Numerical methods for the quadratic hedging problem in Markov models with jumps*

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Abstract

We develop algorithms for the numerical computation of the quadratic hedging strategy in incomplete markets modeled by pure jump Markov process. Using the Hamilton-Jacobi-Bellman approach, the value function of the quadratic hedging problem can be related to a triangular system of parabolic partial integro-differential equations (PIDE), which can be shown to possess unique smooth solutions in our setting. The first equation is non-linear, but does not depend on the pay-off of the option to hedge (the pure investment problem), while the other two equations are linear. We propose convergent finite difference schemes for the numerical solution of these PIDEs and illustrate our results with an application to electricity markets, where time-inhomogeneous pure jump Markov processes appear in a natural manner.

Key words: Quadratic hedging, electricity markets, Markov jump processes, Partial integro-differential equation, Hamilton-Jacobi-Bellman equation, Hölder spaces, discretization schemes for PIDE.

1 Introduction

In an incomplete market setting, where exact replication of contingent claims is not possible, quadratic hedging is the most common approach, among both academics and practitioners. This method consists in minimizing the $\mathbb{L}^2$ distance between the hedging portfolio and the claim. Its popularity is due to the fact that the strategy is linear with respect to the claim, and is relatively easy to compute in a variety of settings.

In its most general form, the quadratic hedging problem can be formulated as follows. Consider a random variable $H \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{P})$ (which stands for the option one wants to hedge) and a set $\mathcal{X}$ of admissible strategies, which is a subset of the set of adapted processes with caglad paths. The quadratic hedging problem becomes

$$\text{minimize } \mathbb{E}^{\mathbb{P}}\left[\left(x + \int_0^T \theta_t dS_t - H\right)^2\right] \text{ over } x \in \mathbb{R} \text{ and } \theta \in \mathcal{X} \tag{1.1}$$

where S is a semimartingale modeling the stock price. If (x^*, θ^*) is a minimizer, we call θ^* the optimal mean-variance hedging strategy and x^* its price. This problem has been extensively studied in the literature, starting with the seminal works of Föllmer and Sondermann (1986) and Föllmer and Schweizer (1991) and until the complete theoretical solution in the general semimartingale setting was given in Černỳ and Kallsen (2007). The case when S is a $\mathbb{P}$-square integrable martingale is particularly simple and can be solved using the well known Galtchouk-Kunita-Watanabe decomposition. The general case is much more involved, and has only been solved in Černỳ and Kallsen

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(2007) by means of introducing a specific non martingale change of measure (the opportunity neutral measure).

The problem of numerical computation of the hedging strategy is an important issue in its own right, since various objects appearing in the theoretical solution (opportunity neutral measure, Galtchouk-Kunita-Watanabe decomposition, Föllmer-Schweizer decomposition) are often not known in explicit form. When the underlying asset is modeled by a Lévy process, a complete semi-explicit solution was obtained in Hubalek et al. (2006) using Fourier methods. Their approach was extended to additive processes in Goutte et al. (2011). Laurent and Pham (1999) and Heath et al. (2001) characterize the optimal strategy via an HJB equation in continuous Markovian stochastic volatility models while Černý and Kallsen (2008) and Kallsen and Vierthauer (2009) treat affine stochastic volatility models using Fourier methods.

In this paper, we propose algorithms for the numerical computation of the quadratic hedging strategy in general Markovian models with jumps. We first review the HJB characterization of the value function, obtained in De Franco (2012). We only give a brief review, referring the readers to De Franco (2012) for full details and proofs because in this paper, we are interested in the numerical schemes for the computation of the hedging strategies and in applications to electricity markets. The value function of the quadratic hedging problem can be related to a triangular system of parabolic PIDEs, which can be shown to possess unique smooth solutions in our setting. The first equation is a non-linear PIDE of HJB type, but does not depend on the pay-off of the option to hedge (the pure investment problem), while the other two equations are linear. We next propose two finite difference schemes for the numerical solution of the linear and the nonlinear PIDEs. The convergence of these schemes is carefully analyzed and we provide an estimate of the global approximation error as function of various truncation and discretization parameters. For the numerical schemes, we concentrate on the infinite variation case, which is more relevant in applications.

Our main motivation comes from hedging problems in electricity markets. These markets are structurally incomplete and often illiquid, owing to a relatively small number of market participants and the particular nature of electricity, which is a non-storable commodity. As pointed out in Geman and Roncoroni (2006) and Meyer-Brandis and Tankov (2008), due to these features, the electricity prices exhibit highly non-Gaussian behavior with jumps and spikes (upward movements followed by quick return to the initial level) and several authors have therefore suggested to model electricity prices by pure jump processes (Benth et al., 2007; Deng and Jiang, 2005).

On the other hand, since the spot electricity is non storable, the main hedging instruments in electricity markets are futures. A typical future contract with maturity T and duration d guarantees to its holder continuous delivery of electricity during the period $[T, T+d]$. Maturities, durations and amounts of electricity are standardized for listed contracts. This continuous delivery feature implies that even if the spot electricity follows a simple model, such as the exponential of a Lévy process, the price of the future contract will be a general Markov process with jumps, non-homogeneous in time and space.[1] Therefore, Fourier methods such as the ones developed in Hubalek et al. (2006) and Goutte et al. (2011) cannot be applied in this setting. For this reason, in Section 5, after introducing a model for the futures prices, where the spot price is described by the exponential of the tempered stable (CGMY) or Normal Inverse Gaussian (NIG) Lévy process, we derive the associated HJB equations and use the finite difference schemes to compute the hedging strategies and analyze their behavior. The numerical results illustrate the performance of our method and show in particular that the computation of the hedging strategies under the true historical probability (as opposed to the martingale probability, which does not require solving non-linear HJB equations) leads to a considerable improvement in the efficiency of the hedge.

The paper is structured as follows. After introducing the model and the quadratic hedging problem in Section 2, we review the HJB characterization of the solution and the regularity results in Section 3. The finite difference schemes for the solution of the HJB equations, which are the main results of this paper, are presented in Section 4. In Section 5 these results are applied to a concrete hedging problem in electricity markets. The proofs of the convergence results are given in the Appendix A.

[1]If a single delivery length is fixed, it is possible to model the future price directly as the exponential of a process with independent increments, as in Goutte et al. (2011). However this approach does not allow to treat problems involving futures of different durations, say, hedging a product with specified duration with the listed contracts.

2 The model and the quadratic hedging problem

Let J be a Poisson random measure on $[0,+\infty) \times \mathbb{R}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $\mathcal{F}_t$ being the natural filtration of J. We suppose that $\mathcal{F}_0$ contains the null sets and also $\mathcal{F} = \mathcal{F}_T$ where $T > 0$ is given. Let also $dt \times \nu(dy)$ be the intensity measure of J where ν satisfies the standard integrability condition $\int_{\mathbb{R}} \left(1 \wedge |y|^2\right) \nu(dy) < \infty$. We denote

$$\tilde{J}(dt \times dy) := J(dt \times dy) - dt \times \nu(dy)$$

the compensated martingale jump measure. On this probability space we introduce a family of real-valued Markov pure jump process as the solution of the following:

$$dZ_r^{t,z} := \mu\left(r, Z_r^{t,z}\right) dr + \int_{\mathbb{R}} \gamma\left(r, Z_{r-}^{t,z}, y\right) \tilde{J}(dr \times dy)\,,\; Z_t^{t,z} = z,\, r \geq t \tag{2.1}$$

for $t \in [0, T)$ and $z \in \mathbb{R}$. The asset price process S is given by $S_u^{t,z} = \exp(Z_u^{t,z})$. We make the following assumptions:

Assumption 2.1.
[C]- The coefficients.

i). There exists $\overline{\mu} \geq 0$ such that $\|\mu\|_\infty \leq \overline{\mu}$.

ii). For all $t \in [0,T]$ and $y \in \mathbb{R}$ the functions $z \to \mu(t,z)$ and $z \to \gamma(t,z,y)$ belong to $\mathcal{C}^1(\mathbb{R},\mathbb{R})$.

iii). There exist $K_{lip}^c \geq 0$, $K_{lip}^d \geq 0$ and a positive locally bounded function $\rho : \mathbb{R} \to \mathbb{R}^+$ such that for all $y \in \mathbb{R}$ and all $t \in [0,T]$ we have

$$|\mu(t,z) - \mu(t,z')| \leq K_{lip}^c |z - z'|$$
$$|\gamma(t,z,y) - \gamma(t,z',y)| \leq K_{lip}^d \rho(y) |z - z'|$$

[L]- The Lévy measure. *The Lévy measures $\nu(dy)$ verifies $\nu(dy) = \nu(y)dy$ where $\nu(y) := g(y)|y|^{-(1+\alpha)}$ for some $\alpha \in (1,2)$, where g is a measurable function bounded in a neighborhood of zero:*

$$0 < m_g \leq g(y) \leq M_g,\; \forall y \in (-y_0, y_0)$$

for some positive constants m_g, M_g and some $y_0 > 0$. We also assume that

$$\lim_{y \to 0^-} g(y) = g(0^-) \quad \text{and} \quad \lim_{y \to 0^+} g(y) = g(0^+) \quad \text{with } g(0^+), g(0^-) > 0.$$

Example: *The tempered stable (CGMY) processes, whose Lévy measure is of the form*

$$\nu(y) := \frac{c_-}{|y|^{1+\alpha}} e^{-\lambda_- |y|} \mathbf{1}_{y<0} + \frac{c_+}{|y|^{1+\alpha}} e^{-\lambda_+ |y|} \mathbf{1}_{y>0}$$

for $c_- > 0, c_+ > 0, \lambda_- > 0$ and $\lambda_+ > 0$ satisfies the above assumption.
[I]- Integrability conditions. *The function*

$$\tau(y) := \max\left(\sup_{t,z}\left(|\gamma(t,z,y)|, e^{|\gamma(t,z,y)|} - 1\right), \rho(y)\right)$$

verifies

$$\sup_{0<|y|\leq y_0} \frac{\tau(y)}{|y|} \leq M \qquad \text{and} \qquad \tau \in \mathbb{L}^4(\{|y| \geq y_0\}, \nu(dy))$$

[ND]- No degeneracy. *The function*

$$\Gamma(y) := \inf_{t,z}\left(e^{\gamma(t,z,y)} - 1\right)^2 \qquad \text{verifies} \qquad |\Gamma| := \int_{\mathbb{R}} \Gamma^2(y)\nu(dy) > 0$$

[RG]- Regularity of the γ function.

i). For any t, z the mapping $y \to \gamma(t, z, y)$ is continuous, strictly increasing and maps $\mathbb{R}$ onto $\mathbb{R}$. Moreover, it is twice continuously differentiable on $|y| \leq y_0$ for some $y_0 > 0$ and there exist positive constants m_1, m_2 such that

$$0 < m_1 \leq \inf_{t,z,|y|\leq y_0} |\gamma_y(t, z, y)| \text{ and } \qquad \sup_{t,z,|y|\leq y_0} |\gamma_{yy}(t, z, y)| \leq m_2$$

In particular γ as function of y is invertible: we call $\gamma^{-1}(t, z, y)$ its inverse.

ii). For all $t, z \in [0, T) \times \mathbb{R}$, $\gamma_y(t, z, 0) = 1$

iii). The function γ_y is Lipschitz continuous in the variable z:

$$\sup_{t,z,|y|\leq y_0} |\gamma_y(t, z + h, y) - \gamma_y(t, z, y)| \leq m_2|h|$$

We denote $K_{max} := \max(K^c_{lip}, K^d_{lip})$,

$$\tilde{\mu} := \mu + \int_{\mathbb{R}} (e^{\gamma} - 1 - \gamma)\nu(dy) \text{ and } \|\tilde{\mu}\| := \sup_{t,z} |\tilde{\mu}(t, z)| \tag{2.2}$$

In the rest of the paper we denote $\|\tau\|_{1,\nu} := \int_{|y|\geq 1} \tau(y)\nu(dy)$ whereas $\|\tau\|^2_{2,\nu} := \int_{\mathbb{R}} \tau^2(y)\nu(dy)$.

It is well known that the SDE (2.1) has a unique strong solution (Jacod and Shiryaev, 2003).

On the Assumption $[RG] - ii)$

Among the assumptions listed above, undoubtedly $[RG] - ii)$ seems to be the most restrictive one: if for example the jump function is of the form $\gamma(t, z, y) = \hat{\gamma}(t, z)y$ then the only possible choice would be $\hat{\gamma}(t, z) = 1$ for all t, z. In this paragraph we prove that this assumption could be relaxed by making a special change of variable. More precisely, given a process Z which does not satisfy this assumption, we look for a process L defined by $L_t = \phi(t, Z_t)$ for some smooth function ϕ such that

$$dL^{t,l}_s := \mu^L(s, L^{t,l}_s)ds + \int \gamma^L(s, L^{t,l}_{s-}, y)\bar{J}(dyds) \tag{2.3}$$

where μ^L and γ^L satisfy Assumptions 2.1, in particular, $\partial_y \gamma^L(t, l, 0) = 1$ for all t, l. If for such ϕ, the function $z \to \phi(t, z)$ is invertible and smooth enough to apply Itô's formula then

$$\gamma^L(t, l, y) := \phi(t, \phi^{-1}(t, l) + \gamma(t, \phi^{-1}(t, l), y)) - l \tag{2.4}$$

$$\begin{aligned} \mu^L(t, l) :=& \frac{\partial \phi}{\partial t}(t, \phi^{-1}(t, l)) + \mu(t, \phi^{-1}(t, l))\frac{\partial \phi}{\partial z}(t, \phi^{-1}(t, l)) \\ &+ \int_{|y|\leq 1} \left(\gamma^L(t, l, y) - \gamma(t, \phi^{-1}(t, l), y)\frac{\partial \phi}{\partial z}(t, \phi^{-1}(t, l))\right)\nu(dy) \end{aligned} \tag{2.5}$$

In particular one has

$$\gamma^L_y(t, l, 0) = \frac{\partial \phi}{\partial z}(t, \phi^{-1}(t, l)))\gamma_y(t, \phi^{-1}(t, l), 0)$$

If we select for example

$$\phi(t, z) := \int_0^z \frac{ds}{\gamma_y(t, s, 0)} \tag{2.6}$$

then trivially $\gamma^L_y(t, l, 0) = 1$ for all t, l. The following Lemma shows that this choice guarantees that the coefficients μ^L and γ^L verify Assumptions 2.1

Lemma 2.2. *Assume that there exist positive constants m_1, m_2 such that* *i). For all $t,z \in [0,T]\times\mathbb{R}$ the mapping $y \to \gamma(t,z,y)$ is differentiable at $y=0$ and*

$$0 < m_1 \le |\gamma_y(t,z,0)| \le m_2 \text{ for all } t,z \in [0,T]\times\mathbb{R}$$

ii). The function $(t,z) \to \gamma_y(t,z,0)$ is differentiable and

$$\left|\frac{d}{dt}\gamma_y(t,z,0)\right| + \left|\frac{d}{dz}\gamma_y(t,z,0)\right| \le m_2 \text{ for all } t,z \in [0,T]\times\mathbb{R}$$

iii). The function $z \to \frac{d}{dt}\gamma_y(t,z,0)$ is Lipschitz continuous:

$$\left|\frac{d}{dt}\gamma_y(t,z,0) - \frac{d}{dt}\gamma_y(t,z',0)\right| \le m_2|z-z'| \text{ for all } t \in [0,T],\ z,z' \in \mathbb{R}$$

Then the functions μ^L and γ^L defined in (2.4)–(2.5) with the choice of ϕ given by (2.6) verify Assumptions 2.1.

The proof of this Lemma can be found in De Franco (2012), Lemma 7.16.
The message of this Lemma is that, up to some regularity of the function γ_y at $y=0$, it is possible to remove the assumption $[RG]-ii)$: we could work with the process L instead of Z and derive all the results for L. By applying the function ϕ we could then obtain the corresponding results for the process Z. We refer to Chapter 7 in De Franco (2012) for further details.

Nevertheless, we prefer here to work with assumption $[RG]-ii)$ because it will make all computations easier to handle.

Admissible strategies and the value functions

To describe the set of admissible strategies in the quadratic hedging problem we follow the ideas developed in Černỳ and Kallsen (2007): we first introduce the sets of simple strategies:

$$\mathcal{D} := \left\{\theta := \sum_i Y_i \mathbf{1}_{[\varsigma_i,\varsigma_{i+1})},\ Y_i \in \mathbb{L}^\infty(\mathcal{F}_{\varsigma_i}),\ \varsigma_i \le \varsigma_{i+1} \text{ stopping times}\right\}$$
$$\mathcal{D}_t := \left\{\theta \mathbf{1}_{(t,T]} \mid \theta \in \mathcal{D}\right\} \tag{2.7}$$

The set of admissible strategies is a subset of the $\mathbb{L}^2(\mathbb{P})$-closure of $\mathcal{D}$:

$$\mathcal{X} := \left\{\theta \in \bar{\mathcal{D}} : \int \theta dS \in \mathbb{L}^2(\mathbb{P})\right\} \tag{2.8}$$

We define the wealth process for all t,z,x by

$$dX_r^{t,z,x,\theta} := \theta_{r-} dS_r^{t,z}, \quad X_t^{t,z,x,\theta} := x \tag{2.9}$$

where θ represents the number of shares in the portfolio at time t. The set of admissible controls is then given by

$$\mathcal{X}(t,z,x) := \left\{\theta \mathbf{1}_{(t,T]} \mid \theta \in \mathcal{X} \quad x + \int_t^T \theta_{r-} dS_r^{t,z} \in \mathbb{L}^2(\mathbb{P})\right\} \tag{2.10}$$

Consider a European option of the form $f(Z_T)$ where f is, for the moment, a bounded and measurable function. The quadratic hedging problem can be formulated as follows:

$$\mathbf{QH}: \quad \text{minimize } \mathbb{E}^{\mathbb{P}}\left[\left(f\left(Z_T^{0,z}\right) - X_T^{0,z,x,\theta}\right)^2\right]$$
$$\text{over } \theta \in \mathcal{X}(0,z,x)$$

The dynamic version of **QH** gives us the value function of the problem:

$$v_f(t,z,x) := \inf_{\theta\in\mathcal{X}(t,z,x)} \mathbb{E}^{\mathbb{P}}\left[\left(f\left(Z_T^{t,z}\right) - X_T^{t,z,x,\theta}\right)^2\right] \tag{2.11}$$
$$v_f(T,z,x) = \left(f(z)-x\right)^2$$

As remarked by several authors, the function v_f has the following structure

$$v_f(t,z,x) = a(t,z)x^2 + b(t,z)x + c(t,z) \tag{2.12}$$

In particular, taking $f=0$, one has

$$v_0(t,z,x) := x^2 \inf_{\theta\in\mathcal{X}(t,z,x)} \mathbb{E}\left[\left(1+\int_t^T \theta_{r-}dS_r^{t,z}\right)^2\right] \tag{2.13}$$

because the set $\mathcal{X}(t,z,x)$ is a cone. Consequently

$$a(t,z) := \inf_{\theta\in\mathcal{X}(t,z,1)} \mathbb{E}\left[\left(1+\int_t^T \theta_{r-}dS_r^{t,z}\right)^2\right] \tag{2.14}$$

This problem is known in the literature as the *pure investment problem.* The dual formulation of this problem relates the function a to the so called variance optimal martingale measure (Černỳ and Kallsen, 2007). We recall here some fundamental properties on the function a, whose proof can be found in Chapter 5 of De Franco (2012).

