Séminaire - FIME

Probabilistic representation of a class of nonconservative nonlinear PDE

ANTHONY LECAVIL [1] , NADIA OUDJANE [2] AND FRANCESCO RUSSO [3].

June, 5th 2015

1. ENSTA-ParisTech, UMA
2. EDF R&D, FiME
3. ENSTA-ParisTech, UMA

Summary

Plan

We consider the following non conservative and nonlinear PDE

$$\begin{cases} \partial_t u = \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{ij}\left((\Phi\Phi^t)_{i,j}(t,x,u)u\right) - div\left(g(t,x,u)u\right) + \Lambda(t,x,u)u \\ \\ u(0,dx) = \zeta_0(dx)\,. \end{cases}$$

- Aim 1 : Find a forward probabilistic representation of the PDE
- Aim 2 : Propose a numerical approximation of the solution which is both
  1. *less sensitive to the dimension* as a *Monte Carlo* scheme ;
  2. able to concentrate the computing efforts in *the region of interest* as a *forward* representation.

Major contributions since the sixties

- Conservative PDE : $\int_{\mathbb{R}^d} u_t(x)dx = 1$ for all $t \in [0, T]$

$$\partial_t u_t = \frac{1}{2}\partial^2_{xx}(\Phi(x, u_t)u_t) - \partial_x(b(x, u_t)u_t) \ , \quad (\Lambda = 0) \quad \text{where}$$

$$\begin{cases} \Phi(x, u_t) & := & \int_{\mathbb{R}^d} K^{\Phi}(x, y)u_t(dy) \ , \\ g(x, u_t) & := & \int_{\mathbb{R}^d} K^{g}(x, y)u_t(dy) \quad , \end{cases}$$

Integral dependence on u and not point dependence on u.

- McKean introduced the notion of nonlinear SDE (NLSDE)

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(Y_s, u_s)dW_s + \int_0^t g(Y_s, u_s)ds \\ u_t \text{ is the density of the law of } Y_t \ , \end{cases} \tag{1.1}$$

- Propose an interacting particle system (IPS) whose the limit is a sol. of PDE : propagation of chaos estimates.

- Méléard et al. have studied, under smooth assumptions, exist./uniqu. of

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s))dW_s + \int_0^t g(u(s, Y_s))ds \\ u_t \text{ is the density of the law of } Y_t \end{cases} \tag{1.2}$$

$\Longrightarrow$ point dependence on u, i.e. $K^{\Phi}(\cdot, y) = K^{g}(\cdot, y) = \delta_y$.

- They also proved that the regularized version

$$\begin{cases} Y_t^{\varepsilon} = Y_0 + \int_0^t \Phi((K_{\varepsilon} * u^{\varepsilon})(s, Y_s^{\varepsilon}))dW_s + \int_0^t g((K_{\varepsilon} * u^{\varepsilon})(s, Y_s^{\varepsilon}))ds \\ u_t^{\varepsilon} \text{ is the density of the law of } Y_t^{\varepsilon} \end{cases}$$

strongly converges to (1.2) when $K_{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \delta$.

- Benachour et al. have proved exist./uniq. of

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s))dW_s \\ u_t \text{ is the density of the law of } Y_t \ , \end{cases} \quad (1.3)$$

with $\Phi : x \in \mathbb{R} \mapsto x^{\frac{k-1}{2}}, \ k \geq 1$.

Russo et al. have extended (1.3) for Φ only bounded and measurable.

- This representation is associated to the Porous Media Equation

$$\partial_t u = \frac{1}{2}\partial^2_{xx}(u\Phi^2(u)) \ .$$

- Framework : Nonconservative nonlinear PDE of the form

$$\begin{cases} \partial_t u = \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{ij}\left((\Phi\Phi^t)_{i,j}(t,x,u)u\right) - div\left(g(t,x,u)u\right) + \boxed{\Lambda(t,x,u)u} \\ u(0,dx) = \zeta_0(dx)\,, \end{cases}$$

where

- ζ_0 is a probability measure on $\mathbb{R}^d$;
- $\Phi : [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^{d\times d}$, $g : [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$, $\Lambda : [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ are bounded and measurable functions ;
- $u(0,dx) = \zeta_0(dx)$ means $u(t,x)dx \xrightarrow[t\to 0]{} \zeta_0(dx)$ weakly

Nonconservative $\Longleftrightarrow \int_{R^d} u(t,x)dx = fct(t) \Longleftrightarrow \Lambda \neq 0$.

