Numerical Methods for non-linear Black-Scholes Equations

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Introduction

Deficiencies of Black-Scholes model

- Black-Scholes model oversimplifies market mechanisms, e.g.: lack of
  - transaction costs,
  - uncertain volatility,
  - market illiquidity,
  - ...

Improvement: Consider models with nonlinear volatility term:

Nonlinear Black-Scholes Equation

$$V_t + \frac{1}{2}\sigma^2(t, S, V_{SS}) \cdot S^2 V_{SS} + (r-q)SV_S - rV = 0$$

Outline

Transaction Costs Model

Leland's model

$$\sigma_{\mathrm{Le}}^2(t, S, V_{SS}) := \sigma_0^2 \cdot (1 + A \cdot \mathrm{sign}(V_{SS}))$$

- $$A := \sqrt{\frac{2}{\pi}} \frac{k}{\sigma_0 \sqrt{\delta t}} \qquad \text{(Leland-number)},$$

  σ_0 volatility of the underlying, k round-trip costs,
  δt time between adjustments of portfolio
- limited applicability: $A \leq 1$

H. E. Leland
Option Pricing and Replication with Transactions Costs.
Journal of Finance, 40 (5), 1985

Transaction Costs Model

Soner's and Barles model

$$\sigma^2_{\mathrm{SB}}(t, S, V_{SS}) := \sigma_0^2 \cdot \left(1 + \Psi(e^{r(T-t)}a^2 S^2 V_{SS})\right),$$

$$\Psi'(x) = \frac{\Psi(x)+1}{2\sqrt{x\Psi(x)} - x}, x \neq 0, \quad \text{and} \quad \Psi(0) = 0$$

- a parameter for the transaction costs and risk aversion,
- equation is obtained by utility maximation

G. Barles and H. M. Soner
Option pricing with transaction costs and a nonlinear Black-Scholes equation.
Finance and Stochastics, 2 (4), 1998

Uncertain Volatility Model

UV model

$$\sigma^2_{\mathrm{ALP}+}(t, S, V_{SS}) := \begin{cases} \sigma^2_{max} & : \quad V_{SS} \leq 0 \\ \sigma^2_{min} & : \quad V_{SS} > 0 \end{cases}$$

- volatility unknown, but assumed to lie between σ_{max} and σ_{min},
- $V(S, t)$ are costs of dynamic hedging under worst-case volatility path

M. Avellaneda, A. Levy and A. Parás
Pricing and hedging derivative securities in markets with uncertain volatilities.
Applied Mathematical Finance, 2 (2), 1995

Market Illiquidity Model

Frey's illiquidity model

$$\sigma_{\mathrm{FP}}^2(t, S, V_{SS}) := \frac{\sigma_0^2}{(1 - \rho S V_{SS})^2}$$

- ρ parameter for the liquidity of the market
- market is not perfectly liquid, thus feedback effect on the underlying by hedging strategy

R. Frey and P. Patie
Risk Management for Derivatives in Illiquid Markets.
Advances in finance and stochastics, Springer, 2002

Preparations

log-money and reversed time

$$x = \log \frac{S}{K},\ \tau = \frac{1}{2}\sigma_0^2(T-t),\ u(x,\tau) = e^{-x}\frac{V(S,t)}{K}$$

transformed problem

$$-u_\tau + \tilde{\sigma}^2(\tau, x, u_x, u_{xx}) \cdot (u_x + u_{xx}) + \frac{2r}{\sigma_0^2}u_x = 0, \text{ for } x \in [A,B], \tau \in \left[0, \frac{\sigma_0^2 T}{2}\right]$$

$$u(x,0) = \Lambda(x),$$

$$u(A,\tau) = \alpha(\tau; A),$$

$$u(B,\tau) = \beta(\tau; B).$$

e.g. for Call: $\alpha_C(\tau; A) = 0, \quad \beta_C(\tau; B) = 1 - \exp\left(-\frac{2r}{\sigma_0^2}\tau - B\right)$

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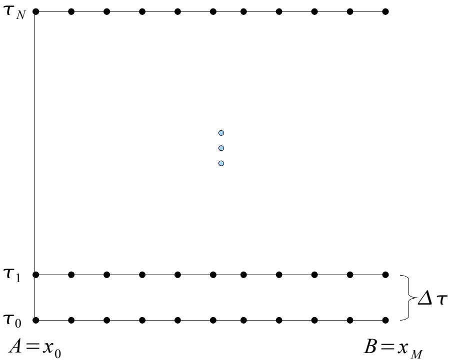
Discretization in space and time



space

$$A = x_0 < \ldots < x_M = B$$

time

$$\tau_j = j \cdot \Delta\tau, \Delta\tau := \frac{\sigma_0^2 T}{2N}$$

finite-differences

$$w_i^j \approx u(x_i, \tau_j)$$
$$\delta_x w_i^j \approx u_x(x_i, \tau_j)$$
$$\delta_{xx} w_i^j \approx u_{xx}(x_i, \tau_j)$$
$$\Gamma_i^j := \delta_x w_i^j + \delta_{xx} w_i^j$$

