Approximation of moving average options and valuation of oil-indexed gas supply contracts

Journée de la Chaire FDD et du Laboratoire FiME

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Outline

1. Introduction and motivations
   - Oil indexed gas supply contracts
   - Difficulties and existing valuation methods
2. An approximation for pricing moving average options
   - A finite-dimensional approximation based on Laguerre decomposition
   - Illustration for approximating uniformly-weighted moving averages
3. Numerical results
   - Black and Scholes framework
   - Oil indexed gas contracts

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Oil indexed gas contracts

Oil indexed gas supply contracts are also known as gas Swing options.

- These contracts give to their buyer the right to purchase an amount of gas with spot price S to a strike price M which is indexed on moving average of various commodities.
- The volume of gas purchased is submitted to local and global constraints.

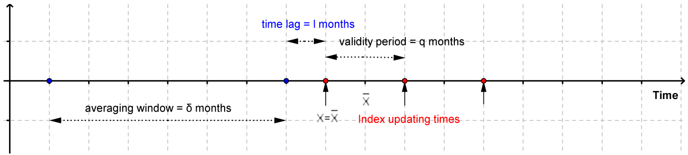
The payoff at exercise of a normalized Swing contract has the form :

$$\phi(S_t, M_t) = (S_t - M_t)^+ \quad \text{with} \quad M_t = K + \sum_{i=1}^{d} \alpha_i \bar{X}_t^i.$$

- K is a fixed cost, standing for the fixed part of delivery.
- $S^i, i = 1, \ldots, d$ are correlated commodity prices : gas oil, fuel oil, coal, etc.
- α_i is the weight attributed to commodity with price S^i.
- $\bar{X}^i$ is the moving average of the price of commodity i over the δ months preceding the l last months before the last updating date : each $\bar{X}^i$ is updated every q months.

Three characteristic numbers of an indexed strike :

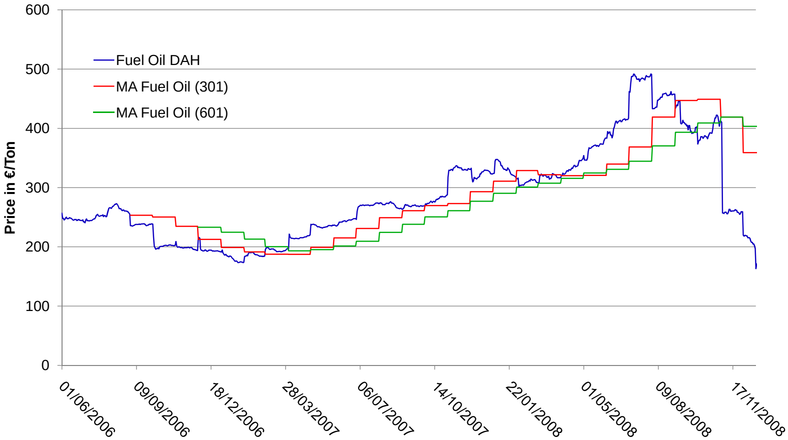
- δ is the length of the averaging period,
- l is the time delay (time lag),
- q is the validity period.



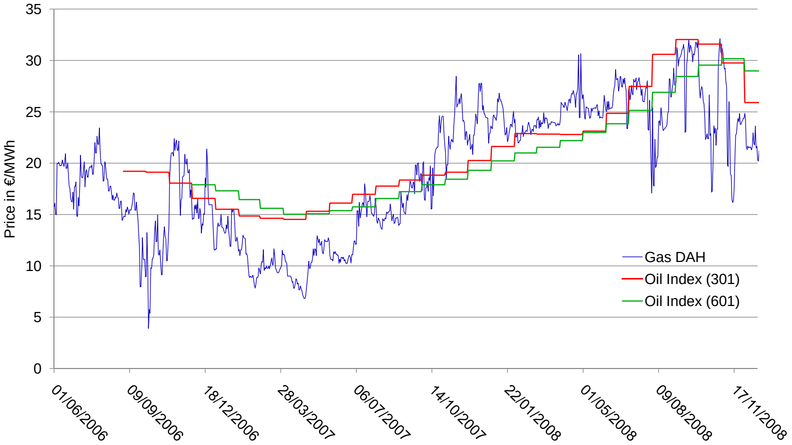
We shall consider typical Swing contracts where :

- strike prices are indexed on gas oil and fuel oil prices,
- the characteristic triple $(\delta l q)$ is equal to (601), (301), (311), etc.

