Plan

- $(X_n)_{n\geq 0}$: a Markov chain taking values in $(E_n, \mathcal{E}_n)_{n\geq 0}$, with
  - initial distribution on E_0: $\eta_0 = \mathrm{Law}(X_0)$;
  - Markov transition from E_{n-1} to E_n: $M_n(x_{n-1}, dx_n)$.

- f_n: a sequence of non-negative measurable *payoff* functions on E_n.

- For any measurable function φ_{k+1} defined on E_{k+1}, $M_{k+1}(\varphi_{k+1})$ stands for the conditional expectation function on E_k:

$$\begin{aligned} M_{k+1}(\varphi_{k+1})(x_k) &= \int_{E_{k+1}} M_{k+1}(x_k, dx_{k+1})\ \varphi_{k+1}(x_{k+1})\ , \quad x_k \in E_k \\ &= \mathbb{E}\left(\varphi_{k+1}(X_{k+1})\,|\,X_k = x_k\right)\ . \end{aligned}$$

Goal: to compute the Snell envelope u_n given by

$$\begin{cases} u_n = f_n \\ u_k = \mathcal{H}_{k+1}(u_{k+1}) := f_k \vee M_{k+1}(u_{k+1})\ , \quad \text{for any } 0 \leq k < n\ , \end{cases} \tag{1}$$

- Backward operator $\mathcal{H}_k$, for $k \leq l \leq n$:

$$\begin{aligned} u_k &= \mathcal{H}_{k+1}(u_{k+1}) = f_k \vee M_{k+1}(u_{k+1}) \\ u_k &= \mathcal{H}_{k,l}(u_l)\ , \quad \text{for any } k \leq l \leq n \\ \mathcal{H}_{k,l} &= \mathcal{H}_{k+1} \circ \mathcal{H}_{k+1,l}\ , \quad \text{with the convention} \quad \mathcal{H}_{k,k} = Id\ . \end{aligned}$$

- Lipschitz property: for any functions u, v on E_k,

$$\left|\mathcal{H}_{k,l}(u) - \mathcal{H}_{k,l}(v)\right| \leq M_{k,l}\left(|u-v|\right)\ . \tag{2}$$

- Backward approximation operator $\widehat{\mathcal{H}}_{k+1}$:

$$\widehat{u}_k = \widehat{\mathcal{H}}_{k+1}(\widehat{u}_{k+1}) = \widehat{f}_k \vee \widehat{M}_{k+1}(\widehat{u}_{k+1})\ .$$

  - Local error $\left|\mathcal{H}_{k+1}(u) - \widehat{\mathcal{H}}_{k+1}(u)\right| \leq |f_k - \widehat{f}_k| + |(M_{k+1} - \widehat{M}_{k+1})(u)|$.
  - Error propagation: $u_k - \widehat{u}_k = \sum_{l=k}^{n} \left[\widehat{\mathcal{H}}_{k,l}(\mathcal{H}_{l+1}(u_{l+1})) - \widehat{\mathcal{H}}_{k,l}(\widehat{\mathcal{H}}_{l+1}(u_{l+1}))\right]$,

Lemma 1: Robustness Lemma

For any $0 \leq k < n$, on the state space $\widehat{E}_k$.

$$|u_k - \widehat{u}_k| \quad \leq \quad \sum_{l=k}^{n} \widehat{M}_{k,l}|f_l - \widehat{f}_l| + \sum_{l=k}^{n-1} \widehat{M}_{k,l}|(M_{l+1} - \widehat{M}_{l+1})u_{l+1}|\ ,$$

Applications to bound the error induced by . . .

- Cut-off type models
- Euler approximation models
- Interpolation type models
- Quantization tree models
- Monte Carlo approximation models . . .