- Our idea : consider the following representation

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, \mathbf{u}(s, Y_s))dW_s + \int_0^t g(s, Y_s, \mathbf{u}(s, Y_s))ds \\ \mathbf{u}(t, \cdot) := \frac{d\nu_t}{dx} \quad \text{such that for any } \varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}) \\ \nu_t(\varphi) := \mathbb{E}\left[\varphi(Y_t) \exp\left\{\int_0^t \Lambda(s, Y_s, \mathbf{u}(s, Y_s))\, ds\right\}\right] \end{cases},$$

Observations :

- $\int_{R^d} u(t,x)dx = \mathbb{E}\left[\exp\left\{\int_0^t \Lambda(s, Y_s, \mathbf{u}(s, Y_s))\, ds\right\}\right].$
- The measure ν_t needs the law of all the process Y ($\in \mathcal{P}(\mathcal{C}([0, T], \mathbb{R}^d))$) and not only marginals laws.
- point dependence on u in Φ and g => technical difficulty.

- Bypass the difficulty : consider a *regularized version of NLSDE*,

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, \mathbf{u}(s, Y_s))dW_s + \int_0^t g(s, Y_s, \mathbf{u}(s, Y_s))ds \\ \mathbf{u}(t, y) = \mathbb{E}\left[K(y - Y_t) \exp\left\{\int_0^t \Lambda(s, Y_s, \mathbf{u}(s, Y_s))ds\right\}\right] \end{cases} .$$

- Integral dependence on $\mathcal{L}(Y_\cdot) \in \mathcal{P}(\mathcal{C}^d)$.
- u depends on itself $\Longrightarrow$ main difference with the cases already covered in the literature.
- Formally, $\Lambda = 0$ and $K = \delta$: cases already developed by Méléard and al. (i.e. conservative case).

Main results of existence and uniqueness

1. *"Lipschitz" case :* If
   - ζ_0 admits a 2nd order moment,
   - Φ, g, Λ are bounded, **uniformly Lipschitz w.r.t.** t,

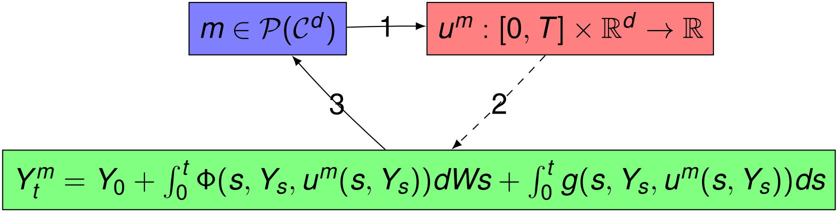
   there is a unique **strong solution** (Y, u).
2. *"Semi-weak" case :* If
   - ζ_0 admits a 2nd order moment,
   - Φ, g are bounded and **uniformly Lipschitz w.r.t.** t,
   - Λ is only continuous,

   there is a (non-unique) **strong solution** (Y, u).
3. *"Weak" case :* If
   - Φ, g, Λ are bounded and continuous

   there is a weak solution (Y, u).

Plan

- Existence/uniqueness of (Y, u) in *Lipschitz case* relies on :



where we recall that

$$u^m(t, y) = \mathbb{E}\left[K(y - Y_t)\, e^{\int_0^t \Lambda\left(s, Y_s, u^m(s, Y_s)\right) ds}\right], \text{ with } m := \mathcal{L}(Y) .$$

- Aim 1 : Existence / uniqueness of the map $m \mapsto u^m$.
- Aim 2 : Existence / uniqueness of the map $\Theta : m \mapsto \mathcal{L}(Y^m)$.

- Aim 1 : Existence and uniqueness of u^m for fixed $m \in \mathcal{P}(\mathcal{C}^d)$

Does it exist a unique function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that for all $(t, y) \in [0, T] \times \mathbb{R}^d$,

$$u(t, y) = \int_{\mathcal{C}^d} K(y - X_t(\omega)) \exp\left\{ \int_0^t \Lambda\big(s, X_s(\omega), u(s, X_s(\omega))\big) ds \right\} dm(\omega)$$

Yes, if Λ is bounded and uniformly Lipschitz w.r.t. t.

- Idea of the proof : fixed-point argument.