Theorem 2.3. *Under Assumptions 2.1-*$[\mathbf{C},\mathbf{I},\mathbf{ND}]$ *the function a verifies*

$$e^{-C(T-t)} \le a(t,z) \le 1 \qquad \textit{where} \qquad C := \frac{\|\tilde{\mu}\|^2}{|\Gamma|}$$

Furthermore, there exists $T^*>0$ and $K^a_{lip}\ge 0$ such that if $T<T^*$ then

$$|a(t,z') - a(t,z)| \le K^a_{lip}|z-z'|$$

for all $t\in[0,T]$ and $z,z'\in\mathbb{R}$. T^* depends on $\overline{\mu}$, τ, C and K_{max} defined in Assumptions 2.1. Moreover $T^*\to+\infty$ when $K_{max}\to 0$ and the other constants remain fixed.

Remark that these results hold true without assuming any particular structure of the Lévy measure $\nu(dy)$. The next goal is to characterize the functions a, b and c as the solutions of certain PIDEs.

3 HJB formulation and main regularity results

Remarks on notation For a function $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ we denote $\|f\|_\infty := \sup_{t\le T, x\in\mathbb{R}}|f(t,x)|$. For a function φ defined on $[0,T]\times\mathbb{R}$ and $k\in\mathbb{N}$ we denote $D^k\varphi := \partial^k\varphi/\partial x^k$ whereas $\partial_t\varphi$ denotes the derivative in the time variable. We adopt the following convention: for any $l\in\mathbb{R}^+$

$$l = \lfloor l\rfloor + \{l\}^-, \text{ where } \{l\}^- \in [0,1)$$
$$l = \lceil l\rceil + \{l\}^+, \text{ where } \{l\}^+ \in (0,1]$$

Let us first introduce the functional spaces in which we will work: for $\beta\in(0,1]$ we define

$$\langle\psi\rangle^{(\beta)} := \sup_{x,0<|h|\le 1}\frac{|\psi(x+h)-\psi(x)|}{|h|^\beta}, \qquad \langle\varphi\rangle^{(\beta)}_{Q_T} := \sup_{t,x,0<|h|\le 1}\frac{|\varphi(t,x+h)-\varphi(t,x)|}{|h|^\beta}$$

The elliptic Hölder space of order l, $H^l(\mathbb{R}^n)$, is defined as the space of continuously differentiable functions ψ for all order $j \leq \lceil l \rceil$ with finite norm

$$\|\psi\|_l := \sum_{j=0}^{\lceil l \rceil} \sum_{(j)} \left\| D_x^j \psi \right\|_\infty + \sum_{(\lceil l \rceil)} \langle D_x^{\lceil l \rceil} \psi \rangle^{(\{l\}^+)} \tag{3.1}$$

where $\sum_{(j)}$ represents the summation over all possible derivative of order j. The parabolic Hölder space $H^l([0,T] \times \mathbb{R}^n)$ is defined as the set of measurable functions $\varphi : [0,T] \to H^l(\mathbb{R}^n)$ with finite norm

$$\|\varphi\|_l := \sum_{j=0}^{\lceil l \rceil} \sum_{(j)} \left\| D_x^j \varphi \right\|_\infty + \sum_{(\lceil l \rceil)} \langle D_x^{\lceil l \rceil} \varphi \rangle_{Q_T}^{(\{l\}^+)} \tag{3.2}$$

The spaces defined above are all Banach spaces equipped with their respective norms. For a complete description see for example Chapter I in Adams and Fournier (2009).

In the spirit of HJB approach we now introduce the operators associated to the process Z:

Definition 3.1. *For a real valued function $\varphi \in H^{\alpha+\delta}([0,T] \times \mathbb{R})$, $\delta > 0$, we define the following linear operators*

$$\begin{aligned}
\mathcal{A}_t\varphi(t,z) &:= -\mu \frac{\partial \varphi}{\partial z}(t,z) \\
\mathcal{B}_t\varphi(t,z) &:= \int_{\mathbb{R}} \left(\varphi(t, z + \gamma(t,z,y)) - \varphi(t,z) - \gamma(t,z,y) \frac{\partial \varphi}{\partial z}(t,z) \right) \nu(dy) \\
\mathcal{Q}_t\varphi(t,z) &:= \tilde{\mu}\varphi(t,z) + \int_{\mathbb{R}} (e^\gamma - 1) \left(\varphi(t, z + \gamma(t,z,y)) - \varphi(t,z) \right) \nu(dy) \\
\mathcal{G}_t\varphi(t,z) &:= \int_{\mathbb{R}} (e^\gamma - 1)^2 \varphi(t, z + \gamma(t,z,y)) \nu(dy)
\end{aligned}$$

where $\tilde{\mu}$ stands for $\tilde{\mu}(t,z)$ and so on. In addition, $\mathcal{H}$ denotes the nonlinear operator

$$\mathcal{H}_t[\varphi](z) := \inf_{|\pi| \leq \bar{\Pi}} \left[2\pi \mathcal{Q}_t \varphi(t,z) + \pi^2 \mathcal{G}_t \varphi(t,z) \right]$$

where

$$\bar{\Pi} := \frac{e^{CT}}{|\Gamma|} \max \left(\|\tilde{\mu}\|_\infty, 2 \left(\|\tau\|_{4,\nu}^4 + \|\tau\|_{2,\nu}^2 \right) \right) \left(1 + K_{lip}^a \right). \tag{3.3}$$

The main result concerning the functions a is:

Theorem 3.2. *Let Assumptions 2.1 hold true and consider $T < T^*$ as in Theorem 2.3. The function a is the unique solution of*

$$0 = -\frac{\partial a}{\partial t} + \mathcal{A}_t a - \mathcal{B}_t a - \mathcal{H}_t[a], \qquad a(T,z) = 1 \tag{3.4}$$

in the Hölder space $H^{\alpha+\delta}([0,T] \times \mathbb{R})$ for $0 < \delta < \alpha - 1$. The function $t \mapsto a(t,z)$ is also differentiable on $(0,T)$. The optimal strategy for the stochastic control problem (2.13) is

$$\theta_t^* = e^{-Z_{t-}} \pi^*(t, Z_{t-}) X_{t-}^{\theta^*}, \qquad X_t^{\theta^*} := x + \int_0^t \theta_{r-}^* dS_r$$

where

$$\pi^*(t,z) := -\frac{\mathcal{Q}a(t,z)}{\mathcal{G}a(t,z)} \tag{3.5}$$

□

For the general value function v_f we have

Theorem 3.3. *Let $T < T^*$ as in Theorem 2.3. Let also Assumptions 2.1 hold true and $f \in H^{\alpha+\delta}(\mathbb{R})$ for some $0 < \delta < \alpha - 1$. The function v_f in (2.11) admits the decomposition*

$$v_f(t, z, x) = a(t, z)x^2 + b(t, z)x + c(t, z)$$

where a is defined in (2.14), so it does not depend on f, and it is the unique solution in $H^{\alpha+\delta}([0, T]\times \mathbb{R})$ of (3.4), whereas b and c are the unique solutions of the following linear parabolic PIDEs

$$0 = -\frac{\partial b}{\partial t} + \mathcal{A}b - \mathcal{B}b - \pi^*\mathcal{Q}b, \qquad b(T, .) = -2f; \tag{3.6}$$

$$0 = -\frac{\partial c}{\partial t} + \mathcal{A}c - \mathcal{B}c + \frac{1}{4}\frac{(\mathcal{Q}b)^2}{\mathcal{G}a}, \qquad c(T, .) = f^2 \tag{3.7}$$

in the Hölder space $H^{\alpha+\delta}([0, T] \times \mathbb{R})$, where π^* is defined in (3.5). The functions $t \mapsto a(t, .), b(t, .), c(t, .)$ are also differentiable on $(0, T)$.
Furthermore the optimal policy in the control problem (2.11) is given by

$$\theta_t^* := e^{-Z_{t-}}\left(\pi^*(t, Z_{t-})X_{t-}^{\theta^*} - \frac{1}{2}\frac{\mathcal{Q}b(t, Z_{t-})}{\mathcal{G}a(t, Z_{t-})}\right), \ X_t^{\theta^*} := x + \int_0^t \theta_{r-}^* dS_r \tag{3.8}$$

□

The proof of these results can be found in Chapter 7 of De Franco (2012). From the decomposition (2.12) we also obtain the optimal price in (2.11):

$$x^*(f) := \arg\inf_{x\in\mathbb{R}} v_f(t, z, x) = -\frac{b^f(t, z)}{2a(t, z)} \tag{3.9}$$

which is a linear function of the payoff f since b^f is.

Non smooth payoff Theorem 3.3 allows us to characterize the value function v_f when the payoff function f is sufficiently smooth, i.e. $f \in H^{\alpha+\delta}(\mathbb{R})$. However, in most cases of interest (for example put options, straddles or bear spreads) this function is not even continuously differentiable. The following lemma proves the stability of the optimal price $x^*(f)$ and the optimal hedging strategy under small perturbations of the function f:

Lemma 3.4. *Let f_1, f_2 be two measurable functions with $f_i(Z_T^{t,z}) \in \mathbb{L}^2(\mathbb{P})$ for all t, z, $i = 1, 2$. Then for any $t < T$ and $z \in \mathbb{R}$*

$$|x^*(f_1)(t, z) - x^*(f_2)(t, z)| \leq a(t, z)^{-1/2} \left\|(f_1 - f_2)(Z_T^{t,z})\right\|_{\mathbb{L}^2(\mathbb{P})}$$

$$\left|\left(v^{f_1} - v^{f_2}\right)(t, z, x)\right| \leq 2\left(x + \left\|(f_1 + f_2)(Z_T^{t,z})\right\|_{\mathbb{L}^2(\mathbb{P})}\right)\left\|(f_1 - f_2)(Z_T^{t,z})\right\|_{\mathbb{L}^2(\mathbb{P})}$$

Fix now (t, z, x) and let f_n such that $\left\|(f_n - f)(Z_T^{t,z})\right\|_2 \to 0$, $n \to \infty$. If θ^n is the optimal control in the problem (2.11) with payoff function f_n then, for all $\varepsilon > 0$, there exists some $N > 0$ such that for any $n \geq N$ one has

$$\left|v_f(t, z, x) - \mathbb{E}^{\mathbb{P}}\left[\left(f(Z_T^{t,z}) - x - \int_t^T \theta_{r-}^n dS_r^{t,z}\right)^2\right]\right| \leq \varepsilon$$

The proof of this Lemma can be found in Chapter 5 of De Franco (2012). One can thus approximate a non-smooth payoff function f with smooth functions f_n, controlling the error on the value function and the cost of the hedging strategy with $\|f - f_n\|_2$. Furthermore the corresponding strategies (θ_n) become ε-optimal for the pay-off f starting from sufficiently large n.

4 Numerical solution schemes

We now present a numerical scheme to solve the PIDE introduced in Section 3 when the Lévy measure ν verifies the Assumption 2.1-[**L**]. From (3.8) and (3.9) we remark that in order to solve the problem (2.11), i.e. to find the optimal strategy θ^* and the optimal price x^*, we only need to compute the functions a and b, solutions, respectively, of PIDEs (3.4) and (3.6).

The finite difference discretization schemes will be constructed using the Markov chain approximation technique developed in Kushner (1976). One of the advantages of the probabilistic treatment is that it allows to estimate the error due to the truncation of the domain in a simple manner. The Markov chain approximation method works as follows. We first construct a discrete-time Markov process $(\widehat{Z}_{t_i})_{i=0,\dots,N_T}$ evolving on a regular space grid $z_j = j\Delta z, -N < j < N$ and regular time grid $t_i = i\Delta t$, $i = 0, \dots, N_T$ with $\Delta t = \frac{T}{N_T}$, approximating the process Z defined in (2.1). We then replace the process Z with the Markov chain $\widehat{Z}$ in the quadratic hedging problem. The dynamic programming algorithm for the discretized hedging problem then provides an approximation scheme for the original control problem and thus for equations (3.4) and (3.6).

4.1 Definition of the approximating Markov chain

The action of the generator of Z on a test function φ is given by

$$\mathcal{L}^Z\phi(t,z) = \mu\frac{\partial\varphi}{\partial z}(z) + \int_{\mathbb{R}}\left(\varphi(z+\gamma(t,z,y)) - \varphi(z) - \gamma(t,z,y)\frac{\partial\varphi}{\partial z}(z)\right)\nu(dy)$$

We now detail the computation of the integral term in this generator. In order to avoid interpolation, we use a space and time dependent grid for discretizing the Lévy density, with discretization points denoted by $(y_i(t,z))_{-I\le i\le I}$. We select the discretization point $y_i(t,z)$, which corresponds to the center of i-th discretization interval, as the unique solution of the equation $\gamma(t,z,y_i(t,z)) = i\Delta z$. The boundaries of the discretization intervals will then correspond to half-integer values of i. Although these discretization points depend on t and z, we will sometimes omit this dependence to simplify notation.

To treat the singularity of the Lévy density at zero, we adapt the methodology of Forsyth et al. (2007) and divide the real line into four disjoint regions, for an integer $\kappa \ge 1$

$$\begin{aligned}
\Omega_0(t,z) &= \left\{y : y_{-\kappa-\frac{1}{2}}(t,z) \le y \le y_{\kappa+\frac{1}{2}}(t,z)\right\},\\
\Omega_1(t,z) &= \left\{y : y_{\kappa+\frac{1}{2}}(t,z) < y \le y_{\zeta+\frac{1}{2}}(t,z)\right\} \cup \left\{y : y_{-\kappa-\frac{1}{2}}(t,z) > y \ge y_{-\zeta'-\frac{1}{2}}(t,z)\right\},\\
\Omega_2(t,z) &= \left\{y : y_{\zeta+\frac{1}{2}}(t,z) < y \le y_{I+\frac{1}{2}}(t,z)\right\} \cup \left\{y : y_{-\zeta'-\frac{1}{2}}(t,z) > y \ge y_{-I-\frac{1}{2}}(t,z)\right\},\\
\Omega_3(t,z) &= \left\{y : y \le y_{-I-\frac{1}{2}}(t,z)\right\} \cup \left\{y : y \ge y_{I+\frac{1}{2}}(t,z)\right\},
\end{aligned}$$

where $\zeta = \inf\{i : y_i(t,z) \ge 1\}$ and $\zeta' = \inf\{i : y_{-i}(t,z) \le -1\}$. Without loss of generality, we shall always assume that $\kappa < \zeta < I$ and $\kappa < \zeta' < I$.

The jumps of the Lévy measure in the regions $\Omega_0, \dots, \Omega_3$ are treated as follows:

- The small jumps in the region Ω_0 are truncated and the corresponding part of the integral operator is replaced by a local operator (e.g., second derivative).
- The jumps in the regions Ω_1 and Ω_2 are discretized.
- The large jumps in the region Ω_3 are truncated.

For reader's convenience the different truncation and discretization parameters of our algorithm are listed in the following table. Theorems 4.1 and 4.7 below show how these different parameters affect the overall approximation error.

Parameter	Meaning
Δt	Time discretization step (with $\Delta t = \frac{T}{N_T}$)
Δz	Space discretization step
N	Space grid size
I	Lévy measure truncation (number of points)
κ	Truncation of small jumps (number of points)

After replacing the small jumps with a local operator and removing the large jumps, we obtain a generator of the form

$$\mu \frac{\partial \varphi}{\partial z} + \frac{D(t,z)}{2}\frac{\partial^2 \varphi}{\partial z^2} + \int_{\Omega_1 \cup \Omega_2} \left(\varphi(z + \gamma(t,z,y)) - \varphi(z) - \gamma(t,z,y)\frac{\partial \varphi}{\partial z}(z) \right) \nu(dy),$$

where the coefficient D is defined by

$$D(t,z) := \int_{\Omega_0} \gamma(t,z,y)^2 \nu(y) dy. \tag{4.1}$$

Discretization of the remaining integral leads to a generator of the form

$$\begin{aligned} &\mu \frac{\partial \varphi}{\partial z} + \frac{D(t,z)}{2}\frac{\partial^2 \varphi}{\partial z^2} + \sum_{\kappa < |j| \le I} \omega_j(t,z) \left(\varphi(z + \gamma(t,z,y_j(t,z))) - \varphi(z) - \gamma(t,z,y_j(t,z))\frac{\partial \varphi}{\partial z}(z) \right), \\ &= \hat{\mu} \frac{\partial \varphi}{\partial z} + \frac{D(t,z)}{2}\frac{\partial^2 \varphi}{\partial z^2} + \sum_{\kappa < |j| \le I} \omega_j(t,z) \left(\varphi(z + j\Delta z) - \varphi(z) \right) \end{aligned} \tag{4.2}$$

where

$$\hat{\mu}(t,z) = \mu(t,z) - \sum_{\kappa < |i| \le I} \omega_i(t,z)\gamma(t,z,y_i(t,z)).$$

and we have introduced the weights :

$$\omega_i(t,z) = \begin{cases} \dfrac{1}{\gamma^2(t,z,y_i(t,z))} \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} \gamma^2(t,z,y)\nu(y)dy & \text{if } y_i \in \Omega_1 \\ \displaystyle\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} \nu(y)dy & \text{if } y_i \in \Omega_2 \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

Depending on the form of γ and the Lévy measure, the integrals in $D(t,z)$ and $\omega_i(t,z)$ may sometimes be calculated explicitly. Usually, they should be calculated numerically, using, e.g., the 5-point trapezoidal rule, which, in the second case above yields

$$\hat{\omega}_i(t,z) = \frac{\Delta y_i}{4} \left(\frac{1}{2}\nu(y_i - \frac{\Delta y_i}{2}) + \nu(y_i - \frac{\Delta y_i}{4}) + \nu(y_i) + \nu(y_i + \frac{\Delta y_i}{4}) + \frac{1}{2}\nu(y_i + \frac{\Delta y_i}{2}) \right) \tag{4.4}$$

with $\Delta y_i(t,z) = y_{i+\frac{1}{2}}(t,z) - y_{i-\frac{1}{2}}(t,z)$. In the following, we shall assume that these integrals are calculated explicitly without error; the additional error introduced by the their numerical evaluation can be easily estimated along the lines of other computations in the following section.

