Stochastic Target Problems with controlled Loss

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Motivation

- **Stock price:** (with large investor's strategy π)

$$\frac{dS^{\pi}(u)}{S^{\pi}(u)} = \mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u$$

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- **Wealth process:** (risk free interest rate $r = 0$)

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$$v(0, S_0) := \inf\left\{x \geq 0 : \ X_x^{\pi}(T) \geq g\left(S^{\pi}(T)\right) \ \mathbb{P} - \text{ps}, \quad \text{for some } \pi \in \mathcal{A}\right\} .$$

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- **Asset management under Quantile hedging constraint:**

$$\sup_\pi \mathbb{E}\left[U(X_T^{x,\pi})\right] \qquad \text{for } \pi \;\; s.t. \;\; \mathbb{P}\left[X_x^\pi(T) \geq g\left(S^\pi(T)\right)\right] \geq p .$$

Explicit Solution in Complete Market

- **Stock price under the (unique) Risk Neutral Measure $\mathbb{Q}$:**

$$\frac{dS(u)}{S(u)} = \sigma\left(u, S(u)\right) dW_u^{\mathbb{Q}} \quad \text{(independent on } \pi\text{)}$$

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Maximize the Probability of Hedge for a given starting wealth x

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$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_x^{\pi}(T) \geq g(S(T))\right]$$

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$$\max_{A \in \mathcal{F}_T} \mathbb{P}\left[A\right] \quad \text{under} \quad \mathbb{Q}^g[A] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}\left[g\left(S(T)\right)\right]},$$

with $\mathbb{Q}^g$ the risk neutral measure under the contingent claim numeraire

$$\frac{d\mathbb{Q}^g}{d\mathbb{Q}} = \frac{g\left(S(T)\right)}{\mathbb{E}^{\mathbb{Q}}\left[g\left(S(T)\right)\right]}$$

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A interprets as a critical region when testing $\mathbb{Q}^g$ against $\mathbb{P}$.

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By Neyman-Pearson Lemma,

$$A^*(x) = \left\{\frac{d\mathbb{P}}{d\mathbb{Q}} > a^* \frac{d\mathbb{Q}^g}{d\mathbb{Q}}\right\}, \text{ with } a^* := \inf\left\{a : \mathbb{Q}^g\left[\frac{d\mathbb{P}}{d\mathbb{Q}} > a \frac{d\mathbb{Q}^g}{d\mathbb{Q}}\right] = \frac{x}{\mathbb{E}^{\mathbb{Q}}\,[g\,(S(T))]}\right\}$$

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and the success region $A^*(x) = \{X_x^{\pi^*(x)}(T) \geq g(S(T))\}$ with

$$\pi^*(x) \quad \text{the hedging strategy of} \quad g(S(T))\mathbf{1}_{A^*(x)}$$

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$\Rightarrow$ Find $x^*(p)$ such that $\mathbb{P}[A^*(x^*(p))] = p$

Solution in General Case

- **Pros:**

- Explicit solution in some simple (but important) cases.

- Generic solution of the form:

$$X_x^\pi(T) = g\left(S_{t,s}(T)\right)\mathbf{1}_A \quad \text{or} \quad X_x^\pi(T) = g\left(S_{t,s}(T)\right)\zeta \text{ with } \zeta \in L^0[0,1]\,.$$

- Similar structure in incomplete markets.

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- **Cons:**

- Explicit solution not known in general (numerics)

- Dual problem in incomplete markets is a control problem: how to solve it ?

- Relies heavily on the duality between super-hedgeable claims and risk neutral measures.

Comparison with the super-hedging problem

$$v(t,s;1) \;\; := \;\; \inf\left\{x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \geq g\left(S_{t,s}(T)\right)\right] = 1\right\}$$

- **Dual approach:**

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- (DP1): $x > v(t,s;1) \Rightarrow \exists\, \pi \in \mathcal{A}$ s.t. for all stopping time $\tau \leq T$

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$\Rightarrow$ is sufficient to derive PDEs associated to $v(\cdot;1)$.