Remark

- *Existence and uniqueness of u is obtained for all $m \in \mathcal{P}(\mathcal{C}^d)$.*
- *Only the hypothesis on Λ are used here(= bounded and uniformly Lipschitz w.r.t. t) and not those of Φ, g.*
- *Uniqueness is lost if Λ is only continuous ! ! !*

Stability properties for $u^m(t,y) := u(m,t,y)$ under various norms :

•$\forall\ (m,m') \in \mathcal{P}(C^d) \times \mathcal{P}(C^d), \forall\ (t,y,y') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$:

$$|u^m(t,y) - u^{m'}(t,y')|^2 \leq \mathfrak{C}_{K,\Lambda}(T)\left[|y-y'|^2 + |\widetilde{W}_t(m,m')|^2\right],$$

where the map

$$(m,m') \in \mathcal{P}(\mathcal{C}^d) \times \mathcal{P}(\mathcal{C}^d) \mapsto |\widetilde{W}_T(m,m')|^2$$

is the 2-Wasserstein distance on the space of Borel probability measures on $\mathcal{C}^d$, s.th. for all $t \in [0,T]$,

$$|\widetilde{W}_t(m,m')|^2 := \inf_{\mu \in \widetilde{\Pi}(m,m')} \int_{\mathcal{C}^d \times \mathcal{C}^d} \left(\sup_{0\leq s\leq t} |X_s(\omega) - X_s(\omega')|^2 \wedge 1\right) d\mu(\omega,\omega')$$

• The function

$$(m,t,x) \mapsto u^m(t,x)$$

is continuous on $\mathcal{P}(\mathcal{C}^d) \times [0,T] \times \mathbb{R}^d$ where $\mathcal{P}(\mathcal{C}^d)$ is endowed with the topology of weak convergence.

• Suppose here that $K \in W^{1,2}(\mathbb{R}^d)$.
For any $t \in [0,T]$, $(m,m') \in \mathcal{P}_2(\mathcal{C}^d) \times \mathcal{P}_2(\mathcal{C}^d)$,

$$\|u^m(t,\cdot) - u^{m'}(t,\cdot)\|_2^2 \le \tilde{\mathfrak{C}}_{K,\Lambda}(T)|W_t(m,m')|^2 ,$$

where $\|\cdot\|_2$ is the standard $L^2(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d,\mathbb{R}^d)$-norms.

- Suppose (additionally) that $\mathcal{F}(K) \in L^1(\mathbb{R}^d)$. Then $\exists\ \bar{\mathfrak{C}}_{K,\Lambda}(t) > 0$ for all $(m,t) \in \mathcal{P}(\mathcal{C}^d) \times [0,T]$,

$$\mathbb{E}[\|u^{S^N(\xi)}(t,\cdot) - u^m(t,\cdot)\|_\infty^2] \leq \bar{\mathfrak{C}}_{K,\Lambda}(T) \sup_{\substack{\varphi \in \mathcal{C}_b(\mathcal{C}^d) \\ \|\varphi\|_\infty \leq 1}} \mathbb{E}[|\langle S^N(\xi) - m, \varphi\rangle|^2]$$

where

$$S^N(\xi) := \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i}$$

for $(\xi^i, 1 \leq i \leq N)$ given continous processes.

- Aim 2 : Existence and uniqueness of the process Y
  - The map $u : m \mapsto u^m$ is defined for all $m \in \mathcal{P}(\mathcal{C}^d)$;
  - We apply a fixed-point argument (see Sznitman) to the map

$$\Theta : \mathcal{P}_2(\mathcal{C}^d) \to \mathcal{P}_2(\mathcal{C}^d) \ ,$$

defined by $\Theta(m) = \mathcal{L}(Y^m)$ s.th.

$$Y_t^m = Y^0 + \int_0^t \Phi(s, Y_s^m, u^m(s, Y_s^m))dW_s + \int_0^t g(s, Y_s^m, u^m(s, Y_s^m))ds$$

with $\mathcal{P}_2(\mathcal{C}^d)$ equipped with 2-Wasserstein distance W_T^2.

Plan

- Existence in *Semi-weak case* and *Weak case* :

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, u(s, Y_s))dW_s + \int_0^t g(s, Y_s, u(s, Y_s))ds \\ u(t, y) = \mathbb{E}\left[K(y - Y_t) \exp\left\{\int_0^t \Lambda\big(s, Y_s, u(s, Y_s)\big)ds\right\}\right] \end{cases},$$

admits a solution in semi-weak and weak case.