Finally, to approximate the local part of the generator, introduce the weights υ and χ defined by

$$\upsilon(t,z) = \frac{D(t,z)}{2\Delta z^2} - \frac{\hat{\mu}(t,z)}{2\Delta z} \tag{4.5}$$

$$\chi(t,z) = \frac{D(t,z)}{2\Delta z^2} + \frac{\hat{\mu}(t,z)}{2\Delta z} \tag{4.6}$$

if both these expressions are positive or and by

$$\begin{aligned}
\upsilon(t,z) &= \frac{D(t,z)}{2\Delta z^2} + \max(0, -\frac{\hat{\mu}(t,z)}{\Delta z}) \\
\chi(t,z) &= \frac{D(t,z)}{2\Delta z^2} + \max(0, \frac{\hat{\mu}(t,z)}{\Delta z})
\end{aligned}$$

otherwise. Approximating the local part of the generator by central or non-central differences in the usual way, we obtain the fully discretized generator

$$\begin{aligned}
\widehat{\mathcal{L}}^Z\varphi(z) = \chi(t,z)\varphi(z+\Delta z) + \upsilon(t,z)\varphi(z-\Delta z) - (\chi(t,z)+\upsilon(t,z))\varphi(z) \\
+ \sum_{\kappa<|j|\le I} \omega_j(t,z)\left(\varphi(z+j\Delta z) - \varphi(z)\right) \quad (4.7)
\end{aligned}$$

Assume that Δt is small enough so that for all (t,z),

$$\frac{D(t,z)}{\Delta z^2} + \frac{\overline{\mu}}{\Delta z} + \sum_{\kappa<|i|\le I} |i|\omega_i(t,z) + \sum_{\kappa<|i|\le I} \omega_i(t,z) \le \frac{1}{\Delta t}, \quad (4.8)$$

where $\overline{\mu}$ is defined in Assumption 2.1-[**C**]-(i). Under this condition, we may introduce a discrete-time Markov chain $(\widehat{Z}_{t_i})_{i=1,\dots,N_T}$ defined as follows:

$$\widehat{Z}_{t_{i+1}} = \begin{cases}
\widehat{Z}_{t_i} + j\Delta z, j = \kappa+1,\dots,I, \quad \text{with probabilities} \quad \omega_j(t,\widehat{Z}_{t_i})\Delta t \\
\widehat{Z}_{t_i} + j\Delta z, j = -I,\dots,-\kappa-1, \quad \text{with probabilities} \quad \omega_j(t,\widehat{Z}_{t_i})\Delta t \\
\widehat{Z}_{t_i} + \Delta z \quad \text{with probability} \quad \chi(t,\widehat{Z}_{t_i})\Delta t \\
\widehat{Z}_{t_i} - \Delta z \quad \text{with probability} \quad \upsilon(t,\widehat{Z}_{t_i})\Delta t \\
\widehat{Z}_{t_i} \quad \text{with probability} \quad 1 - \left(\upsilon(t,\widehat{Z}_{t_i}) + \chi(t,\widehat{Z}_{t_i}) + \sum_{\kappa<|j|\le I} \omega_j(t,\widehat{Z}_{t_i})\right)\Delta t
\end{cases}$$

When the chain starts from the point z at time t_n, its value will be denoted by $\widehat{Z}^{z,t_n}_{t_i}$, $i \ge n$. This Markov chain is related to the discretized generator (4.7) in the following way:

$$\mathbb{E}[\varphi(\widehat{Z}^{z,t_n}_{t_{n+1}})] = \varphi(z) + \Delta t\, \widehat{\mathcal{L}}^Z\varphi(z)$$

For this reason, we choose this Markov chain as the discretized approximation to the process Z.

To make the notation more compact, we shall denote the probability of transition from $\widehat{Z}_{t_i}$ to $\widehat{Z}_{t_i} + j\Delta z$ by $p_j(t_i, \widehat{Z}_{t_i})$, for $-I \le j \le I$. Also, in order to restrict the values of the chain to the space grid, it shall be stopped at the first moment when it exits the grid, defined by

$$\beta^{z,t_n} = \inf\{i \ge n : \widehat{Z}^{z,t_n}_{t_i} \notin [-N\Delta z, N\Delta z]\}.$$

4.2 A finite-difference scheme for the function $a(t,z)$

The values of the approximations of the functions a and b at points (t_i, z_j) will be denoted, respectively, by a^i_j and b^i_j. We also introduce the functions $a^i(z)$ and $b^i(z)$, defined only for $z = j\Delta z$ with $j \in \mathbb{Z}$, and given by $a^i(z_j) = a^i_j$ and $b^i(z_j) = b^i_j$.

Consider the following discrete-time control problem for $\widehat{Z}$:

$$\hat{v}_f(t_n, z_j, x) = \inf_{\pi_{n+1},\dots,\pi_{N_T}} \mathbb{E}\left[\left(f(\widehat{Z}^{t_n,z_j}_{\beta^{t_n,z_j}\wedge N_T}) - x\prod_{i=n+1}^{\beta^{t_n,z_j}\wedge N_T}\left(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)\right)^2\right]$$

where $\pi_i \in [-\overline{\Pi}, \overline{\Pi}]$ for $i = n+1,\dots,N_T$. This control problem is the discretized version of the original control problem (2.11). It is obtained by replacing the continuous time process $Z = \log S$

with the discrete-time Markov chain $\widehat{Z}$; assuming that the amount invested into the risky asset $\pi_t = \theta_t S_t$ remains constant and equal to π_i on the i-the discretization interval, and stopping the Markov chain at the time β^{z,t_n} when it first exits the space grid. Similarly to the original continuous-time problem, it is easy to show that $\hat{v}_f$ has a quadratic structure:

$$\hat{v}_f(t_n, z_j, x) = x^2 \hat{a}(t_n, z_j) + x\hat{b}(t_n, z_j) + \hat{c}(t_n, z_j) \tag{4.9}$$

for some functions $\hat{a}$, $\hat{b}$ and $\hat{c}$ defined on the grid. In particular, the function $\hat{a}$ is the solution of the pure investment problem for the discrete process $\hat{Z}$ and satisfies

$$\hat{a}(t_n, z_j) = \inf_{\pi_{n+1},\dots,\pi_{N_T}} \mathbb{E}\left[\prod_{i=n+1}^{\beta^{t_n,z_j}\wedge N_T} \left(1 + \pi_i (e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2\right], \tag{4.10}$$

For greater generality[2], we introduce an arbitrary boundary / terminal condition and define the function a^n, approximating the function a, by

$$a^n(z_j) = \inf_{\pi_{n+1},\dots,\pi_{N_T}} \mathbb{E}\left[\prod_{i=n+1}^{\beta^{t_n,z_j}\wedge N_T} \left(1 + \pi_i (e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2 q^a(\Delta t(\beta^{t_n,z_j} \wedge N_T), \widehat{Z}^{t_n,z_j}_{\beta^{t_n,z_j}\wedge N_T})\right],$$

where the function q^a is measurable, bounded from above and below by positive constants, and satisfies $q^a(T, z) = 1$ for all z. In the numerical examples, we take $q^a \equiv 1$ and in Theorems 4.1 and 4.7 we shall see that the effect of the boundary conditions on the approximation becomes negligible for $\Delta z N$ sufficiently large.

The dynamic programming principle for this discrete-time control problem writes

$$\begin{aligned} a^n(z_j) &= \inf_{\pi\in[-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\left[\left(1 + \pi (e^{\widehat{Z}^{t_n,z_j}_{t_{n+1}} - z_j} - 1)\right)^2 a^{n+1}(\widehat{Z}^{t_n,z_j}_{t_{n+1}})\right], \quad \text{if} \quad j \in (-N, N), \\ a^n(z_j) &= q^a(t_n, z_j), \quad \text{if} \quad j \notin (-N, N), \end{aligned} \tag{4.11}$$

or in other words

$$a^n_j = \inf_{\pi\in[-\bar{\Pi},\bar{\Pi}]} \sum_{-I\leq l\leq I} p_l(t_n, z_j)(1 + \pi(e^{l\Delta z} - 1))^2 a^{n+1}_{j+l}, \tag{4.12}$$

which gives a fully explicit finite difference scheme for approximating the function a. Using the explicit form of the transition probabilities, it can be rewritten as

$$\begin{aligned} &\frac{a^{n+1}_j - a^n_j}{\Delta t} - (\upsilon(t_n, z_j) + \chi(t_n, z_j))\, a^{n+1}_j + \chi(t_n, z_j) a^{n+1}_{j+1} + \upsilon(t_n, z_j) a^{n+1}_{j-1} + \sum_{\kappa<|l|\leq I} \omega_l(t_n, z_j)(a^{n+1}_{j+l} - a^{n+1}_j) \\ &+ \inf_{\pi\in[-\bar{\Pi},\bar{\Pi}]} \Big\{\pi^2 \Big(\sum_{\kappa<|l|\leq I} \omega_l(t_n, z_j)(e^{l\Delta z} - 1)^2 a^{n+1}_{j+l} + \chi(t_n, z_j)(e^{\Delta z} - 1)^2 a^{n+1}_{j+1} + \upsilon(t_n, z_j)(e^{-\Delta z} - 1)^2 a^{n+1}_{j-1}\Big) \\ &+ 2\pi \left(\sum_{\kappa<|l|\leq I} \omega_l(t_n, z_j)(e^{l\Delta z} - 1) a^{n+1}_{j+l} + \chi(t_n, z_j)(e^{\Delta z} - 1) a^{n+1}_{j+1} + \upsilon(t_n, z_j)(e^{-\Delta z} - 1) a^{n+1}_{j-1}\right)\Big\} = 0. \end{aligned} \tag{4.13}$$

Hence, this scheme uses the following approximations for the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{G}$ appearing

[2]For example, if some a priori approximation for the function a is available, it may be used as boundary condition to improve the accuracy of the scheme.

in Equation (3.4):

$$
\begin{aligned}
(\mathcal{B}_t-\mathcal{A}_t)a(t_n,z_j) &\approx -\left(\upsilon(t_n,z_j)+\chi(t_n,z_j)\right)a_j^{n+1}+\chi(t_n,z_j)a_{j+1}^{n+1}+\upsilon(t_n,z_j)a_{j-1}^{n+1}\\
&\quad+\sum_{\kappa<|l|\le I}\omega_l(t_n,z_j)(a_{j+l}^{n+1}-a_j^{n+1})\\
\mathcal{G}_t a(t_n,z_j) &\approx \sum_{\kappa<|l|\le I}\omega_l(t_n,z_j)(e^{l\Delta z}-1)^2a_{j+l}^{n+1}+\chi(t_n,z_j)(e^{\Delta z}-1)^2a_{j+1}^{n+1}+\upsilon(t_n,z_j)(e^{-\Delta z}-1)^2a_{j-1}^{n+1}\\
\mathcal{Q}_t a(t_n,z_j) &\approx \sum_{\kappa<|l|\le I}\omega_l(t_n,z_j)(e^{l\Delta z}-1)a_{j+l}^{n+1}+\chi(t_n,z_j)(e^{\Delta z}-1)a_{j+1}^{n+1}+\upsilon(t_n,z_j)(e^{-\Delta z}-1)a_{j-1}^{n+1}
\end{aligned}
$$

Stability analysis Under the condition (4.8), the transition probabilities of the Markov chain are positive. Therefore, from Equation (4.12), by choosing $\pi=0$, it follows that whenever $a_j^{n+1}\ge 0$ for $-N<j<N$ and the boundary condition satisfies $q^a(t,z)\ge 0$ for all t,z, the elements of a^n satisfy

$$0\le a_i^n\le \max\{\max_{-N<j<N}a_j^{n+1},\|q^a\|_\infty\},\quad -N<i<N.$$

In other words, the scheme is L^∞-stable as long as the terminal and boundary data are non-negative. In the numerical examples we choose $q^a(t,z)\equiv 1$ which means that $0\le a_j^n\le 1$ for $n=0,\dots,N_T$ and $-N<j<N$. Therefore, condition (4.8) plays the role of the CFL condition in this setting, ensuring the stability of the numerical scheme for a.

Let us now derive a more tractable sufficient condition of stability. By definition of the weights D and ω,

$$
\begin{aligned}
&\frac{D(t,z)}{\Delta z^2}+\frac{\bar\mu}{\Delta z}+\sum_{\kappa<|i|\le I}|i|\omega_i(t,z)+\sum_{\kappa<|i|\le I}\omega_i(t,z) \qquad (4.14)\\
&=\frac{1}{\Delta z^2}\int_{\Omega_0}\gamma(t,z,y)^2\nu(dy)+\sum_{i:y_i\in\Omega_1}\frac{(1+|i|)}{i^2\Delta z^2}\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}\gamma^2(t,z,y)\nu(dy)\\
&\qquad+\sum_{i:y_i\in\Omega_2}(1+|i|)\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}\nu(dy)+\frac{\bar\mu}{\Delta z}\\
&\le\frac{1}{\Delta z^2}\int_{\Omega_0}\gamma(t,z,y)^2\nu(dy)+\sum_{i:y_i\in\Omega_1}\frac{(1+|i|)(1/2+|i|)}{i^2\Delta z}\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}|\gamma(t,z,y)|\nu(dy)\\
&\qquad+\sum_{i:y_i\in\Omega_2}\frac{1+|i|}{(|i|-1/2)\Delta z}\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}|\gamma(t,z,y)|\nu(dy)+\frac{\bar\mu}{\Delta z}\\
&\le\frac{1}{\Delta z^2}\int_{\Omega_0}\gamma(t,z,y)^2\nu(dy)+\frac{2}{\Delta z}\frac{(\kappa+2)^2}{(\kappa+1)^2}\int_{\Omega_1\cup\Omega_2}|\gamma(t,z,y)|\nu(dy)+\frac{\bar\mu}{\Delta z} \qquad (4.15)
\end{aligned}
$$

where the inequalities follow from the fact that γ is increasing in y and $\gamma(t,z,y_{i+1/2})=(i+1/2)\Delta z$, the fact that $\inf\{|i|:y_i\in\Omega_1\}=\kappa+1$ and some straightforward simplifications.

We now proceed to estimate each of the above terms. For the first term, since Δz is small and κ is a constant, we may assume that $(\kappa+1/2)(1\vee\frac{1}{m_1})\Delta z<y_0$ with y_0 and m_1 defined in Assumption 2.1-[**RG**] $-$ **i**). Then, from this assumption we may deduce that for all t,z,

$$\gamma^{-1}(t,z,(\kappa+1/2)\Delta z)\le\frac{(\kappa+1/2)\Delta z}{m_1}\quad\text{and}\quad\gamma^{-1}(t,z,-(\kappa+1/2)\Delta z)\ge-\frac{(\kappa+1/2)\Delta z}{m_1}$$

and on the other hand, $|\gamma(t,z,y)|\le(1+m_2y_0)|y|$ for $y\in\Omega_0$. This implies:

$$\frac{1}{\Delta z^2}\int_{\Omega_0}\gamma(t,z,y)^2\nu(dy)\le\frac{(1+m_2y_0)^2}{\Delta z^2}\int_{|y|\le\frac{(\kappa+1/2)\Delta z}{m_1}}y^2\nu(dy)\le\frac{2M_g(1+m_2y_0)^2}{2-\alpha}\left(\frac{\kappa+\frac12}{m_1}\right)^{2-\alpha}\Delta z^{-\alpha}$$

The second term satisfies

$$\frac{1}{\Delta z}\int_{\Omega_1\cup\Omega_2}|\gamma(t,z,y)|\nu(dy) \leq \frac{1}{\Delta z}\int_{(\Omega_1\cup\Omega_2)\cap\{|y|\leq y_0\}}|\gamma(t,z,y)|\nu(dy) + \frac{1}{\Delta z}\int_{|y|>y_0}\sup_{t,z}|\gamma(t,z,y)|\nu(dy)$$
$$\leq \frac{1+m_2y_0}{\Delta z}\int_{|y|>\frac{(\kappa+1/2)\Delta z}{1+m_2y_0}}|y|\nu(dy) + \frac{1}{\Delta z}\int_{|y|>y_0}\sup_{t,z}|\gamma(t,z,y)|\nu(dy)$$
$$\leq \frac{(\kappa+\frac{1}{2})M_g}{\alpha-1}\left(\frac{\kappa+1/2}{1+m_2y_0}\right)^{-\alpha}\Delta z^{-\alpha} + \frac{1}{\Delta z}\int_{|y|>y_0}\sup_{t,z}|\gamma(t,z,y)|\nu(dy),$$

where we remark that the integral in the right-hand side is finite by Assumption 2.1-[**I**].

Finally, we conclude that the stability condition (4.8) is implied by the bound

$$\Delta t \leq \frac{\Delta z^\alpha}{C_1 + C_2\Delta z^{\alpha-1}} \quad \text{with} \tag{4.16}$$
$$C_1 = \frac{2M_g(1+m_2y_0)^2}{2-\alpha}\left(\frac{\kappa+\frac{1}{2}}{m_1}\right)^{2-\alpha} + \frac{2(\kappa+2)^2(\kappa+\frac{1}{2})M_g}{(\kappa+1)^2(\alpha-1)}\left(\frac{\kappa+1/2}{1+m_2y_0}\right)^{-\alpha},$$
$$C_2 = \bar{\mu} + 2\frac{(\kappa+2)^2}{(\kappa+1)^2}\int_{|y|>y_0}\sup_{t,z}|\gamma(t,z,y)|\nu(dy).$$

When Δz is sufficiently small, to satisfy this condition, it is enough to take $\Delta t \leq \frac{\Delta z^\alpha}{C_1+\varepsilon}$ for some small value $\varepsilon > 0$.

In practice, (4.16) is only an upper bound and not a necessary condition for stability. We recommend using the sharper condition (4.8), for example, by decreasing the time step in the computational procedure whenever this condition is violated.

Accuracy analysis The following theorem analyzes the error and convergence of our approximation algorithm.

