Revealed Preference Theory for Indivisible Goods

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Working Paper: Afriat's Theorem for Indivisible Goods.

Outlines

- Introduction: Revealed Preference Theory and Afriat's Theorem
  - Motivation
  - GARP
  - Afriat's constructive approach
  - Extensions
- The case of indivisible goods
  - Intuition
  - (new) Axiom
  - Main results and sketch of the proof
  - Related literature

Introduction - Revealed Preference Theory

- Samuelson (1937):

  " [Revealed Preference] ... is a topic that people will be discussing a hundred years from now "

  (from Mas-Colell (1982))

- Samuelson's general postulate:

  the consumer theory based on preferences (or utility functions) must be operational, i.e., refutable by observable data generated from feasible experiments.

- Conversely: given observable data, underlying preferences consistent with data.

- Importance for Welfare Economics: from what is observed, one wants to evaluate and predict what would happen with a change in the environment

- $\Rightarrow$ very basic axiom on demand data: WARP

Introduction - Revealed Preference Theory

- General formulation: Richter (1966) based on Houthakker (1950)
  - Possibly abstract framework
  - In the competitive framework he defines SARP
  - SARP reinforces WARP to a chain of observations (transitive closure).
  - SARP is necessary and sufficient for the existence of underlying preferences (assuming single-valued demand functions)

- Another Approach: Integrability (or differentiable approach)
  - The whole demand is observed.
  - Testable restrictions are based on Slutsky relations.
  - See Chiappori and Ekeland (2009)

Introduction - Revealed Preference Theory

Richter's approach is general but not constructive.

Alternative approach: **Afriat (1967)** (Diewert (1973) and Varian (1982) in its modern formulation)

Main ingredients:

- Fully constructive and operational
- Competitive framework: standard consumer problem with linear prices.
- Finite set of observations
- axiom on demand data: GARP
- **Result:** the data satisfy GARP iff there exists a well-behaved utility function consistent with the data

Model and Afriat's Theorem

An analyst observing at each date $t = 1, \ldots, n$

the bundle $x_t \in \mathbb{R}_+^K$ purchased by a single consumer,

and positive prices $p_t \in \mathbb{R}_{++}^K$.

The consumption set is $X \subseteq \mathbb{R}_+^K$.

The budget set at any date t is:

$$B_t := \{x \in \mathbb{R}_+^K \ : \ p_t \cdot x \leq p_t \cdot x_t\}$$

Definition 1 *A utility function $u : X \to \mathbb{R}$ is called a rationalization of the observations $(x_t, p_t)_{t=1,\ldots,n}$ if, at each date t, x_t solves*

$$\max u(x) \text{ subject to } x \in B_t \cap X \tag{1}$$

Model and Afriat's Theorem

The bundle x_i is said to be *directly revealed preferred* to x_j, $x_i R x_j$, if $x_j \in B_i$.

The transitive closure of R is denoted by H,

$\Big($ That is, $x_i H x_s$ if there exists an ordered subset $\{i, j, k, \ldots, r, s\} \subset \{1, \ldots, n\}$ such that $x_i R x_j$, $x_j R x_k$, ..., $x_r R x_s$.$\Big)$

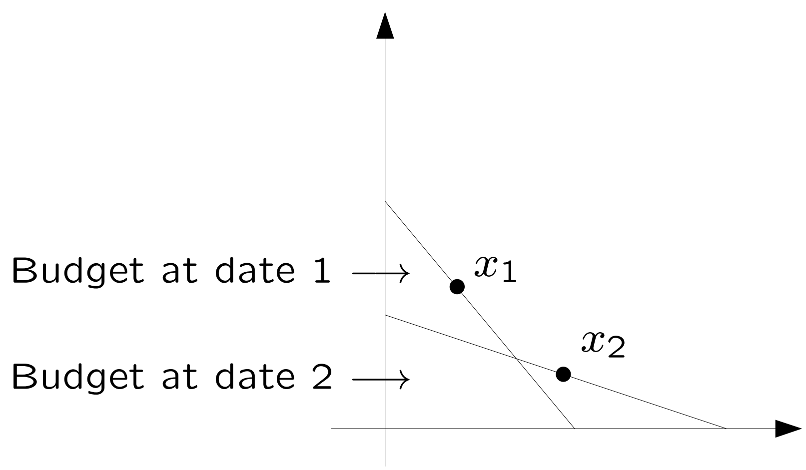
x_i is said to be *revealed preferred* to x_s if $x_i H x_s$.