The proof consists in

1. regularizing the coefficients Φ, g, Λ with a mollifier $(\varphi_n)_{n\in\mathbb{N}}$.
2. using the *Lipschitz / Semi-weak case* result for mollified coefficients $\Longrightarrow$ existence of $(Y^n, u^n)_{n\in\mathbb{N}}$.
3. convergence of $(u^n)_n$ and identification of the limit.
4. identify the limit of (Y^n) (stability of SDEs / martingale formulation).

Plan

- Link with PIDE : Ito's formula implies that (Y, u) solution of (regularized) NLSDE is related to the partial integro-differential equation (PIDE)

$$
\begin{cases}
\partial_t v = \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left((\Phi\Phi^t)_{i,j}(t,x,K*v)v \right) - div\left(g(t,x,K*v)v\right) \\
+\Lambda(t,x,K*v)v \\
v_0 = \zeta_0 \ ,
\end{cases}
$$

by the relation

$$
v_t(\cdot) = (K*v_t)(\cdot) = \int_{R^d} K(\cdot - y)v_t(dy)\ .
$$

Plan

- In all the sequel, only assumptions of *Lipschitz case* will be satisfied.

- Interacting Particle System (IPS)

For fixed i.i.d. r.v. $(Y_0^i)_{i=1,\cdots,N}$ and $(W^i)_{i=1,\cdots,N}$ a family of independent Brownian motions, the IPS $\xi := (\xi^{i,N})_{i=1,\cdots,N}$ is defined by

$$\begin{cases} \xi_t^{i,N} = Y_0^i + \int_0^t \Phi_s(\xi_s^{i,N}, u_s^{S^N(\xi)}(\xi_s^{i,N}))dW_s^i + \int_0^t g_s(\xi_s^{i,N}, u_s^{S^N(\xi)}(\xi_s^i))ds \\ u_t^{S^N(\xi)}(x) = \frac{1}{N}\sum_{j=1}^N K(x-\xi_t^{j,N}) \exp\left(\int_0^t \Lambda(r,\xi_r^{j,N}, u_r^{S^N(\xi)}(\xi_r^{j,N}))dr \right), \end{cases}$$

with $S^N(\xi) := \dfrac{1}{N}\displaystyle\sum_{j=1}^N \delta_{\xi^{j,N}}$, empirical measure associated to ξ.

For such systems, propagation of chaos $\equiv$ *"asymptotic independence"* of the components $(\xi^i)_{i=1,\cdots,N}$ when the size N (=number of components) goes to $+\infty$.

Main ideas :

- Transform a d-dimensional (regularized) NLSDE into a $d \times N$-dimensional classical SDEs.
- The function $u^{S^N(\xi)}$ can be seen as the "mixing/interaction term". It can be written

$$u_t^{S^N(\xi)}(x) = F(t, x, \xi_t^1, \cdots, \xi_t^N, \underbrace{(\xi_{\cdot\wedge t}^1), \cdots, (\xi_{\cdot\wedge t}^N)}_{\text{past of the trajectories}}) .$$

- Dimension of the state space (=$(\mathbb{R}^d)^N$) depends on $N \neq \omega \mapsto S^N(\xi(\omega)) \in \mathcal{P}(\mathcal{C}^d)$ with $\dim(\mathcal{P}(\mathcal{C}^d)) = \infty$.

- If $\xi := (\xi^i)_{i=1,\cdots,N}$ are i.i.d. $\mathbb{R}^d$-valued r.v. according to $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$\left\langle S^N(\xi), \varphi \right\rangle = \frac{1}{N}\sum_{i=1}^{N} \varphi(\xi^i) \xrightarrow[N\to+\infty]{p.s.} \left\langle \mu, \varphi \right\rangle ,$$

by the Strong law of large numbers.

Existence/uniqueness of such IPS

• Non-anticipative property : $\forall (s,x) \in [0,T] \times \mathbb{R}^d$,

$$u_s^{S^N(\xi)}(x) = u_s^{S^N((\xi_r,\, 0 \le r \le s))}(x).$$

$\Longrightarrow$ all integrands of IPS are adapted and so, Itô's integral is well-defined.