Theorem 4.1. *Assume that the following technical conditions hold in addition to Assumptions 2.1:*

1. *The Lévy density ν is twice continuously differentiable with bounded derivatives outside any neighborhood of zero;*
2. *The function g has bounded derivatives near zero;*
3. *Assumption 2.1-[**RG**] $-$ **i**) holds true with $y_0 = +\infty$;*
4. *The function $\gamma(t,z,y)$ is 3 times continuously differentiable with respect to y with bounded derivatives;*
5. *The data of the problem are such that the functions a and b together with their derivatives up to order 4 with respect to z and up to order 2 with respect to t are bounded.*

Let the condition (4.8) *be satisfied and assume that $N\Delta z > I\Delta z + \bar{\mu}T$. Then, there exist two positive constants $c, C < \infty$, which do not depend on truncation or discretization parameters, such that for all $\kappa > c$,*

$$|a_n(z_j) - a(t_n,z_j)| \leq C\Bigg\{\frac{(1+|z_j|)}{(N-I)\Delta z - \bar{\mu}T} + \Delta t + \Delta z^{3-\alpha}\kappa^{3-\alpha} + \Delta z^{3-\alpha}\kappa^{1-\alpha}$$
$$+ \int_{|y|\geq\frac{I\Delta z}{\|\gamma_y'\|_\infty}}(1+|y|+\tau(y)+\tau^2(y))\nu(dy)\Bigg\},$$

for $-N \leq j \leq N$ and $0 \leq n \leq N_T$;

The proof of this result is given in Appendix A.

Remark 4.2. *This theorem gives the decomposition of the global approximation error into four sources: the domain truncation error (first term in brackets), the time discretization error (second term), the error coming from space discretization and truncation of small jumps (third and fourth term) and the big jump truncation (last term). It may be used for the optimal choice of the parameters of our algorithm.*

Remark 4.3. *The constant c is needed to ensure that for Δz sufficiently small, the weights υ and χ may be assumed to be given by (4.5)-(4.6). If the Lévy measure is locally symmetric near zero, that is, $c_+ = c_-$, one can take $c = 1$. Indeed, in this case, the integral $\int_{\varepsilon<x\leq 1} x\nu(dx)$ remains bounded as $\varepsilon \to 0$, which means that $\hat{\mu}(t,z)$ remains bounded as well. On the other hand, $\frac{D(t,z_j)}{\Delta z^2}$ is of order of $(\kappa\Delta z)^{-\alpha}$, which means that, eventually $\frac{D(t,z_j)}{\Delta z^2}$ becomes bigger than $\frac{\hat{\mu}(t,z)}{\Delta z}$ for all $\kappa \geq 1$ (since under our assumptions $\alpha > 1$).*

Remark 4.4 (Implicit-explicit scheme)**.** *In practice, it may be preferable to use an implicit scheme for the convection diffusion part and an explicit one for the integral part:*

$$\begin{aligned}&\frac{a_j^{n+1}-a_j^n}{\Delta t} - (\upsilon(t_n,z_j)+\chi(t_n,z_j))\,a_j^n + \chi(t_n,z_j)a_{j+1}^n + \upsilon(t_n,z_j)a_{j-1}^n + \sum_{\kappa<|l|\leq I}\omega_l(t_n,z_j)(a_{j+l}^{n+1}-a_j^{n+1})\\
&+\inf_{\pi\in[-\bar\Pi,\bar\Pi]}\Big\{\pi^2\Big(\sum_{\kappa<|l|\leq I}\omega_l(t_n,z_j)(e^{l\Delta z}-1)^2a_{j+l}^{n+1}+\chi(t_n,z_j)(e^{\Delta z}-1)^2a_{j+1}^{n+1}+\upsilon(t_n,z_j)(e^{-\Delta z}-1)^2a_{j-1}^{n+1}\Big)\\
&+2\pi\Big(\sum_{\kappa<|l|\leq I}\omega_l(t_n,z_j)(e^{l\Delta z}-1)a_{j+l}^{n+1}+\chi(t_n,z_j)(e^{\Delta z}-1)a_{j+1}^{n+1}+\upsilon(t_n,z_j)(e^{-\Delta z}-1)a_{j-1}^{n+1}\Big)\Big\}=0.\end{aligned}\tag{4.17}$$

Assume that Δt is small enough so that for all (t,z),

$$\sum_{\kappa<|i|\leq I}\omega_i(t,z)\leq\frac{1}{\Delta t}.\tag{4.18}$$

Taking $\pi = 0$ in the expression to be minimized, we get

$$\frac{a_j^{n+1}-a_j^n}{\Delta t}-(\upsilon(t_n,z_j)+\chi(t_n,z_j))\,a_j^n+\chi(t_n,z_j)a_{j+1}^n+\upsilon(t_n,z_j)a_{j-1}^n+\sum_{\kappa<|l|\leq I}\omega_l(t_n,z_j)(a_{j+l}^{n+1}-a_j^{n+1})\geq 0,$$

or equivalently

$$a_j^n \leq a_j^{n+1}\frac{1-\Delta t\sum_{\kappa<|l|\leq I}\omega_l(t_n,z_j)}{1+\Delta t(\chi+\upsilon)}+\frac{\Delta t\chi a_{j+1}^n}{1+\Delta t(\chi+\upsilon)}+\frac{\Delta t\upsilon a_{j-1}^n}{1+\Delta t(\chi+\upsilon)}+\frac{\Delta t\sum_{\kappa<|l|\leq I}\omega_l(t_n,z_j)a_{j+l}^{n+1}}{1+\Delta t(\chi+\upsilon)}$$

Under (4.18) this implies

$$a_j^n \leq \frac{\|a^{n+1}\|_\infty\vee\|q^a\|_\infty}{1+\Delta t(\chi+\upsilon)}+\frac{\Delta t\chi a_{j+1}^n}{1+\Delta t(\chi+\upsilon)}+\frac{\Delta t\upsilon a_{j-1}^n}{1+\Delta t(\chi+\upsilon)},$$

which means that

$$a_j^n \leq \|a^{n+1}\|_\infty\vee\|q^a\|_\infty,\quad \textit{all } j.$$

This means that the implicit-explicit scheme is L^∞-stable if the boundary condition is non-negative, and the non-negativity of the solution is imposed at each step of the scheme.

From the above argument we see that (4.18) plays the role of the CFL condition for the implicit-explicit scheme. An a priori upper bound for Δt in terms of model parameters is therefore given by the bound on the second term in (4.15). We conclude that the stability of the implicit-explicit scheme is guaranteed by the condition

$$\begin{aligned}\Delta t &\leq \frac{\Delta z^\alpha}{C_1+C_2\Delta z^{\alpha-1}}\quad \textit{with}\\
C_1 &= \frac{2(\kappa+2)^2(\kappa+\frac12)M_g}{(\kappa+1)^2(\alpha-1)}\left(\frac{\kappa+1/2}{1+m_2y_0}\right)^{-\alpha},\\
C_2 &= 2\frac{(\kappa+2)^2}{(\kappa+1)^2}\int_{|y|>y_0}\sup_{t,z}|\gamma(t,z,y)|\nu(dy).\end{aligned}$$

When Δz is sufficiently small, to satisfy this condition, it is enough to take $\Delta t \leq \frac{\Delta z^\alpha}{C_1+\varepsilon}$ for some small value $\varepsilon > 0$. The stability condition for the implicit-explicit scheme is therefore similar to the one for the fully explicit scheme, but with a smaller constant C_1.

Computing the optimal strategy The optimal hedging (pure investment) strategy may be computed as the value of the minimizer in (4.11): with

$$\pi_j^{*n} \equiv \pi^{*n}(z_j) = -\bar{\Pi} \vee \left(-\frac{\mathbb{E}\left[\left(e^{\widehat{Z}_{t_{n+1}}^{t_n,z_j}-z_j}-1\right) a_{n+1}(\widehat{Z}_{t_{n+1}}^{t_n,z_j})\right]}{\mathbb{E}\left[\left(e^{\widehat{Z}_{t_{n+1}}^{t_n,z_j}-z_j}-1\right)^2 a_{n+1}(\widehat{Z}_{t_{n+1}}^{t_n,z_j})\right]} \right) \wedge \bar{\Pi},$$

The error of evaluating the hedging strategy may be greater than that of the evaluating the value function. The following corollary gives an estimate of this error.

Corollary 4.5. *Under the assumptions of Theorem 4.1,*

$$|\pi^{*n}(z_j) - \pi(t_{n-1}, z_j)| \leq \frac{C}{\sqrt{\Delta t}} \left\{ \frac{(1+|z_j|)}{(N-I)\Delta z - \bar{\mu} T} + \Delta t + \Delta z^{3-\alpha}\kappa^{3-\alpha} + \Delta z^{3-\alpha}\kappa^{1-\alpha} \right. \tag{4.19}$$

$$\left. + \int_{|y| \geq \frac{I\Delta z}{\|\gamma_y'\|_\infty}} (1+|y|+\tau(y)+\tau^2(y))\nu(dy) \right\}. \tag{4.20}$$

The proof of this Corollary can be found in Appendix A.

Remark 4.6. *For the error of approximating the optimal strategy to tend to zero, the space discretization step Δz must therefore be sufficiently small compared to Δt. At the same time the CFL condition imposes a lower bound on Δz, which may not tend to zero faster than $\Delta t^{1/\alpha}$. Letting $\Delta z \sim \Delta t^{1/\alpha}$, we get a convergence rate of $\Delta t^{\frac{1}{2} \wedge (\frac{3}{\alpha} - \frac{3}{2})}$ for the optimal strategy. This convergence rate for the strategy is worse than the rate for the value function since the computation of the strategy involves a "derivative-like" operator. However, is in applications one is not so much interested in computing the exact value of the strategy as in obtaining any strategy which gives a hedging error close to the optimum.*

4.3 A finite-difference scheme for the function b

The function $\hat{b}$ solution appearing in the representation (4.9) for the discretized control problem is given by

$$\hat{b}(t_n, z_j) = -2\mathbb{E}\left[\prod_{i=n+1}^{\beta^{t_n,z_j} \wedge N_T} \left(1 + \pi_i^*(e^{\widehat{Z}_{t_i}^{t_n,z_j} - \widehat{Z}_{t_{i-1}}^{t_n,z_j}} - 1)\right) f(\widehat{Z}_{\beta^{t_n,z_j} \wedge N_T}^{t_n,z_j}) \right],$$

where π^* is the optimal strategy for (4.10).

For greater generality we introduce an arbitrary boundary / terminal condition and define the function b^n, approximating the function b, by

$$b^n(z_j) = \mathbb{E}\left[\prod_{i=n+1}^{\beta^{t_n,z_j} \wedge N_T} \left(1 + \pi^{*n}(\widehat{Z}_{t_i}^{t_n,z_j})(e^{\widehat{Z}_{t_i}^{t_n,z_j} - \widehat{Z}_{t_{i-1}}^{t_n,z_j}} - 1)\right) q^b(\Delta t(\beta^{t_n,z_j} \wedge N_T), \widehat{Z}_{\beta^{t_n,z_j} \wedge N_T}^{t_n,z_j}) \right], \tag{4.21}$$

where the function q^b is measurable bounded and satisfies $q^b(T, z) = -2f(z)$.

The finite-difference approximation for the function b is therefore given by the solution of the following linear dynamic programming problem:

$$b^n(z_j) = \mathbb{E}\left[\left(1 + \pi^{*n}(z_j)(e^{\widehat{Z}_{t_{n+1}}^{t_n,z_j} - z_j} - 1)\right) b_{n+1}(\widehat{Z}_{t_{n+1}}^{t_n,z_j})\right]$$

for $j \in (-N, N)$ and $b^n(z_j) = q^b(t_n, z_j)$ for $j \notin (-N, N)$. In other words,

$$b_j^n = \sum_{-I \le l \le I} p_l(t_n, z_j)(1 + \pi_j^{*n}(e^{l\Delta z} - 1))b_{j+l}^{n+1}. \tag{4.22}$$

The terminal condition is given by $b_j^{N_T} = -2f(z_j)$ and in the numerical examples we take $q^b(t, z) = -2f(z_j)$ as well.

Equation (4.22) defines a fully explicit finite difference scheme for computing the function b, which can be rewritten as

$$\begin{aligned}
&\frac{b_j^{n+1} - b_j^n}{\Delta t} - (\upsilon(t_n, z_j) + \chi(t_n, z_j))b_j^{n+1} + \chi(t_n, z_j)b_{j+1}^{n+1} + \upsilon(t_n, z_j)b_{j-1}^{n+1} \\
&+ \sum_{\kappa < |l| \le I} \omega_l(t_n, z_j)(b_{j+l}^{n+1} - b_j^{n+1}) \\
&+ \pi_j^{*n}\left(\sum_{\kappa < |l| \le I} \omega_l(t_n, z_j)(e^{l\Delta z} - 1)b_{j+l}^{n+1} + \chi(t_n, z_j)(e^{\Delta z} - 1)b_{j+1}^{n+1} + \upsilon(t_n, z_j)(e^{-\Delta z} - 1)b_{j-1}^{n+1}\right) = 0.
\end{aligned} \tag{4.23}$$

This scheme uses the same approximations for the operators appearing in (3.6) as the scheme for the function a defined in section 4.2. An implicit-explicit scheme for the function b can be defined along the lines of Remark 4.4.

Stability analysis The numerical scheme for b is L^∞-stable under the condition (4.8). Indeed, under this condition, and using the Cauchy-Schwarz inequality, it follows from the representation (4.21) that for every n,

$$\begin{aligned}
\|b^n\|_\infty &\le \|q^b\|_\infty \mathbb{E}\left[\prod_{i=n+1}^{\beta^{t_n,z_j} \wedge N_T}\left(1 + \pi^{*n}(\widehat{Z}_{t_i}^{t_n,z_j})(e^{\widehat{Z}_{t_i}^{t_n,z_j} - \widehat{Z}_{t_{i-1}}^{t_n,z_j}} - 1)\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{\|q^b\|_\infty}{(\min_{t,z} q^a(t,z))^{\frac{1}{2}}}\mathbb{E}\Big[\prod_{i=n+1}^{\beta^{t_n,z_j} \wedge N_T}\left(1 + \pi^{*n}(\widehat{Z}_{t_i}^{t_n,z_j})(e^{\widehat{Z}_{t_i}^{t_n,z_j} - \widehat{Z}_{t_{i-1}}^{t_n,z_j}} - 1)\right)^2 \\
&\qquad \times q^a(\Delta t(\beta^{t_n,z_j} \wedge N_T), \widehat{Z}_{\beta^{t_n,z_j} \wedge N_T}^{t_n,z_j})\Big]^{\frac{1}{2}} \\
&= \|q^b\|_\infty \left(\frac{a_j^n}{\min_{t,z} q^a(t,z)}\right)^{\frac{1}{2}} \le \|q^b\|_\infty \left(\frac{\|q^a\|_\infty}{\min_{t,z} q^a(t,z)}\right)^{\frac{1}{2}},
\end{aligned}$$

by definition of π^* and a_j^n. With the boundary condition $q^a \equiv 1$ for the function a, one simply has $\|b^n\|_\infty \le \|q^b\|_\infty$.

Accuracy analysis The algorithm for the function b has the same accuracy as the one for the function a.

Theorem 4.7. *Under the assumptions of Theorem 4.1,*

$$\begin{aligned}
|b_n(z_j) - b(t_n, z_j)| \le C\Bigg\{&\frac{(1 + |z_j|)}{(N - I)\Delta z - \bar{\mu}T} + \Delta t + \Delta z^{3-\alpha}\kappa^{3-\alpha} + \Delta z^{3-\alpha}\kappa^{1-\alpha} \\
&+ \int_{|y| \ge \frac{I\Delta z}{\|\gamma_y'\|_\infty}} (1 + |y| + \tau(y) + \tau^2(y))\nu(dy)\Bigg\}.
\end{aligned}$$

The proof of this result is provided in Appendix A.

5 Application to the electricity market

Many studies have shown that price spikes in electricity and gas markets are incompatible with Gaussian dynamics (Geman and Roncoroni, 2006; Meyer-Brandis and Tankov, 2008) and several models based on Lévy processes have been developed to fit the observed fat tails (Deng and Jiang, 2005; Benth et al., 2007). Most of them model the price under the martingale measure, or make some assumption on the change of probability resulting in a similar model under the martingale measure (Benth et al., 2007). However, these martingale models are not adapted for the evaluation of hedging strategies, since the hedging error should be computed under the historical measure.

In this paper, we propose a model which describes the deformation of the forward curve directly under the historical probability, and satisfies Assumptions 2.1 so that we can apply Theorems 3.2–3.3. We start by introducing a Lévy process $\hat{L}$ as follows

$$\hat{L}_s = \zeta s + \int_0^s \int_{\mathbb{R}} y\tilde{J}(ds \times dy) \tag{5.1}$$

where $\zeta \in \mathbb{R}$ and $\tilde{J}$ is a compensated Poisson random measure, whose Lévy measure is denoted by $\nu(dy)$. Fix $c \in \mathbb{R}^+$, $l(s) = e^{-cs}$ and

$$A_t := \int_0^t e^{cs} d\hat{L}_s \tag{5.2}$$

We model the price at time t of the future contract with maturity T and instantaneous delivery as a random perturbation of the initial forward curve ψ which is supposed to be known. Using the above notation we have

$$\bar{F}_{0,T,t} = \psi(0,T)e^{l(T)A_t}$$

By no arbitrage in the futures market, the price at time t of a future contract with duration d is equal to the average over the time period $[T, T+d]$ of the future contract prices with instantaneous delivery. We therefore model the price at time t of a future contract with delivery time T and duration $d > 0$ by

$$F_{d,T,t} = \frac{1}{d}\int_T^{T+d} \bar{F}_{0,s,t} ds = \frac{1}{d}\int_T^{T+d} \psi(0,s)e^{l(s)A_t} ds$$

For reasons which will become clear in the sequel, we prefer the following notation:

$$F_{d,T,t} := \exp(\Phi(A_t)) \quad \text{where} \quad \Phi(A) := \log\left(\frac{1}{d}\int_T^{T+d} \psi(0,s)e^{l(s)A} ds\right) \tag{5.3}$$

The model (5.3) essentially states that the price of a future contract $F_{d,T,t}$ is the average price on the interval $[T, T+d]$ of the future contract with instantaneous delivery up to the random perturbation $e^{l(s)A}$. In this context, the problem of hedging a European option on $F_{d,T,t}$ with the quadratic hedging approach becomes

$$\textbf{minimize} \quad \mathbb{E}\left[\left(H(F_{d,T,t}) - x - \int_t^T \theta_{u-} dF_{d,T,u}\right)^2\right] \quad \text{over } \theta \text{ and } x \in \mathbb{R} \tag{5.4}$$

for a given map H. The process $F_{d,T,t}$ corresponds to S in the formulation (1.1). The following results proves that $Z = \log(F)$ is a Markov jump process satisfying our assumptions.