Plan

- Path-space Markov chain
  - X'_n Markov chain with transitions $M'_k(x_{k-1}, dx'_k)$ from E'_{k-1} into E'_k.
  - $X_n = (X'_0, \ldots, X'_n) \in E_n = (E'_0 \times \ldots \times E'_n)$ Markov chain with transition kernels

$$M_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1})\, M'_k(y'_{k-1}, dy'_k)\ , \text{ for any } \begin{cases} x_{k-1} = & (x'_0, \ldots, x'_{k-1}) \in E_{k-1} \\ y_k = & (y'_0, \ldots, y'_k) \in E_k\ . \end{cases}$$

- Change of measure to replace the conditional expectation by a simple expectation

$$u'_k(x'_k) = f'_k(x'_k) \vee \eta_{k+1}\Big(R_{k+1}(x'_k, \cdot)\ u'_{k+1}\Big)\ ,$$

where the weighting factor R_{k+1} is given by

$$R_{k+1}(x'_k, x'_{k+1}) = \frac{dM'_{k+1}(x'_k, .)}{d\eta_{k+1}}(x'_{k+1})\ .$$

- Approximation with the random Monte Carlo operator $\widehat{M}'_k$ from E'_k into $\widehat{E}'_{k+1}$

$$\widehat{M}'_{k+1}(x'_k, dx'_{k+1}) = \widehat{\eta}_{k+1}(dx'_{k+1})\ R_{k+1}(x'_k, x'_{k+1})\ , \quad \text{where} \quad \widehat{\eta}_k = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi^i_k}\ .$$

Broadie-Glasserman models ([2] (2004)) (I)

- Monte carlo approximation $\widehat{\eta}_k = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_k^i}$, associated with
  - $\xi_k := (\xi_k^i)_{1\leq i\leq N}$ iid $\sim \eta_k$ on E'_k.
  - $(\xi_k)_{0\leq k\leq n}$ are independent.
- Local error

$$(M'_{k+1} - \widehat{M}'_{k+1})(x'_k, dx'_{k+1}) = [\eta_{k+1} - \widehat{\eta}_{k+1}](dx'_{k+1})\ R_{k+1}(x'_k, x'_{k+1}) \ ,$$

Theorem 1: BG1

For any integer $p \geq 1$, we denote by p' the smallest even integer greater than p. Then for any time horizon $0 \leq k \leq n$, and any $x'_k \in E'_k$, we have

$$\sqrt{N}\widehat{\mathbb{E}}_{\eta_0}\left(\left|u'_k(x'_k) - \widehat{u}'_k(x'_k)\right|^p\right)^{\frac{1}{p}} \leq 2a(p) \sum_{k\leq l<n} \left\{\int M'_{k,l}(x'_k, dx'_l)\eta_{l+1}\left[(R_{l+1}(x'_l,\cdot)u'_{l+1})^{p'}\right]\right\}^{\frac{1}{p'}} .$$

Broadie-Glasserman models ([2] (2004)) (II)

- Alternative BG model Suppose R_{l+1} is not known (i.e. cannot be evaluated at any point of E_{l+1}). This may occur if we chose $\eta_{l+1} = \mathrm{law}(X'_{l+1})$ as the sampling distribution.
- Assumption: $M'_n(x'_{n-1}, \cdot) \ll \lambda_n$ with

$$(H)_0 \qquad H_n(x'_{n-1}, x'_n) = \frac{dM'_n(x'_{n-1}, \cdot)}{d\lambda_n}(x'_n) > 0 \ , \quad \forall (x'_{n-1}, x'_n) \in \left(E'_{n-1} \times E'_n\right) \ ,$$

where H_n is supposed to be known up to a normalizing constant.

- $\mathcal{L}(X'_{k+1}) =: \eta_{k+1} \ll \lambda_{k+1}$, with the Radon-Nikodym derivative

$$\eta_{k+1}(dx'_{k+1}) = \eta_k M'_{k+1}(dx'_{k+1}) = \eta_k \left(H_{k+1}(\cdot, x'_{k+1})\right) \ \lambda_{k+1}(dx'_{k+1}) \ .$$