Definition 2 *The observations $(x_t, p_t)_{t=1,\ldots,n}$ satisfy GARP if for any $i, j = 1, \ldots, n$*

$$x_i H x_j \Rightarrow p_j \cdot x_i \geq p_j \cdot x_j$$

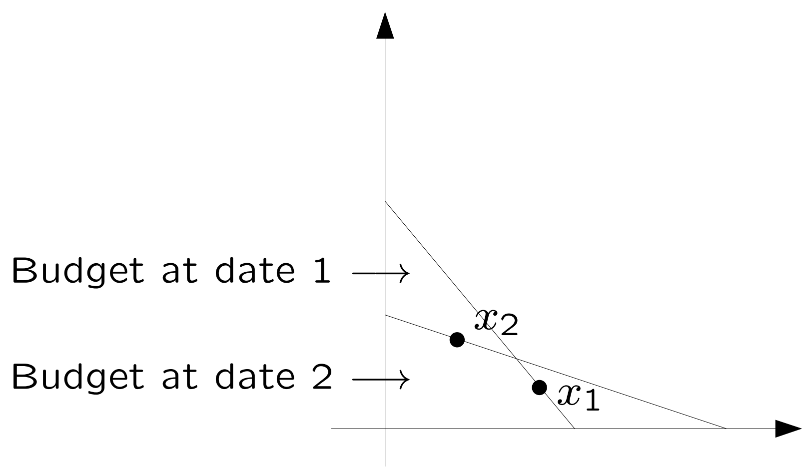
Theorem 1 (Afriat) *Let $X = \mathbb{R}^K_+$. The observations $(x_t, p_t)_{t=1,\ldots,n}$ satisfy GARP if, and only if, there exists a continuous, concave and strictly monotonic rationalization of the observations.*

GARP



Budget at date 1 $\longrightarrow$
Budget at date 2 $\longrightarrow$
x_1
x_2

Violation of GARP



Afriat's constructive approach

(Afriat's inequalities) If the observations satisfy GARP it can be shown that the following system admit a solution with $\bar{\psi}_1, \ldots, \bar{\psi}_n$ and $\bar{\delta}_1, \ldots, \bar{\delta}_n > 0$:

$$\bar{\psi}_k \leq \bar{\psi}_j + \bar{\delta}_j \alpha_{jk} \quad \forall j, k = 1, \ldots, n \qquad (*)$$

where $\alpha_{jk} = p_j \cdot x_k - p_j \cdot x_k$.

Can be shown by using linear programming (Fostel et al. (2004)) or graph theory (Fujishige and Yang (2013))

The following function provides a well-behaved rationalization of the observations:

$$\bar{u}(x) = \min\left\{\bar{\psi}_1 + \bar{\delta}_1 p_1 \cdot (x - x_1), \ldots, \bar{\psi}_n + \bar{\delta}_n p_n \cdot (x - x_n)\right\}$$

Afriat's Theorem - Extensions

Other models:

- General Equilibrium (multi-agent): Brown and Matzkin (1996)
- Household consumption: Cherchye, De Rock and Vermeulen (2007)
- Production: Varian (1982) and Cournot competition: Carvajal, Deb, Fenske and Quah (2013)

Other specifications of the consumer model (on p, u or X):

- General budget sets: Forges and Minelli (2009)
- Additive preferences: Quah (2012)
- Indivisible goods (this paper)

What we do: indivisible goods

$$X = \mathbb{N}^K$$

Consumption bundles, not items (Ekeland Galichon (2012))!

In practice, goods are often indivisible, in the field or in the laboratory.

Local nonsatiation becomes meaningless so that GARP, in its usual form, is no longer a necessary condition of rationalization.