Lemma 5.1. *The process $Z_t := \log(F_{d,T,t})$ verifies*

$$dZ_t = \mu(t, Z_t)dt + \int \gamma(t, Z_{t-}, y)\tilde{J}(dydt)$$

where

$$\begin{aligned}\gamma(t,z,y) &:= \Phi(\Phi^{-1}(z) + ye^{ct}) - z \\ \mu(t,z) &:= \zeta e^{ct}\Phi'(\Phi^{-1}(z)) + \int_{\mathbb{R}} \left(\gamma(t,z,y) - ye^{ct}\Phi'(\Phi^{-1}(z))\right)\nu(dy)\end{aligned}$$

Assume that the Lévy measure $\nu(dy)$ is given by $\nu(dy) = g(y)|y|^{-(1+\alpha)}dy$, for some $\alpha \in (1,2)$ and a bounded, strictly positive and measurable g such that the following conditions hold true:

i). There exists $m < \infty$ such that for all $y, y' \in (-y_0, 0) \cup (0, y_0)$ with $yy' > 0$, $|g(y) - g(y')| \leq m|y - y'|$

ii). $\lim_{y \to 0^-} g(y) = g(0^-)$ *and* $\lim_{y \to 0^+} g(y) = g(0^+)$ *with* $g(0^+), g(0^-) > 0$

iii). $\int_{y \leq -1} y^4 \nu(dy) + \int_{1<y} e^{4y} \nu(dy) < +\infty$

Then the functions μ and γ verify the Assumptions 2.1-$[\mathbf{C}, \mathbf{L}, \mathbf{I}, \mathbf{ND}, \mathbf{RG_i}, \mathbf{RG_{iii}}]$, where the function τ is given by

$$\tau(y) = \max\left(|y|, |e^y - 1|\right)$$

The proof of this result is given in Appendix B.

In order to apply our results (Theorems 3.2 and 3.3) we also need to verify Assumption 2.1-$[\mathbf{RG_{ii}}]$ and it is easy to see that the function γ does not verify it: however, as we have already said in Section 2, this can be avoided by making a change of variable $L_t = \phi(t, Z_t)$. We refer to Chapter 7 and 8 in De Franco (2012) for further details.

In terms of the process Z, the problem (5.4) becomes

$$v_f(t,z,x) = \inf_{\theta \in \mathcal{X}(t,z,x)} \mathbb{E}\left[\left(f(Z_T^{t,z}) - x - \int_t^T \theta_{u-} d\exp(Z_u^{t,z})\right)^2\right] \tag{5.5}$$

where $\mathcal{X}(t,z,x)$ is defined in (2.10) and $f(z) = H(e^z)$.

We now describe a special class of pay-offs which is of interest in problem (5.5). Let $p(x) := (K - x)^+$ and define

$$h(A) := \frac{1}{d'} \int_T^{T+d'} \psi(0,s) e^{l(s)A} ds$$

for $d' \neq d$. From (5.3) it follows that $h \circ \Phi^{-1}(Z_t) = F_{d',T,t}$, and then, by defining $f := p(h \circ \Phi^{-1})$, we obtain $f(Z_t) = (K - F_{d',T,t})^+$, which is a put option written on a future contract with different duration d'. Using this specific option we can rewrite problem (5.5) as follows

$$\inf_{\theta \in \mathcal{X}(t,z,x)} \mathbb{E}^{t,z,x}\left[\left((K - F_{d',T,t})^+ - x - \int_t^T \theta_{u-} dF_{d,T,u}\right)^2\right]$$

The financial meaning of the above problem is particularly interesting: one tries to hedge (in the quadratic sense) a put option written on a future contract with duration $d' \neq d$ using as hedging instrument the future contract with duration d. This may be useful when, for example, one sells a future contract with a non-standardized duration in the OTC market and hedges the resulting position using instruments which are liquidly traded.

5.1 Numerical example: the CGMY model

In this section we study the problem (5.5) when L in (5.1) is a CGMY process (Carr et al. (2002)) We can write then

$$L_t = \left(\mu + C\Gamma(1-Y)(M^{Y-1} - G^{Y-1})\right)t + \int_0^t \int_{\mathbb{R}} y\tilde{J}(dyds)$$

where $\tilde{J}$ is a compensated Poisson random measure with intensity

$$\nu(dy) = \nu(y)dy, \quad \nu(y) = C\frac{e^{-My}}{y^{1+Y}} 1_{y>0} + C\frac{e^{Gy}}{|y|^{1+Y}} 1_{y<0},$$

The goal of this paragraph is to solve numerically the equations (3.4) and (3.6) when the model for Z is given in Lemma 5.1 and the source of randomness L in (5.1) is given by the CGMY process introduced above. We first apply the implicit-explicit variant (4.17) of the scheme (4.13) for the function a with maturity $T = 7$. The coefficients μ and γ given in Lemma 5.1:

$$\begin{aligned}\mu(t,z) =& \Phi^{'}(\Phi^{-1}(z))\left(\mu + C\Gamma(1-Y)(M^{Y-1} - G^{Y-1})\right)e^{ct} \\ &+ \int_{\mathbb{R}}\left[\Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z - ye^{ct}\Phi^{'}(\Phi^{-1}(z))\right]\nu(dy) \\ \gamma(t,z,y) =& \Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z,\end{aligned}$$

and the function $D(t, z)$ introduced in (4.1), are computed numerically. To this end, we compute the integration points y_i, as in Section 4, such that $\gamma(t, z, y_i(t, z)) = i\Delta z$, or equivalently

$$y_i(t,z) := e^{-ct}\left(\Phi^{-1}(z + i\Delta z) - \Phi^{-1}(z)\right)$$

By expanding γ around zero, we obtain

$$\begin{aligned}D(t,z) :=& \int_{y_{-\kappa-1/2}(t,z)}^{y_{\kappa+1/2}(t,z)}\gamma(t,z,y)^2\nu(dy) \simeq e^{2ct}(\Phi^{'}(\Phi^{-1}(z)))^2\int_{y_{-\kappa-1/2}(t,z)}^{y_{\kappa+1/2}(t,z)}y^2\nu(dy) \\ \simeq& e^{ct}\frac{C}{2-Y}\left((\Phi^{-1}(z+\frac{1}{2}\Delta z) - \Phi^{-1}(z))^{2-Y} + (\Phi^{-1}(z) - \Phi^{-1}(z-\frac{1}{2}\Delta z))^{2-Y}\right)\end{aligned}$$

Using an approach similar to the one of Section 4, we approximate

$$\int_{\mathbb{R}}\left[\Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z - ye^{ct}\Phi^{'}(\Phi^{-1}(z))\right]\nu(dy)$$

with

$$\begin{aligned}&\frac{1}{2}\Delta z(\Phi^{'}(\Phi^{-1}(z))e^{ct})^2\frac{\delta}{\pi} + \sum_{y_i,\,|i|>\kappa}\hat{\omega}_i(t,z)\left[\Phi\left(\Phi^{-1}(z) + y_ie^{ct}\right) - z - y_ie^{ct}\Phi^{'}(\Phi^{-1}(z))\right] \\ =&\frac{1}{2}\Delta z(\Phi^{'}(\Phi^{-1}(z))e^{ct})^2\frac{\delta}{\pi} + \sum_{y_i,\,|i|>\kappa}\hat{\omega}_i(t,z)\left[i\Delta z - \left(\Phi^{-1}(z+i\Delta z) - \Phi^{-1}(z)\right)\Phi^{'}(\Phi^{-1}(z))\right]\end{aligned}$$

where the weights $\hat{\omega}_i(t, z)$ are given in (4.4). The effect of approximating the coefficient of the PIDE on the solution of the original problem, appearing for example in the pricing of European options, has been studied in Jakobsen and Karlsen (2005).

We solve the problem (5.5) for European options f, with maturity one week and delivery for the 7 days of the week. We recall that the future contract in this case is given by

$$F_{7days,1week,t} = \frac{1}{7}\int_7^{14}\psi(0,s)e^{l(s)A_t}ds \tag{5.6}$$

and A_t is given in (5.2) with $\hat{L}$ being the CGMY process defined above. The initial forward curve for the seven days of delivery is given in Table 1, indicating that prices are lower for the week end. This case with continuous long delivery corresponds to a non stationary process where the hedge cannot be calculated efficiently as in Hubalek et al. (2006) or Goutte et al. (2011).

We use the scheme (4.17) and the corresponding implicit-explicit version of the scheme (4.23) to obtain a numerical approximation of the functions a and b. The C, G, M parameters of the CGMY model are $C = 0.01$, $G = 1.1$, and $M = 1.1$. The mean reverting coefficient c is equal to 0.1. In all experiments, the resolution domain is $[-10, 10]$ and the integration domain for the Levy density is $[-2, 2]$ so that $I = N/5$ and we take $\kappa = 0$.

Taking the trend μ equal to zero, table 2 gives for an at-the-money call option and for a number of time steps equal to 800 the calculated values of the a and b with $Y = 1.2$, $Y = 1.9$ and $Y = 1.98$, depending on N (the parameter Y of the CGMY model corresponds to the α of our

Day	s	Price ($\psi(0,s)$)
Monday	$s \in [7,8)$	80
Tuesday	$s \in [8,9)$	90
Wednesday	$s \in [9,10)$	70
Thursday	$s \in [10,11)$	90
Friday	$s \in [11,12)$	80
Saturday	$s \in [12,13)$	70
Sunday	$s \in [13,14]$	60

Table 1: The forward curve. Prices are given in Eur.

Table 2: Space discretization convergence ($N_T = 800$) for $CGMY$ model (the function b is computed for an at-the-money call option)

	$Y = 1.2$					
N	25	50	100	200	400	800
a value	0.8858	0.8639	0.8517	0.8454	0.8423	0.8411
k_a	–	–	0.85	0.94	1.02	1.36
b value	3.5081	4.4930	4.8889	4.9196	4.8990	4.8984

	$Y = 1.9$				
N	25	50	100	200	400
a value	0.8359	0.8270	0.8249	0.8241	0.8240
k_a	–	–	1.967	2.26	2.4
b value	17.598	18.940	19.244	19.311	19.326
k_b	–	–	2.15	2.13	2.2

	$Y = 1.98$				
N	25	50	100	200	400
a value	0.5598	0.5446	0.5407	0.5398	0.5396
k_a	–	–	1.96	2.03	2.09
b value	40.225	41.221	41.471	41.533	41.548
k_b	–	–	1.99	2.02	2.

main assumptions and must belong to the interval $(1,2)$). Expecting to have an error with respect to the time discretized equation in $c\Delta z^{\xi}$, the order of convergence for k_a and k_b are computed as

$$k_u^N = \frac{log((u_{N/2} - u_{N/4})/(u_N - u_{N/2}))}{log(2)}, \tag{5.7}$$

where u_N is the estimation of u with N discretization points. We have checked in all calculations that the CFL condition (4.18) condition is satisfied. We find that for $Y = 1.9$ and $Y = 1.98$ the order of convergence as $N \to \infty$ for the functions a and b is somewhat better than the one predicted by our theoretical result (theorems 4.1 and 4.7). In fact this computed order of convergence clearly goes up to 2 as Y goes to 2. For $Y = 1.2$ the convergence of the function a is very fast but the convergence of the function b exhibits oscillations, which make it difficult to estimate the order of convergence. So the order of convergence is not computed.

Table 3 displays the same convergence results for time discretization as function of N_T, taking $N = 200$.

For all three values of the parameter Y, and for both functions, we find first-order convergence in time. This is perfectly in line with the theoretical result (theorem 4.1 and 4.7).

Practitioners usually price options of this type and calculate the hedging strategy assuming that the underlying process F is a martingale. It is therefore interesting to evaluate the loss of efficiency when using the hedging strategy computed in the martingale model. Assuming that F

Table 3: Time discretization convergence ($N = 200$) for $CGMY$ model (the function b is computed for an at-the-money call option)

	$Y = 1.2$				
N_T	100	200	400	800	1600
a value	0.8646	0.8633	0.8626	0.8623	0.8622
k_a	–	–	1.	0.99	1.
b value	4.7266	4.6985	4.6843	4.6771	4.89388
k_b	–	–	0.99	0.96	1.03

	$Y = 1.9$				
N_T	100	200	400	800	1600
a value	0.82463	0.82437	0.82424	0.82418	0.82415
k_a	–	–	1.	1.	1.
b value	19.2728	19.2951	19.306	19.3114	19.3141
k_b	–	–	1.02	1.01	1.

	$Y = 1.98$				
N_T	100	200	400	800	1600
a value	0.53855	0.53929	0.53965	0.53984	0.53993
k_a	–	–	1.	1.	0.99
b value	41.245	41.288	41.493	41.5334	41.5534
k_b	–	–	1.03	1.01	1.

is a martingale means that we should have

$$F_{d,T,t} := \frac{1}{d}\int_T^{T+d} \psi(0,s)\exp\left(M(s,t) + l(s)A_t\right) ds$$

for some M that makes F a martingale under the historical probability $\mathbb{P}$. Using Lemma 15.1 in Cont and Tankov (2004) we obtain

$$\begin{aligned} M(t,s) = &-\int_0^t \left(\mu + C\Gamma(1-Y)(M^{Y-1} - G^{Y-1})\right) e^{-c(s-r)} dr \\ &+ \int_0^t C\Gamma(-Y)((M - e^{-c(s-t)})^Y - M^Y + (G + e^{-c(s-t)})^Y - G^Y) dr \end{aligned}$$

First remark that when the underlying process F is a martingale, $a \equiv 1$: indeed, from PIDE (3.4), we have

$$\begin{aligned} &0 = -\frac{\partial a}{\partial t} - \mu\frac{\partial a}{\partial z} - \int_{\mathbb{R}} \left(a(t, z+\gamma) - a(t,z) - \gamma\frac{\partial a}{\partial z}(t,z) \right) \nu(dy) - \inf_{|\pi|\le\bar{\Pi}} \left\{ 2\pi\mathcal{Q}a(z) + \pi^2\mathcal{G}a \right\} \\ &a(T,z) = 1 \end{aligned}$$

On the other hand, from Definition 3.1, we have

$$\mathcal{Q}a(z) := \int_{\mathbb{R}} (e^{\gamma} - 1)\left(a(t, z + \gamma(t,z,y)) - a(t,z)\right)\nu(dy)$$

since $\tilde{\mu}$, given in (2.2), is equal to zero (it is the drift of the process F which is now a martingale). From this, it is straightforward to deduce that the function $a = 1$ is the unique solution of PIDE (3.4). So that, when F is a martingale, one only needs to compute the function b.

Taking now μ equal to 0.01, we evaluate the loss of efficiency when using the martingale hedging strategies compared to the quadratic hedging strategies under the true historical measure. Our efficiency comparison criterion is the following: if H is the (put) option and $\theta^{true}, \theta^{mart}$ are,

Y	Option H	Moneyness	Option value	efficiency(θ^{true})	efficiency(θ^{mart}) Reduction	Variance
1.2	Call	1	4.76	4.93	5.22	-5.6 %
1.2	Put	1	4.75	4.93	5.22	-5.6 %
1.2	Call	1.1	1.17	6.33	6.60	-4.2 %
1.2	Put	1.1	8.88	6.33	6.60	-4.2 %
1.2	Call	0.9	10.52	3.58	3.81	-6.4 %
1.2	Put	0.9	2.36	3.61	3.82	-5.8 %
1.98	Call	1	41.54	2.22	3.048	-27.6 %
1.98	Put	1	41.63	2.19	3.03	-27.7 %
1.98	Call	1.1	39.91	2.44	3.335	-26.7 %
1.98	Put	1.1	47.71	2.41	3.32	-27.4 %
1.98	Call	0.9	43.35	1.98	2.737	-27.6 %
1.98	Put	0.9	35.71	1.95	2.72	-28.3 %

Table 4: Pricing and standard deviation of hedged portfolio in the CGMY case ($N = 400$, $N_T = 800$

respectively, the optimal quadratic hedging strategy and the martingale strategy, then the efficiency is measured in terms of the standard deviation of the hedged portfolios:

$$\text{efficiency}(\theta^{true})^2 := \text{Var}\left(H(F_{d,T,t}) - x^{true} - \int_t^T \theta_{r-}^{true} dF_{d,T,r}\right) \tag{5.8}$$

where x^{true} is the true optimal price given in (3.9). Similarly

$$\text{efficiency}(\theta^{mart})^2 := \text{Var}\left(H(F_{d,T,t}) - x^{mart} - \int_t^T \theta_{r-}^{mart} dF_{d,T,r}\right) \tag{5.9}$$

where x^{mart} is the price given in (3.9) when one uses the functions a and b computed in the martingale model., i.e. x^{mart} is the risk neutral price of H. The variances are computed by Monte Carlo over 100000 paths using the rejection method algorithm described in Madan and Yor (2005), with 800 rebalancing dates in each path. The number of Monte Carlo trajectories used is limited due to the cost of the simulation algorithm. The trajectories of $F_{d,T,t}$ are simulated using the true model in both cases. Table 4 summarizes the results of simulations with a number of time steps equal to 800 and a space mesh size equal to 400. The numerical experiment proves that one loses efficiency when using the martingale hedging strategy. This is consistent with the fact that θ^{true} achieves the minimum in problem (5.5) and outperforms the strategy θ^{mart}.

5.2 Numerical example: the Normal Inverse Gaussian process

In this last paragraph we study the problem (5.5) when $\hat{L}$ in (5.1) is a Normal Inverse Gaussian process with parameters α, β, δ, u: $\hat{L}_t \sim NIG(\alpha, \beta, \delta t, ut)$.