=> The backward recursion of the Snell envelope u'_k becomes

$$\begin{aligned} u'_k(x'_k) &= f'_k(x'_k) \vee \left(\int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1}) \ \frac{dM'_{k+1}(x'_k, .)}{d\eta_{k+1}}(x'_{k+1}) \ u'_{k+1}(x'_{k+1}) \right) \\ &= f'_k(x'_k) \vee \left(\int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1}) \ \frac{H_{k+1}(x'_k, x'_{k+1})}{\eta_k(H_{k+1}(\cdot, x'_{k+1}))} \ u'_{k+1}(x'_{k+1}) \right) \ . \end{aligned}$$

- Approximation model $\widehat{u}'_k(x'_k) = f'_k(x'_k) \vee \widehat{M}'_{k+1}(x'_k, \widehat{u}'_{k+1})$, where

$$\widehat{M}'_{k+1}(x'_k, dx'_{k+1}) = \widehat{\eta}_{k+1}(dx'_{k+1})\ \widehat{R}_{k+1}(x'_k, x'_{k+1}) \quad \text{with} \quad \widehat{R}_{k+1}(x'_k, x'_{k+1}) := \frac{H_{k+1}(x'_k, x'_{k+1})}{\widehat{\eta}_k(H_{k+1}(\cdot, x'_{k+1}))}.$$

- Local error For any even integer $p \geq 1$, and measurable function f on E_l

$$\sqrt{N}\ \widehat{\mathbb{E}}_{\eta_0}\left(\left|\left[M'_{l+1} - \widehat{M}'_{l+1}\right](f)(x'_l)\right|^p |\mathcal{F}_l\right)^{\frac{1}{p}} \leq 2\ a(p)\ \widehat{\eta}_l M'_{l+1}\left[(\widehat{R}_{l+1}(x'_l, \cdot) f)^p\right]^{\frac{1}{p}} .$$

- Assumption $(H)_1$

$$(H)_1 \quad \begin{cases} \|M'_{l+1}(u^{2p}_{l+1})\| < \infty \\ \sup_{x'_l, y'_l \in E'_l} \frac{H_{l+1}(x'_l, x'_{l+1})}{H_{l+1}(y'_l, x'_{l+1})} \leq h_{l+1}(x'_{l+1}) \text{ with } \|M'_{l+1}(h^{2p}_{l+1})\| < \infty , \end{cases}$$

Theorem 2: BG2

Under the conditions $(H)_0$ and $(H)_1$, for any even integer $p > 1$, any $0 \leq k \leq n$, and $x'_k \in E'_k$, we have

$$\sqrt{N}\ \mathbb{E}\left(\left|u'_k(x'_k) - \widehat{u}'_k(x'_k)\right|^p\right)^{\frac{1}{p}} \leq 2a(p) \sum_{k \leq l < n} \left(\|M'_{l+1}(h^{2p}_{l+1})\|\ \|M'_{l+1}((u'_{l+1})^{2p})\|\right)^{\frac{1}{2p}} .$$

Plan

Neutral genetic models $(\xi_k) := \left(\xi_k^i\right)_{1\leq i\leq N} \in E_k^N$

$$\xi_k \in E_k^N \xrightarrow{\text{Selection}} \widehat{\xi}_k := \left(\widehat{\xi}_k^i\right)_{1\leq i\leq N} \in E_k^{\widehat{N}} \xrightarrow{\text{Mutation}} \xi_{k+1} \in E_{k+1}^N . \tag{3}$$

1. Initialization: $\xi_0 = \left(\xi_0^i\right)_{0\leq i\leq N_0}$, i.i.d. random copies of X_0.
2. (Neutral) selection: select randomly N path valued particles $\widehat{\xi}_k := \left(\widehat{\xi}_k^i\right)_{1\leq i\leq N}$ among the N path valued particles $\xi_k = (\xi_k^i)_{1\leq i\leq N}$.
3. Mutation: $\widehat{\xi}_k \rightsquigarrow \xi_k$, every selected path valued individual $\widehat{\xi}_k^i$ evolves randomly to a new path valued individual $\xi_{k+1}^i = x$ randomly chosen with the distribution $M_{k+1}(\widehat{\xi}_k^i, x)$, with $1 \leq i \leq \widehat{N}$.