• Lipschitz property of integrands :

Lipschitz property of $m \mapsto u^m$ implies that

$$(s,\bar{\xi}) \in [0,T] \times (\mathcal{C}^d)^N \mapsto \Phi(s, \bar{\xi}_s^{i,N}, u_s^{S^N(\bar{\xi})}(\bar{\xi}_s^{i,N}))$$

and

$$(s,\bar{\xi}) \in [0,T] \times (\mathcal{C}^d)^N \mapsto g(s, \bar{\xi}_s^{i,N}, u_s^{S^N(\bar{\xi})}(\bar{\xi}_s^{i,N}))$$

are Lipschitz.

Consequently, classical results for path-dependent SDEs give existence/uniqueness.

Coupling technique :

Let $(Y^i)_{i=1,\cdots,N}$ be solutions of

$$\begin{cases} Y_t^i = Y_0^i + \int_0^t \Phi(s, Y_s^i, u^{m_i}(s, Y_s^i))dW_s^i + \int_0^t g(s, Y_s^i, u^{m^i}(s, Y_s^i))ds \\ u^{m^i}(t,x) = \mathbb{E}\Big[K(x - Y_t^i) \exp\Big(\int_0^t \Lambda(r, Y_r^i, u^{m^i}(s, Y_s^i)\Big)\Big] , \end{cases}$$

where $(W^i)_{i=1,\cdots,N}$ is the same family of independent Brownian motions driving the IPS $(\xi^{i,N})_{i=1,\cdots,N}$. Then,

- $(Y^1, \cdots, Y^N)$ are i.i.d. and their common law will be denoted by m^0.

Theorem

Under some assumptions, the following inequalities hold :

$$\begin{aligned}
\mathbb{E}[\| u_t^{S^N(\xi)} - u_t^{m_0} \|_\infty^2] &\leq \frac{C}{N} \\
\mathbb{E}[\sup_{0\leq s\leq t} |\xi_s^{i,N} - Y_s^i|^2] &\leq \frac{C}{N} \\
\mathbb{E}[\| u_t^{S^N(\xi)} - u_t^{m_0} \|_2^2] &\leq \frac{C}{N},
\end{aligned}$$

for all $i \in \{1, \cdots, N\}$ and where C does not depend on N.

Time discretization version of the IPS

To simplify notations, we set $g \equiv 0$.

- Euler Scheme : for $k = 1, \cdots, n$

$$
\begin{cases}
\tilde{\xi}^{i,N}_{t_{k+1}} = \tilde{\xi}^{i,N}_{t_k} + \Phi(t_k, \tilde{\xi}^{i,N}_{t_k}, \tilde{v}_{t_k}(\tilde{\xi}^{i,N}_{t_k}))\mathcal{N}(0, \delta t) \\
\tilde{\xi}^{i,N}_0 = Y^i_0 \\
\tilde{v}_{t_{k+1}}(y) = \frac{1}{N}\sum_{j=1}^N K(y - \tilde{\xi}^{j,N}_{t_{k+1}}) e^{\left\{\sum_{p=0}^k \Lambda(t_p, \tilde{\xi}^{j,N}_{t_p}, \tilde{v}_{t_p}(\tilde{\xi}^{j,N}_{t_p}))\delta t\right\}}
\end{cases},
$$

where $0 \leq t_0 < \cdots < t_k = k * \delta t < \cdots < t_n \leq T$ is a regular time grid.

Theorem

Under Lipschitz continuity assumption, we have

$$\mathbb{E}[\| \tilde{v}_t - u_t^{S^N(\xi)} \|_\infty^2] + \sup_{i=1,\cdots N} \mathbb{E}\left[\sup_{s \le t} |\tilde{\xi}_s^{i,N} - \xi_s^{i,N}|^2 \right] \le C_{K,\Lambda,T}\, \delta t \,.$$

and the Mean Integrated Squared Error (MISE) verifies

$$\mathbb{E}[\| \tilde{v}_t - u_t^{S^N(\xi)} \|_2^2] \le C_{K,\Lambda,T}\, \delta t \,.$$

Remark

- *The constant $C_{K,\Lambda,T}$ does not depend on N and δt.*
- *By the two preivous Theorems, we have for all $t \in [0, T]$*

$$\mathbb{E}\left[\|\tilde{v}_t - u_t^{m^0}\|_\infty^2\right] \;\leq\; C\left(\delta t + \frac{1}{N}\right).$$