Remark 5.2. *The parameter α should not be mistaken for the parameter in Lemma 5.1. We use this notation because it is standard in the literature.*

We can write then

$$\hat{L}_t = \left(u + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}}\right)t + \int_0^t \int_{\mathbb{R}} y\tilde{J}(dyds)$$

where $\tilde{J}$ is a compensated Poisson random measure with intensity

$$\nu(dy) = \nu(y)dy, \quad \nu(y) = \frac{\alpha\delta}{\pi|y|}K_1(\alpha|y|)e^{\beta y},$$

where K_1 is the modified Bessel function of the second kind (Section 4.4.3 in Cont and Tankov (2004)). The Lévy density $\nu(y)$ satisfies

$$\nu(y) \overset{y\to 0}{\sim} \frac{\delta}{\pi |y|^2}, \qquad \nu(y) \overset{y\to +\infty}{\sim} \frac{1}{|y|^{3/2}} e^{-(\alpha-\beta)y}, \qquad \nu(y) \overset{y\to -\infty}{\sim} \frac{1}{|y|^{3/2}} e^{-(\alpha+\beta)|y|}.$$

Remark 5.3. *The NIG is a infinite variation Lévy process with stable-like behavior of small jumps, and since the Blumenthal-Getoor index is equal to* 1*, we cannot formally apply Lemma 5.1 and Theorems 3.2–3.3. It is nevertheless a case of interest because the NIG model is popular among practitioners, and we shall see in the sequel that our numerical schemes yield acceptable results for this model.*

We want to solve numerically the equations (3.4) and (3.6) where the model for Z is given in Lemma 5.1 and the source of randomness $\hat{L}$ in (5.1) is given by the NIG process introduced above. We apply the scheme (4.17) for the function a with maturity $T = 7$. Once again, the coefficients

$$\begin{aligned} \mu(t,z) =& \Phi^{'}(\Phi^{-1}(z)) \left(u + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \right) e^{ct} + \int_{\mathbb{R}} \left[\Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z - ye^{ct}\Phi^{'}(\Phi^{-1}(z)) \right] \nu(dy) \\ \gamma(t,z,y) =& \Phi\left(\Phi^{-1}(z) + ye^{ct}\right) - z \end{aligned}$$

and the function $D(t,z)$ introduced in (4.1) are computed by numerical integration over the points y_i such that $\gamma(t,z,y_i(t,z)) = i\Delta z$, or equivalently

$$y_i(t,z) := e^{-ct}\left(\Phi^{-1}(z+i\Delta z) - \Phi^{-1}(z)\right)$$

By expanding γ around zero we obtain

$$\begin{aligned} D(t,z) :=& \int_{y_{-\kappa-1/2}(t,z)}^{y_{\kappa+1/2}(t,z)} \gamma(t,z,y)^2 \nu(dy) \simeq e^{2ct}(\Phi^{'}(\Phi^{-1}(z)))^2 \int_{y_{-\kappa-1/2}(t,z)}^{y_{\kappa+1/2}(t,z)} y^2\nu(dy) \\ \simeq& e^{ct}\left(\Phi^{-1}(z+(\kappa+\frac{1}{2})\Delta z) - \Phi^{-1}(z-(\kappa+\frac{1}{2})\Delta z) \right) (\Phi^{'}(\Phi^{-1}(z)))^2 \frac{\delta}{\pi} \end{aligned}$$

since, around zero, we have $y^2\nu(dy) \simeq \frac{\delta}{\pi} + \frac{\delta\beta}{\pi}y + O(y^2)$. (See for example Raible (2000)). We proceed as for the CGMY case, even though we do not have a priori results on the existence of a smooth solution.

We consider once again the problem (5.5) for European options f, with maturity one week and delivery for the 7 days of the week as in (5.6). The parameters of the NIG process are $u = 0.08$, $\alpha = 6.23$, $\beta = 0.06$, $\delta = 0.1027$. The mean reverting coefficient c is taken equal to 0.19. In all experiments, the resolution domain is $[-10, 10]$ and the integration domain for the Levy density is $[-2, 2]$ so that $I = N/5$ and $\kappa = 0$. Once again, we have checked in all calculations that the CFL condition (4.18) condition is satisfied.

Table 5 displays the values of a and b for an at-the-money call option as function of the space mesh size N for the number of time steps $N_T = 800$ and as function of the number of time steps for the space mesh size $N = 800$. Orders of convergence k_a for a and k_b for b are computed with equation (5.7).

In this case, there are no theoretical results to which the simulations may be compared. Numerically we do observe convergence in time and in space, but it seems that the convergence as $N \to \infty$ is slower than for the CGMY model, in particular, for the function a the space discretization error seems to be much higher than the time discretization error.

As in the CGMY case, we estimate the loss in the efficiency of the hedge when using a martingale model in terms of the standard deviation of the hedged portfolios as in (5.8)–(5.9). For F to be a martingale, we should have

$$F_{d,T,t} := \frac{1}{d}\int_T^{T+d} \psi(0,s)\exp\left(M(s,t) + l(s)A_t\right) ds$$

where

$$M(t,s) = -\int_0^t \left(\left(u + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \right) e^{-c(s-r)} + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2 - (\beta + e^{-c(s-r)})^2} \right) \right) dr$$

Table 5: Space and time discretization convergence for NIG model (The value b is computed for an at-the-money call option)

	Space					
N	100	200	400	800	1600	3200
a value	0.398	0.266	0.182	0.136	0.113	0.1012
k_a	–	–	0.65	0.87	1.00	0.96
b value	4.716	4.608	4.446	4.324	4.247	4.204
k_b	–	–	0.56	0.40	0.66	0.84

	Time					
N_T	100	200	400	800	1600	3200
a value	0.136290	0.136438	0.1365111	0.136548	0.136566	0.136575
k_a value	-	-	1.02	0.98	1.04	1.
b value	4.475	4.388	4.345	4.324	4.313	4.308
k_b	–	–	1.02	1.03	0.93	1.13

Option H	Moneyness	Option value	efficiency(θ^{true})	efficiency(θ^{mart})	Variance Reduction
Call	1	4.247	1.084	1.343	-19,28%
Put	1	4.230	1.084	1.343	-19,28%
Call	1.5	0.120	0.142	0.171	-16,9 %
Put	1.5	38.809	0.145	0.171	-15,2 %

Table 6: Pricing and standard deviation of hedged portfolio in the NIG ($N = 1600$, $N_T = 800$) case

We already know that in this case we only need to compute the function b ($a = 1$ in the martingale case). Table 6 summarizes the results of simulations, for $t = 0$. The numerical experiment proves that using the martingale hedging strategy is inaccurate. The loss of efficiency is of order -20%.

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A Convergence analysis

Proof of Theorem 4.1. Under the additional assumptions of this theorem,

$$D(t,z) = \int_{\Omega_0} \gamma^2(t,z,y)\nu(dy) \geq m_1 \int_{\Omega_0} y^2 \nu(dy) \geq m_1 \int_{|y| \leq \frac{(\kappa+1/2)\Delta z}{\|\gamma_y\|_\infty}} y^2 \nu(dy) \sim C(\kappa \Delta z)^{2-\alpha}, \quad \Delta z \to 0.$$

where, throughout this appendix, C denotes a finite constant, which does not depend on any truncation / discretization parameters and whose exact definition may change from line to line. On the other hand,

$$|\hat{\mu}(t,z)| \leq \bar{\mu} + \Delta z \sum_{k<|i|\leq I} |i| \omega_i(t,z).$$

Using the mean value theorem and our assumptions on γ, we can show that

$$|\hat{\mu}(t,z)| \leq \bar{\mu} + C\int_{\Omega_1\cup\Omega_2} |y|\nu(dy) \leq \bar{\mu} + C|\kappa\Delta z|^{1-\alpha}.$$

Therefore, one may choose a constant c which does not depend on Δz (for Δz small enough), such that for $\kappa > c$, the weights υ and χ are both positive and given by (4.5)–(4.6). Throughout this proof we shall assume that such a choice of κ has been made.

Let

$$\hat{\beta}^{z,t_n} = \inf\{i \geq n : \widehat{Z}^{z,t_n}_{t_i} + I\Delta z \geq N\Delta z \text{ or } \widehat{Z}^{z,t_n}_{t_i} - I\Delta z \leq -N\Delta z\}. \tag{A.1}$$

We have,

$$\begin{aligned}
a^n(z_j) &= \inf_{\pi_i\in[-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\Bigg[\prod_{i=n+1}^{\hat{\beta}^{t_n,z_j}\wedge N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^2 \\
&\times \prod_{i=\hat{\beta}^{t_n,z_j}\wedge N_T+1}^{\beta^{t_n,z_j}\wedge N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^2 q^a(t_{\beta^{t_n,z_j}\wedge N_T}, \widehat{Z}^{t_n,z_j}_{t_{\beta^{t_n,z_j}\wedge N_T}})\Bigg] \\
&= \inf_{\pi_i\in[-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\Bigg[\mathbf{1}_{\hat{\beta}^{t_n,z_j}<N_T} \prod_{i=n+1}^{\beta^{t_n,z_j}\wedge N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^2 q^a(\beta^{t_n,z_j}\wedge N_T, \widehat{Z}^{t_n,z_j}_{\beta^{t_n,z_j}\wedge N_T}) \\
&+ \mathbf{1}_{\hat{\beta}^{t_n,z_j}\geq N_T} \prod_{i=n+1}^{N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^2 a(t_{N_T}, \widehat{Z}^{t_n,z_j}_{t_{N_T}})\Bigg]
\end{aligned}$$

Let

$$\hat{a}^n(z_j) = \inf_{\pi_i\in[-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\left[\mathbf{1}_{\hat{\beta}^{t_n,z_j}>N_T-1} \prod_{i=n+1}^{N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^2 a(t_{N_T}, \widehat{Z}^{t_n,z_j}_{t_{N_T}})\right].$$

Clearly, $a_n(z_j) \geq \hat{a}_n(z_j)$. On the other hand, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
a_n(z_j) - \hat{a}_n(z_j) &\leq \sup_{\pi_i\in[-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\left[\mathbf{1}_{\hat{\beta}^{t_n,z_j}<N_T} \prod_{i=n+1}^{N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^2\right] \|q^a\|_\infty. \\
&\leq \|q^a\|_\infty \mathbb{P}[\hat{\beta}^{t_n,z_j} < N_T]^{\frac{1}{2}} \sup_{\pi_i\in[-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\left[\prod_{i=n+1}^{N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^4\right]^{\frac{1}{2}}
\end{aligned}$$

The second factor can be bounded using the fact, that by Lemma A.1, for every $\pi \in [-\bar{\Pi},\bar{\Pi}]]$

$$\begin{aligned}
&\mathbb{E}[(1+\pi(e^{\widehat{Z}^{z,t_n}_{t_{n+1}}-z}-1))^4] = 1 + \Delta t\Big[4\pi\Big\{\mu(t_n,z) + \int_{\mathbb{R}} (e^{\gamma(t_n,z,y)} - 1 - \gamma(t_n,z,y))\nu(dz)\Big\} \\
&+ 6\pi^2\int_{\mathbb{R}} (e^{\gamma(t_n,z,y)}-1)^2\nu(dz) + 4\pi^3\int_{\mathbb{R}} (e^{\gamma(t_n,z,y)}-1)^3\nu(dz) + \pi^4\int_{\mathbb{R}} (e^{\gamma(t_n,z,y)}-1)^4\nu(dz) + \mathcal{E}\Big],
\end{aligned}$$

where $\mathcal{E}$ is the error term. By our assumptions, the terms in square brackets are bounded, and therefore, by applying iterated conditional expectations,

$$\sup_{\pi_i\in[-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\left[\prod_{i=n+1}^{N_T} \Big(1+\pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\Big)^4\right] \leq (1+C\Delta t)^{N_T-n} \leq e^{C\Delta T N_T} = e^{CT}. \tag{A.2}$$

Together with the estimate of Lemma A.2 for the first factor, this yields

$$|a_n(z_j) - \hat{a}_n(z_j)| \leq \frac{C(1+|z_j|)}{(N-I)\Delta z - \bar{\mu}T}, \quad N\Delta z > I\Delta z + \bar{\mu}T. \tag{A.3}$$

The remaining error is decomposed as follows:

$$\hat{a}^n(z_j) - a(t_n, z_j)$$
$$= \sum_{k=n+1}^{N_T} \Bigg\{ \inf_{\pi_i \in [-\bar{\Pi},\bar{\Pi}], i=n+1,\dots,k} \mathbb{E}\left[\mathbf{1}_{\hat{\beta}^{t_n,z_j} > k-1} \prod_{i=n+1}^{k} \left(1 + \pi_i (e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2 a(t_k, \widehat{Z}^{t_n,z_j}_{t_k}) \right]$$
$$- \inf_{\pi_i \in [-\bar{\Pi},\bar{\Pi}], i=n+1,\dots,k-1} \mathbb{E}\left[\mathbf{1}_{\hat{\beta}^{t_n,z_j} > k-1} \prod_{i=n+1}^{k-1} \left(1 + \pi_i (e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2 a(t_{k-1}, \widehat{Z}^{t_n,z_j}_{t_{k-1}}) \right] \Bigg\} \tag{A.4}$$

The first term inside the brackets satisfies

$$\inf_{\pi_i \in [-\bar{\Pi},\bar{\Pi}], i=n+1,\dots,k} \mathbb{E}\left[\mathbf{1}_{\hat{\beta}^{t_n,z_j} > k-1} \prod_{i=n+1}^{k} \left(1 + \pi_i (e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2 a(t_k, \widehat{Z}^{t_n,z_j}_{t_k}) \right]$$
$$= \inf_{\pi_i \in [-\bar{\Pi},\bar{\Pi}], i=n+1,\dots,k-1} \mathbb{E}\Bigg[\mathbf{1}_{\hat{\beta}^{t_n,z_j} > k-1} \prod_{i=n+1}^{k-1} \left(1 + \pi_i (e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2$$
$$\times \inf_{\pi_k \in [-\bar{\Pi},\bar{\Pi}]} \mathbb{E}\left[\left(1 + \pi_k (e^{\widehat{Z}^{t_n,z_j}_{t_k} - \widehat{Z}^{t_n,z_j}_{t_{k-1}}} - 1)\right)^2 a(t_k, \widehat{Z}^{t_n,z_j}_{t_k}) \Big| \widehat{Z}^{t_n,z_j}_{t_{k-1}} \right] \Bigg]$$

By Lemma A.1, for every $\pi \in [-\bar{\Pi}, \bar{\Pi}]$,

$$\mathbb{E}\left[\left(1 + \pi (e^{\widehat{Z}^{t_n,z_j}_{t_k} - \widehat{Z}^{t_n,z_j}_{t_{k-1}}} - 1)\right)^2 a(t_k, \widehat{Z}^{t_n,z_j}_{t_k}) \Big| \widehat{Z}^{t_n,z_j}_{t_{k-1}} = z \right]$$
$$= \mathbb{E}\left[\left(1 + \pi (e^{\widehat{Z}^{t_n,z_j}_{t_k} - \widehat{Z}^{t_n,z_j}_{t_{k-1}}} - 1)\right)^2 a(t_{k-1}, \widehat{Z}^{t_n,z_j}_{t_k}) \Big| \widehat{Z}^{t_n,z_j}_{t_{k-1}} = z \right]$$
$$+ \int_{t_{k-1}}^{t_k} \mathbb{E}\left[\left(1 + \pi (e^{\widehat{Z}^{t_n,z_j}_{t_k} - \widehat{Z}^{t_n,z_j}_{t_{k-1}}} - 1)\right)^2 \frac{\partial a}{\partial t}(t, \widehat{Z}^{t_n,z_j}_{t_k}) \Big| \widehat{Z}^{t_n,z_j}_{t_{k-1}} = z \right] dt$$
$$= a(t_{k-1}, z) + \Delta t \mathcal{L}_\pi a(t_{k-1}, z) + \Delta t \int_{t_{k-1}}^{t_k} \mathcal{L}_\pi \frac{\partial a}{\partial t}(t, z) dt + a(t_k, z) - a(t_{k-1}, z) + \Delta t \tilde{\mathcal{E}}_k,$$

where

$$|\tilde{\mathcal{E}}_k| \le C \Delta z^{3-\alpha} (\kappa^{3-\alpha} + \kappa^{1-\alpha}) + C \int_{|y| \ge \frac{I \Delta z}{\|\gamma'_y\|_\infty}} (1 + |y| + \tau(y) + \tau^2(y)) \nu(dy),$$

and the operator $\mathcal{L}_\pi$ is defined by

$$\mathcal{L}_\pi f(t,z) = \mu(t,z) f'(z) + \int_{\mathbb{R}} (f(z + \gamma(t,z,y)) - f(z) - f'(z)\gamma(t,z,y)) \nu(dy)$$
$$+ 2\pi \left\{ \mu(t,z) f(z) + \int_{\mathbb{R}} ((e^{\gamma(t,z,y)} - 1) f(z + \gamma(t,z,y)) - f(z)\gamma(t,z,y)) \nu(dy) \right\}$$
$$+ \pi^2 \int_{\mathbb{R}} (e^{\gamma(t,z,y)} - 1)^2 f(z + \gamma(t,z,y)) \nu(dy).$$

From the regularity of a and the integrability conditions on the Lévy measure, we deduce that $\mathcal{L}_\pi \frac{\partial a}{\partial t}(t,z)$ is uniformly bounded on $\pi \in [-\bar{\Pi}, \bar{\Pi}]$, which means that

$$\mathbb{E}\left[\left(1 + \pi (e^{\widehat{Z}^{t_n,z_j}_{t_k} - \widehat{Z}^{t_n,z_j}_{t_{k-1}}} - 1)\right)^2 a(t_k, \widehat{Z}^{t_n,z_j}_{t_k}) \Big| \widehat{Z}^{t_n,z_j}_{t_{k-1}} = z \right]$$
$$= a(t_{k-1}, z) + \Delta t \left\{ \mathcal{L}_\pi a(t_{k-1}, z) + \frac{\partial a}{\partial t}(t_{k-1}, z) \right\} + \Delta t \mathcal{E}_k$$

with

$$|\mathcal{E}_k| \leq C\overline{\mathcal{E}}, \quad \overline{\mathcal{E}} := \Delta t + \Delta z^{3-\alpha}(\kappa^{3-\alpha} + \kappa^{1-\alpha}) + \int_{|y| \geq \frac{I\Delta z}{\|\gamma'_y\|_\infty}} (1 + |y| + \tau(y) + \tau^2(y))\nu(dy), \quad \text{(A.5)}$$

Using the equation (3.4) satisfied by a, we finally get

$$\left| \inf_{\pi \in [-\bar{\Pi}, \bar{\Pi}]} \mathbb{E}\left[\left(1 + \pi(e^{\widehat{Z}^{t_n,z_j}_{t_k} - \widehat{Z}^{t_n,z_j}_{t_{k-1}}} - 1)\right)^2 a(t_k, \widehat{Z}^{t_n,z_j}_{t_k}) \middle| \widehat{Z}^{t_n,z_j}_{t_{k-1}} = z \right] - a(t_{k-1}, z) \right| \leq C\Delta t \overline{\mathcal{E}}.$$

Plugging this estimate back into (A.4) yields

$$\begin{aligned}
&\hat{a}^n(z_j) - a(t_n, z_j) \\
&\leq \sum_{k=n+1}^{N_T} \Bigg\{ \inf_{\pi_i \in [-\bar{\Pi}, \bar{\Pi}], i=n+1,\dots,k-1} \mathbb{E}\left[\mathbf{1}_{\hat{\beta}^{t_n,z_j} > k-1} \prod_{i=n+1}^{k-1} \left(1 + \pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2 (a(t_{k-1}, \widehat{Z}^{t_n,z_j}_{t_{k-1}}) + C\Delta t \overline{\mathcal{E}}) \right] \\
&- \inf_{\pi_i \in [-\bar{\Pi}, \bar{\Pi}], i=n+1,\dots,k-1} \mathbb{E}\left[\mathbf{1}_{\hat{\beta}^{t_n,z_j} > k-1} \prod_{i=n+1}^{k-1} \left(1 + \pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2 a(t_{k-1}, \widehat{Z}^{t_n,z_j}_{t_{k-1}}) \right] \Bigg\},
\end{aligned}$$

which implies:

$$\hat{a}^n(z_j) - a(t_n, z_j) \leq C\Delta t \overline{\mathcal{E}} \sum_{k=n+1}^{N_T} \sup_{\pi_i \in [-\bar{\Pi}, \bar{\Pi}], i=n+1,\dots,k-1} \mathbb{E}\left[\prod_{i=n+1}^{k-1} \left(1 + \pi_i(e^{\widehat{Z}^{t_n,z_j}_{t_i} - \widehat{Z}^{t_n,z_j}_{t_{i-1}}} - 1)\right)^2 \right]$$