$$\begin{aligned}
\xi_k^i &:= \left(\xi_{0,k}^i, \xi_{1,k}^i, \ldots, \xi_{k,k}^i\right)\\
\widehat{\xi}_k^i &:= \left(\widehat{\xi}_{0,k}^i, \widehat{\xi}_{1,k}^i, \ldots, \widehat{\xi}_{k,k}^i\right) \in E_k := (E_0' \times \ldots \times E_k')\\
\xi_{k+1}^i &:= \Big(\underbrace{(\xi_{0,k+1}^i, \xi_{1,k+1}^i, \ldots, \xi_{k,k+1}^i)}_{||}, \ \xi_{k+1,k+1}^i\Big)\\
&:= \Big(\overbrace{(\widehat{\xi}_{0,k}^i, \ \widehat{\xi}_{1,k}^i, \ldots, \ \widehat{\xi}_{k,k}^i)}, \ \xi_{k+1,k+1}^i\Big) = \left(\widehat{\xi}_k^i, \xi_{k+1,k+1}^i\right) .
\end{aligned}$$

Neutral genetic models

• Let η_k^N and $\widehat{\eta}_k^N$ be the occupation measures of the genealogical tree model after the mutation and the selection steps;

$$\eta_k^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_k^i} \quad \text{and} \quad \widehat{\eta}_k^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\widehat{\xi}_k^i} .$$

Forward algorithm

Initialization At time step $k = 0$, generate N i.i.d. random copies of X_0 and set $\xi_0 = \left(\xi_0^i\right)_{0 \leq i \leq N}$.

At each time step $k = 1, \cdots, n$

1. **Selection**: For each $i = 1, \cdots, N$, generate independently an indice $I_i \in \{1, \cdots, N\}$ with probability $\mathbb{P}(I_i = j) = 1/N$. Then set $\hat{\xi}_{k-1}^i = \xi_{k-1}^{I_i}$.
2. **Mutation**: For each $i = 1, \cdots, N$, generate independently N i.i.d. random variables $(\xi_{k,k}^i)_{0 \leq i \leq N}$ according to the transition kernel $M_k'(\hat{\xi}_{k-1,k-1}^i, \cdot)$. Then set $\xi_k^i = (\hat{\xi}_{k-1}^i, \xi_{k,k}^i)$.

Convergence of the occupation measures to the underlying measure

Lemma 2: Convergence of the particle approximation of the measures

For any $p \geq 1$, we denote by p' the smallest even integer greater than p. In this notation, for any $k \geq 0$ and any $f \in \mathbb{L}_{p'}(\eta_n)$, we have the non asymptotic estimates

$$\sqrt{N}\,\mathbb{E}\left(\left|[\eta_n^N - \eta_n](f)\right|^p\right)^{1/p} \leq 2\; a(p)\; \|f\|_{p',\eta_n}\;(n+1)\;. \tag{4}$$

- Particle approximation of the Markov transitions M'_k

$$\widehat{M}'_{k+1}(f)(x) := \frac{\eta_n^N((1_x \circ \pi_k)\ (f \circ \pi_{k+1}))}{\eta_n^N((1_x \circ \pi_k))} := \frac{\sum_{1\le i\le N}\ 1_x(\xi^i_{k,n})\ f(\xi^i_{k+1,n})}{\sum_{1\le i\le N} 1_x(\xi^i_{k,n})}\ ,$$

for every state x in the support $\widehat{E}_{k,n}$ of the measure $\eta_n^N \circ \pi_k^{-1}$.

- Approximation model of the Snell envelope

$$\widehat{u}_k(x) = \begin{cases} f_k(x) \vee \widehat{M}'_{k+1}(u_{k+1})(x) & \forall x \in \widehat{E}_{k,n} \\ 0 & \text{otherwise}\ . \end{cases}$$

In terms of the ancestors at level k, this recursion takes the following form

$$\widehat{u}_k\left(\xi^i_{k,n}\right) = f_k\left(\xi^i_{k,n}\right) \vee \widehat{M}'_{k+1}(\widehat{u}_{k+1})\left(\xi^i_{k,n}\right)\ , \qquad \forall 1 \le i \le N.$$

Backward algorithm

Initialization At time step $k = n$, for all $i = 1, \cdots, N$, set $\hat{u}_n(\xi^i_{n,n}) = f(\xi^i_{n,n})$.