BDF, Crank-Nicolson and BDF2

Replace differential-quotients by difference-quotients:

BDF ($\theta = 0$) and Crank-Nicolson ($\theta = 1/2$)

$$-w_i^{j+1} + w_i^j + \Delta\tau \cdot (1-\theta)\left[\tilde{\sigma}^2(\tau_{j+1}, x_i, \Gamma_i^{j+1}) \cdot \Gamma_i^{j+1} + \frac{2r}{\sigma_0^2}\delta_x w_i^{j+1}\right]$$
$$+\Delta\tau \cdot \theta \left[\tilde{\sigma}^2(\tau_j, x_i, \Gamma_i^j) \cdot \Gamma_i^j + \frac{2r}{\sigma_0^2}\delta_x w_i^j\right] = 0$$

BDF2

$$-3w_i^{j+1} + 4w_i^j - w_i^{j-1} + 2\Delta\tau\left[\tilde{\sigma}^2(\tau_{j+1}, x_i, \Gamma_i^{j+1}) \cdot \Gamma_i^{j+1} + \frac{2r}{\sigma_0^2}\delta_x w_i^{j+1}\right] = 0$$

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BDF2

$$-3w_i^{j+1} + 4w_i^j - w_i^{j-1} + 2\Delta\tau\left[\tilde{\sigma}^2(\tau_{j+1}, x_i, \Gamma_i^{j+1}) \cdot \Gamma_i^{j+1} + \frac{2r}{\sigma_0^2}\delta_x w_i^{j+1}\right] = 0$$

A non-linear (non-smooth) System

- Write these equations neatly as

$$\mathbf{F}(\mathbf{w}^{j+1}; \mathbf{w}^{j}) = 0,$$

respectively,

$$\mathbf{F}(\mathbf{w}^{j+1}; \mathbf{w}^{j}, \mathbf{w}^{j-1}) = 0,$$

where $\mathbf{F} = (F_0, \ldots, F_M)^T : \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$

- Boundary data are incorporated by

$$F_0(\mathbf{w}^{j+1}) := w_0^{j+1} - \alpha(\tau_{j+1}; A)$$
$$F_M(\mathbf{w}^{j+1}) := w_M^{j+1} - \beta(\tau_{j+1}; B)$$

- Caution: System might be non-smooth.

Generalized Jacobian

While $\frac{\partial \tilde{\sigma}^2(\tau,x,\Gamma)}{\partial \Gamma}$ does not necessarily exist, the *generalized derivative* $\frac{\partial (\tilde{\sigma}^2(\tau,x,\Gamma)\cdot\Gamma)}{\partial \Gamma}$ exists.

Generalized Derivatives

$$\text{Le:}\quad \frac{\partial(\tilde{\sigma}^2(\tau,x,\Gamma)\cdot\Gamma)}{\partial\Gamma} = \begin{cases} 1+A & : \quad \Gamma \geq 0 \\ 1-A & : \quad \Gamma < 0 \end{cases}$$

$$\text{SB:}\quad \frac{\partial(\tilde{\sigma}^2(\tau,x,\Gamma)\cdot\Gamma)}{\partial\Gamma} = \left(1+\Psi(\alpha_i\Gamma)+\alpha_i\Gamma\Psi'(\alpha_i\Gamma)\right)$$

$$\text{with}\quad \alpha_i := e^{2r\tau/\sigma_0^2}a^2Ke^{x_i}$$

$$\text{FP:}\quad \frac{\partial(\tilde{\sigma}^2(\tau,x,\Gamma)\cdot\Gamma)}{\partial\Gamma} = \frac{1+\rho\cdot\Gamma}{(1-\rho\cdot\Gamma)^3}$$

$$\text{ALP+:}\quad \frac{\partial(\tilde{\sigma}^2(\tau,x,\Gamma)\cdot\Gamma)}{\partial\Gamma} = \begin{cases} \sigma_{max}^2 & : \quad \Gamma \leq 0 \\ \sigma_{min}^2 & : \quad \Gamma > 0 \end{cases}$$