Fuel oil price from June 2006 to December 2008 observed on the ARA oil market for West Europe (Amsterdam-Rotterdam-Anvers).



Gas price from June 2006 to December 2008 observed on the Zeebrugge market. Indexed strike with oil prices from June 2006 to December 2009 observed on the ARA market and $M_t = 2.525 + 0.0286\bar{X}_t^{fo} + 0.0259\bar{X}_t^{go}$.



Why is the valuation of such contrats difficult ?

- Valuation of Swing options, see e.g. Jaillet et al. (2004), Bardou et al. (2009)
- The difficulty comes from the indexed strike M :
  - stochastic,
  - average of prices, on a rolling period in time.
- Do not confuse with the simpler problem of Asian American-style options.
- Even (S, S^{fo}, S^{go}) is Markovian, the process $(S, \bar{X}^{fo}, \bar{X}^{go})$ is not.
  $\Longrightarrow$ Infinite-dimensional stochastic control problem (in continuous time)
- In discrete time $\Longrightarrow$ Computational challenge due to high dimensionality

The dimension of the problem is related to the number of time steps within the averaging window :

$$\text{Dimension} = 1 + 2\,(N_\delta + N_l)\,.$$

If the time step = 1 day :

Index type (δlq)	Dimension
(601)	$1 + 2\,(6 \times 30) = 361$
(311)	$1 + 2\,(3 \times 30 + 1 \times 30) = 241$

- Due to high dimensionality, the Markovian resolution of the problem seems unfeasible in practice.
- The computational difficulty comes from the estimation of conditional expectations involved in the optimal exercise rule.

Two main approximations are used in practice :

1. **Deterministic index** The oil indexed strike is assumed to be exogenous.
   - Dimension = 1, see e.g. Bardou et al. (2009)
   - Equivalent to assume a zero volatility coefficient of the oil prices
   - Do not take into account the correlation between the prices of gas and oil
2. **Non-Markovian approximation** Practioners compute conditional expectations estimators by using only explanatory variables (S, M).
   - Dimension = 2, see e.g. Broadie et Cao (2008)
   - Forget the whole history of oil prices on the averaging window (path-dependence)
   - Introduce a bias : the resulting solution is suboptimal

Our motivations were twofold :

1. Propose a new (Markovian) method for pricing moving average options
2. Quantify the error made by the non-Markovian approximation most often used in practice

Basic idea of our approach :

- Find a finite-dimensional approximation of the moving average process M
- Reduce the problem dimension (< 8) so that the Markovian resolution becomes feasible by using Monte Carlo techniques

A Finite-Dimensional Approximation for Pricing Moving Average Options, Marie Bernhart, Peter Tankov and Xavier Warin, SIAM Journal on Financial Mathematics, Vol. 2, pp. 989-1013, November 2011

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We consider the simplified problem of moving average American option pricing :

$$\sup_{\tau \leq T} \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] \quad \text{with} \quad M_t = \int_0^{+\infty} S_{t-u} h(u) du, t \geq 0.$$

- ϕ is the payoff function.
- S is a Markov process. We set $S_t := S_0, \forall t \leq 0$.
- h is a density on $[0, +\infty)$: M arithmetic $\Rightarrow$ uniform density $h = \frac{1}{\delta}\mathbb{I}_{[l, l+\delta]}$.

Nota Bene Results directly generalizable to multi-asset models and valuation of multiple-exercise options (Swing).

We would like to find a finite-dimensional approximation of the moving average M :

- find n processes $Y := (X^0, \ldots, X^{n-1})$
- such that $(S, X^0, \ldots, X^{n-1})$ are jointly Markov
- and M_t is approximated by some M_t^n which depends deterministically on $(S_t, X_t^0, \ldots, X_t^{n-1})$.