Initialization for $k = 0$

1. Generate $(\xi_0^i)_{i=1,..,N}$ i.i.d.$\sim v(0,x)dx$;
2. Set $G_0^i = 1$, $i = 1, \cdots, N$;
3. Set $\tilde{v}_0(\cdot) := v(0,\cdot)$;

Iterations for k = 1, ..., n-1

- Independently for each particle $\tilde{\xi}_k^{j,N}$ for $j = 1, \cdots N$,

$$\tilde{\xi}_{k+1}^{j,N} = \tilde{\xi}_k^{j,N} + \Phi(t_k, \tilde{\xi}_k^{j,N}, \tilde{v}_k(\tilde{\xi}_k^{j,N}))\mathcal{N}(0,\delta t)$$

- Set

$$G_{k+1}^j := G_k^j \times \exp\left(\Lambda(t_k, \tilde{\xi}_k^{j,N}, \tilde{v}_k(\tilde{\xi}_k^{j,N}))\delta t\right) ;$$

- Set $\tilde{v}_k(\cdot) = \frac{1}{N}\sum_{j=1}^{N} G_k^j \times K_h(\cdot - \tilde{\xi}_{k-1}^{j,N})$.

- ***In all the sequel, we expose an empirical analysis for which the assumptions of the theorems are not necessarily satisfied.***
- ***Since***

$$\mathbb{E}\left[\|\tilde{v}_t - u_t^{S^N(\xi)}\|_\infty^2\right] \leq C\,\delta t$$

where $C := C(\|K\|_\infty, \|\Lambda\|_\infty, L_K, L_\Lambda, \|\nabla K\|_2, T)$***, notice that we neglect the time discretization in the present empirical analysis.***

Aim : show how the particle system can be used to estimate u, solution of the PDE

$$\begin{cases} \partial_t u = \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left((\Phi\Phi^t)_{i,j}(t,x,u)u\right) - div\left(g(t,x,u)u\right) + \Lambda(t,x,u)u \\ u(0,dx) = \zeta_0(dx) \ . \end{cases}$$

Let us consider the interacting particle system $\xi^{i,N,\varepsilon}$, where $K = K_\varepsilon$ for $\varepsilon > 0$.

$$\xi_t^{i,N,\varepsilon} = Y_0^i + \int_0^t \Phi_s(\xi_s^{i,N,\varepsilon}, u_s^{S^N(\xi^\varepsilon)}(\xi_s^{i,N,\varepsilon}))dW_s^i + \int_0^t g_s(\xi_s^{i,N,\varepsilon}, u_s^{S^N(\xi^\varepsilon)}(\xi_s^{i,N,\varepsilon}))ds \ ,$$

where

$$u_t^{S^N(\xi^\varepsilon)}(x) = \frac{1}{N}\sum_{j=1}^{N} K_\varepsilon(x - \xi_t^{j,N,\varepsilon}) e^{\int_0^t \Lambda(r, \xi_r^{j,N,\varepsilon}, u_r^{S^N(\xi^\varepsilon)}(\xi_r^{j,N,\varepsilon}))dr} \ .$$

We are going to try to show this empirically. To this end, we consider the Mean Integrated Squared Error (MISE) that we decompose as the **Variance** and the **Bias**,

$$\begin{aligned} MISE_t(\varepsilon, N) &:= V_t(\varepsilon, N) + B^2(\varepsilon, N) \\ &= \mathbb{E}\Big[\|u_t^{S^N(\xi^\varepsilon)} - \mathbb{E}[u_t^{S^N(\xi^\varepsilon)}]\|_2^2\Big] + \mathbb{E}\Big[\|\mathbb{E}[u_t^{S^N(\xi^\varepsilon)}] - v_t\|_2^2\Big]. \end{aligned}$$

Ideally, we would like that

$$\mathbb{E}[u_t^{S^N(\xi^\varepsilon)}] \sim u_t^{m^0,\varepsilon} \ .$$

If the propagation of chaos holds, it means that the particles $\xi^{N,\varepsilon}$ are close to an i.i.d. system according to m^0, which is the common law of the processes Y^i, $1 \leq i \leq N$. Then, in the particular case where $\Lambda(t,x,u) := \Lambda(t,x)$,

$$\begin{aligned}
\mathbb{E}[u_t^{S^N(\xi^\varepsilon)}] &= \frac{1}{N}\mathbb{E}\Big[\sum_{j=1}^{N} K(\cdot - \xi_t^{j,N,\varepsilon}) \exp\Big(\int_0^t \Lambda(r,\xi^{j,N,\varepsilon})dr\Big)\Big] \\
&\approx \mathbb{E}\Big[K_\varepsilon(\cdot - Y_t^{1,\varepsilon}) \exp\Big(\int_0^t \Lambda(r, Y^{1,\varepsilon})dr\Big)\Big] \\
&= u_t^{m^0,\varepsilon}
\end{aligned}$$

We expect to observe the same behavior for the case $\Lambda = \Lambda(t,x,u)$ depends on u.

