An application of Mean Field Games to Oil Production

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(joint work with PN Giraud, O Guéant, PL Lions)

Research done with the support of the FDD and CFE.

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- 2009: Two generalized frameworks : congestion and planning (J.M. Lasry + P.L. Lions).
- 2010-?: Applications are being developed in many places (Dauphine, Chicago, Austin, Cambridge, ...). Planning will have applications to design incentives in economics (*mechanism design*).

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- Labor Market (*PhD Dissertation*)
- Portfolio Management (*PhD Dissertation*)
- Economic Growth (*PhD Dissertation*)
- Oil Production in the long run* (*Paris-Princeton Lectures on Mathematical Finance*)

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- Spatial Distribution of Populations (*PhD Dissertation, Journal de Mathématiques Pures et Appliquées*)
- Mexican Wave (*Paris-Princeton Lectures on Mathematical Finance*)
- People arrival times at a Meeting (*Paris-Princeton Lectures on Mathematical Finance*)
- Viruses Propagation

Framework

- Optimal oil extraction in the long run (100 to 150 years).
- Continuum of producers: perfect competition and MFG.
- The goal is to characterize the problem with 2 PDEs (HJB for the Value function of holding a certain quantity of oil - Transport equation for the distribution of oil reserves)
- Generalization to unusual optimization criteria

Setup

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Supply side

- A continuum of producers. Same technology. Different oil reserves.
- Uniform instantaneous production cost for a quantity qdt: $C(q) = \alpha q + \frac{\beta}{2} q^2$.
- Optimization problem:

$$Max \mathbb{E} \int_0^\infty \left(p(t)q(t) - C(q(t))\right) e^{-rt} dt$$

$$dR(t) = -q(t)dt + \nu R(t) dW(t), \quad \forall t > 0, R(t), q(t) \geq 0$$

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Demand side

Isoelastic demand and economic growth: $D(t,p) = We^{\rho t} p^{-\sigma}$ or $D(t,p) = We^{\rho t} p^{-\sigma} - \delta$

The MFG framework

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$\nu > 0$

Mean field games are necessary in general:

- Bellman function

$$u(t,R) = \sup_{q(\cdot)} \int_t^{\infty} \left(p(s)q(s) - C(q(s))\right) e^{-r(s-t)} ds$$

$$s.t. \quad R(t) = R, \quad dR(s) = -q(s)ds + \nu R(s)dW(s)$$

$$\forall s > t, R(s), q(s) \geq 0$$

- Distribution of reserves $m(t,R)$

PDEs I

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Hamilton-Jacobi-Bellman

$$\partial_t u(t,R) - ru(t,R) + \frac{\nu^2}{2} R^2 \partial^2_{RR} u(t,R) + \frac{\beta}{2} q^*(t,R)^2 = 0$$

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Kolmogorov

$$\partial_t m(t,R) + \partial_R \left(-q^*(t,R) m(t,R)\right) = \partial^2_{RR} \left[\frac{\nu^2}{2} R^2 m(t,R)\right]$$

where $q^*(t,R)$ is the optimal extraction function.
HJB is backward. Kolmogorov is forward with $m(0,R)$ given.

PDEs II

PDEs II

Optimal extraction

$$q^*(t,R) = \frac{(p(t) - \alpha - \partial_R u(t,R))_+}{\beta}$$

We see that $\partial_R u(t,R)$ is the Hotelling rent.

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Price

Demand = Supply $\Rightarrow$

$$p(t) = D(t, \cdot)^{-1} \left(-\frac{d}{dt} \int Rm(t, R) dR \right)$$

Price is a (complex) function of m

Evolution of production

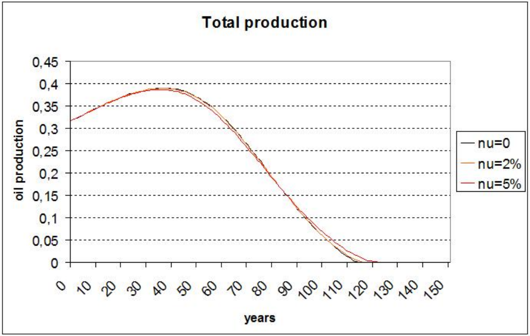
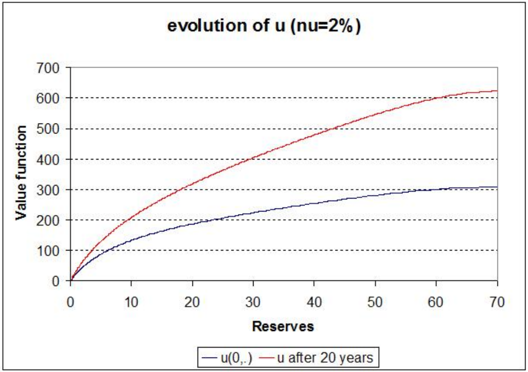


Figure: $r = 5\%, \rho = 2\%, \alpha = 10, \beta = 100, \sigma = 1.2, \delta = 0.1$

Evolution of u



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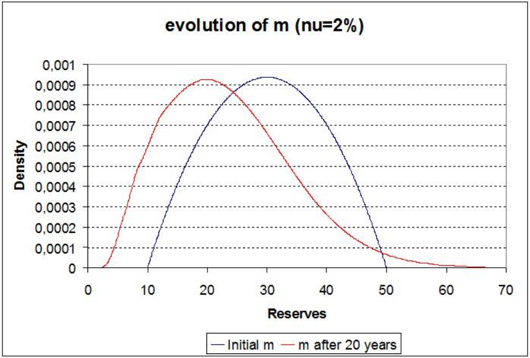
Evolution of m



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Generalization

- p was a function of m. Why not considering more general functions?
- Example: Producers do not want to be the last ones to extract oil (risk of nationalization, ...)
- Slightly modified model: only the HJB equation changes.

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HJB with ranking

$$\partial_t u(t,R) + \frac{\nu^2}{2} R^2 \partial^2_{RR} u(t,R) - ru(t,R)$$

$$-\epsilon \int_0^R m(t,\phi)d\phi + \frac{1}{2\beta}\left[(p(t) - \alpha - \partial_R u(t,R))_+\right]^2 = 0$$

Effect of the additional term:

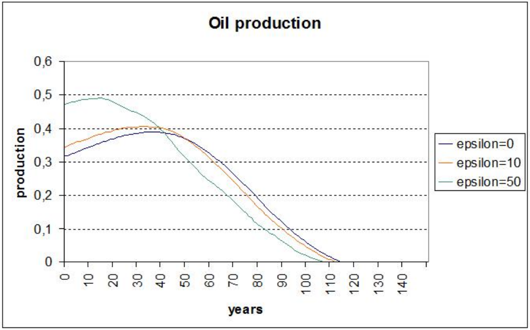

Figure: $r = 5\%, \rho = 2\%, \alpha = 10, \beta = 100, \sigma = 1.2, \delta = 0.1, \nu = 2\%$