Direct approach for quantile hedging ?

- **Form of the DP:**

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PDE derivation (formally)

- Dynamics of the wealth

$$dX^{\pi}_{t,s,x}(u) \;=\; \pi_u \left[\mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u\right]$$

- Dynamics of the quantile price at point $Y_u = (u, S^{\pi}_{t,s}(u); P^{\alpha}_{t,p}(u))$

$$dv(Y_u) \;=\; \mathcal{L}^{\pi,\alpha} v(Y_u) du + \left[D_s v(Y_u) \sigma\left(u, S^{\pi}(u), \pi_u\right) + D_p v(Y_u) \alpha_u\right] dW_u$$

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- This formally leads to the PDE

$$\max_{(\pi,\alpha)\in\mathcal{G}(t,s,p)} \pi\mu\left(t, s, \pi\right) - \mathcal{L}^{\pi,\alpha} v(t, s; p) = 0$$

where $\mathcal{G}(t, s, p) := \{(\pi, \alpha) : \; \pi\sigma\left(t, s, \pi\right) = D_s v(t, s; p)\sigma\left(t, s, \pi\right) + D_p v(t, s; p)\alpha\}$

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- α is not bounded

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- Viscosity approach

- α is not bounded

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Boundary in time

$$v(T^-,s,p) = p\,g(s)$$

Example: Quantile Hedging in Black Scholes

- **The Dynamics:**

$$dS_{t,s}(r) = S_{t,s}(r)\left(\mu dt + \sigma dW_r\right) \quad \text{and} \quad dX^{\pi}_{t,x,s}(r) = \pi_r dS_{t,s}(r)$$

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$$v(t,s;p) \ := \ \inf\left\{x \in \mathbb{R}_+ \ : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,x,s}(T) \geq g(S_{t,s}(T))\right] \geq p\right\} \ .$$

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$$0 = \sup_{\pi\sigma s = \sigma s v_s + \alpha v_p}\left(\pi\mu s - \mu s v_s - \frac{1}{2}\sigma^2 s^2 v_{ss} - \alpha\sigma s v_{sp} - \alpha^2 v_{pp}\right)$$

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with the controls

$$\hat{\pi} := v_s + \frac{\hat{\alpha}}{s\sigma}v_p \quad \text{and} \quad \hat{\alpha} := \frac{\frac{\mu}{\sigma}v_p - \sigma s v_{sp}}{v_{pp}} .$$

Verification in the quantile hedging problem

- **Associated PDE:** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$

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Verification in the quantile hedging problem

- **Associated PDE (bis):** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$
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c- Feynman-Kac:

$$u(t,s,q) = \mathbb{E}_t^{\mathbb{Q}}\left[\left(Q_{t,q}(T) - g\left(S_{t,s}(T)\right)\right)^+\right] \quad \text{where} \quad \frac{dQ(r)}{Q(r)} = (\mu/\sigma)dW_r^{\mathbb{Q}}$$

Extensions

- **On the Dynamics:**

$$
\begin{aligned}
S^{\pi} &= s + \int_t^{\cdot} \mu\left(S^{\pi}(u), \pi_u\right) du + \int_t^{\cdot} \sigma\left(S^{\pi}(u), \pi_u\right) dW_u \\
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- **Applications**

$$\ell(x, s) = \mathbf{1}\{x \geq g(s)\} \quad \Rightarrow \quad \text{Quantile Hedging}$$

$$\ell(x, s) = U([x - g(s)]^+) \text{ with } U \nearrow \text{ concave} \quad \Rightarrow \quad \text{Loss function}$$

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- **Dynamic programming based on the reformulation**

$$
v(t,s;p) = \text{inf}\left\{x \in \mathbb{R}_+ : \ \exists\,(\pi,\alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq P^\alpha_{t,p}(T)\right\}
$$

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$\Rightarrow$ **Stochastic Target problems (with unbounded controls)**

Optimal Control with Stochastic Target Constraints

General framework

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- **Problems:** Given $F, \bar\ell$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

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V(t,s,x) := \sup_{\pi \in \bar{\mathcal{A}}^{\bar\ell}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]
$$

where

$$
\bar{\mathcal{A}}^{\bar\ell}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\} .
$$

Example 1: Moment constraints

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

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- **Reformulation:** We have

$$\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \exists\, \alpha \in L^2 \text{ s.t. } \ell\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right) \geq P^{\alpha}_{t,p}(T)\right\} ,$$

where $P^{\alpha}_{t,p}(r) := p + \int_t^r \alpha_u dW_u$.