Generalized Jacobian

In this sense define

Jacobian for BDF and CN

$$\frac{\partial F_i}{\partial w_k^{j+1}} = -\frac{w_i^{j+1}}{\partial w_k^{j+1}} + \Delta\tau \cdot (1-\theta)\left(\frac{\partial(\tilde{\sigma}^2(\tau_{j+1}, x_i, \Gamma_i^{j+1})\Gamma_i^{j+1})}{\partial \Gamma_i^{j+1}} \cdot \frac{\partial \Gamma_i^{j+1}}{\partial w_k^{j+1}} + \frac{2r}{\sigma_0^2} \cdot \frac{\partial(\delta_x w_i^{j+1})}{\partial w_k^{j+1}}\right)$$

and

Jacobian for BDF2

$$\frac{\partial F_i}{\partial w_k^{j+1}} = -3\frac{w_i^{j+1}}{\partial w_k^{j+1}} + 2\Delta\tau \cdot \left(\frac{\partial(\tilde{\sigma}^2(\tau_{j+1}, x_i, \Gamma_i^{j+1})\Gamma_i^{j+1})}{\partial \Gamma_i^{j+1}} \cdot \frac{\partial \Gamma_i^{j+1}}{\partial w_k^{j+1}} + \frac{2r}{\sigma_0^2} \cdot \frac{\partial(\delta_x w_i^{j+1})}{\partial w_k^{j+1}}\right)$$

Algorithm

Data: model; option parameters; payoff $\Lambda(x)$

Input: θ, temporal discretization N; spatial discretization M; cut-off interval $[A, B]$

Output: w_i^j for $j = 1, \ldots, N$ and $i = 0, \ldots, M$

Set $\mathbf{w}^0 \longleftarrow \Lambda(\mathbf{x})$;

for $j = 0, \ldots, N-1$ **do**

  Set $\tau \longleftarrow (j+1) \cdot \Delta\tau$;

  Set $\mathbf{w}^{j+1} \longleftarrow \mathbf{w}^j$;

  repeat `/* Newton iteration */`

    Compute $\mathbf{F}(\mathbf{w}^{j+1})$;

    Compute $D\mathbf{F}(\mathbf{w}^{j+1})$;

    Solve $D\mathbf{F}(\mathbf{w}^{j+1})\Delta\mathbf{w} = -\mathbf{F}(\mathbf{w}^{j+1})$;

    Set $\mathbf{w}^{j+1} \longleftarrow \mathbf{w}^{j+1} + \Delta\mathbf{w}$;

  until $\|\Delta\mathbf{w}\| < \epsilon$;

Convergence Results

Theorem (Convergence of BDF)

The fully implicit BDF scheme converges to the viscosity solution, whenever $\tilde{\sigma}^2(\tau, x, \Gamma)\Gamma$ satisfies the following conditions:

1. $\tilde{\sigma}^2(\tau, x, \Gamma) \cdot \Gamma$ *is continuous and monotone increasing in* Γ
2. *there exists a constant* $c_+ > 0$ *so that for all* $\Gamma \in I$ *and* $\varepsilon > 0$

$$\tilde{\sigma}^2(\tau, x, \Gamma + \varepsilon) \cdot (\Gamma + \varepsilon) \geq \tilde{\sigma}^2(\tau, x, \Gamma) \cdot \Gamma + c_+ \cdot \varepsilon$$

3. $$c_+ \frac{2-h}{h} \geq \frac{2r}{\sigma_0^2}.$$

Convergence Results

1. $\tilde{\sigma}^2(\tau, x, \Gamma) \cdot \Gamma$ is continuous and monotone increasing in Γ
2. $\exists c_+ > 0 \, \forall \Gamma \in I, \; \varepsilon > 0 : \tilde{\sigma}^2(\tau, x, \Gamma + \varepsilon) \cdot (\Gamma + \varepsilon) \geq \tilde{\sigma}^2(\tau, x, \Gamma) \cdot \Gamma + c_+ \cdot \varepsilon$
3. $c_+ \frac{2-h}{h} \geq \frac{2r}{\sigma_0^2}$

Theorem (Convergence of CN)

The Crank-Nicolson - scheme converges to the viscosity solution, whenever $\tilde{\sigma}^2(\Gamma)\Gamma$ satisfies conditions (1)-(3) and:

5. *there exists a constant $c_- > 0$ so that for all $\Gamma \in I$ and $\varepsilon > 0$*

$$\tilde{\sigma}^2(\tau, x, \Gamma - \varepsilon) \cdot (\Gamma - \varepsilon) \geq \tilde{\sigma}^2(\tau, x, \Gamma)\Gamma - c_- \cdot \varepsilon$$

6.

$$\Delta\tau \leq \frac{h^2}{c_-}$$

Assumptions on Volatility term

With the generalized derivative one can compute:

$$c_+ = \min_{\Gamma \in I} (\tilde{\sigma}^2(\Gamma) \cdot \Gamma)'$$
$$c_- = \max_{\Gamma \in I} (\tilde{\sigma}^2(\Gamma) \cdot \Gamma)'$$