- M linear in $S \Rightarrow$ Assume that $Y := (X^0, \ldots, X^{n-1})$ satisfies a linear SDE and M^n depends deterministically on Y :

$$dY_t = -AY_t dt + \mathbf{1}\left(\alpha S_t dt + \beta dS_t\right) \quad \text{and} \quad M_t^n = B^{\perp} Y_t,$$

$$\Longrightarrow \quad M_t^n = \int_{-\infty}^{t} B^{\perp} e^{-A(t-u)} \mathbf{1}\left(\alpha S_u du + \beta dS_u\right) = K_n S_t + \int_0^{+\infty} S_{t-u} h_n(u) du.$$

- This implies that h_n is of the form (Hankel approximation) :

$$h_n(u) = \sum_{k=1}^{K} e^{-p_k u} \sum_{i=0}^{n_k} c_i^k u^i, \quad n_1 + \ldots + n_K + K = n.$$

- We focus on a subclass of solutions for which h_n has the form :

$$h_n(u) = e^{-pu} \sum_{i=0}^{n-1} c_i u^i \text{ : known as Laguerre approximation.}$$

- Finding a finite-dimensional approximation M^n for $M \iff$ Finding an approximation of the form $K_n\delta_0(du) + h_n(u)du$ for the weighting measure of M, $h(u)du$.
- Laguerre functions are used e.g. in signal processing for approximating infinite-dimensional systems, see Mäkilä (1990).

Let $p > 0$ be a scale parameter. The scaled Laguerre functions $(L_k^p)_{k\geq 0}$ defined by

$$L_k^p(x) := \sqrt{2p}\; P_k(2px)e^{-px}, \quad \forall x \geq 0, \forall k \geq 0$$

form an orthonormal basis of $\left(\mathrm{L}^2([0,\infty)), \langle\cdot,\cdot\rangle\right)$ with P_k, simple Laguerre polynomial.

- Let $H(x) := \int_x^{+\infty} h(u)du$. Decomposition of H in a Laguerre series :

$$H_n^p(x) = \sum_{k=0}^{n-1} A_k^p L_k^p(x), \quad A_k^p := \langle H, L_k^p\rangle.$$

- Setting $h_n^p(x) = -\frac{d}{dx}H_n^p(x)$, we get $h_n^p(x) = \sum_{k=0}^{n-1} a_k^p L_k^p(x)$.

- Approximation of the moving average M_t by

$$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \int_0^{+\infty} S_{t-u} h_n^p(u)du, \quad \forall t \geq 0$$
$$= (H(0) - H_n^p(0))S_t + \sum_{k=0}^{n-1} a_k^p X_t^{p,k},$$

where $X_t^{p,k} := \int_0^{+\infty} S_{t-u} L_k^p(u)du$, $\forall k \leq n-1$ Laguerre processes.

- The Laguerre processes have Markovian dynamics :

$$\begin{cases} dX_t^{p,0} = \left(\sqrt{2p}S_t - pX_t^{p,0}\right) dt, \\ \vdots \\ dX_t^{p,k} = \left(\sqrt{2p}S_t - 2p\sum_{i=0}^{k-1} X_t^{p,i} - pX_t^{p,k}\right) dt, \end{cases}$$

with initial values $X_0^{p,k} = S_0(-1)^k \frac{\sqrt{2p}}{p}, \forall k \geq 0$.

Proposition : Finite-dimensional approximation of the problem

The process $(S, X^{p,0}, X^{p,1}, \dots, X^{p,n-1})$ is Markovian so that the approximate problem

$$\sup_{\tau \leq T} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right]$$

is $(n+1)$-dimensional where :

- p is the scale parameter of the Laguerre functions,
- n is the number of Laguerre functions used in the decomposition.

Assumption (A) : The price process S is a continuous Itô process :

$$S_t = S_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \text{ with } \mathbb{E}\left[\sup_{0 \leq t \leq T} |b_s|\right] + \mathbb{E}\left[\sup_{0 \leq t \leq T} |\sigma_s|^{1+\gamma}\right] < \infty$$

for some $\gamma > 0$. Then, it can be shown, see Fischer and Nappo (2010), that :

$$\mathbb{E}\left[\sup_{t,u \in [0,T]:|t-u| \leq h} |S_t - S_u|\right] \leq C\varepsilon(h), \quad \varepsilon(h) := \sqrt{h \ln\left(\frac{2T}{h}\right)}.$$

Theorem : Convergence of the finite-dimensional approximation

- Let (A) be satisfied and assume that h has compact support, finite variation and is constant in the neighborhood of zero. Then

$$\mathcal{E}^{n,p} := \mathbb{E}\left[\sup_{0\leq t\leq T} |M_t - M_t^{n,p}|\right] \leq C\varepsilon(n^{-\frac{3}{4}}).$$