At each time step $k = n-1, \cdots, 0$, for all $i = 1, \cdots, N$ set

$$\hat{u}_k(\xi^i_{k,n}) = f_k\left(\xi^i_{k,n}\right) \vee \frac{\sum_{j=1}^N \hat{u}_{k+1}(\xi^j_{k+1,n})\, 1_{\xi^j_{k,n}=\xi^i_{k,n}}}{\sum_{j=1}^N 1_{\xi^j_{k,n}}}\ .$$

Assumption: finite state spaces

Lemma 3:

For any $p \geq 1$, and $0 \leq i \leq N$ we have the following uniform estimate

$$\sup_{0 \leq l \leq n} \left\| \widehat{M}'_{l+1}(f)(\xi^i_{l,n}) - M'_{l+1}(f)(\xi^i_{l,n}) \right\|_p \leq c_p(n)/\sqrt{N} , \tag{5}$$

with some finite constants $c_p(n) < \infty$ depending on the parameters p and n.

$$\sup_{N \geq 1} \sup_{0 \leq l \leq k \leq n} \left\| \eta^N_k (1_{\xi^i_{l,k}} \circ \pi_l)^{-1} \right\|_p < \infty . \tag{6}$$

Theorem 4:

For any $p \geq 1$, and $0 \leq i \leq N$ we have the following uniform estimate

$$\sup_{0 \leq k \leq n} \left\| (u_k - \widehat{u}_k)(\xi^i_{k,n}) \right\|_p \leq c_p(n)/\sqrt{N} , \tag{7}$$

Moreover, the bias of the genealogical tree based estimator is always positive.

Price dynamics and options (Examples from Bouchard and Warin [4])

- Asset prices $(\tilde{X}_t)$ follows a geometric Brownian motion under the risk-neutral measure,
$$\frac{d\tilde{X}_t(i)}{\tilde{X}_t(i)} = rdt + \sigma_i dz_t^i \ , \quad \text{for assets } i = 1, \cdots, d \ , \tag{8}$$
where z^i, for $i = 1, \cdots, d$ are independent standard Brownian motions.
- Interest rate r is set to 5% annually, $\tilde{X}_{t_0}(i) = 1$, for all $i = 1, \cdots, d$, volatilities $\sigma_i = 20\%$ annually, maturity $T = 1$ year, 11 equally distributed exercise opportunities.
- Two different payoffs:
  1. a geometric average put option with strike $K = 1$ and payoff $(K - \prod_{i=1}^d \tilde{X}_T(i))_+$,
  2. an arithmetic average put option with strike $K = 1$ and payoff $(K - \frac{1}{d}\sum_{i=1}^d \tilde{X}_T(i))_+$.
- Benchmark values for the geometric and arithmetic put options (taken from [4])

Number of assets	1	2	3	4	5	6
Geometric Payoff	0.06033	0.07815	0.08975	0.09837	0.10511	0.11073
Arithmetic Payoff	0.06033	0.03882	0.02947	0.02403	0.02046	0.01830

Finite state space approximation

• State space partitioning: quantization-like approach
At each time t_k, $k = 1, \cdots, n$, we build a finite state space

$$E_0 = \{X_{t_0}\}\ , \quad \text{and} \quad E_k = \{S_k^1, \cdots, S_k^{N'}\}\ ,$$

composed of discrete points $S_k^1, \cdots, S_k^{N'}$ that will be referred to as *sites*.