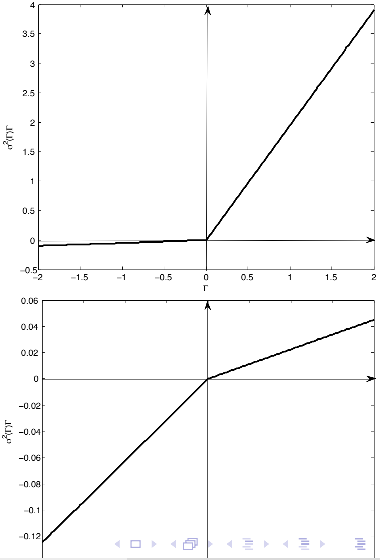
Assumptions on Volatility term

Transaction costs (Leland)

$$c_+ = 1 - A$$
$$c_- = 1 + A$$

Uncertain volatility model

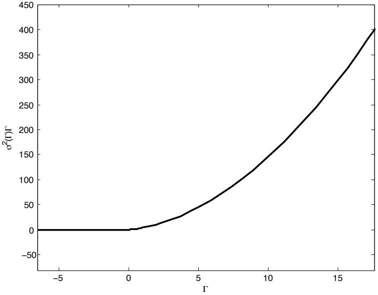
$$c_+ = \sigma^2_{min}$$
$$c_- = \sigma^2_{max}$$



Assumptions on Volatility term

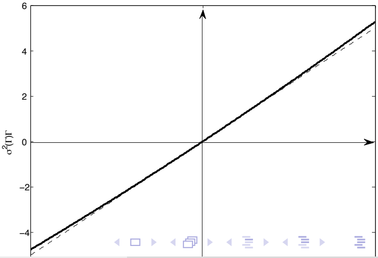
Transaction costs (S. & B.)

$$c_+ = O\left(\Gamma^{-1}\right), \quad \Gamma \to -\infty$$
$$c_- = O(\Gamma), \quad \Gamma \to \infty$$



Illiquidity model

$$c_+ = O\left(\Gamma^{-2}\right), \quad \Gamma \to -\infty$$
$$c_- \to \infty, \quad \Gamma \to 1/\rho$$



Proof of the Theorems (Sketch)

Theorem (Barles, Souganidis)

Any monotone, stable and consistent scheme converges to the unique viscosity solution.

Proof of the Theorems (Sketch)

1. *Monotonicity*:

Definition

A discretization is monotone if

- for all $i = 0, \dots, M$

$$F_i(\mathbf{w}^{j+1} + \varepsilon^{j+1}, \mathbf{w}^j + \varepsilon^j) \geq F_i(\mathbf{w}^{j+1}, \mathbf{w}^j)$$

with $\varepsilon^{j+1} = (0, \dots, 0, \varepsilon_{i-1}^{j+1}, 0, \varepsilon_{i+1}^{j+1}, 0, \dots, 0) \geq \mathbf{0}$,
$\varepsilon^j = (0, \dots, 0, \varepsilon_{i-1}^j, \varepsilon_i^j, \varepsilon_{i+1}^j, 0, \dots, 0) \geq \mathbf{0}$ and

$$F_i(\mathbf{w}^{j+1} + \varepsilon^{j+1}, \mathbf{w}^j) \leq F_i(\mathbf{w}^{j+1}, \mathbf{w}^j)$$

with $\varepsilon^{j+1} = (0, \dots, \varepsilon_i^{j+1}, \dots, 0) \geq \mathbf{0}$,

Proof of the Theorems (Sketch)

1. *Monotonicity*:
   - E.g.: Perturb $w_{i-1}^{j+1} \mapsto w_{i-1}^{j+1} + \varepsilon$, $\varepsilon > 0$
   - $\delta_x w_i^{j+1} \mapsto \delta_x w_i^{j+1} - \frac{\varepsilon}{2h}$
   - $\delta_{xx} w_i^{j+1} \mapsto \delta_{xx} w_i^{j+1} + \frac{\varepsilon}{h^2}$
   - $\Gamma_i^{j+1} \mapsto \Gamma_i^{j+1} + \varepsilon \frac{2-h}{2h^2}$
   -

$$
\begin{aligned}
&F_i(w_i^{j+1}, w_{i-1}^{j+1} + \varepsilon, w_{i+1}^{j+1}, w_i^j) = \\
&- w_i^{j+1} + w_i^j + \Delta\tau \left[\tilde{\sigma}^2 \left(\tau_{j+1}, x_i, \Gamma_i^{j+1} + \varepsilon \frac{2-h}{2h^2} \right) \left(\Gamma_i^{j+1} + \varepsilon \frac{2-h}{2h^2} \right) + \frac{2r}{\sigma_0^2} \delta_x w_i^{j+1} - \frac{2r}{\sigma_0^2} \frac{\varepsilon}{2h} \right] \geq \\
&- w_i^{j+1} + w_i^j + \Delta\tau \left[\tilde{\sigma}^2 \left(\tau_{j+1}, x_i, \Gamma_i^{j+1} \right) \Gamma_i^{j+1} + c_+ \cdot \varepsilon \frac{2-h}{2h^2} + \frac{2r}{\sigma_0^2} \delta_x w_i^{j+1} - \frac{2r}{\sigma_0^2} \frac{\varepsilon}{2h} \right] \geq \\
&F_i(w_i^{j+1}, w_{i-1}^{j+1}, w_{i+1}^{j+1}, w_i^j)
\end{aligned}
$$