The expectation can be estimated as in (A.2), and we finally get

$$\hat{a}^n(z_j) - a(t_n, z_j) \leq C\overline{\mathcal{E}}.$$

The upper bound can be obtained in a similar manner. □

Proof of Corollary 4.5. We first estimate the error of approximating the numerator and the denominator in (4.20). For the numerator, we get:

$$\begin{aligned}
&\left| \mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}} - z_j} - 1\right) a_{n+1}(Z^{t_n,z_j}_{t_{n+1}}) \right] - \Delta t \mathcal{Q}_t a(t_n, z_j) \right| \\
&\leq \left| \mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}} - z_j} - 1\right) (a_{n+1}(Z^{t_n,z_j}_{t_{n+1}}) - a(t_{n+1}, Z^{t_n,z_j}_{t_{n+1}})) \right] \right| \\
&+ \left| \mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}} - z_j} - 1\right) (a(t_{n+1}, Z^{t_n,z_j}_{t_{n+1}}) - a(t_n, Z^{t_n,z_j}_{t_{n+1}})) \right] \right| \\
&+ \left| \mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}} - z_j} - 1\right) a(t_n, Z^{t_n,z_j}_{t_{n+1}}) \right] - \Delta t \mathcal{Q}_t a(t_n, z_j) \right|
\end{aligned}$$

Using Equation (A.9), we can show that the second term is bounded by $C\Delta t^2$ and the third term is bounded by $C\Delta t \overline{\mathcal{E}}$, with $\overline{\mathcal{E}}$ defined in (A.5). The first term makes the main contribution to the error, which can be estimated using Theorem 4.1 and the Cauchy-Schwarz inequality:

$$\left| \mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}} - z_j} - 1\right) (a_{n+1}(Z^{t_n,z_j}_{t_{n+1}}) - a(t_{n+1}, Z^{t_n,z_j}_{t_{n+1}})) \right] \right| \leq C\overline{\mathcal{E}} \mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}} - z_j} - 1\right)^2 \right]^{\frac{1}{2}} \leq C\sqrt{\Delta t} \overline{\mathcal{E}}$$

Similarly, for the denominator we have the estimate

$$\left| \mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}} - z_j} - 1\right)^2 a_{n+1}(Z^{t_n,z_j}_{t_{n+1}}) \right] - \Delta t \mathcal{G}_t a(t_n, z_j) \right| \leq C\Delta t \overline{\mathcal{E}}.$$

Using the fact that for $b, b' > 0$ and all a, a',

$$\left| \frac{a'}{b'} - \frac{a}{b} \right| \leq \frac{|a' - a|}{b} + \frac{|a'||b' - b|}{bb'},$$

we obtain the estimate

$$\left|\frac{\mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}}-z_j}-1\right)a_{n+1}(Z^{t_n,z_j}_{t_{n+1}})\right]}{\mathbb{E}\left[\left(e^{Z^{t_n,z_j}_{t_{n+1}}-z_j}-1\right)^2 a_{n+1}(Z^{t_n,z_j}_{t_{n+1}})\right]}-\frac{\mathcal{Q}_t a(t_n,z_j)}{\Delta t\mathcal{G}_t a(t_n,z_j)}\right|$$
$$\leq \frac{C\sqrt{\Delta T}\overline{\mathcal{E}}}{\Delta t\mathcal{G}_t a(t_n,z_j)}+\frac{C\Delta t^{\frac{3}{2}}\overline{\mathcal{E}}}{\Delta t^2\mathcal{G}_t a(t_n,z_j)(\mathcal{G}_t a(t_n,z_j)-C\overline{\mathcal{E}})}\leq C\Delta t^{-\frac{1}{2}}\overline{\mathcal{E}}$$

since $\mathcal{G}_t a$ is bounded from below (this follows from Theorem 2.3). We conclude by observing that projecting both the optimal strategy and its approximation on the interval $[-\bar{\Pi},\bar{\Pi}]$ does not increase the error. □

Proof of Theorem 4.7. As in the proof of Theorem 4.1, we may and will assume that υ and χ are positive and given by (4.5)–(4.6). To deal with the domain truncation, we also, similarly to the proof of Theorem 4.1, take $\hat{\beta}^{z,t_n}$ as in (A.1), and use the representation

$$b^n(z_j)=\mathbb{E}\Bigg[\mathbf{1}_{\hat{\beta}^{t_n,z_j}<N_T}\prod_{i=n+1}^{\beta^{t_n,z_j}\wedge N_T}\left(1+\pi^{*i}(\widehat{Z}^{t_n,z_j}_{t_i})(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\right)q^b(t_{\beta^{t_n,z_j}\wedge N_T},\widehat{Z}^{t_n,z_j}_{t_{\beta^{t_n,z_j}\wedge N_T}})$$
$$+\mathbf{1}_{\hat{\beta}^{t_n,z_j}\geq N_T}\prod_{i=n+1}^{N_T}\left(1+\pi^{*i}(\widehat{Z}^{t_n,z_j}_{t_i})(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\right)f(\widehat{Z}^{t_n,z_j}_{t_{N_T}})\Bigg]$$

Letting

$$\hat{b}^n(z_j)=\mathbf{1}_{\hat{\beta}^{t_n,z_j}>N_T-1}\prod_{i=n+1}^{N_T}\left(1+\pi^{*i}(\widehat{Z}^{t_n,z_j}_{t_i})(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\right)f(\widehat{Z}^{t_n,z_j}_{t_{N_T}})\Bigg],$$

the remainder can be estimated as follows:

$$|b_n(z_j)-\hat{b}_n(z_j)|\leq\|q^b\|_\infty\mathbb{P}[\hat{\beta}^{t_n,z_j}<N_T]^{\frac{1}{2}}\mathbb{E}\left[\prod_{i=n+1}^{N_T}\left(1+\pi^{*i}(\widehat{Z}^{t_n,z_j}_{t_i})(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\right)^2\right]^{\frac{1}{2}}$$
$$\leq\|q^b\|_\infty\mathbb{P}[\hat{\beta}^{t_n,z_j}<N_T]^{\frac{1}{2}},$$

where the last inequality uses the fact that π^* is the minimizer of the expectation in the first line, and that the expectation equals one when substituting the value $\pi^*=0$. Therefore, $|b_n(z_j)-\hat{b}_n(z_j)|$ admits the same bound as $|a_n(z_j)-\hat{a}_n(z_j)|$, given by (A.3).

The remaining error is decomposed as follows:

$$\hat{b}^n(z_j)-b(t_n,z_j)=\sum_{k=n+1}^{N_T}\Bigg\{\mathbb{E}\Bigg[\mathbf{1}_{\hat{\beta}^{t_n,z_j}>k-1}\prod_{i=n+1}^{k-1}\left(1+\pi^{*i}(\widehat{Z}^{t_n,z_j}_{t_i})(e^{\widehat{Z}^{t_n,z_j}_{t_i}-\widehat{Z}^{t_n,z_j}_{t_{i-1}}}-1)\right)$$
$$\times\mathbb{E}\left[\left(1+\pi^{*k}(\widehat{Z}^{t_n,z_j}_{t_k})(e^{\widehat{Z}^{t_n,z_j}_{t_k}-\widehat{Z}^{t_n,z_j}_{t_{k-1}}}-1)\right)b(t_k,\widehat{Z}^{t_n,z_j}_{t_k})-b(t_{k-1},\widehat{Z}^{t_n,z_j}_{t_{k-1}})\middle|\widehat{Z}^{t_n,z_j}_{t_{k-1}}\right]\Bigg]\Bigg\}\quad\text{(A.6)}$$

For any function $f(z)$, it is easy to see that

$$
\begin{aligned}
&\mathbb{E}\left[\left(1+\pi^{*k}(\widehat{Z}_{t_k}^{t_n,z_j})(e^{\widehat{Z}_{t_k}^{t_n,z_j}-\widehat{Z}_{t_{k-1}}^{t_n,z_j}}-1)\right)f(\widehat{Z}_{t_k}^{t_n,z_j})\Big|\widehat{Z}_{t_{k-1}}^{t_n,z_j}\right]\\
&=\frac{1}{2}\inf_{\pi}\mathbb{E}\Big[\left(1+\pi(e^{\widehat{Z}_{t_k}^{t_n,z_j}-\widehat{Z}_{t_{k-1}}^{t_n,z_j}}-1)\right)^2 a(t_k,\widehat{Z}_{t_k}^{t_n,z_j})\\
&\qquad\qquad+\left(1+\pi(e^{\widehat{Z}_{t_k}^{t_n,z_j}-\widehat{Z}_{t_{k-1}}^{t_n,z_j}}-1)\right)f(\widehat{Z}_{t_k}^{t_n,z_j})\Big|\widehat{Z}_{t_{k-1}}^{t_n,z_j}=z\Big]\\
&-\frac{1}{2}\inf_{\pi}\mathbb{E}\Big[\left(1+\pi(e^{\widehat{Z}_{t_k}^{t_n,z_j}-\widehat{Z}_{t_{k-1}}^{t_n,z_j}}-1)\right)^2 a(t_k,\widehat{Z}_{t_k}^{t_n,z_j})\\
&\qquad\qquad-\left(1+\pi(e^{\widehat{Z}_{t_k}^{t_n,z_j}-\widehat{Z}_{t_{k-1}}^{t_n,z_j}}-1)\right)f(\widehat{Z}_{t_k}^{t_n,z_j})\Big|\widehat{Z}_{t_{k-1}}^{t_n,z_j}=z\Big]
\end{aligned}\tag{A.7}
$$

Using this and Lemma A.1, the last factor in (A.6) be alternatively written as follows:

$$
\begin{aligned}
&\mathbb{E}\left[\left(1+\pi^{*k}(z)(e^{\widehat{Z}_{t_k}^{t_n,z_j}-\widehat{Z}_{t_{k-1}}^{t_n,z_j}}-1)\right)b(t_k,\widehat{Z}_{t_k}^{t_n,z_j})-b(t_{k-1},\widehat{Z}_{t_{k-1}}^{t_n,z_j})\Big|\widehat{Z}_{t_{k-1}}^{t_n,z_j}=z\right]\\
&=\frac{\Delta t}{2}\inf_{\pi}\mathbb{E}[\mathcal{L}_\pi a(t_{k-1,z})+\pi\mathcal{Q}_t b(t_{k-1},z)]-\frac{\Delta t}{2}\inf_{\pi}\mathbb{E}[\mathcal{L}_\pi a(t_{k-1,z})-\pi\mathcal{Q}_t b(t_{k-1},z)]\\
&+\Delta t\left\{(\mathcal{B}_t-\mathcal{A}_t)b(t_{k-1},z)+\frac{\partial b}{\partial t}(t_{k-1},z)\right\}+\Delta t\mathcal{E}_k
\end{aligned}
$$

with $\mathcal{E}_k$ satisfying (A.5), where $\mathcal{L}_\pi$ is defined in the proof of Theorem 4.1. Evaluating explicitly the inf over π, and using equation (3.6), this leads to

$$
\mathbb{E}\left[\left(1+\pi^{*k}(z)(e^{\widehat{Z}_{t_k}^{t_n,z_j}-\widehat{Z}_{t_{k-1}}^{t_n,z_j}}-1)\right)b(t_k,\widehat{Z}_{t_k}^{t_n,z_j})-b(t_{k-1},\widehat{Z}_{t_{k-1}}^{t_n,z_j})\Big|\widehat{Z}_{t_{k-1}}^{t_n,z_j}=z\right]=\Delta t\mathcal{E}_k.
$$

We conclude as in the proof of Theorem 4.1. □

Auxiliary lemmas

Lemma A.1. *Let f be 4 times continuously differentiable with bounded derivatives. Then,*

$$
\begin{aligned}
&\left|\frac{\mathbb{E}[f(\widehat{Z}_{t_{n+1}}^{z,t_n})]-f(z)}{\Delta t}-\mu(t_n,z)f'(z)-\int_{\mathbb{R}}(f(z+\gamma(t_n,z,y))-f(z)-f'(z)\gamma(t_n,z,y))\nu(dy)\right|\\
&\leq C\Delta z^{3-\alpha}(k^{3-\alpha}+k^{1-\alpha})+C\int_{|y|\geq\frac{I\Delta z}{\|\gamma_y\|_\infty}}(1+|y|)\nu(dy)
\end{aligned}\tag{A.8}
$$

$$
\begin{aligned}
&\left|\frac{\mathbb{E}[(e^{\widehat{Z}_{t_{n+1}}^{z,t_n}-z}-1)f(\widehat{Z}_{t_{n+1}}^{z,t_n})]}{\Delta t}-\mu(t_n,z)f(z)-\int_{\mathbb{R}}((e^{\gamma(t_n,z,y)}-1)f(z+\gamma(t_n,z,y))-f(z)\gamma(t_n,z,y))\nu(dy)\right|\\
&\leq C\Delta z^{3-\alpha}(k^{3-\alpha}+k^{1-\alpha})+C\int_{|y|\geq\frac{I\Delta z}{\|\gamma_y\|_\infty}}\tau(y)\nu(dy)
\end{aligned}\tag{A.9}
$$

$$
\begin{aligned}
&\left|\frac{\mathbb{E}[(e^{\widehat{Z}_{t_{n+1}}^{z,t_n}-z}-1)^p f(\widehat{Z}_{t_{n+1}}^{z,t_n})]}{\Delta t}-\int_{\mathbb{R}}(e^{\gamma(t_n,z,y)}-1)^p f(z+\gamma(t_n,z,y))\nu(dy)\right|\\
&\leq C\Delta z^{3-\alpha}(k^{3-\alpha}+k^{1-\alpha})+C\int_{|y|\geq\frac{I\Delta z}{\|\gamma_y\|_\infty}}\tau^p(y)\nu(dy),\quad p=2,3,4,
\end{aligned}\tag{A.10}
$$

where τ is defined in Assumption 2.1-I.

Proof. We begin with (A.8). By definition of $\widehat{Z}$, the expression inside the absolute value on the left-hand side may be decomposed into the following terms:

$$\frac{f(z+\Delta z)+f(z-\Delta z)-2f(z)-f''(z)\Delta z^2}{2\Delta z^2}D(t_n,z) \tag{A.11}$$

$$-\int_{\Omega_0(t_n,z)}\left\{f(z+\gamma(t_n,z,y))-f(z)-f'(z)\gamma(t_n,z,y)-\frac{1}{2}f''(z)\gamma^2(t_n,z,y)\right\}\nu(dy) \tag{A.12}$$

$$+\frac{f(z+\Delta z)-f(z-\Delta z)-2f'(z)\Delta z}{2\Delta z}\hat{\mu}(t_n,z) \tag{A.13}$$

$$-\int_{\Omega_3(t_n,z)}(f(z+\gamma(t_n,z,y))-f(z)-\gamma(t_n,z,y)f'(z))\nu(dy) \tag{A.14}$$

$$+\sum_{i:y_i(t_n,z)\in\Omega_2}\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}(F(z,\gamma(t_n,z,y_i))-F(z,\gamma(t_n,z,y)))\nu(dy) \tag{A.15}$$

$$+\sum_{i:y_i(t_n,z)\in\Omega_1}\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}(\widetilde{F}(z,\gamma(t_n,z,y_i))-\widetilde{F}(z,\gamma(t_n,z,y)))y^2\nu(dy), \tag{A.16}$$

where we denote

$$F(z,y)=f(z+y)-f(z)-yf'(z)$$
$$\text{and}\quad \widetilde{F}(z,y)=\frac{f(z+y)-f(z)-yf'(z)}{\gamma^{-1}(t,z,y)^2},$$

The function F clearly satisfies

$$\|F'_y\|_\infty\le 2\|f'\|_\infty,\quad \|F''_y\|_\infty\le\|f''\|_\infty.$$

As for the function $\widetilde{F}$, we can write it as $\widetilde{F}=u(z,y)v^2(t,z,y)$ with

$$u(z,y)=\frac{f(z+y)-f(z)-yf'(z)}{y^2}=\int_0^1 f''(z+\theta y)(1-\theta)d\theta,$$
$$\text{and}\quad v(t,z,y)=\frac{y}{\gamma^{-1}(t,z,y)}=\int_0^1\gamma_y(\theta\gamma^{-1}(y))d\theta.$$

It is easy to see that under our assumptions the functions u and v are bounded together with their first and second derivatives. It follows that $\|\widetilde{F}'_y\|_\infty<\infty$ and $\|\widetilde{F}''_y\|_\infty<\infty$.

Remark also that by the mean value theorem and our assumptions on γ,

$$\left|\sum_{i:y_i(t,z)\in\Omega_1\cup\Omega_2}\omega_i(t,z)\gamma(t,z,y_i(t,z))\right|\le C\int_{\Omega_1\cup\Omega_2}|\gamma(t,z,y)|\nu(dy)$$

for some constant C which does not depend on truncation / discretization parameters.