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- Setting $\bar{\ell}(s,x,p) := \ell(s,x) - p$, we get

$$\text{then} \ \ V(t,s,x;p) \ := \ \sup_{(\pi,\alpha) \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x,p}} \mathbb{E}\left[F\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right] \ .$$

where $\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x,p} := \left\{(\pi,\alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \bar{\ell}\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T), P^{\alpha}_{t,p}(T)\right) \geq 0\right\}$

Example 2: Constraints in probability

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

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where $\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,x,s}(T) \geq g(S^{\pi}_{t,s}(T))\right] \geq p\right\}$,

for $\ell(s,x) := \mathbf{1}_{x \geq g(x)}$.

(see Boyle and Tian 07 for dual approach in complete market)

Example 3: Index tracking constraint

- $F(s,x) = U(x)$: utility function.
- $S^{\pi,1}$ an index. and X^π: wealth process.
- Portfolio optimization problem

$$V(t,s,x) \;\; := \sup_{\pi \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x}} \mathbb{E}\left[U\left(X^\pi_{t,x,s}(T)\right)\right]$$

where

$$\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } X^\pi_{t,x,s}(T)/x_0 \geq 90\% \times S^{\pi,1}_{t,s}(T)/s^1_0\right\} \; .$$

Here, $\bar{\ell}(s,x) := x/x_0 - 90\% \times s/s_0$.

Example 4: Mean variance

- **Problems:**

$$V(t,s,x;p) \;:=\; \inf_{\pi \in \mathcal{A}_{t,s,x,p}} \mathrm{Var}\left[X^{\pi}_{t,x,s}(T)\right]$$

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"is equivalent to"

$$V(t,s,x;p) \;:=\; \inf_{\pi\in\mathcal{A}_{t,s,x,p}} \mathbb{E}\left[\left(X^{\pi}_{t,x,s}(T)\right)^2\right] - p^2$$

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- Set $D := \{(t,s,x) : \ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$ and $v(t,s) := \inf\{x \in \mathbb{R} : \ (t,s,x) \in D\}$.

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\begin{aligned}
S^\pi &= s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u \\
X^\pi &= x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u
\end{aligned}
$$

- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$
V(t,s,x) := \sup_{\pi \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]
$$

where $\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar{\ell}\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\}$.

- Set $D := \{(t,s,x) : \ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$ and $v(t,s) := \inf\{x \in \mathbb{R} : \ (t,s,x) \in D\}$.

- If $\bar{\ell}$ is non-decreasing in x and v is smooth, then

$$
\mathrm{cl}(D) = \mathrm{int}_p D \cup \partial_p D \cup \partial_T D \quad \text{with} \quad
\begin{aligned}
\mathrm{int}_p D &:= \{t < T \ , \ x > v(t,s)\}, \\
\partial_p D &:= \{t < T \ , \ x = v(t,s)\} \ , \\
\partial_T D &:= \{t = T \ , \ x \geq v(T,s)\} \ .
\end{aligned}
$$

PDE Derivation

- **Dynamics:**

$$S^\pi = s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u$$
$$X^\pi = x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u$$

- **Problems:** Given $F, \bar\ell$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x) := \sup_{\pi \in \bar{\mathcal{A}}^{\bar\ell}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]$$

where $\bar{\mathcal{A}}^{\bar\ell}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\}$.

- Set $D := \{(t,s,x) :\ \bar{\mathcal{A}}^{\bar\ell}_{t,s,x} \neq \emptyset\}$ and $v(t,s) := \inf\{x \in \mathbb{R} :\ (t,s,x) \in D\}$.