Proof of the Theorems (Sketch)

2. *Stability:*
   - Stability guarantees that $\|\mathbf{w}^j\|_\infty$ is bounded for any $j = 0, \ldots, N$
   - Follows from the monotonicity of the scheme and the maximum principle
3. *Consistency:*
   - local discretization error of BDF vanishes as $\Delta\tau, h \to 0$

Complementary and penalty formulation

Complementary formulation

As in linear Black-Scholes case:

$$V_t + \frac{1}{2}\sigma^2(t, S, V_S, V_{SS})S^2 V_{SS} + (r-q)SV_S - rV \leq 0$$

$$V - V^* \geq 0$$

$$\left(V_t + \frac{1}{2}\sigma^2(t, S, V_S, V_{SS})S^2 V_{SS} + (r-q)SV_S - rV\right) = 0 \quad \vee \quad (V - V^*) = 0$$

Penalty formulation

Idea: add a positive penalty term $\tilde{p}$ to ensure $V(S,t) \geq V^*(S) - \varepsilon$.

$$V_t + \frac{1}{2}\sigma^2(t, S, V_{SS})S^2 V_{SS} + (r-q)SV_S - rV + \tilde{p} \cdot \max(V^* - V, 0) = 0.$$

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New co-ordinates

log money and revesed time

$$x = \log \frac{S}{K},\ \tau = \frac{\sigma_0^2}{2}(T - t),\ u(\tau, x) = e^{-x}\frac{V(t, S)}{K}$$

transformed complementary formulation

With $\Gamma := u_x + u_{xx}$,

$$\left.\begin{aligned} u_\tau - \widetilde{\sigma}^2(\tau, x, \Gamma)\Gamma - \frac{2(r-q)}{\sigma_0^2}u_x + \frac{2q}{\sigma_0^2}u \geq 0 \\ u(\tau, x) - u^*(x) \geq 0 \quad \text{for all } \tau \in \left[0, \frac{\sigma_0^2 T}{2}\right] \\ u_\tau - \widetilde{\sigma}^2(\tau, x, \Gamma)\Gamma - \frac{2(r-q)}{\sigma_0^2}u_x + \frac{2q}{\sigma_0^2}u = 0 \vee u(\tau, x) - u^*(x) = 0 \end{aligned}\right\}$$

New co-ordinates

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$$x = \log\frac{S}{K},\ \tau = \frac{\sigma_0^2}{2}(T-t),\ u(\tau,x) = e^{-x}\frac{V(t,S)}{K}$$

transformed penalty formulation

$$u_\tau - \widetilde{\sigma}^2(\tau,x,\Gamma)\Gamma - \frac{2(r-q)}{\sigma_0^2}u_x + \frac{2q}{\sigma_0^2}u - p(u^* - u)^+ = 0$$

- Transformed volatility $\tilde{\sigma}$ explicitly known by model
- Transformed payoff u^* also known explicitly

Finite Differences for Penalty Formulation

Replace differential quotients by difference quotients:

BDF ($\theta = 0$) and Crank-Nicolson ($\theta = 1/2$)

$$w_i^{j+1} - w_i^j = \Delta\tau(1-\theta)\left[\tilde{\sigma}^2(\tau_{j+1}, x_i, \Gamma_i^{j+1})\Gamma_i^{j+1} + \frac{2(r-q)}{\sigma_0^2}\delta_x w_i^{j+1} - \frac{2q}{\sigma_0^2} w_i^{j+1}\right]$$
$$+ \Delta\tau\theta\left[\tilde{\sigma}^2(\tau_j, x_i, \Gamma_i^j)\Gamma_i^j + \frac{2(r-q)}{\sigma_0^2}\delta_x w_i^j - \frac{2q}{\sigma_0^2} w_i^j\right] + p(w_i^* - w_i^{j+1})^+$$

finite differences

$$w_i^j \approx u(x_i, \tau_j) \qquad \delta_x w_i^j \approx u_x(x_i, \tau_j)$$
$$\delta_{xx} w_i^j \approx u_{xx}(x_i, \tau_j) \qquad \Gamma_i^j := \delta_x w_i^j + \delta_{xx} w_i^j$$

Finite Differences for Penalty Formulation

- Write this equation for fixed j compactly

$$\mathbf{F}^j(W^{j+1}; W^j) = 0$$

with $\mathbf{F}^j(W^{j+1}) := (F_0^j(W^{j+1}), \ldots, F_M^j(W^{j+1}))^T$

- boundaries are included by

$$F_0^j(W^{j+1}) := w_0^{j+1} - \mu(\tau_{j+1}, x_{\min})$$
$$F_M^j(W^{j+1}) := w_M^{j+1} - \nu(\tau_{j+1}, x_{\max})$$

- Caution: The system is not differentiable!

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- Caution: The system is not differentiable!