• Finite state space Markov chain
We define a finite state space Markov chain $(X'_k)_{k=0,\cdots,n}$ on (E_k) such that

$$\begin{cases} X'_0 = \tilde{X}_{t_0}\ , \quad \text{and for} \quad k = 1, \cdots, n\ , \\ \mathbb{P}\left(X'_k = S_k^j \,|\, X'_{k-1} = S_{k-1}^i\right) = \mathbb{P}\left(\tilde{X}_{t_k} \in V_k^j \,|\, \tilde{X}_{t_{k-1}} = S_{k-1}^i\right)\ , \end{cases}$$

where V_k^j denotes the Voronoi cell associated to the site S_k^j in the the discrete set E_k and $(\tilde{X}_{t_k})$ is the Markov process verifying (8) observed at the discrete times $t_0, \cdots, t_n$.
To simulate a transition of $(X'_k)_{k=0,\cdots,n}$ from $S_{k-1}^i \in E_{k-1}$ to the time step k:

1. Simulate $\tilde{X}_{t_k}$ according to $\tilde{M}_k(S_{k-1}^i, \cdot)$, where $\tilde{M}_k$ denotes the transition kernel of the continuous state space Markov chain verifying (8)
2. Set $X'_k = S_k^{i^*}$, where $S_k^{i^*}$ is the nearest neighbor of $\tilde{X}_{t_k}$ among elements of E_k.

Complexity

• The major part of the computing time is spent in the *forward step* for simulating the discrete space Markov chain (X'_k).
One has to compute a nearest neighbour among N' sites which finally leads to a complexity of order $O(NN')$ by time step, for the whole set of N particles.

• The approximation error can be decomposed into two terms:

1. The state space discretization error bounded, according to [8], by $\frac{c}{N'^{1/d}}$;
2. The error induced by the genealogical tree algorithm, could be bounded, by $c\frac{N'^{\beta}}{N^{1/2}}$, for a given positive real $\beta > 0$.

• Trade-off between the size of the space N' and the number of particles N minimizing the global error, $N' = O(N^{\frac{d}{2\beta d+2}})$.

$$=> \begin{cases} \text{Complexity} & O(N^{\frac{(1+2\beta)d+2}{2\beta d+2}}) \ , \\ \text{Error bound} & \frac{c}{N^{\frac{1}{2\beta d+2}}} \ . \end{cases}$$

In our numerical simulations, $\beta = 1/2$ so that the complexity grows with the dimension from $N^{4/3}, N^{3/2}, N^{8/5}, \cdots, N^2$ for dimensions $d = 1, 2, 3, \cdots, \infty$.

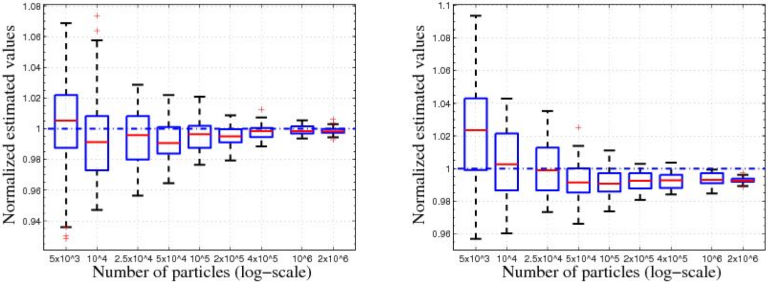
Comparison with the quantization algorithm

Error (in % of the option value) for the geometric and arithmetic put options of

- Genealogical algorithm with $N = 25000$ particles and $N' = N^{\frac{d}{d+2}}$ sites
  => complexity $N^{\frac{2d+2}{d+2}}$
- Quantization algorithm with $N = 25600$ quantization points, (taken from [4]) within parenthesis.
  => complexity N^2.

Number of assets	$d = 3$	$d = 4$	$d = 5$	$d = 6$
Geometric Put error (in % of the option value)	2 (2)	7 (8)	14 (15)	17 (22)
Arithmetic Put error (in % of the option value)	3.5 (3.5)	10 (8)	15 (16)	14 (17)

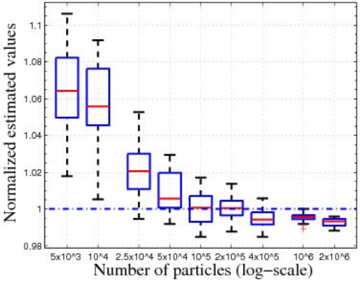
Geometric put



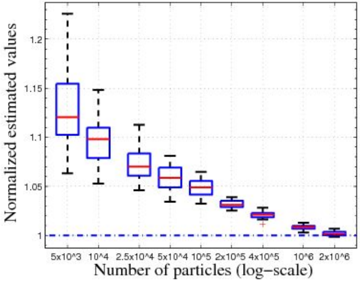
(a) $d = 1$ (b) $d = 2$

Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **geometric** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.