- If $\bar\ell$ is non-decreasing in x, then

$$\mathrm{cl}(D) = \mathrm{int}_p D \cup \partial_p D \cup \partial_T D \quad \text{with} \quad \begin{array}{l} \mathrm{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}, \\ \partial_p D := \{t < T\ ,\ x \in [v_*(t,s), v^*(t,s)]\}\ , \\ \partial_T D := \{t = T\ ,\ x \geq v_*(T,s)\}\ . \end{array}$$

PDE in the domain $\text{int}_p D$

- Recall that

$$\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\} \quad \text{with} \quad v(t,s) := \inf\{x \in \mathbb{R} \ : \ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$$

- $x > v^*(t,s) \Rightarrow X^{\pi}_{t,x,s}(\tau) > v^*(\tau, S^{\pi}_{t,s}(\tau))$ for $\tau > t$ well chosen and $\pi \in \mathcal{A}$ given.

- Locally can choose any control !

- Associated PDE

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x)\right) = 0$$

On the boundary $\partial_T D$

- Recall that

$$\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\} \quad \text{with} \quad v(t,s) := \inf\{x \in \mathbb{R} \ : \ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$$

- We have the natural boundary condition: $V(T-,s,x) = F(s,x)$.

PDE on the spacial boundary $\partial_p D$

- Recall that

$$\partial_p D := \{t < T \;,\; x \in [v_*(t,s), v^*(t,s)]\} \text{ with } v(t,s) := \inf\{x \in \mathbb{R} : \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$$

- Assume v is smooth.

If $x = v(t,s)$, we should have $dX^{\pi}_{t,x,s}(t) \geq dv(t, S^{\pi}_{t,s}(t))$.

This implies that

$$\begin{aligned}\pi_t \in \mathcal{N}(t,s,x,v) \;&:=\; \{\pi \in A \;:\; \beta(s,x,\pi) = \sigma(s,\pi)Dv(t,s)\;, \\ &\qquad \rho(s,x,\pi) - \mathcal{L}^{\pi}_{S} v(t,s) \geq 0\}\;.\end{aligned}$$

- PDE on $\partial_p D$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x)\right) = 0\;.$$

PDE formulation: sum up

- On $\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

PDE formulation: sum up

- On $\text{int}_p D := \{t < T \;,\; x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \; .$$

- On $\partial_T D := \{t = T \;,\; x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x)\right) = 0 \ .$$

- On $\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T \ , \ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x)\right) = 0 \ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A \ : \ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s) \ , \ \rho(s,x,\pi) - \mathcal{L}^{\pi}_{S} v(t,s) \geq 0\}$.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

- On $\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T \ , \ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

with $\mathcal{N}(t,s,x,v) :=$
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- Already proved:

On $\text{int}_p D$ after relaxing the operator (A may be unbounded).

PDE formulation: sum up

- On $\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

- On $\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T \ , \ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A \ : \ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s) \ , \ \rho(s,x,\pi) - \mathcal{L}^{\pi}_S v(t,s) \geq 0\}$.

- Already proved:

On $\partial_T D$ after relaxing the operator (A may be unbounded).

PDE formulation: sum up

- On $\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\}$:

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- On $\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T \ , \ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A \ : \ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s) \ , \ \rho(s,x,\pi) - \mathcal{L}^{\pi}_S v(t,s) \geq 0\}$.

- Already proved:

On $\partial_p D$ when v is continuous (need to express the constraint $\mathcal{N}$ in terms of test functions for v).

PDE formulation: sum up

- On $\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

- On $\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T \ , \ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A \ : \ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s) \ , \ \rho(s,x,\pi) - \mathcal{L}^{\pi}_S v(t,s) \geq 0\}$.

- Already proved:

On $\partial_p D$ when v is not continuous: the constraint does not appear in the subsolution property.

Remaining points to study

1. Comparison principle

2. Numerical schemes on PDE

3. Examples

4. Better understanding of what happens on the boundary $\partial_p D$