Finite Differences for Complementary Formulation

- With
$$G_i^j(W^{j+1}) := F_i^j(W^{j+1}) + p \cdot (w_i^* - w_i^{j+1})^+$$
- define the function $\mathbf{G}^j : \mathbb{R}^{(M+1)} \to \mathbb{R}^{(M+1)}$ by
$$\mathbf{G}^j(W^{j+1}) := (G_0^j(W^{j+1}), \dots, G_M^j(W^{j+1}))^T.$$

discrete complementary problem

$$\left.\begin{aligned} \mathbf{G}^j(W^{j+1}) &\geq 0 \\ W^{j+1} - W^* &\geq 0 \\ \mathbf{G}^j(W^{j+1}) = 0 \quad &\vee \quad W^{j+1} - W^* = 0. \end{aligned}\right\}$$

Numerical Scheme

Set W^0 by $w_i^0 \longleftarrow u^*(x_i)$ for $i = 0, \ldots, M$;
for $j = 0, \ldots, N-1$ **do**
  Set $\tau \longleftarrow (j+1) \cdot \Delta\tau$;
  Set $W^{j+1} \longleftarrow W^j$;
  repeat /* Newton iteration */
    Solve $D\mathbf{F}^j(W^{j+1})\Delta W = -\mathbf{F}^j(W^{j+1})$;
    Set $W^{j+1} \longleftarrow W^{j+1} + \Delta W$;
  until $\|\Delta W\| / \|W^{j+1}\| < \epsilon$;

- $D\mathbf{F}^j(W)$ is the *generalized* Jacobi-matrix and is known explicitly
- $D\mathbf{F}^j(W)$ is tridiagonal, usually 2-3 Newton iterations per level

First Results

General assumptions

Assume that $\Delta x, \Delta\tau$ are sufficiently small, such that

- $$\widetilde{\sigma}^2(\tau_j, x_i, \Gamma_i^j)\Big(-\frac{1}{2\Delta x} + \frac{1}{\Delta x^2}\Big) - \frac{r-q}{\sigma_0^2 \Delta x} \geq 0$$
- $$1 - \theta\Delta\tau \left(\widetilde{\sigma}^2(\tau_j, x_i, \Gamma_i^j)\frac{2}{\Delta x^2} + \frac{2q}{\sigma_0^2} \right) \geq 0$$

and assume that

- the transformed volatility $\tilde{\sigma}(\tau, x, \Gamma)$ is bounded

First Results

Satz (M-matrix)

The Jacobian matrix $DF^j(W^{j+1})$ is a M-matrix. Consequently, $DF^j(W^{j+1})$ is regular and the linear system

$$DF^j(W^{j+1})\Delta W = -F^j(W^{j+1})$$

can be solved by Gaussian elimination without pivoting.

First Results

Lemma (Stability)

Let W^{j+1} be a solution of $\mathbf{F}^j(W^{j+1}) = 0$. Then,

$$\|W^{j+1}\|_\infty \leq C_*$$

with a positive constant C_* which depends only on the payoff $u^*(x)$.

Penalty formulation $\longrightarrow$ Complementary formulation

Theorem

Assume that the stability condition

$$\frac{\Delta\tau}{\Delta x^2} < c_1 .$$

Then, for $\Delta\tau, \Delta x \to 0$

$$\mathbf{G}^j(W^{j+1}) \geq 0$$
$$W^{j+1} - W^* \geq -\frac{C}{p}$$
$$\mathbf{G}^j(W^{j+1}) = 0 \;\vee\; \left|W^{j+1} - W^*\right| \leq \frac{C}{p}$$

The constant $C > 0$ is independent of the penalty term p, $\Delta\tau$ and Δx.

Hence, W^{j+1} solves the discrete complementary formulation for $p \to \infty$.

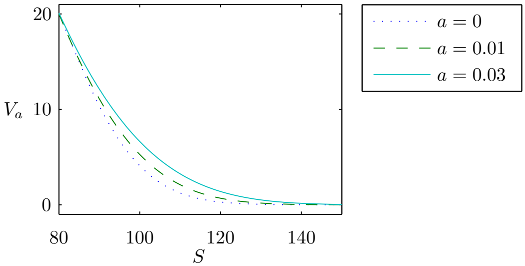
Case Study 1 - Transaction costs (Soner and Barles)