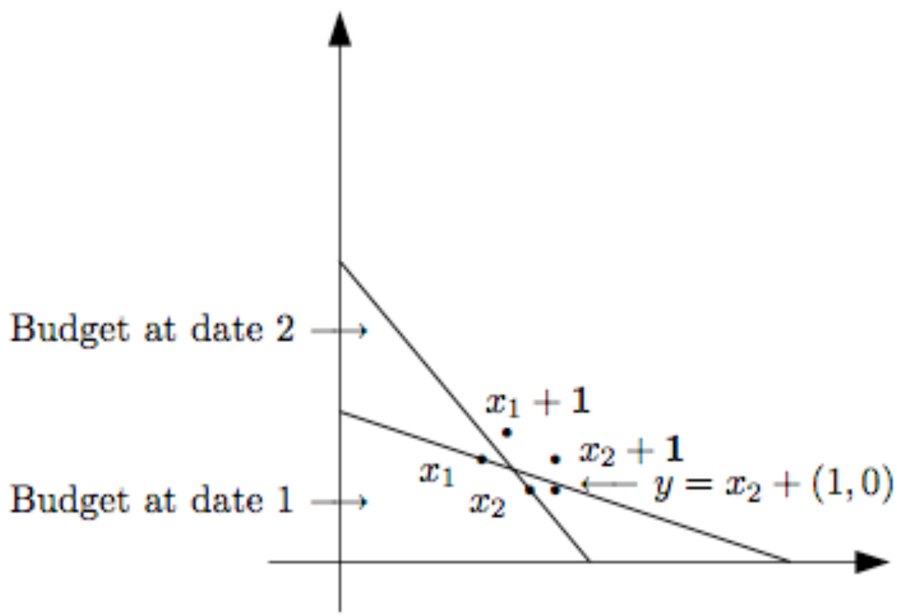
We identify a natural counterpart of the standard GARP for demand data in which goods are all indivisible.

We show that the new axiom (DARP, for "discrete axiom of revealed preference") is necessary and sufficient for the rationalization of the data by a well-behaved utility function.

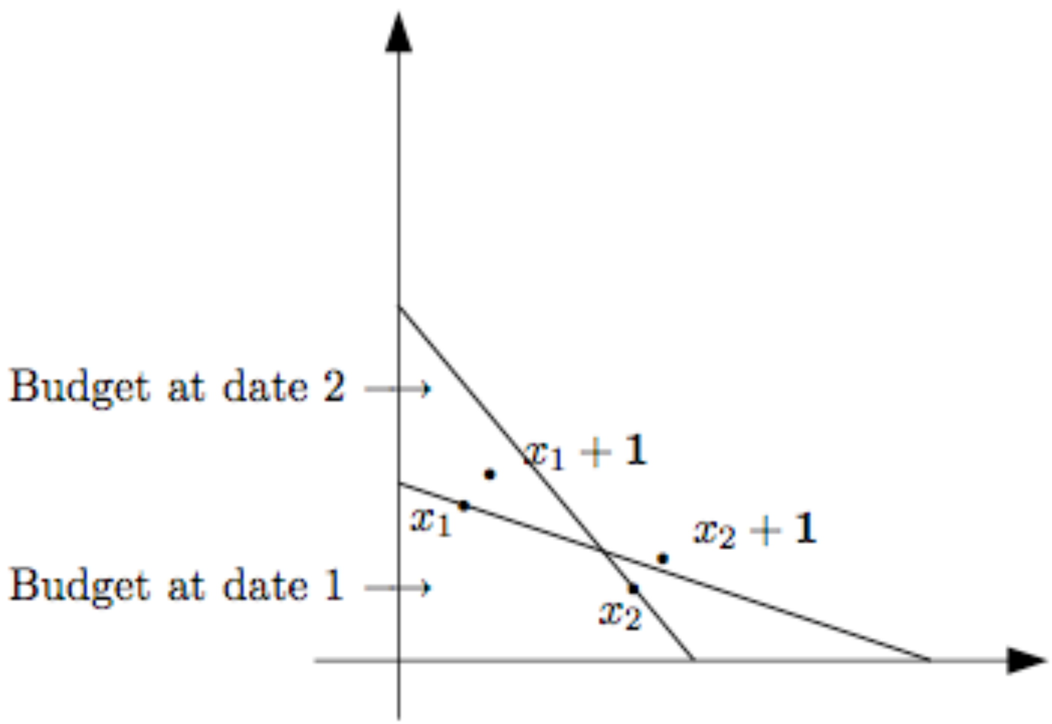
Illustration

Problem:

$$\max u(x) \text{ subject to } x \in B_t \cap \mathbb{N}^K \tag{2}$$



Illustration



Main result

Definition 3 *The observations* $(x_t, p_t)_{t=1,\ldots,n}$ *satisfy the discrete axiom of revealed preference (DARP) if for any* $i, j = 1, \ldots, n$

$$x_i H x_j \Rightarrow x_i + \mathbf{1} \notin B_j$$

where $\mathbf{1} = (1, \ldots, n)$*.*

Proposition 1 *The observations* $(x_t, p_t)_{t=1,\ldots,n}$ *satisfy DARP if, and only if, there exists a discrete quasi-concave and monotonic rationalization of the observations.*

Strict monotonicity

One shortcoming of the previous result is that we do not obtain a strictly monotonic rationalization.

Given the set of observations, let $c(t)$ be one of the cheapest goods at date t, that is, $c(t) \in \text{argmin}\{p_t^g \,:\, g = 1, \ldots, K\}$.

Definition 4 *The observations* $(x_t, p_t)_{t=1,\ldots,n}$ *satisfy DARP* if for any* $i, j = 1, \ldots, n$

$$x_i H x_j \;\Rightarrow x_i + \mathbf{e}^{c(j)} \notin B_j$$

where $\mathbf{e}^{c(j)} = (1, \ldots, 0, 1, 0, \ldots, 0)$*, with* 1 *in the* $c(j)$*-th component.*

Proposition 2 *The observations* $(x_t, p_t)_{t=1,\ldots,n}$ *satisfy DARP* if, and only if, there exists a discrete quasi-concave and strictly monotonic rationalization of the observations.*

About the proof

An interesting (and perhaps unexpected) feature of the proof of our main results (propositions 1 and 2) is that the construction of an explicit utility function from Afriat's inequalities goes through in our discrete framework.

We deduce the existence of a solution for adequately chosen Afriat's inequalities. We obtain then $\psi_1, \ldots, \psi_n$ and $\delta_1, \ldots, \delta_n > 0$ such that $u : \mathbb{N}^K \to \mathbb{R}$ defined as follows is a well-behaved rationalization of the observations:

$$u(x) = \min\Big\{\psi_1 + \delta_1 p_1 \cdot (x - x_1)\mathbb{1}_{A_1^c}(x), \ldots, \psi_n + \delta_n p_n \cdot (x - x_n)\mathbb{1}_{A_n^c}(x)\Big\}$$

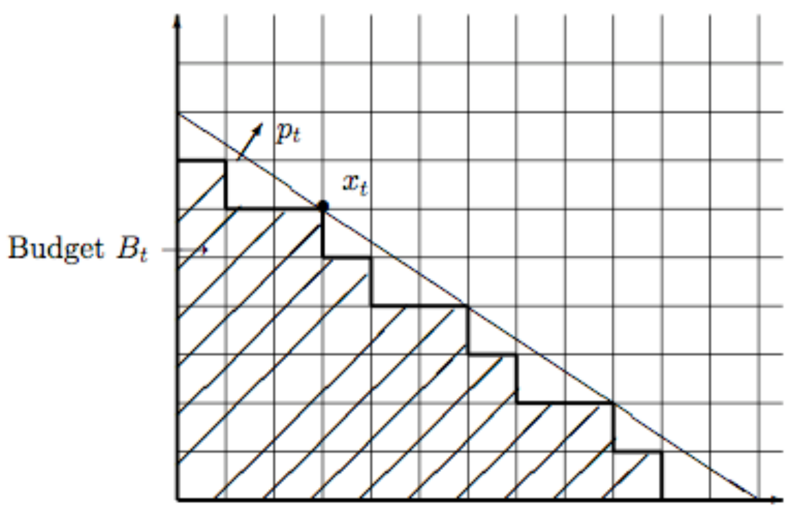
where $A_t = \{x \in \mathbb{N}^K \; : \; p_t \cdot (x_t - \mathbf{1}) < p_t \cdot x \leq p_t \cdot x_t\big\}$.

A potential difficulty with our construction is that the desirable properties of a utility function are not *a priori* granted here...