- If in addition ϕ is Lipschitz in its second variable, then the pricing error is such that :

$$\left|\sup_{\tau}\mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] - \sup_{\tau}\mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right]\right| \leq C\varepsilon(n^{-\frac{3}{4}}).$$

Idea of the proof : We show that $\mathcal{E}^{n,p} \leq C\varepsilon\left(\left\|H - H_n^p\right\|_2\right)$. Then the properties of the Laguerre coefficients imply $\left\|H - H_n^p\right\|_2 = \left(\sum_{k\geq n}\left|A_k^p\right|^2\right)^{\frac{1}{2}} = \mathcal{O}(n^{-\frac{3}{4}})$.

In the case of a (delayed) arithmetic moving average

$$M_t = \frac{1}{\delta}\int_{t-l-\delta}^{t-l} S_u du, \quad \forall t \geq \delta + l,$$

the weighting measure admits an uniform density :

$$h(t)dt = \frac{1}{\delta}\mathbb{I}_{[l,l+\delta]}(t)dt \quad \Longrightarrow \quad H(x) = \frac{1}{\delta}\left\{(\delta + l - x)^+ - (l-x)^+\right\}.$$

We optimize the approximation by scaling the Laguerre functions $\left\{L_0^p, \ldots, L_{n-1}^p\right\}$.

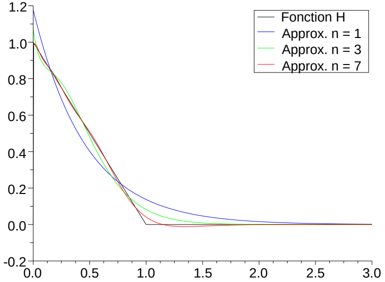
- The optimal scale parameter $p_{\mathrm{opt}}(n,\delta,l)$ is such that :

$$p_{\mathrm{opt}}(n,\delta,l) = \underset{p>0}{\arg\min}\, \|H - H_n^p\|_2^2 = \underset{p>0}{\arg\min} \left\{ \left(\frac{\delta}{3} + l\right) - \sum_{k=0}^{n-1} \left|A_k^p\right|^2 \right\}.$$

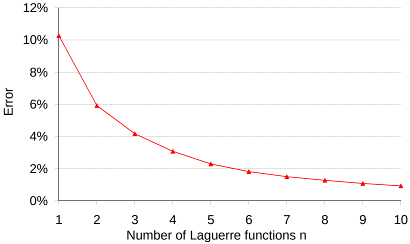
  where the coefficients $A_k^p = \langle H, L_k^p \rangle$ can be computed explicitly.
- p_{opt} satisfies a scaling relation : $\forall \lambda > 0, p_{\mathrm{opt}}(n,\delta,l) = \frac{p_{\mathrm{opt}}(n,\delta/\lambda,l/\lambda)}{\lambda}$.

Example when $\delta = 1$ and $l = 0$: density $h(u) = \mathbb{I}_{[0,1]}(u) \Rightarrow H(x) = (1-x)^+$.

n	1	2	3	4	5	6	7	8	9	10
$p_{\text{opt}}(n)$	2.15	4.07	6.00	4.23	5.83	7.47	9.15	10.86	9.15	10.73

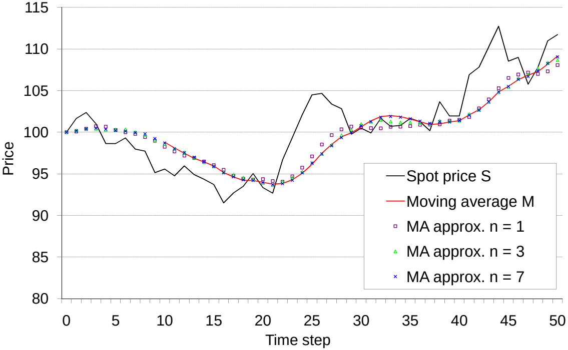


Approx. of H by $H_n^{p_{\text{opt}}(n)}$



Error $||H - H_n^{p_{\text{opt}}(n)}||_2$

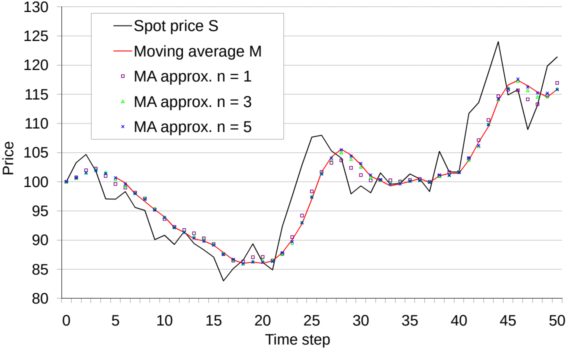
Black-Scholes model with $S_0 = 100$, $r = 5\%$ and $\sigma = 30\%$.
Parameters : $T = 50$ days, $\delta = 10$ days and $l = 0$.