Plain put

$K = 100, T = 0.25$

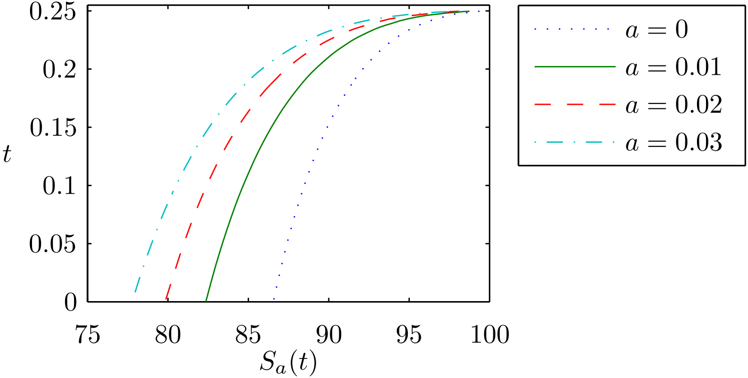
$r = 0.1, q = 0.01$

$\sigma = 0.25, p = 10^6$



- optimal convergence rates for BDF and Crank-Nicolson
- penalty parameter $p \approx 10^6$ sufficient

Case Study 1 - Transaction costs (Soner and Barles)



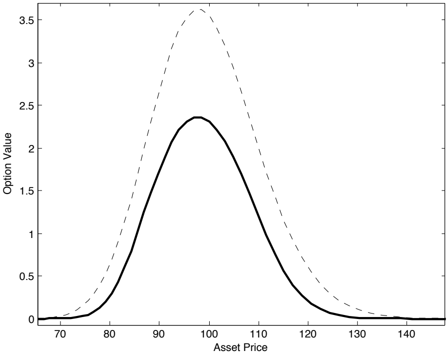
Case Study 2 - Uncertain Volatility

Butterfly Spread

$$K_1 = 90, K_2 = 110$$
$$r = 0.1, T = .25$$

and

$$\sigma_{max} = .25, \sigma_{min} = .15$$



Case Study 2 - Uncertain Volatility

Choose:

$M = 1000, A = -5, B = 3$

for stability:

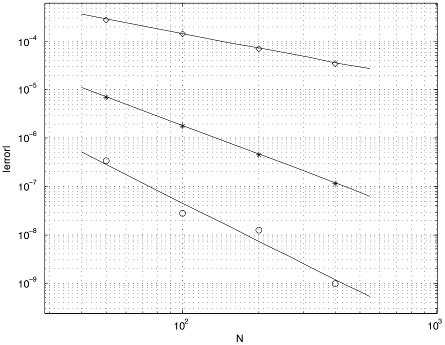
$$\Delta\tau \leq \frac{(0.008)^2}{\sigma^2_{min}},$$

thus

$$N \geq 44$$

Conv. Rates

BDF (diamonds) : -1
CN (1) (stars) : -1.99
BDF2 (circles) : -2.63



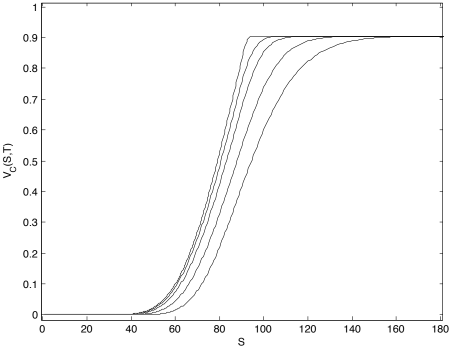
Case Study 3 - Transaction costs model (Leland)

Binary Option

$$K = 100, T = 1$$
$$\sigma_0 = .2, r = .1$$

and

$$A = 0, .5, .8, .9, .99$$



Case Study 3 - Transaction costs model (Leland)

Choose:

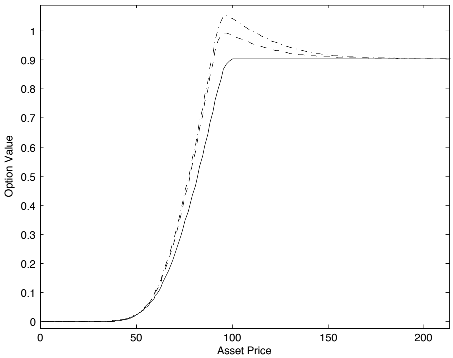
$M = 512, A = .95$

for stability:

$N \geq 160$

Plots are for $N = 10$:

wrong solution for CN and BDF2!



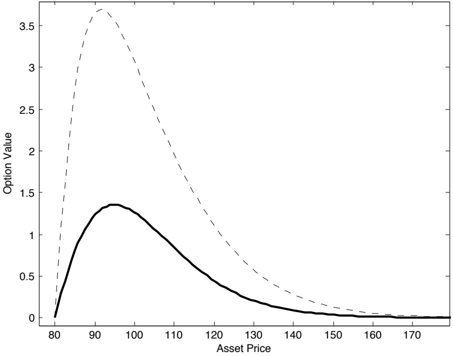
Case Study 4 - Transaction costs (Soner and Barles)

Barrier Option - DO

$B = 80, K = 100,$
$\sigma_0 = 0.2, r = 0.1,$
$T = 1$

and

$a = 0, 0.01$



Plot with BDF, $M = 300$

Case Study 4 - Transaction costs (Soner and Barles)

BUT

Case Study 4 - Transaction costs (Soner and Barles)

		BDF, $a = 0$			BDF, $a = 0.01$		
M	N	Value	Difference	Ratio	Value	Difference	Ratio
150	40	0.01367	0.00027		0.03303		
300	80	0.01355	0.00015	1.86	0.03394	0.00009	
600	160	0.01348	0.00007	2.01	no conv.		
1200	320	0.01344	0.00004	2.03	no conv.		

no convergence for large M and $a > 0$

Case Study 4 - Transaction costs (Soner and Barles)

WHY?