Remark: nonconvexities

The next picture describes a nonconvex and continuous budget generated by the revenue $p_t \cdot x_t$.

By using existing result of the literature (Forges and Minelli, 2008), one cannot get particular property beyond monotonicity.



Slight generalization: (possibly) non binding budgets

Under monotonic preferences, the rational consumer holds an unobserved revenue r_t at date t such that $p_t \cdot x_t \leq r_t$ and $p_t \cdot x > r_t$ for every $x \gg x_t$, with $x \in \mathbb{N}^K$.

Contrary to the perfectly divisible case, this does not imply in our framework that $p_t \cdot x_t = r_t$.

It follows that, instead of B_t, the analyst may be willing to consider larger budget sets, which are compatible with such typical losses.

Formally, a family of budget gap parameters is $\boldsymbol{\theta} = (\theta_t)_{t=1,\ldots,n}$, with $\theta_t \in [0, 1)$.

Given $\boldsymbol{\theta}$, the budget at date t is:

$$B_t^{\boldsymbol{\theta}} := \left\{ x \in \mathbb{R}_+^K \,:\, p_t \cdot x \leq p_t \cdot (x_t + \theta_t \mathbf{1}) \right\}$$

Our previous results are virtually not affected by allowing for such budgets: $\boldsymbol{\theta}$-DARP iff $\boldsymbol{\theta}$-rationalization. (idem with $\boldsymbol{\theta}$-DARP*)

Related literature with $X = \mathbb{N}^K$

Polisson and Quah (2013): unobserved continuous good

Consumer Model: at each date $t = 1, \ldots, n$, there exist $M_t \geq 0, q_t > 0$ such that

$(x_t, \frac{M_t - p_t \cdot x_t}{q_t})$ solves

$$\max_{(x,y) \in X \times \mathbb{R}_+} u(x) + y \text{ subject to } p_t \cdot x + q_t y \leq M_t \tag{3}$$

Axiom: GARP.

Fujishige and Yang (2012): cost efficiency

Consumer Model: at each date $t = 1, \ldots, n$,

x_t solves

$$\max_{x \in X} u(x) \text{ subject to } p_t \cdot x \leq p_t \cdot x_t \tag{4}$$

and

$$\min_{x \in X} p_t \cdot x \text{ subject to } u(x) \geq u(x_t) \tag{5}$$

Axiom: GARP.

Brown and Calsamiglia (2007) and Sákovics (2013): money

Consumer Model: there exists $M \geq 0$ such that at each date $t = 1, \ldots, n$, $(x_t, M - p_t \cdot x_t)$ solves

$$\max_{(x,y) \in X \times \mathbb{R}_+} u(x) + y \text{ subject to } p_t \cdot x + y \leq M \tag{6}$$

Axiom: ARV (axiom of revealed valuation), stronger than GARP.

Considering continuous or indivisible goods is innocuous here.

Cosaert and Demuynck (2013): finite choice sets

The data consist, at every date $t = 1, \ldots, n$, of a choice among finitely many consumption bundles $\left\{b_t^1, \ldots, b_t^{N_t}\right\}$ with $b_t^k \in \mathbb{R}_+^K$, $k = 1, \ldots, N_t$.

a utility function $u : X \to \mathbb{R}$ is a rationalization if at each date $t = 1, \ldots, n$,

x_t solves

$$\max_{x \in X} u(x) \text{ subject to } x \in \left\{b_t^1, \ldots, b_t^{N_t}\right\} \tag{7}$$

Axiom: WMARP

Can be reformulated by considering auxiliary budget sets

$$B_t' = \left\{x \in \mathbb{R}_+^K : \exists b_t^k \text{ such that } x \leq b_t^k\right\}$$

Then WMARP is just GARP for the general budgets B_t' as in Forges and MInelli (2009)

If the finite budget sets are generated by discrete linear budget sets, $B_t \cap \mathbb{N}^K$, then WMARP and DARP are equivalent.

Concluding remarks

- Watch out: GARP in its standard form is not always adequate
- Indivisibilities: DARP
- Need to distinguish between monotonicity and strict monotonicity
- Need to account for possibly non binding budgets
- Overall our results complete the picture on the role of indivisibilities in RP theory