The terms in (A.11-A.16) admit the following bounds:

$$|(\text{A.11})|\le\frac{\|f^{(4)}\|_\infty}{24}\Delta z^2\int_{\Omega_0}\gamma^2(t_n,z,y)\nu(dy)\le\frac{\|f^{(4)}\|_\infty}{24}\Delta z^2\|\gamma_y\|_\infty^2\int_{y\le\frac{(k+1/2)\Delta z}{m_1}}y^2\nu(dy)$$
$$\le C\Delta z^2(k\Delta z)^{2-\alpha},$$

$$|(\mathrm{A}.12)| \leq \frac{\|f^{(3)}\|}{6}\int_{\Omega_0}\gamma^3(t_n,z,y)\nu(dy) \leq C(k\Delta z)^{3-\alpha},$$

$$|(\mathrm{A}.13)| \leq \frac{\|f^{(3)}\|}{6}\Delta z^2(\bar{\mu}+\int_{\Omega_1\cup\Omega_2}|\gamma(t_n,z,y)|\nu(dy))$$

$$\leq \frac{\|f^{(3)}\|}{6}\Delta z^2(\bar{\mu}+\|\gamma_y\|\int_{|y|\geq\frac{(k+1/2)\Delta z}{\|\gamma_y\|_\infty}}|y|\nu(dy)) \leq C\Delta z^2(k\Delta z)^{1-\alpha},$$

$$|(\mathrm{A}.14)| \leq 2\|f\|_\infty\nu(\Omega_3)+\|f'\|_\infty\int_{\Omega_3}|\gamma(t,z,y)|\nu(dy) \leq C\int_{|y|\geq\frac{I\Delta z}{\|\gamma_y\|_\infty}}(1+|y|)\nu(dy),$$

$$|(\mathrm{A}.15)| = \left|\sum_{i:y_i(t_n,z)\in\Omega_2}\int_{(i-1/2)\Delta z}^{(i+1/2)\Delta z}(F(z,\zeta)-F(z,i\Delta z))\frac{\nu(\gamma^{-1}(\zeta))}{\gamma_y(\gamma^{-1}(\zeta))}d\zeta\right|$$

$$\leq \sum_{i:y_i(t_n,z)\in\Omega_2}\left\{\frac{\Delta z^2}{4}\|F_{yy}\|_\infty\int_{y_{i-1/2}}^{y_{i+1/2}}\nu(dy)+\|F_y\|_\infty\left|\int_{(i-1/2)\Delta z}^{(i+1/2)\Delta z}(\zeta-i\Delta z)\frac{\nu(\gamma^{-1}(\zeta))}{\gamma_y(\gamma^{-1}(\zeta))}d\zeta\right|\right\}$$

$$\leq C\Delta z^2\int_{\Omega_2}\nu(dy)+\sum_{i:y_i(t_n,z)\in\Omega_2}\|F_y\|_\infty\frac{\Delta z^2}{4}\int_{(i-1/2)\Delta z}^{(i+1/2)\Delta z}\left|\frac{d}{d\zeta}\frac{\nu(\gamma^{-1}(\zeta))}{\gamma_y(\gamma^{-1}(\zeta))}\right|d\zeta$$

$$\leq C\Delta z^2\left(\int_{\Omega_2}\nu(dy)+\int_{\Omega_2}|\nu'(y)|dy\right) \leq C\Delta z^2.$$

Similarly,

$$|(\mathrm{A}.16)| = \left|\sum_{i:y_i(t_n,z)\in\Omega_1}\int_{(i-1/2)\Delta z}^{(i+1/2)\Delta z}(\widetilde{F}(z,\zeta)-\widetilde{F}(z,i\Delta z))\gamma^{-1}(\zeta)^2\frac{\nu(\gamma^{-1}(\zeta))}{\gamma_y(\gamma^{-1}(\zeta))}d\zeta\right|$$

$$\leq \frac{\Delta z^2}{4}\sum_{i:y_i(t_n,z)\in\Omega_1}\left\{\|\widetilde{F}_{yy}\|_\infty\int_{y_{i-1/2}}^{y_{i+1/2}}y^2\nu(dy)+\|\widetilde{F}_y\|_\infty\int_{(i-1/2)\Delta z}^{(i+1/2)\Delta z}\left|\frac{d}{d\zeta}\left(\gamma^{-1}(\zeta)^2\frac{\nu(\gamma^{-1}(\zeta))}{\gamma_y(\gamma^{-1}(\zeta))}\right)\right|d\zeta\right\}$$

$$\leq C\Delta z^2\left(\int_{\Omega_1}y^2\nu(dy)+\int_{\Omega_1}|y|\nu(dy)+\int_{\Omega_1}|y^2\nu'(y)|dy\right) \leq C\Delta z^2(k\Delta z)^{1-\alpha}.$$

We finish the proof of (A.8) by observing that given that $k\Delta z$ is small (this is the small jump truncation level), the leading contribution is made by terms (A.12), (A.13), (A.14) and (A.16).

We next prove (A.9). The expression inside the absolute value on the left-hand side may be decomposed into the following terms:

$$\frac{(e^{\Delta z}-1)f(z+\Delta z)+(e^{-\Delta z}-1)f(z-\Delta z)-(f(z)+2f'(z))\Delta z^2}{2\Delta z^2}D(t_n,z) \tag{A.17}$$

$$-\int_{\Omega_0(t_n,z)}\left\{(e^{\gamma(t,z,y)}-1)f(z+\gamma(t_n,z,y))-f(z)\gamma(t_n,z,y)-\frac{1}{2}(f(z)+2f'(z))\gamma^2(t_n,z,y)\right\}\nu(dy) \tag{A.18}$$

$$+\frac{(e^{\Delta z}-1)f(z+\Delta z)-(e^{-\Delta z}-1)f(z-\Delta z)-2f(z)\Delta z}{2\Delta z}\hat{\mu}(t_n,z) \tag{A.19}$$

$$-\int_{\Omega_3(t_n,z)}((e^{\gamma(t_n,z,y)}-1)f(z+\gamma(t_n,z,y))-\gamma(t_n,z,y)f(z))\nu(dy) \tag{A.20}$$

$$+\sum_{i:y_i(t_n,z)\in\Omega_2}\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}(G(z,\gamma(t_n,z,y_i))-G(z,\gamma(t_n,z,y)))\nu(dy) \tag{A.21}$$

$$+\sum_{i:y_i(t_n,z)\in\Omega_1}\int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)}(\widetilde{G}(z,\gamma(t_n,z,y_i))-\widetilde{G}(z,\gamma(t_n,z,y)))y^2\nu(dy), \tag{A.22}$$

where now the functions

$$G(z,y) := (e^y-1)f(z+y)-yf(z) \quad \text{and} \quad \widetilde{G}(z,y) := \frac{(e^y-1)f(z+y)-yf(z)}{\gamma^{-1}(t,z,y)^2}$$

satisfy

$$|G_y'| + |G_y''\| \leq C(e^y + 1), \quad y \in \mathbb{R}$$

and

$$|\widetilde{G}_y'| + |\widetilde{G}_y''| \leq C, \quad |y| \leq 1.$$

The latter estimate, in particular, follows from the Taylor formula representation

$$\frac{(e^y-1)f(z+y) - yf(z)}{y^2} = \int_0^1 \{e^{\theta y}(f(z+\theta y)) + 2e^{\theta y}(f'(z+\theta y)) + (e^{\theta y}-1)(f''(z+\theta y))\}(1-\theta)d\theta.$$

The different terms can be estimated in a manner, similar to the first part of the proof:

$$\begin{aligned}
&|(\text{A.17})| \leq C\Delta z^2 (k\Delta z)^{2-\alpha}, \qquad |(\text{A.18})| \leq C(k\Delta z)^{3-\alpha}, \qquad |(\text{A.19})| \leq C\Delta z^2 (k\Delta z)^{1-\alpha},\\
&|(\text{A.20})| \leq C \int_{|y| \geq \frac{I\Delta z}{\|\gamma_y\|_\infty}} \tau(y)\nu(dy),\\
&|(\text{A.21})| \leq C\Delta z^2 \left(\int_{\Omega_2} (e^{\gamma(t_n,z,y)} + 1)\nu(dy) + \int_{\Omega_2} (e^{\gamma(t_n,z,y)} + 1)|\nu'(y)|dy \right) \leq C\Delta z^2\\
&|(\text{A.22})| \leq C\Delta z^2 (k\Delta z)^{1-\alpha}.
\end{aligned}$$

Finally, (A.10) can be proven in a similar manner. For example, for $p = 2$, we decompose the expression inside the absolute value on the left-hand side into the following terms:

$$\begin{aligned}
&\frac{(e^{\Delta z}-1)^2 f(z+\Delta z) + (e^{-\Delta z}-1)^2 f(z-\Delta z) - 2f(z)\Delta z^2}{2\Delta z^2} D(t_n, z)\\
&- \int_{\Omega_0(t_n,z)} \left\{ (e^{\gamma(t,z,y)}-1)^2 f(z+\gamma(t_n,z,y)) - f(z)\gamma^2(t_n,z,y) \right\} \nu(dy)\\
&+ \frac{(e^{\Delta z}-1)^2 f(z+\Delta z) - (e^{-\Delta z}-1)^2 f(z-\Delta z)}{2\Delta z} \hat{\mu}(t_n, z)\\
&- \int_{\Omega_3(t_n,z)} (e^{\gamma(t_n,z,y)}-1)^2 f(z+\gamma(t_n,z,y))\nu(dy)\\
&+ \sum_{i:y_i(t_n,z)\in\Omega_2} \int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} (H(z,\gamma(t_n,z,y_i)) - H(z,\gamma(t_n,z,y)))\nu(dy)\\
&+ \sum_{i:y_i(t_n,z)\in\Omega_1} \int_{y_{i-1/2}(t,z)}^{y_{i+1/2}(t,z)} (\widetilde{H}(z,\gamma(t_n,z,y_i)) - \widetilde{H}(z,\gamma(t_n,z,y)))y^2\nu(dy),
\end{aligned}$$

with

$$H(z,y) := (e^y-1)^2 f(z+y) \quad \text{and} \quad \widetilde{H}(z,y) := \frac{(e^y-1)^2 f(z+y)}{\gamma^{-1}(t,z,y)^2}.$$

□

Lemma A.2. *There exists a constant $C < \infty$ which does not depend on the truncation / discretization parameters such that for all $n \in \mathbb{N}$ and all $K > \bar{\mu}n\Delta t$, the approximating Markov chain $\widehat{Z}$ satisfies*

$$\mathbb{P}[\max_{0\leq i\leq n} |Z_{t_i}^{z,t_0}| \geq K] \leq \frac{z^2 + Cn\Delta t}{(K - \bar{\mu}n\Delta t)^2}.$$

Proof. By construction, the approximating chain $\widehat{Z}$ satisfies

$$\begin{aligned}
\mathbb{E}[\widehat{Z}_{t_{i+1}}|\widehat{Z}_{t_i}] &= \mu(t_i, Z_{t_i})\Delta t\\
\text{Var}\,[\widehat{Z}_{t_{i+1}}|\widehat{Z}_{t_i}] &= D(t_i,\widehat{Z}_{t_i})\Delta t - \mu^2(t_i, Z_{t_i})\Delta t^2 + \Delta t \sum_{l:y_l(t_i,\widehat{Z}_{t_i})\in\Omega_1\cup\Omega_2} \omega_l(t_i,\widehat{Z}_{t_i})\gamma^2(t_i,\widehat{Z}_{t_i}, y_l(t_i,\widehat{Z}_{t_i})).
\end{aligned}$$

Therefore, the process $M_{t_i} = \widehat{Z}^{z,t_0}_{t_i} - \sum_{0\le j<i} \mu(t_j, \widehat{Z}^{z,t_0}_{t_j})\Delta t$ is a discrete-time martingale. By Doob's martingale inequality we then get

$$
\begin{aligned}
\mathbb{P}[\max_{0\le i\le n} |Z^{z,t_0}_{t_i}| \ge K] \le \mathbb{P}[\max_{0\le i\le n} |M_{t_i}| \ge K - \bar{\mu}\Delta tn] &\le \frac{\mathbb{E}[M^2_{t_n}]}{(K-\bar{\mu}\Delta tn)^2} \\
&\le \frac{z^2 + \mathbb{E}[\sum_{0<j\le n} \mathrm{Var}\,[\widehat{Z}^{z,t_0}_{j+1}|\widehat{Z}^{z,t_0}_{j}]]}{(K-\bar{\mu}\Delta tn)^2} \\
&\le \frac{z^2 + \mathbb{E}[\sum_{0<j\le n}(D(t_j,\widehat{Z}_{t_j}) + \sum_{l:y_l\in\Omega_1\cup\Omega_2}\gamma^2(t_j,\widehat{Z}_{t_j},y_l)\omega_l(t_j,\widehat{Z}_{t_j}))]\Delta t}{(K-\bar{\mu}\Delta tn)^2}.
\end{aligned}
$$

We finish the proof by applying the mean value theorem and using the boundedness of the derivatives of γ. □

B Proof of Proposition 5.1

Proof. Before we start, remark that the function $A \mapsto F_{d,T}(A)$ is strictly increasing, so invertible, and infinitely differentiable: in particular

$$
\begin{aligned}
\Phi'(A) =& \frac{\int_T^{T+d} \psi(0,s)l(s)e^{l(s)A}ds}{\int_T^{T+d} \psi(0,s)e^{l(s)A}ds} \\
\Phi''(A) =& \frac{\left(\int_T^{T+d} \psi(0,s)l^2(s)e^{l(s)A}ds\right)\left(\int_T^{T+d} \psi(0,s)e^{l(s)A}ds\right) - \left(\int_T^{T+d} \psi(0,s)l(s)e^{l(s)A}ds\right)^2}{\left(\int_T^{T+d} \psi(0,s)e^{l(s)A}ds\right)^2}
\end{aligned}
$$

from which we deduce

$$
e^{-c(T+d)} \le \Phi'(A) \le e^{-cT} \qquad \text{and} \qquad e^{-2c(T+d)} - e^{-2cT} \le \Phi''(A) \le e^{-2cT} - e^{-2c(T+d)}
$$

From Itô's formula, we obtain

$$
\begin{aligned}
dZ_t =& \left(\Phi'(A_t)e^{ct}\zeta + \int_{\mathbb{R}} \left(\Phi(A_{t-}+e^{ct}y) - \Phi(A_{t-}) - ye^{ct}\Phi'(A_{t-})\right)\nu(dy)\right)dt \\
&+ \int_{\mathbb{R}} \left(\Phi(A_{t-}+e^{ct}y) - \Phi(A_{t-})\right)\tilde{J}(dydt)
\end{aligned}
$$

or equivalently

$$
dZ_t = \mu(t,Z_t)dt + \int \gamma(t,Z_{t-},y)\tilde{J}(dydt)
$$

We can now prove that μ and γ verify the Assumptions 2.1. We detail the computations only for the function γ, since similar computations can be done for μ. First we remark that $z \mapsto \gamma(t,z,y)$ is differentiable and we can compute this derivative to obtain

$$
\begin{aligned}
\partial_z\gamma(t,z,y) =& -1 + (\Phi'(\Phi^{-1}(z)))^{-1}\Phi'(\Phi^{-1}(z) + ye^{ct}) \\
=& e^{ct}y(\Phi'(\Phi^{-1}(z)))^{-1}\int_0^1 \Phi''(\Phi^{-1}(z) + re^{ct})dr
\end{aligned}
$$

so that

$$
\begin{aligned}
|\partial_z\gamma(t,z,y)| =& \left|e^{ct}y(\Phi'(\Phi^{-1}(z)))^{-1}\int_0^1 \Phi''(\Phi^{-1}(z) + re^{ct})dr\right| \\
\le& |y|e^{cT}(\inf_A |\Phi(A)|)^{-1}\|\Phi''\|_\infty \le e^{cT}e^{-c(T+d)}\|\Phi''\|_\infty |y| \\
\le& |y|e^{cT}e^{c(T+d)}\left(e^{-2cT} - e^{-2c(T+d)}\right) \le e^{cd}|y|
\end{aligned}
$$

From the bounds on the first and second derivative of Φ we obtain $\sup_{t,z} |\partial_z \gamma(t,z,y)| \leq e^{cd}|y|$, which gives us the function ρ introduced in Assumptions 2.1. Again by the definition of Φ in (5.3) we have

$$\exp(e^{-c(T+d)}y) - 1 \leq e^{\gamma(t,z,y)} - 1 \leq e^y - 1$$

if $y > 0$ and the inverse inequality stands in force if $y < 0$ which yield $\sup_{t,z} |e^{\gamma(t,z,y)} - 1| \leq |e^y - 1|$. According to the definition of the function τ given in Assumptions 2.1 and the estimations above we deduce that

$$\tau(y) := \max\left(\sup_{t,z}\left(|\gamma(t,z,y)|, \left|e^{\gamma(t,z,y)} - 1\right|\right), \rho(y)\right) = e^{cd}\max\left(|y|, |e^y - 1|\right)$$

If follows then that Assumptions 2.1-[$\mathbf{C}, \mathbf{I}, \mathbf{L}$] hold true. For Assumption 2.1-[$\mathbf{ND}$] we have, from the definition of γ

$$\left(e^{\gamma(t,z,y)} - 1\right)^2 \geq \left(\exp(e^{-c(T+d)}y) - 1\right)^2$$

so then, for some positive $M > 0$ we have

$$\begin{aligned}\Gamma(y) :=& \int_{\mathbb{R}} \inf_{t,z} \left(e^{\gamma(t,z,y)} - 1\right)^2 \nu(dy) \geq \int_{\mathbb{R}} \inf_{t,z} \left(\exp(e^{-c(T+d)}y) - 1\right)^2 \nu(dy) \\ \geq& M \int_{|y|\leq\epsilon} |y|^{1-\alpha} g(y) dy > 0\end{aligned}$$

since $g(0^+)$ and $g(0^-)$ are strictly positive, we can select ϵ small enough and obtain

$$\Gamma(y) \geq M \int_{|y|\leq\epsilon} |y|^{1-\alpha} dy > 0$$

We can derive γ w.r.t y to obtain

$$\begin{aligned}\gamma_y(t,z,y) =& e^{ct}\Phi'(\Phi^{-1}(z) + e^{ct}y) \\ \gamma_{yy}(t,z,y) =& e^{2ct}\Phi''(\Phi^{-1}(z) + e^{ct}y)\end{aligned}$$

so then $e^{-c(T+d)} \leq |\gamma_y(t,z,y)| \leq e^{cT}$ and $|\gamma_{yy}(t,z,y)| \leq e^{2cT}$, which proves that Assumptions 2.1-[$\mathbf{RG_i}$] holds true. For Assumption 2.1-[$\mathbf{RG_{iii}}$], one can differentiate γ_y w.r.t. z and give for it an upper bound to prove that indeed $z \to \gamma_y(t,z,y)$ is Lipschitz continuous uniformly in t, y. The Assumption 2.1-[$\mathbf{RG_{ii}}$] does not hold true since trivially $\gamma_y(t,z,y) = e^{ct}\Phi'(\Phi^{-1}(z) + e^{ct}y) \neq 1$. The last thing we need to prove is the condition 4 assumed in Theorem 4.1, i.e. that γ is 3 times differentiable w.r.t. y with bounded derivatives. From the definition of γ, this is equivalent to prove that Φ is 3 times differentiable with bounded derivatives. Let us introduce

$$p_i(A) := \int_T^{T+d} \psi(0,s) l^i(s) e^{l(s)A} ds$$

so that $\Phi'(A) = p_1(A)/p_0(A)$ and $\Phi''(A) = (p_2(A)p_0(A) - p_1(A)^2)/p_0(A)^2$. By remarking that $p_i'(A) = p_{i+1}(A)$, we can differentiate Φ'' to obtain

$$\Phi^{(iii)}(A) = \frac{p_3(A)p_0(A) + p_2(A)p_1(A) - 2p_1(A)p_2(A)}{p_0(A)^2} - 2\Phi''(A)\Phi'(A)$$

with Φ' and Φ'' bounded as already proved. With the same type of computation it is straightforward to prove that $\Phi^{(iii)}(A)$ is also bounded.

□