Case Study 4 - Transaction costs (Soner and Barles)

- Jump discontinuity at x_{i^*} at $\tau = 0$: thus $\Gamma^0_{i^*} = O(-1/h^2)$
- Remember: $c_+ = O(\Gamma^{-1}) = O(h^2)$ for $\Gamma \to -\infty$
- stability condition $c_+ \frac{2-h}{h} \geq \frac{2r}{\sigma_0^2}$ will be violated for $h \to 0$

Consider a Call:

- $\Gamma^0_{i^*} = O(1/h)$
- $c_+ = O(\Gamma^{-1}) = O(h)$ for $\Gamma \to -\infty$
- stability condition $c_+ \frac{2-h}{h} \geq \frac{2r}{\sigma_0^2}$ can be satisfied for $h \to 0$

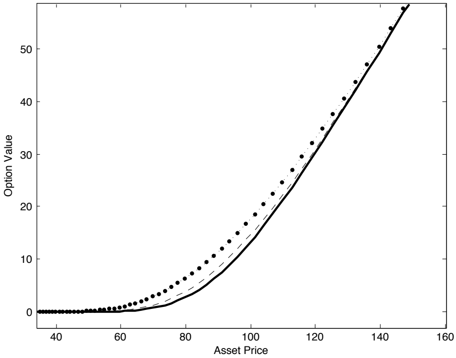
Case Study 4 - Transaction costs (Soner and Barles)

Call option

$$K = 100, \sigma_0 = 0.2$$
$$r = 0.1, T = 1$$

and

$$a = 0, 0.01, 0.05$$



		BDF			BDF2			CN (1)		
M	N	Value	Difference	Ratio	Value	Difference	Ratio	Value	Difference	Ratio
64	150	0.14269			0.14280			0.14280		
128	300	0.14554	0.00285		0.14560	0.00280		0.14560	0.00280	
256	600	0.14622	0.00068	4.21	0.14629	0.00065	4.29	0.14625	0.00065	4.29
512	1200	0.14639	0.00017	3.93	0.14641	0.00016	4.12	0.14641	0.00016	4.11

Case Study 5 - Illiquidity model

- Consider: Bull Spread as before, $\rho = 0.01$
- We get $c_+ = O(h^2)$
- Expect convergence problems for large M

M	N	BDF Value	BDF Difference	BDF Ratio	BDF iter	BDF2 Value	BDF2 Difference	BDF2 Ratio	BDF2 iter
64	25	0.02671			3.0	0.02624			3.0
128	50	0.02118	-0.00552		2.86	0.02118	-0.00505		2.78
256	100	0.01982	-0.00137	4.04	2.59	0.01991	-0.00127	3.97	2.53
512	200	0.01975	-0.00007	19.79	2.39	0.01984	-0.00007	18.15	2.34
1024	400	0.01991	0.00016	-0.44	2.28	0.01998	0.00014	-0.51	2.26
2048	800	0.01947	-0.00004	-0.38	2.77	0.01491	-0.00506	-0.03	2.59

M	N	CN (1) Value	CN (1) Difference	CN (1) Ratio	CN (1) iter
64	25	0.02624			3.0
128	50	0.02117	-0.00507		2.74
256	100	0.01990	-0.00127	3.98	2.49
512	200	0.01983	-0.00007	19.54	2.33
1024	400	0.01998	0.00014	-0.46	2.245
2048	800	1.18652	1.16654		4.84

schemes destabilize for large M

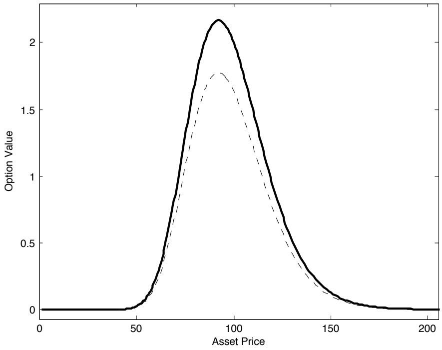
Case Study 5 - Illiquidity model

Butterfly Spread

$$K_1 = 90, K_2 = 110$$
$$r = 0.1, T = 1.$$

and

$$\rho = 0.01$$



Summary

1. Convergent, implicit schemes for non-linear BS-equations
2. Stability of the scheme can be checked *a priori*
3. Stability **must** be checked to avoid spurious solutions
4. Behavior of the scheme depends strongly on the model equation
5. Behavior depends on the payoff profile

Disclaimer