Optimal investment under relative performance concerns

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Joint work with N.Touzi

May 6th, 2009

- Classical portfolio optimization: maximization of one's utility with respect to one's personal wealth or consumption
- Economical literature: relative wealth concerns

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Aim: Try to derive a portfolio optimization theory with such relative wealth concerns.

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We will first assume that all agents are similar.

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Optimization criterion for agent i:

- exponential utility function with risk sensitivity parameter $\eta > 0$
- relative performance sensitivity parameter $\lambda \in [0, 1]$
- average wealth of other agents $\bar{X}^i = \frac{1}{N-1} \sum_{j \neq i} X^j$

Thus agent i wants to maximize upon admissible π^i:

$$-\mathbb{E}e^{-\eta[(1-\lambda)X_T^i+\lambda(X_T^i-\bar{X}_T^i)]}$$

given other π^j $(j \neq i)$

By symmetry, at the equilibrium, it is the same as:

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So the optimal portfolio is (for deterministic θ, $\lambda < 1$):

$$\hat{\pi}_t^i = \frac{1}{\eta(1-\lambda)}\sigma_t^{-1}\theta_t$$

- $|\hat{\pi}^i|$ is increasing in λ
- if $\lambda \to 1$, $|\hat{\pi}^i| \to \infty$ a.s.

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At the equilibrium, its dynamics is given by:

$$d\bar{X}_t = \frac{1}{\eta(1-\lambda)} [|\theta_t|^2 dt + \theta_t . dW_t]$$

Specific parameters:

- risk sensitivity parameter $\eta_i > 0$
- relative performance sensitivity parameter $\lambda_i \in [0, 1]$

Thus agent i wants to maximize upon admissible π^i:

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Portfolio constraints:

Each agent has an area of investment. π^i must stay in a certain A_i that will be assumed to be a vector sub-space of $\mathbb{R}^d$.

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So finally we are looking for:

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And then look for Nash equilibria between the N agents.

Using a result by Hu-Imkeller-Muller for optimal investment in incomplete markets, we can relate the single agent optimization problem with the following (quadratic) BSDE:

$$dY_t^i = (\frac{|\theta_t|^2}{2\eta} - \frac{\eta}{2}|Z_t^i + \frac{\theta_t}{\eta} - P_{\sigma_t A_i}(Z_t^i + \frac{\theta_t}{\eta})|^2)dt + Z_t^i.dB_t$$

$$Y_T^i = \lambda(\bar{X}_T^i - \bar{x}_i) = \frac{\lambda}{N-1}\sum_{j \neq i}\int_0^T \pi_u^j.\sigma_u dB_u$$

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Remark: there is no need for S to be a Markov process.

Putting them together it brings:

$$Y_0^i = -\frac{1}{\eta} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{\eta}{2} \int_0^T |Q_t^i(Z_t^i)|^2 dt - \int_0^T (Z_t^i - \frac{\lambda}{N-1} \sum_{j \neq i} P_t^j(Z_t^j)).dB_t$$

where P_i is the orthogonal projection on σA_i and $Q_i = I - P_i$, $\mathbb{Q}$ is the martingale probability and B a Brownian motion under $\mathbb{Q}$.

After showing the regularity of the operator (under some assumptions), it can be rewritten as:

$$Y_0^i = -\frac{1}{\eta} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{\eta}{2} \int_0^T |Q_t^i([\psi_t(\zeta_t)]^i)|^2 dt - \int_0^T \zeta_t^i . dB_t$$

where $Y \in \mathbb{R}^N$, $\zeta \in M_{N,d}(\mathbb{R})$ and $\psi \in GL(M_{N,d}(\mathbb{R}))$.

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$\rightarrow$ N-dimensional system of coupled quadratic BSDEs.

Assume the following:

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Theorem: There exists a unique equilibrium and an optimal portfolio for agent i is given by:

$$\pi^i = \frac{1}{\eta}\sigma^{-1} P_i([I - \frac{\frac{\lambda}{N-1}}{1 + \frac{\lambda}{N-1}}(\sum_{j \neq i} P_j)(I + \frac{\lambda}{N-1} P_i)]^{-1}\theta)$$

(P_i is the orthogonal projection on σA_i)

In the simple case where d is fixed we have:

Theorem: Let d be fixed, and assume moreover that $\frac{1}{N}\sum_{i=1}^{N} P_i \to U$ in $\mathcal{L}(\mathbb{R}^d)$ with $||\lambda U|| < 1$. Then $\pi^i_N \to \pi^i_\infty$ uniformly where:

$$\pi^i_\infty = \frac{1}{\eta}\sigma^{-1} P_i[(I - \lambda U)^{-1}\theta]$$

Once again the market index is: $\bar{X}_t^N = \frac{1}{N}\sum_{i=1}^N X_t^i$
And we find:

$$d\bar{X}_t^\infty = \frac{1}{\eta}U(I - \lambda U)^{-1}\theta_t.[\theta_t dt + dW_t]$$

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Moreover, $U(I-\lambda U)^{-1}$ is diagonalizable with eigenvalues

$$0 < \frac{\mu_1}{1-\lambda\mu_1} < ... < \frac{\mu_d}{1-\lambda\mu_d} < 1$$

and with the same orthonormal eigenvectors as U (independent of λ).

$\longrightarrow$ The risk (volatility) of the market increases with λ.

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Under the assumption $\prod_{j=1}^{N} \lambda_j < 1$, there is a unique equilibrium.

First case: $\forall i,\ \lambda_i = \lambda$, then:

$$\hat{\pi}_t^i = [\frac{N-1}{N+\lambda-1} + \frac{\lambda N}{(1-\lambda)(N+\lambda-1)}\frac{\eta_i}{\eta^N}]\pi_t^{0,i}$$

η^N is the harmonic average of the η^j.

As $N \to \infty$, if $\eta^N \to \eta > 0$ then the equilibrium portfolio of agent i converges uniformly to:

$$\hat{\pi}_t^{\infty,i} = (1 + \frac{\lambda}{1-\lambda}\frac{\eta_i}{\eta})\pi_t^{0,i}$$

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Same conclusions as in the beginning.

Second case: $\forall j \neq i_0,\ \lambda_j = 1,\ \lambda_{i_0} < 1\ (\forall i,\ \eta_i = \eta)$, then:

$$\hat{\pi}_t^{i_0} = \left[\frac{1}{1-\lambda_{i_0}} + \frac{\lambda_{i_0}(N-1)}{1-\lambda_{i_0}} \right] \pi_t^0$$

Second case: $\forall j \neq i_0,\ \lambda_j = 1,\ \lambda_{i_0} < 1$ ($\forall i,\ \eta_i = \eta$), then:

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$\to$ Impact of surrounding "stupidity".

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- we also assume $\forall i,\ \theta_i = \theta$.

As $N \to \infty$ we find:

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So:

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- the more correlated the assets are (ρ^2 close to 1)

the more risk you take.

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For independent investments ($\rho = 0$), we find the classical optimal portfolio: no impact of λ.

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- $\sigma = \sigma I$ and $\forall i,\ \theta_i = \theta$.

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Same kind of conclusions as for investment on the whole market, but smaller impact of λ, especially for small N.

Short Bibliography

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Special thanks to J.Lebuchoux - Reech Aim