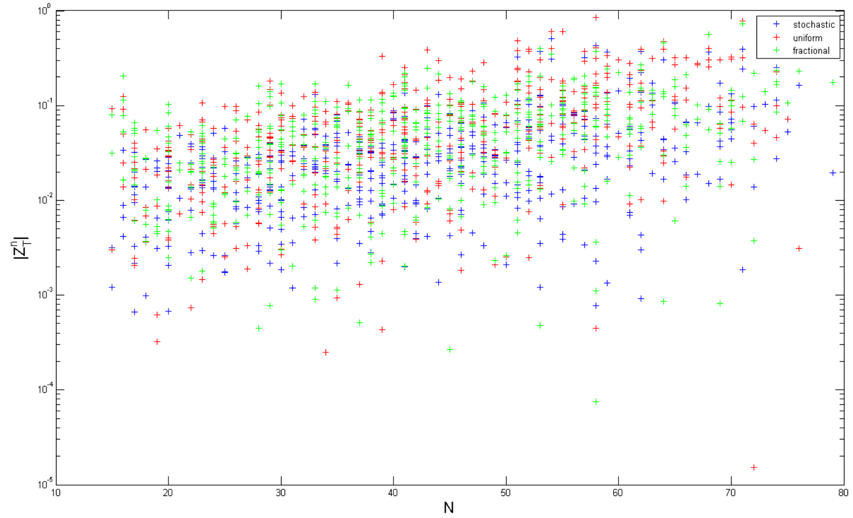
2d - Call Exchange Option: $g(S_T) = (S_T^1 - S_T^2)_+$



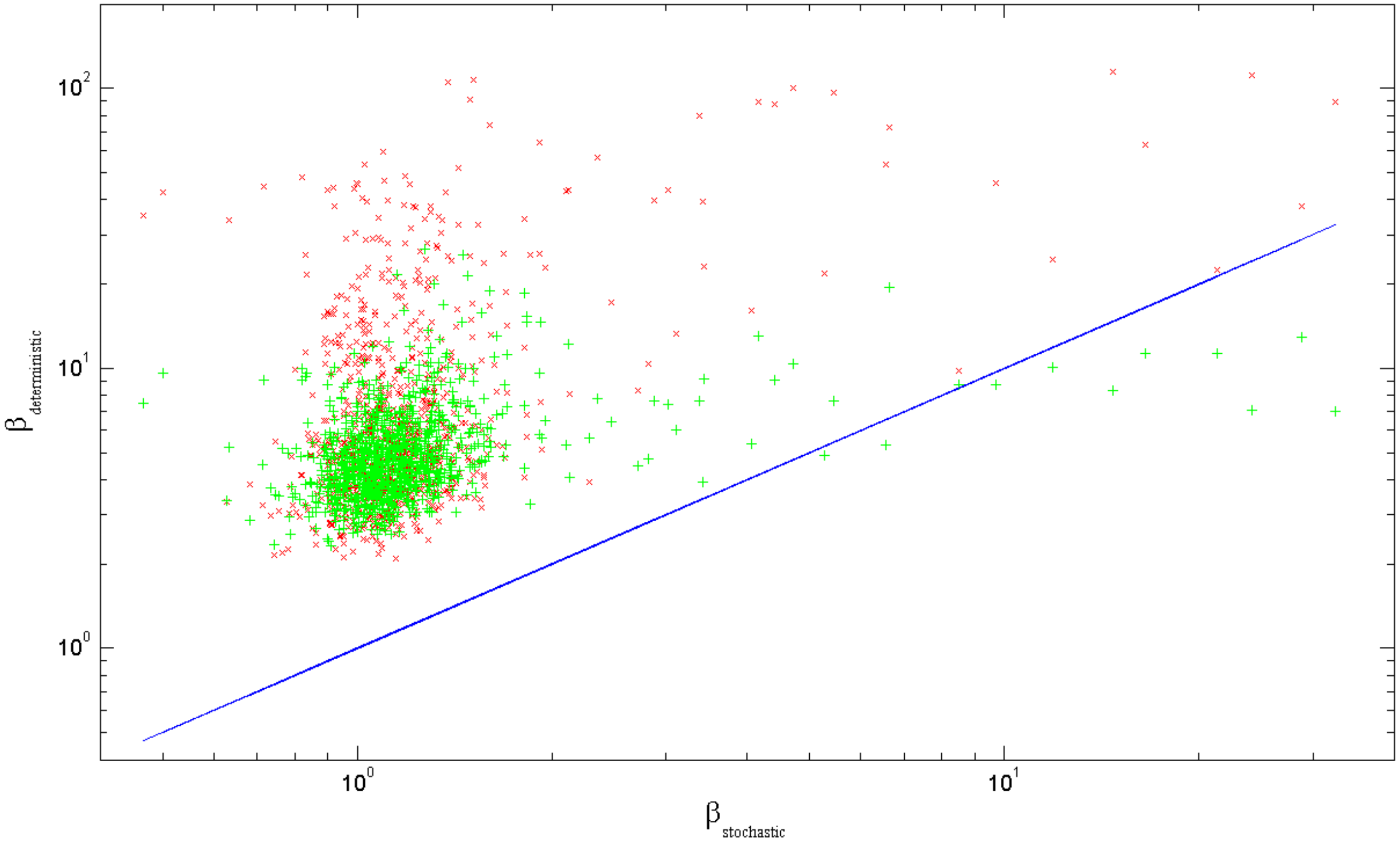


2d - Binary Exchange Option: $g(S_T) = \mathbf{1}_{S_T^1 \geq S_T^2}$





Comparison of $\beta_{.}(\omega) = \frac{N_T^n(\omega)\langle Z^n\rangle_T(\omega)}{\left(\int_0^T \mathrm{Tr}(X_t)\mathrm{d}t\right)^2(\omega)}$ for different strategies



"×", "+" and the blue line correspond respectively to "$(\beta_{\text{stochastic}}, \beta_{\text{uniform}})$", "$(\beta_{\text{stochastic}}, \beta_{\text{fractional}})$" and the identity function.

5. Extensions

- The volatility process can be only locally Hölder continuous;
- For some results (lower bound), ellipticity in one direction is sufficient;
- We can extend results to exotic options, using extra state variables:
  1. $Y = (Y^i)_{1 \leq i \leq d'}$ is a vector of adapted continuous non-decreasing processes
     - (a) Asian options : $Y_t^j := \int_0^t S_s^j \mathrm{d}s$ and $g(x, y) := (\sum_{1 \leq j \leq d} \pi_j y^j - K)_+$, for some weights π_j and a given $K \in \mathbb{R}$.
     - (b) Lookback options : $Y_t^j := \max_{0 \leq s \leq t} S_s^j$ and $g(x, y) := \sum_{1 \leq j \leq d} (\pi_j y^j - \pi'_j x^j)$
  2. price process: $u(t, S_t, Y_t) = u(0, S_0, Y_0) + \int_0^t D_x u(s, S_s, Y_s) \cdot \mathrm{d}S_s$
  3. for some $\rho_Y > 4(\rho_N - 1)$

$$\sup_{n \geq 0} \left(\varepsilon_n^{-\rho_Y} \sup_{1 \leq i \leq N_T^n} |\Delta Y_{\tau_i^n}| \right) < +\infty \quad a.s..$$

- Asymptotic analysis can be extended to take into account transaction costs.
- (with M. Rosembaum): *a.s.* statistical inference.