Geometric put



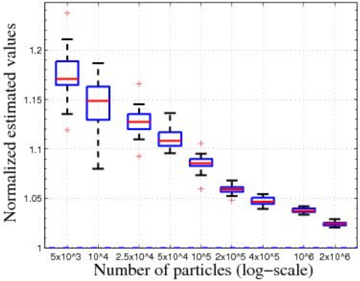
(c) $d = 3$



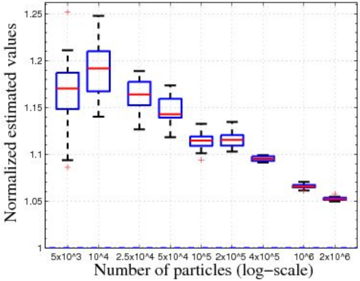
(d) $d = 4$

Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **geometric** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.

Geometric put



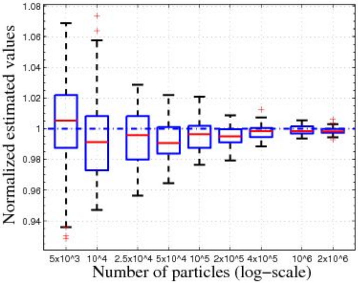
(e) $d = 5$



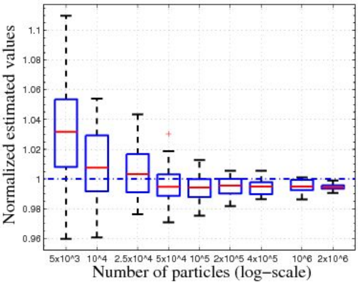
(f) $d = 6$

Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **geometric** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.

Arithmetic put



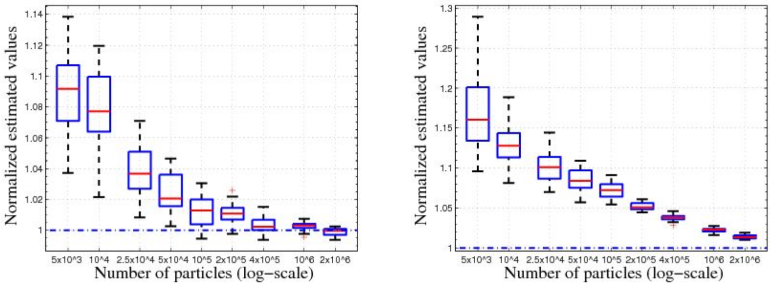
(g) $d = 1$



(h) $d = 2$

Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **arithmetic** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.

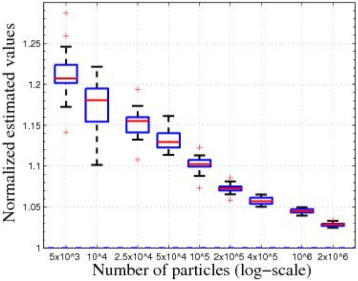
Geometric put



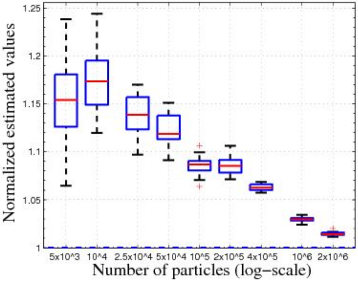
(i) $d = 3$ (j) $d = 4$

Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **arithmetic** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.

Geometric put



(k) $d = 5$



(l) $d = 6$

Boxplots for estimated option values (divided by the benchmark values) as a function of the number of particles for the **arithmetic** put-payoff. The box stretches from the 25th percentile to the 75th percentile, the median is shown as a line across the box, the whiskers extend from the box out to the most extreme data value within 1.5 IQR (Interquartile Range) and red crosses indicates outliers.

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