Simulated trajectory of S, M and its Laguerre approximations $M^{n,p_{\mathrm{opt}}(n)}$

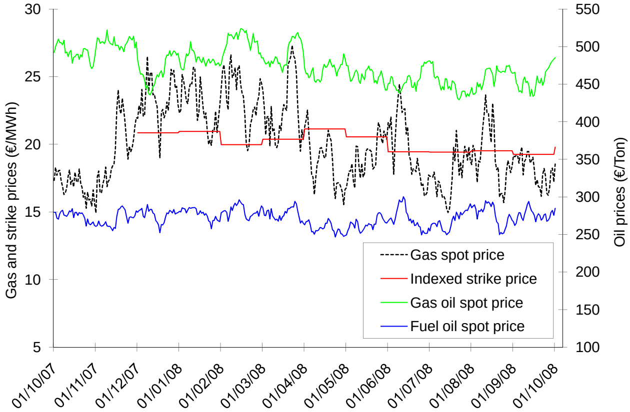
$n = 3$ Laguerre functions (when $l = 0$) and $n = 5$ functions ($l > 0$) are sufficient to provide very accurate dynamics approximation.

Same parameters but $\delta = 5$ days and a volatility $\sigma = 60\%$.



Simulated trajectory of S, M and its Laguerre approximations $M^{n,p_{\mathrm{opt}}(n)}$

Oil indexed strike price of type (111) such that $M_t = 0.025\bar{X}_t^{go} + 0.030\bar{X}_t^{fo}$.
MRG1 models for gas and oil prices : calibration on the Zeebrugge and ARA markets Oct. 2007-Oct. 2008 ($a_g = 50$, $a_{go} = 37$, $a_{fo} = 37$, $\sigma_{go} = 1$, $\sigma_{go} = 0.30$, $\sigma_{fo} = 0.40$, $\rho_{g,go} = \rho_{g,fo} = 0$, $\rho_{go,fo} = 0.80$).



Updating time	Oil indexed strike of type (111)	Laguerre approximation $n = 1$	$n = 3$	$n = 5$	$n = 7$
01/12/07	20.852	21.096	20.815	20.915	20.848
01/01/08	20.954	20.412	20.864	21.052	20.994
01/02/08	19.977	20.390	20.278	20.106	20.020
01/03/08	20.372	20.817	20.370	20.286	20.369
01/04/08	21.132	21.107	20.768	21.096	21.059
01/05/08	20.548	20.257	20.519	20.595	20.646
01/06/08	19.450	19.826	19.605	19.656	19.561
01/07/08	19.427	19.545	19.555	19.362	19.424
01/08/08	19.524	19.279	19.521	19.519	19.514
01/09/08	19.259	19.613	19.423	19.195	19.254
01/10/08	19.789	19.286	19.701	19.705	19.754
Maximal relative error		**2.59%**	**1.72%**	**1.06%**	**0.57%**

Simulated trajectory of M and its Laguerre approximations $M^{n,p_{\mathrm{opt}}(n)}$

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Valuation of two kinds of moving average American options

1. Options in a Black-Scholes framework : impact when varying δ and l
2. Oil indexed gas contracts with realistic oil indexed strike

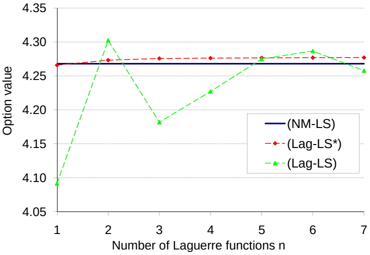
Some details on the Monte Carlo numerical methods used :