Finally,

$$V_t(\varepsilon, N) \approx \mathbb{E}\Big[\|u_t^{S^N(\xi^\varepsilon)} - u_t^{m^0,\varepsilon}\|_2^2\Big]$$

and

$$B_t^2(\varepsilon, N) \approx B_t^2(\varepsilon) := \mathbb{E}\Big[\|u_t^{m^0,\varepsilon} - v_t\|_2^2\Big] \ .$$

In other words,

$V_t(\varepsilon, N)$ corresponds to the convergence of the particles system (i.e. when $N \to +\infty$, for fixed $\varepsilon > 0$) ,

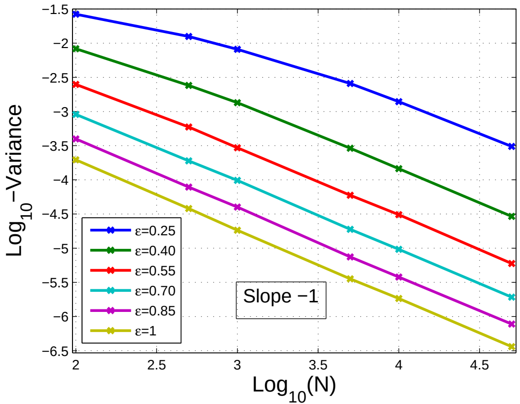
and

$B_t^2(\varepsilon, N)$ corresponds to the convergence of the regularized NLSDE (i.e. when $\varepsilon \to 0$) .

Variance Analysis (1) : Behavior w.r.t. N

Simulations given below for $d = 5$, $T = 1$ give us

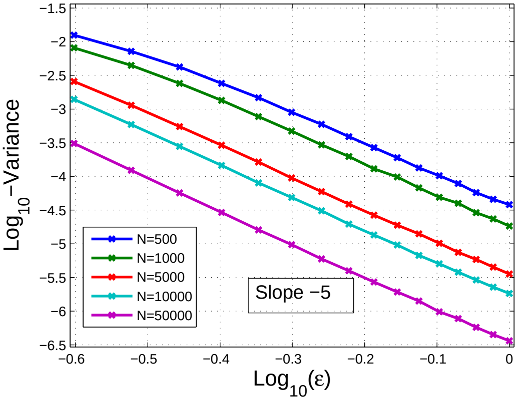
$$V_t(\varepsilon, N) \sim \frac{1}{N\varepsilon^d}$$



Variance Analysis (2) : Behavior w.r.t. ε :

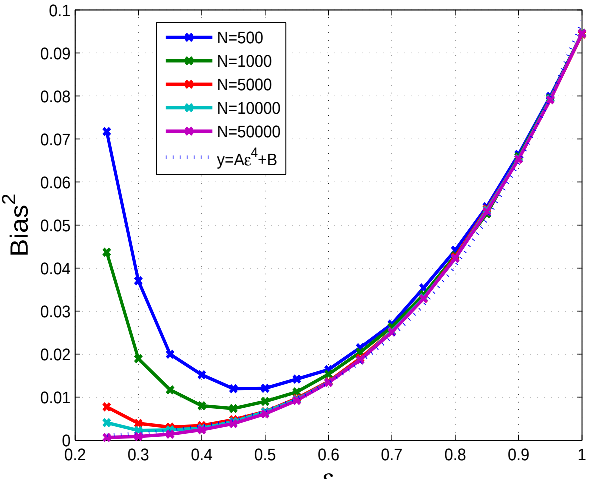
Simulations given below for $d = 5$, $T = 1$ give us

$$V_t(\varepsilon, N) \sim \frac{1}{N\varepsilon^d}$$



Bias Analysis (2) :

Simualtions give us $B^2(\varepsilon) \sim \varepsilon^4$



Thank you for your attention

References I