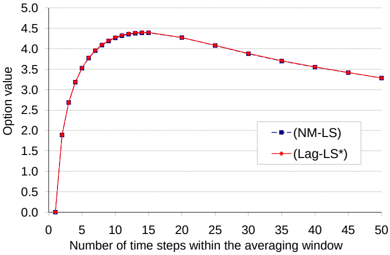
- The Laguerre approximation allows to exactly compute the Laguerre processes $(X^{p,k})_{k\geq 0}$ and $M^{n,p}$ on the time grid.
- Approach of Longstaff and Schwartz (2001), maximal state dimension $= 8$.
- **Non-Markovian method (NM-LS)** The conditional expectations are approximated by
$$\mathbb{E}\left[\cdot|\,(S_t, M_t)\right].$$
- **Laguerre based method** Laguerre approximation and
$$\mathbb{E}\Big[\cdot\,|\mathcal{F}_t\Big] = \mathbb{E}\left[\cdot|\left(S_t, X_t^{p_{\mathrm{opt}},0}, \dots, X_t^{p_{\mathrm{opt}},n-1}\right)\right].$$

(Lag-LS) The moving average M is approximated by $M^{n,p_{\mathrm{opt}}}$.
(Lag-LS*) Improvement by using M in the optimal exercise rule.

- (Lag-LS*) provides better valuation results than (Lag-LS).
- (Lag-LS*) : converged option prices with $n = 3$ functions ($l = 0$).
- For standard moving average options ($l = 0$), the relative error between (Lag-LS*) and (NM-LS) is small : less than 1% in our experiments.

Moving average Call option with $T = 50$ days, $\delta = 10$ days and $l = 0$.
Number of Monte Carlo paths used : 10 million (Lag-LS*) and 5 million (NM-LS).

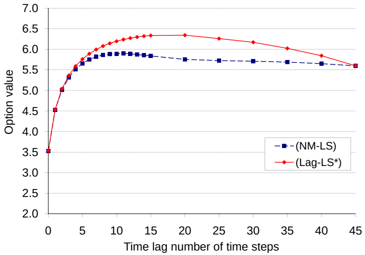


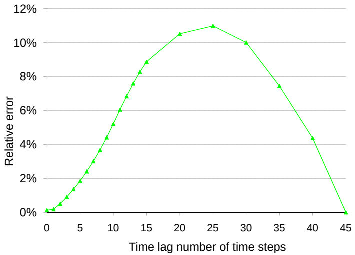


Moving average option prices without time lag ($l = 0$)

- (Lag-LS*) : converged option prices with $n = 5$ functions ($l > 0$).
- For delayed moving average options, (Lag-LS*) gives option prices up to 10% above the suboptimal prices given by (NM-LS).
  $\Longrightarrow$ The Non-Markovian approximation is irrelevant when $l \gg \delta$.

Moving average Call option with $T = 50$ days and $\delta = 5$ days.





Moving average option prices when varying the time lag l

Oil indexed gas contracts with MRG1 models for gas and oil prices : (same parameters as before : calibration on the Zeebrugge and ARA markets).
Parameters for (Lag-LS*) : 20 million of Monte Carlo paths and $n = 5$ functions.

Contract strike type	Deterministic strike	(NM-LS)	(Lag-LS*)
(601)	3.490	3.512	3.513
(131)	6.195	6.215	6.226
(311)	6.254	6.270	6.277
(111)	7.243	7.313	7.321

Oil indexed gas contract prices

- Accurate pricing results obtained by the Laguerre approximation based method.
- The non-Markovian approximate method constitutes a relevant approximation :
  - Large averaging periods δ and relatively small time delays l
  - Monthly-updated strike prices : smoothing effect
  - Mean-reverting behavior of oil prices

The Laguerre approximation based method introduced :

- Very accurate approximation of moving average dynamics
- Only few Laguerre functions needed ($n = 3$ to 5) : reduction of dimension
- Can be used for relatively general weighting measures

Non-Markovian (NM-LS) vs. Laguerre approximation based method (Lag-LS*) :

Kind of moving average option	**Pricing method**
Large time delay l	(NM-LS) is significantly suboptimal
	$\Longrightarrow$ Better use (Lag-LS*)
No time delay ($l = 0$)	(NM-LS) is a relevant approximation
Indexed gas contracts	(NM-LS) is a relevant approximation
	(Lag-LS*) is less competitive (running time $\times 10$)

Perspectives for further research :

- Theoretical justification of the non-Markovian approximation